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Research article

Non-separated inclusion problem via generalized Hilfer and Caputo operators

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Abstract: We aimed to analyze a new class of sequential fractional differential inclusions that involves a combination of ς-Hilfer and ς-Caputo fractional derivative operators, along with non-separated boundary conditions. Two cases of convex-valued and non-convex-valued set-valued maps are considered. Our outcomes are obtained from some famous theorems of fixed point method within the framework of the set-valued analysis. Additionally, some examples are provided to illustrate the applicability of our outcomes.

Keywords: sequential inclusions problem; ς-Hilfer and ς-Caputo fractional derivatives; existence; nonlinear analysis
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1. Introduction

Fractional differential inclusions (FDIs), as an extension of fractional differential equations (FDEs), have gained popularity among mathematical researchers due to their importance and value in optimization and stochastic processes in economics [1, 2] and finance [3]. In addition to their applications in understanding engineering [4] and dynamic systems [5, 6] in biological [7, 8], medical [9], physics [10] and chemical sciences [11], FDIs are also relevant in various other scientific fields [12].

Sousa and Oliveira [13] introduced the new fractional derivative ς-Hilfer to unify different types of fractional derivatives into a single operator, expanding fractional derivatives to the types of operators with potentially applicable value. After that, Asawasamrit et al. [14] investigated the following Hilfer-FDE under nonlocal integral boundary conditions (BCs):

$$\begin{cases} {}^{\mathrm{H}}\mathfrak{D}^{r_1,r_2}\varkappa(\upsilon) = \beth(\upsilon,\varkappa(\upsilon)), & 1 < r_1 < 2,\ 0 \le r_2 \le 1,\ \upsilon \in \mho := [\tilde{s},\tilde{b}], \\ \varkappa(\tilde{s}) = 0,\ \varkappa(\tilde{b}) = \sum_{i=1}^{m} \delta_i\ {}^{\mathrm{RL}}\mathfrak{I}^{\varsigma_i}\varkappa(\tilde{\xi}_i), & \varphi_i > 0,\ \delta_i \in \mathbb{R},\ \tilde{\xi}_i \in \mho. \end{cases} \tag{1.1}$$

In [15], an existence outcome was shown by employing the FPTs (fixed-point theorems) for a sequential FDE of the type,

$$\begin{cases} \left({}^{\mathrm{H}}\mathfrak{D}^{r_1,r_2,\varsigma}\left({}^{\mathrm{C}}\mathfrak{D}^{r_3,\varsigma}\varkappa\right)\right)(\mathfrak{s}) = \beth\left(\mathfrak{s},\varkappa(\mathfrak{s}),{}^{\mathrm{RL}}\mathfrak{I}^{r_5,\varsigma}\varkappa(\upsilon),\int_0^{\tilde{b}}\varkappa(\nu)\,\mathrm{d}\nu\right), \quad \upsilon \in \mathfrak{B} := [0,\tilde{b}], \\ \varkappa(0) + \eta_1\varkappa(\tilde{b}) = 0, \\ {}^{\mathrm{C}}\mathfrak{D}^{r_4+r_3-1,\varsigma}\varkappa(0) + \eta_2\,{}^{\mathrm{C}}\mathfrak{D}^{\delta+r_3-1,\varsigma}\varkappa(\tilde{b}) = 0, \end{cases}$$

where $r_i \in (0,1)$, $i = 1,2,3$, $r_4 = r_1 + r_2(1-r_1)$, $r_4 + r_3 > 1$, $\eta_1,\eta_2 \in \mathbb{R}$, $r_5 > 0$, and $\beth \in C(\mathfrak{B}\times\mathbb{R}^3)$ is a nonlinear function. In 2024, Ahmed et al. [16] investigated a class of separated boundary value problems of the form

$$\begin{cases} \left({}^{\mathrm{H}}\mathfrak{D}_q^{r_1,r_2}\left({}^{\mathrm{C}}\mathfrak{D}_q^{r_3}\varkappa\right)\right)(\upsilon) = \beth(\upsilon,\varkappa(\upsilon)), & q \in (0,1),\ \upsilon \in \mathfrak{B}, \\ \varkappa(0) + \lambda_1\,{}^{\mathrm{C}}\mathfrak{D}_q^{r_3+r_4-1}\varkappa(0) = \varkappa(\tilde{b}) + \lambda_2\,{}^{\mathrm{C}}\mathfrak{D}_q^{r_3+r_4-1}\varkappa(\tilde{b}) = 0, & \lambda_1,\lambda_2 \in \mathbb{R}, \end{cases}$$

where $0 < r_1, r_3 < 1$, $r_2 \in [0,1]$ with $r_4 = r_1 + r_2(1-r_1)$, $r_3 + r_4 > 1$ and $\beth \in C(\mathfrak{B}\times\mathbb{R})$. Lachouri et al., in [17], established the existence of solutions to the nonlinear neutral FDI involving ς-Caputo fractional derivative with ς-Riemann–Liouville (RL) fractional integral boundary conditions:

$$\begin{cases} {}^{\mathrm{C}}\mathfrak{D}^{r_1,\varsigma}\left({}^{\mathrm{C}}\mathfrak{D}^{r_2,\varsigma}\varkappa(\upsilon) - \mathrm{y}(\upsilon,\varkappa(\upsilon))\right) \in \beth(\upsilon,\varkappa(\upsilon)), & \upsilon \in [0,\tilde{b}), \\ \varkappa(a) = {}^{\mathrm{RL}}\mathfrak{I}^{r_3,\varsigma}\varkappa(\tilde{b}) = 0, \qquad a \in (0,\tilde{b}), \end{cases}$$

where $\beth : \mathfrak{B}\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map. Surang et al., in [18], studied the ς-Hilfer type sequential FDEs and FDIs subject to integral multi-point BCs of the form

$$\begin{cases} \left({}^{\mathrm{H}}\mathfrak{D}^{r_1,r_2,\varsigma} + k\ {}^{\mathrm{H}}\mathfrak{D}^{r_1-1,r_2,\varsigma}\right)\varkappa(\upsilon) = \beth(\upsilon,\varkappa(\upsilon)), & \upsilon \in \mho, \\ \varkappa(\tilde{s}) = 0,\ \varkappa(\tilde{b}) = \sum_{i=1}^{n}\mu_i\int_{\tilde{s}}^{\eta_i}\psi'(s)\varkappa(s)\,\mathrm{d}s + \sum_{j=1}^{m}\theta_j\varkappa\left(\xi_j\right), \end{cases} \tag{1.2}$$

where $r_1 \in (1,2)$, $r_2 \in [0,1]$, $\beth \in C(\mho\times\mathbb{R})$, $k,\mu_i,\theta_j \in \mathbb{R}$, and $\eta_i,\xi_j \in (\tilde{s},\tilde{b}]$, $i = \overline{1,n}$, $j = \overline{1,m}$. Etemad et al. [19] introduced and studied a novel existence technique based on some special set-valued maps (SVMs) to guarantee the existence of a solution for the following fractional jerk inclusion problem involving the derivative operator in the sense of Caputo–Hadamard

$$\begin{cases} \left({}^{\mathrm{CH}}\mathfrak{D}_{1^+}^{r_1}\left({}^{\mathrm{CH}}\mathfrak{D}_{1^+}^{r_2}\left({}^{\mathrm{CH}}\mathfrak{D}_{1^+}^{r_3}\varkappa\right)\right)\right)(\upsilon) \in \beth\left(\upsilon,\varkappa(\upsilon),{}^{\mathrm{CH}}\mathfrak{D}_{1^+}^{r_3}\varkappa(\upsilon),\left({}^{\mathrm{CH}}\mathfrak{D}_{1^+}^{r_2}\left({}^{\mathrm{CH}}\mathfrak{D}_{1^+}^{r_3}\varkappa\right)\right)(\upsilon)\right), \\ \varkappa(1) + \varkappa(\exp(1)) = {}^{\mathrm{CH}}\mathfrak{D}_{1^+}^{r_3}\varkappa(\eta) = \left({}^{\mathrm{CH}}\mathfrak{D}_{1^+}^{r_2}\left({}^{\mathrm{CH}}\mathfrak{D}_{1^+}^{r_3}\varkappa\right)\right)(\exp(1)) = 0, \end{cases}$$

for $\upsilon \in [1, \exp(1)]$, where $r_i \in (0,1]$, $i = 1,2,3$, $\eta \in (1, \exp(1))$, and the operator $\beth : [1, \exp(1)] \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$ is an SVM, where $\mathcal{P}(\mathbb{R})$ denotes all nonempty subsets of $\mathbb{R}$.

The boundary conditions (BCs) used in (1.1) and (1.2), share a common feature: the requirement of a zero initial condition, which is essential for the solution to be well-defined. Consequently, certain classes of Hilfer FDEs cannot be addressed, including cases with BCs such as,

- $\varkappa(0) = -\varkappa(\tilde{b})$, $\varkappa'(0) = -\varkappa'(\tilde{b})$ (anti-periodic),
- $\varkappa(0) + \eta_1 \varkappa'(0) = 0$, $\varkappa(\tilde{b}) + \eta_2 \varkappa'(\tilde{b}) = 0$ (separated),
- $\varkappa(0) + \eta_1 \varkappa(\tilde{b}) = 0$, $\varkappa'(0) + \eta_2 \varkappa'(\tilde{b}) = 0$ (non-separated), etc.

To address this limitation and study Hilfer FDEs with such BCs, regardless of whether they are anti-periodic, separated, or non-separated, we propose a novel approach in this research. Specifically, we combine the Hilfer and Caputo fractional derivatives, enabling the study of boundary value problems under these conditions. More specifically, we aimed to analyze a class of FDEs for FDI, subject to non-separated BCs of the form,

$$\begin{cases} {}^{\mathrm{H}}\mathfrak{D}^{r_1,r_2,\varsigma}\left({}^{\mathrm{C}}\mathfrak{D}^{r_3,\varsigma}\varkappa(\upsilon) - \mathrm{y}(\upsilon,\varkappa(\upsilon))\right) \in \beth(\upsilon,\varkappa(\upsilon)), \quad \upsilon \in \mathfrak{B}, \\ \varkappa(0) + \eta_1 \varkappa(\tilde{b}) = 0, \\ {}^{\mathrm{C}}\mathfrak{D}^{\delta+r_3-1,\varsigma}\varkappa(0) + \eta_2\, {}^{\mathrm{C}}\mathfrak{D}^{\delta+r_3-1,\varsigma}\varkappa(\tilde{b}) = 0, \end{cases} \tag{1.3}$$

where $r_i \in (0,1)$, $i = 1,2,3$, $\delta = r_1 + r_2(1 - r_1)$, $\delta + r_3 > 1$, $\eta_1, \eta_2 \in \mathbb{R}$, $\mathrm{y} \in C(\mathfrak{B} \times \mathbb{R})$ and $\beth : \mathfrak{B} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ denotes a SVM, with power set $\mathcal{P}(\mathbb{R})$ of $\mathbb{R}$.

The paper is structured as follows. Section 2 is devoted to discussing the fundamental concepts fractional calculus and set-valued analysis, while Section 3 presents important findings on the qualitative properties of solutions to the ς-Hilfer inclusion FDI (1.3) utilizing FPTs. Finally, Section 4, includes three illustrative examples.

2. Background concepts

2.1. Fractional calculus

We outline the background material that is pertinent to our study. We consider the Banach spaces $\mathbb{E} = C(\mathfrak{B})$ and $L^1(\mathfrak{B})$ of the Lebesgue integrable functions equipped with the norms $\|\varkappa\| = \sup\{|\varkappa(\upsilon)| : \upsilon \in \mathfrak{B}\}$ and

$$\|\varkappa\|_{L^1} = \int_{\mathfrak{B}} |\varkappa(\upsilon)|\, \mathrm{d}\upsilon,$$

respectively. Let $\varsigma \in C^n(\mathfrak{B})$ be an increasing function such that $\varsigma'(\upsilon) \neq 0$, for any $\upsilon \in \mathfrak{B}$.

Definition 2.1 ([20])**.** *The ς-RL fractional integral and derivative of order r_1 for a given function $\varkappa$ are expressed by*

$${}^{\mathrm{RL}}\mathfrak{I}^{r_1,\varsigma}\varkappa(\upsilon) = \int_0^{\upsilon} \frac{\varsigma'(u)}{\Gamma(r_1)} (\varsigma_u(\upsilon))^{r_1-1} \varkappa(u)\, \mathrm{d}u, \qquad \varsigma_u(\upsilon) := \varsigma(\upsilon) - \varsigma(u),$$

and

$${}^{\mathrm{RL}}\mathfrak{D}^{r_1,\varsigma}\varkappa(\upsilon) = D_{\varsigma}^{[n]}\, {}^{\mathrm{RL}}\mathfrak{I}^{n-r_1,\varsigma}\varkappa(\upsilon), \qquad D_{\varsigma}^{[n]} := \left(\frac{1}{\varsigma'(\upsilon)}\frac{\mathrm{d}}{\mathrm{d}\upsilon}\right)^n,$$

where $n = [r_1] + 1$, $n \in \mathbb{N}$, respectively.

Definition 2.2 ([21, 22]). *The Caputo sense of ς-fractional derivative of the $\varkappa \in C^n(\mathfrak{B})$ of order r_1 is given as,*

$$^{\mathrm{C}}\mathfrak{D}^{r_1,\varsigma}\varkappa(\mathfrak{s}) = {}^{\mathrm{RL}}\mathfrak{J}^{(n-r_1),\varsigma}\varkappa^{[n]}(\mathfrak{s}), \qquad \varkappa^{[n]}(\mathfrak{s}) = \left(\frac{1}{\varsigma'(\mathfrak{s})}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{s}}\right)^n \varkappa(\mathfrak{s}).$$

Lemma 2.3 ([20, 22]). *Let $r_1, r_2 > 0$. Then*

$$\begin{aligned} &\text{i)} \quad {}^{\mathrm{RL}}\mathfrak{J}^{r_1,\varsigma}(\varsigma_0(\upsilon))^{r_2-1} = \frac{\Gamma(r_2)}{\Gamma(r_1+r_2)}(\varsigma_0(\upsilon))^{r_1+r_2-1}, \\ &\text{ii)} \quad {}^{\mathrm{C}}\mathfrak{D}^{r_1,\varsigma}(\varsigma_0(\upsilon))^{r_2-1} = \frac{\Gamma(r_2)}{\Gamma(r_2-r_1)}(\varsigma_0(\upsilon))^{r_1+r_2-1}. \end{aligned}$$

Lemma 2.4 ([20]). *For $\varkappa \in C^n(\mathfrak{B})$, we have*

$$^{\mathrm{RL}}\mathfrak{J}^{r_1,\varsigma}\,{}^{\mathrm{C}}\mathfrak{D}^{r_1,\varsigma}\varkappa(\mathfrak{s}) = \varkappa(\mathfrak{s}) - \sum_{k=0}^{n-1}\frac{\varkappa^{[n]}(0^+)}{k!}(\varsigma_0(\mathfrak{s}))^k, \qquad n-1 < r_1 < n,$$

and $0 < r_2 < 1$. Furthermore, if $r_1 \in (0,1)$, then ${}^{\mathrm{RL}}\mathfrak{J}^{r_1,\varsigma}\,{}^{\mathrm{C}}\mathfrak{D}^{r_1,\varsigma}\varkappa(\upsilon) = \varsigma_0(\upsilon)$.

Definition 2.5 ([13]). *The ς-Hilfer fractional derivative for $\varkappa \in C^n(\mathfrak{B})$, of order $n-1 < r_1 < n$ and type $0 \le r_2 \le 1$, is defined by*

$$^{\mathrm{H}}\mathfrak{D}^{r_1,r_2,\varsigma}\varkappa(\upsilon) = \left({}^{\mathrm{RL}}\mathfrak{J}^{r_2(n-r_1),\varsigma}\left(D_{\varsigma}^{[n]}\left({}^{\mathrm{RL}}\mathfrak{J}^{(1-r_2)(n-r_1),\varsigma}\varkappa\right)\right)\right)(\upsilon).$$

Lemma 2.6 ([13, 23]). *Let $r_1, r_2, \mu > 0$. Then*

$$\begin{aligned} &\text{i)} \quad {}^{\mathrm{RL}}\mathfrak{J}^{r_1,\varsigma}\,{}^{\mathrm{RL}}\mathfrak{J}^{r_2,\varsigma}\varkappa(\upsilon) = {}^{\mathrm{RL}}\mathfrak{J}^{r_1+r_2,\varsigma}\varkappa(\upsilon), \\ &\text{ii)} \quad {}^{\mathrm{RL}}\mathfrak{J}^{r_1,\varsigma}(\varsigma_0(\upsilon))^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(r_1+\mu)}(\varsigma_0(\upsilon))^{r_1+\mu-1}. \end{aligned}$$

Lemma 2.7 ([13]). *For $\mu > 0$, $r_1 \in (n-1, n)$, and $0 \le r_2 \le 1$,*

$$^{\mathrm{H}}\mathfrak{D}^{r_1,r_2,\varsigma}(\varsigma_0(\upsilon))^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu-r_1)}(\varsigma_0(\upsilon))^{\mu-r_1-1}, \qquad \mu > n.$$

In particular, if $r_1 \in (1,2)$ and $1 < \mu \le 2$, then ${}^{\mathrm{H}}\mathfrak{D}^{r_1,r_2,\varsigma}(\varsigma_0(\upsilon))^{\mu-1} = 0$.

Lemma 2.8 ([13]). *If $\varkappa \in C^n(\mathfrak{B})$, $n-1 < r_1 < n$ and type $0 < r_2 < 1$, then*

$$\begin{aligned} &\text{i)} \quad {}^{\mathrm{RL}}\mathfrak{J}^{r_1,\varsigma}\,{}^{\mathrm{H}}\mathfrak{D}^{r_1,r_2,\varsigma}\varkappa(\upsilon) = \varkappa(\upsilon) - \sum_{k=1}^{n}\frac{(\varsigma_0(\upsilon))^{\delta-k}}{\Gamma(\delta-k+1)}D_{\varsigma}^{[n-k]}\,{}^{\mathrm{RL}}\mathfrak{J}^{(1-r_2)(n-r_1),\varsigma}\varkappa(0), \\ &\text{ii)} \quad {}^{\mathrm{H}}\mathfrak{D}^{r_1,r_2,\varsigma}\,{}^{\mathrm{RL}}\mathfrak{J}^{r_1,\varsigma}\varkappa(\upsilon) = \varkappa(\upsilon). \end{aligned}$$

2.2. *Auxiliary notions of set-valued maps*

Consider the Banach space $(\mathbb{E}, \|\cdot\|)$ and SVM $\Theta : \mathbb{E} \to \mathcal{P}(\mathbb{E})$. Θ is a) closed (convex), b) bounded and c) measurable, whenever $\Theta(\varkappa)$ is closed (convex) for every $\varkappa \in \mathbb{E}$, $\Theta(B) = \cup_{\varkappa\in B}\Theta(\varkappa)$ is bounded for any bounded set $B \subseteq \mathbb{E}$, that is

$$\sup_{\varkappa\in B}\left\{\sup|\rho| \,:\, \rho \in \Theta(\varkappa)\right\} < \infty,$$

and $\forall \rho \in \mathbb{R}$, the function

$$\upsilon \to d(\rho, \Theta(\upsilon)) = \inf\left\{|\rho - \lambda| \,:\, \lambda \in \Theta(\upsilon)\right\},$$

is measurable, respectively. One can find the definitions of completely continuous and upper semi-continuous in [24]. Additionally, the set of selections of $\beth$ is described as

$$\mathcal{R}_{\beth,\rho} = \left\{\sigma \in L^1(\mathfrak{B}) \,:\, \sigma(\upsilon) \in \beth(\upsilon,\rho),\, \forall \upsilon \in \mathfrak{B}\right\}.$$

Next, we take

$$\mathcal{P}_\beta(\mathbb{E}) = \left\{\Omega \in \mathcal{P}(\mathbb{E}) \,:\, \Omega \neq \varnothing \text{ with has a property } \beta\right\},$$

where $\mathcal{P}_{\rm cl}$, $\mathcal{P}_{\rm c}$, $\mathcal{P}_{\rm b}$, and $\mathcal{P}_{\rm cp}$ represent the classes of every compact, bounded, closed, and convex subset of $\mathbb{E}$, respectively.

Definition 2.9 ([25])**.** *An SVM $\beth : \mathfrak{B} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called Carathéodory if the mapping $\upsilon \to \beth(\upsilon, \varkappa)$ is measurable for all $\varkappa \in \mathbb{R}$, and $\varkappa \to \beth(\upsilon, \varkappa)$ is upper semicontinuous for almost every $\upsilon \in \mathfrak{B}$. Additionally, we say $\beth$ is L^1-Carathéodory whenever for all $m > 0$, exists $z \in L^1(\mathfrak{B}, \mathbb{R}^+)$ such that for a.e. $\upsilon \in \mathfrak{B}$,*

$$\|\beth(\upsilon, \varkappa)\| = \sup\left\{|\sigma| \,:\, \sigma \in \beth(\upsilon, \varkappa)\right\} \leq z(\upsilon), \qquad \forall\ \|z\| \leq m.$$

To achieve the intended outcomes in this search, the following lemmas are necessary.

Lemma 2.10 ([25], Proposition 1.2)**.** *Consider SVM $\Theta : \mathbb{E} \to \mathcal{P}_{\rm cl}(Z)$ with the graph,* $\mathrm{Gr}(\Theta) = \{(\varkappa, \rho) \in \mathbb{E} \times Z \,:\, \rho \in \Theta(\varkappa)\}$. $\mathrm{Gr}(\Theta)$ *is a closed subset of $\mathbb{E} \times Z$ whenever Θ is upper semi-continuous. Conversely, Θ is upper semi-continuous, when it has a closed graph and is completely continuous.*

Lemma 2.11 ([26])**.** *Consider a separable Banach space $\mathbb{E}$ along with a L^1-Carathéodory SVM $\beth : \mathfrak{B} \times \mathbb{R} \to \mathcal{P}_{\rm cp,c}(\mathbb{E})$ and a linear continuous map $\Upsilon : L^1(\mathfrak{B}, \mathbb{E}) \to C(\mathfrak{B}, \mathbb{E})$. Then, the composition*

$$\begin{cases} \Upsilon \circ \mathcal{R}_\beth : C(\mathfrak{B}, \mathbb{E}) \to \mathcal{P}_{\rm cp,c}(C(\mathfrak{B}, \mathbb{E})), \\ \varkappa \to (\Upsilon \circ \mathcal{R}_\beth)(\varkappa) = \Upsilon(\mathcal{R}_{\beth,\varkappa}), \end{cases}$$

is a closed graph map in $C(\mathfrak{B}, \mathbb{E}) \times C(\mathfrak{B}, \mathbb{E})$.

3. Existence results for fractional differential inclusion (1.3)

In relation to the FDI (1.3), the auxiliary Lemma 3.1 is required.

Lemma 3.1. *For $\mathrm{y}, \beth \in C(\mathfrak{B})$, the solution of linear-type problem*

$$\begin{cases} {}^{\rm H}\mathfrak{D}^{r_1,r_2,\varsigma}\left({}^{\rm C}\mathfrak{D}^{r_3,\varsigma}\varkappa(\upsilon) - \mathrm{y}(\upsilon)\right) = \beth(\upsilon), \qquad \upsilon \in \mathfrak{B} \setminus \{\tilde{b}\}, \\ \varkappa(0) + \eta_1\varkappa(\tilde{b}) = 0, \\ {}^{\rm C}\mathfrak{D}^{\delta+r_3-1,\varsigma}\varkappa(0) + \eta_2\, {}^{\rm C}\mathfrak{D}^{\delta+r_3-1,\varsigma}\varkappa(\tilde{b}) = 0, \end{cases} \tag{3.1}$$

is obtained as follows:

$$\varkappa(\upsilon) = {}^{RL}\mathfrak{I}^{r_3,\varsigma}\mathrm{y}(\upsilon) + {}^{RL}\mathfrak{I}^{r_1+r_3,\varsigma}\daleth(\upsilon)$$
$$+ \left[\Lambda_3\left(\varsigma_0(\tilde{b})\right)^{r_3+\delta-1} - \Lambda_1(\varsigma_0(\upsilon))^{r_3+\delta-1}\right]\left({}^{RL}\mathfrak{I}^{1-\delta,\varsigma}\mathrm{y}(\tilde{b}) + {}^{RL}\mathfrak{I}^{r_1-\delta+1,\varsigma}\daleth(\tilde{b})\right)$$
$$- \Lambda_2\left({}^{RL}\mathfrak{I}^{r_3,\varsigma}\mathrm{y}(\tilde{b}) + {}^{RL}\mathfrak{I}^{r_1+r_3,\varsigma}\daleth(\tilde{b})\right), \tag{3.2}$$

where for $\eta_1, \eta_2 \neq -1$,

$$\Lambda_1 = \frac{\eta_2}{(\eta_2+1)\Gamma(r_3+\delta)}, \quad \Lambda_2 = \frac{\eta_1}{\eta_1+1}, \quad \Lambda_3 = \frac{\eta_1\eta_2}{(\eta_2+1)\Gamma(r_3+\delta)}. \tag{3.3}$$

Proof. Applying the ς-fractional integral ${}^{RL}\mathfrak{I}^{r_1,\varsigma}$ to the first equation of (1.3), and using Lemma 2.4, we get

$${}^{C}\mathfrak{D}^{r_3,\varsigma}\varkappa(\upsilon) = \mathrm{y}(\upsilon) + {}^{RL}\mathfrak{I}^{r_1,\varsigma}\daleth(\upsilon) + c_1(\varsigma_0(\upsilon))^{\delta-1}, \qquad \upsilon \in \mathfrak{B},\ c_1 \in \mathbb{R}, \tag{3.4}$$

where $\delta = r_1 + r_2(1 - r_1)$. Now, by taking ${}^{RL}\mathfrak{I}^{r_3,\varsigma}$ in (3.4) from Lemma 2.3, we get

$$\varkappa(\upsilon) = {}^{RL}\mathfrak{I}^{r_3,\varsigma}\mathrm{y}(\upsilon) + {}^{RL}\mathfrak{I}^{r_1+r_3,\varsigma}\daleth(\upsilon) + \frac{c_1\Gamma(\delta)}{\Gamma(r_3+\delta)}(\varsigma_0(\upsilon))^{r_3+\delta-1} + c_2, \qquad c_2 \in \mathbb{R}. \tag{3.5}$$

According to Lemma 2.3, we can obtain

$${}^{C}\mathfrak{D}^{\delta+r_3-1,\varsigma}\varkappa(\upsilon) = {}^{RL}\mathfrak{I}^{1-\delta,\varsigma}\mathrm{y}(\upsilon) + {}^{RL}\mathfrak{I}^{r_1-\delta+1,\varsigma}\daleth(\upsilon) + c_1\Gamma(\delta). \tag{3.6}$$

Next, by combining the BCs $\varkappa(0) + \eta_1\varkappa(\tilde{b}) = 0$ and

$${}^{C}\mathfrak{D}^{\delta+r_3-1,\varsigma}\varkappa(0) + \eta_2\,{}^{C}\mathfrak{D}^{\delta+r_3-1,\varsigma}\varkappa(\tilde{b}) = 0$$

with (3.6), we get

$$c_2(1+\eta_1) + \eta_1\,{}^{RL}\mathfrak{I}^{r_3,\varsigma}\mathrm{y}(\tilde{b}) + \eta_1\,{}^{RL}\mathfrak{I}^{r_1+r_3,\varsigma}\daleth(\upsilon) + c_1\frac{\eta_1\Gamma(\delta)}{\Gamma(r_3+\delta)}\left(\varsigma_0(\tilde{b})\right)^{r_3+\delta-1} = 0, \tag{3.7}$$
$$c_1(1+\eta_2)\Gamma(\delta) + \eta_2\,{}^{RL}\mathfrak{I}^{1-\delta,\varsigma}\mathrm{y}(\tilde{b}) + \eta_2\,{}^{RL}\mathfrak{I}^{r_1-\delta+1,\varsigma}\daleth(\tilde{b}) = 0. \tag{3.8}$$

From (3.7) and (3.8), we find

$$c_1 = \frac{-\eta_2}{(1+\eta_2)\Gamma(\delta)}\left({}^{RL}\mathfrak{I}^{1-\delta,\varsigma}\mathrm{y}(\tilde{b}) + {}^{RL}\mathfrak{I}^{r_1-\delta+1,\varsigma}\daleth(\tilde{b})\right),$$
$$c_2 = \frac{\eta_1\eta_2}{(1+\eta_2)\Gamma(r_3+\delta)}(\varsigma_0(\upsilon))^{r_3+\delta-1}\left({}^{RL}\mathfrak{I}^{1-\delta,\varsigma}\mathrm{y}(\varsigma_0(\upsilon)) + {}^{RL}\mathfrak{I}^{r_1-\delta+1,\varsigma}\daleth(\tilde{b})\right)$$
$$- \frac{\eta_1}{(1+\eta_1)}\left({}^{RL}\mathfrak{I}^{r_3,\varsigma}\mathrm{y}(\tilde{b}) + {}^{RL}\mathfrak{I}^{r_1+r_3,\varsigma}\daleth(\upsilon)\right).$$

By substituting the values of c_1 and c_2 into (3.5), we arrive at the fractional integral equation (3.2). □

Definition 3.2. *An element* $\varkappa \in C^1(\mathfrak{B})$ *can be a solution of* (1.3)*, if there is* $\sigma \in L^1(\mathfrak{B})$ *with* $\sigma(\upsilon) \in \daleth(\upsilon, \varkappa)$ *for every* $\upsilon \in \mathfrak{B}$ *fulfilling the non-separated BC's,* $\varkappa(0) + \eta_1\varkappa(\tilde{b}) = 0$,

$${}^{C}\mathfrak{D}^{\delta+r_3-1,\zeta}\varkappa(0) + \eta_2\,{}^{C}\mathfrak{D}^{\delta+r_3-1,\zeta}\varkappa(\tilde{b}) = 0,$$

and

$$\varkappa(\upsilon) = {}^{RL}\mathfrak{I}^{r_3,\varsigma}\mathrm{y}(\upsilon, \varkappa(\upsilon)) + {}^{RL}\mathfrak{I}^{r_1+r_3,\varsigma}\sigma(\upsilon) + \left[\Lambda_3\left(\varsigma_0(\tilde{b})\right)^{r_3+\delta-1}\right.$$
$$\left. - \Lambda_1(\varsigma_0(\upsilon))^{r_3+\delta-1}\right]\left({}^{RL}\mathfrak{I}^{1-\delta,\varsigma}\mathrm{y}\left(\tilde{b}, \varkappa(\tilde{b})\right) + {}^{RL}\mathfrak{I}^{r_1-\delta+1,\varsigma}\sigma(\tilde{b})\right)$$
$$- \Lambda_2\left({}^{RL}\mathfrak{I}^{r_3,\varsigma}\mathrm{y}\left(\tilde{b}, \varkappa(\tilde{b})\right) + {}^{RL}\mathfrak{I}^{r_1+r_3,\varsigma}\sigma(\tilde{b})\right). \tag{3.9}$$

3.1. The upper semi-continuous case

The first consequence addresses the convex-valued $\gimel$ using the nonlinear alternative for contractive maps [27].

Theorem 3.3. *Suppose that*

P1) $\gimel : \mathfrak{B} \times \mathbb{R} \to \mathcal{P}_{\mathrm{cp,c}}(\mathbb{R})$ *is a* L^1*-Carathéodory SVM;*
P2) *There is a exist* $\widetilde{\varpi}_1 \in C(\mathfrak{B}, \mathbb{R}^+)$ *and a nondecreasing* $\widetilde{\varpi}_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ *with,*

$$\|\gimel(\upsilon, \varkappa)\|_{\mathcal{P}} = \sup\left\{ |\rho| \,:\, \rho \in \gimel(\upsilon, \varkappa) \right\} \le \widetilde{\varpi}_1(\upsilon)\, \widetilde{\varpi}_2(\|\varkappa\|), \quad \forall (\upsilon, \varkappa) \in \mathfrak{B} \times \mathbb{R};$$

P3) *There is a constant* $l_{\mathrm{y}} < \lambda_2^{-1}$ *such that* $|\mathrm{y}(\upsilon, \varkappa_1) - \mathrm{y}(\upsilon, \varkappa_2)| \le l_{\mathrm{y}} |\varkappa_1 - \varkappa_2|$*;*
P4) *There is a exist* $\vartheta_{\mathrm{y}} \in C(\mathfrak{B}, \mathbb{R}^+)$ *such that* $|\mathrm{y}(\upsilon, \varkappa)| \le \vartheta_{\mathrm{y}}(\upsilon)$*, for each* $(\upsilon, \varkappa) \in \mathfrak{B} \times \mathbb{R}$*;*
P5) *There is an* $\mathcal{N} > 0$ *satisfying*

$$\frac{\mathcal{N}}{\lambda_1 \|\widetilde{\varpi}_1\| \widetilde{\varpi}_2(\mathcal{N}) + \lambda_2 \left\|\vartheta_{\mathrm{y}}\right\|} > 1, \tag{3.10}$$

where

$$\begin{aligned} \lambda_1 &= \left(\varsigma_0(\tilde{b})\right)^{r_3 + r_1} \left[\tfrac{|\Lambda_3| + |\Lambda_1|}{\Gamma(r_1 - \delta + 2)} + \tfrac{1 + |\Lambda_2|}{\Gamma(r_1 + r_3 + 1)} \right], \\ \lambda_2 &= \left(\varsigma_0(\tilde{b})\right)^{r_3} \left[\tfrac{|\Lambda_3| + |\Lambda_1|}{\Gamma(2 - \delta)} + \tfrac{1 + |\Lambda_2|}{\Gamma(r_3 + 1)} \right]. \end{aligned} \tag{3.11}$$

Then, (1.3) *admits a solution of* $\mathfrak{B}$*.*

Proof. At first, to convert the sequential-type FDI (1.3) into a problem of the FP type, we write $\Theta : \mathbb{E} \to \mathcal{P}(\mathbb{E})$ as follows:

$$\Theta(\varkappa) = \left\{ \begin{array}{l} z \in C(\mathfrak{B}) : \\ z(\upsilon) = \left\{ \begin{array}{l} {}^{\mathrm{RL}}\mathfrak{J}^{r_3, \varsigma} \mathrm{y}(\upsilon, \varkappa(\upsilon)) + {}^{\mathrm{RL}}\mathfrak{J}^{r_1 + r_3, \varsigma} \sigma(\upsilon) \\ \quad + \left(\Lambda_3 \left(\varsigma_0(\tilde{b})\right)^{r_3 + \delta - 1} - \Lambda_1 (\varsigma_0(\upsilon))^{r_3 + \delta - 1} \right) \\ \quad \times \left({}^{\mathrm{RL}}\mathfrak{J}^{1 - \delta, \varsigma} \mathrm{y}\left(\tilde{b}, \varkappa(\tilde{b})\right) + {}^{\mathrm{RL}}\mathfrak{J}^{r_1 - \delta + 1, \varsigma} \sigma(\tilde{b}) \right) \\ \quad - \Lambda_2 \left({}^{\mathrm{RL}}\mathfrak{J}^{r_3, \varsigma} \mathrm{y}\left(\tilde{b}, \varkappa(\tilde{b})\right) + {}^{\mathrm{RL}}\mathfrak{J}^{r_1 + r_3, \varsigma} \sigma(\tilde{b}) \right) \end{array} \right. \end{array} \right\}, \tag{3.12}$$

for $\sigma \in \mathcal{R}_{\gimel, \varkappa}$. Consider two operators $\Psi_1 : \mathbb{E} \to \mathbb{E}$ and $\Psi_2 : \mathbb{E} \to \mathcal{P}(\mathbb{E})$ as follows:

$$\begin{aligned} \Psi_1 \varkappa(\upsilon) = &\left[\Lambda_3 \left(\varsigma_0(\tilde{b})\right)^{r_3 + \delta - 1} - \Lambda_1 (\varsigma_0(\upsilon))^{r_3 + \delta - 1} \right] {}^{\mathrm{RL}}\mathfrak{J}^{1 - \delta, \varsigma} \mathrm{y}\left(\tilde{b}, \varkappa(\tilde{b})\right) \\ &+ {}^{\mathrm{RL}}\mathfrak{J}^{r_3, \varsigma} \mathrm{y}(\upsilon, \varkappa(\upsilon)) - \Lambda_2 \, {}^{\mathrm{RL}}\mathfrak{J}^{r_3, \varsigma} \mathrm{y}\left(\tilde{b}, \varkappa(\tilde{b})\right), \end{aligned}$$

and

$$\Psi_2(\varkappa) = \left\{ \begin{array}{l} z \in \mathbb{E} : \\ z(\upsilon) = \left\{ \begin{array}{l} \left[\Lambda_3 \left(\varsigma_0(\tilde{b})\right)^{r_3 + \delta - 1} - \Lambda_1 (\varsigma_0(\upsilon))^{r_3 + \delta - 1} \right] {}^{\mathrm{RL}}\mathfrak{J}^{r_1 - \delta + 1, \varsigma} \sigma(\tilde{b}) \\ \quad + {}^{\mathrm{RL}}\mathfrak{J}^{r_1 + r_3, \varsigma} \sigma(\upsilon) - \Lambda_2 \, {}^{\mathrm{RL}}\mathfrak{J}^{r_1 + r_3, \varsigma} \sigma(\tilde{b}) \end{array} \right. \end{array} \right\}.$$

Obviously, $\Theta = \Psi_1 + \Psi_2$. In the following, we demonstrate that Ψ_1 and Ψ_2 fulfill the conditions of the nonlinear alternative for contractive maps [27, Corollary 3.8]. Initially, we consider the set,

$$\Omega_{\gamma^*} = \left\{\varkappa \in \mathbb{E} : \|\varkappa\| \leq \gamma^*\right\}, \qquad \gamma^* > 0, \tag{3.13}$$

which is bounded, and show that $\Psi_\jmath$ define the SVMs $\Psi_\jmath : \Omega_{\gamma^*} \to \mathcal{P}_{\text{cp,c}}(\mathbb{E})$, $\jmath = 1, 2$. To achieve this, we need to prove that Ψ_1 and Ψ_2 are compact and convex-valued. The proof will proceed in five steps.

Step 1. Ψ_2 is bounded on bounded sets of $\mathbb{E}$. Let Ω_{γ^*} be a bounded set in $\mathbb{E}$. Then for every $z \in \Psi_2(\varkappa)$ and $\varkappa \in \Omega_{\gamma^*}$, $\sigma \in \mathcal{R}_{\daleth,\varkappa}$ exists such that,

$$\begin{aligned} z(\upsilon) = &\left[\Lambda_3 \left(\varsigma_0(\tilde{b})\right)^{r_3+\delta-1} - \Lambda_1 \left(\varsigma_0(\upsilon)\right)^{r_3+\delta-1}\right] {}^{\mathrm{RL}}\mathfrak{I}^{r_1-\delta+1,\varsigma}\sigma(\tilde{b}) \\ &+ {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma}\sigma(\upsilon) - \Lambda_2\, {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma}\sigma(\tilde{b}). \end{aligned}$$

Let (P1) holds. For any $\upsilon \in \mathfrak{B}$, we obtain,

$$\begin{aligned} |z(\upsilon)| \leq &\left[|\Lambda_3| \left(\varsigma_0(\tilde{b})\right)^{r_3+\delta-1} + |\Lambda_1| \left(\varsigma_0(\upsilon)\right)^{r_3+\delta-1}\right] {}^{\mathrm{RL}}\mathfrak{I}^{r_1-\delta+1,\varsigma} \left|\sigma(\tilde{b})\right| \\ &+ {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma} |\sigma(\upsilon)| + |\Lambda_2|\, {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma} \left|\sigma(\tilde{b})\right| \\ \leq &\, \|\widetilde{\varpi}_1\|\, \widetilde{\varpi}_2(\gamma^*) \left(\varsigma_0(\tilde{b})\right)^{r_3+r_1} \left[\tfrac{|\Lambda_3|+|\Lambda_1|}{\Gamma(r_1-\delta+2)} + \tfrac{1+|\Lambda_2|}{\Gamma(r_1+r_3+1)}\right]. \end{aligned}$$

Indeed, $\|z\| \leq \lambda_1 \|\widetilde{\varpi}_1\|\, \widetilde{\varpi}_2(\gamma^*)$.

Step 2. Ψ_2 maps bounded sets of $\mathbb{E}$ into equicontinuous sets. Let $\varkappa \in \Omega_{\gamma^*}$ and $z \in \Psi_2(\varkappa)$. In this case, an element $\sigma \in \mathcal{R}_{\daleth,\varkappa}$ exists such that

$$\begin{aligned} z(\upsilon) = &\left[\Lambda_3 \left(\varsigma_0(\tilde{b})\right)^{r_3+\delta-1} - \Lambda_1 \left(\varsigma_0(\upsilon)\right)^{r_3+\delta-1}\right] {}^{\mathrm{RL}}\mathfrak{I}^{r_1-\delta+1,\varsigma}\sigma(\tilde{b}) \\ &+ {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma}\sigma(\upsilon) - \Lambda_2\, {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma}\sigma(\tilde{b}), \qquad \upsilon \in \mathfrak{B}. \end{aligned}$$

Let $\upsilon_1, \upsilon_2 \in \mathfrak{B}$, $\upsilon_1 < \upsilon_2$. Then

$$\begin{aligned} |z(\upsilon_2) - z(\upsilon_1)| \leq &\frac{\|\widetilde{\varpi}_1\|\widetilde{\varpi}_2(\gamma^*)\left(\varsigma_0(\tilde{b})\right)^{r_1-\delta+1}}{\Gamma(r_1-\delta+2)} \left(|\Lambda_1| \left(\varsigma_0(\upsilon_2)\right)^{r_3+\delta-1} - \left(\varsigma_0(\upsilon_1)\right)^{r_3+\delta-1}\right) \\ &+ \tfrac{\|\widetilde{\varpi}_1\|\widetilde{\varpi}_2(\gamma^*)}{\Gamma(r_1+r_3+1)} \left[\left(\varsigma_0(\upsilon_2)\right)^{r_1+r_3} - \left(\varsigma_0(\upsilon_1)\right)^{r_1+r_3}\right]. \end{aligned}$$

As $\upsilon_1 \to \upsilon_2$, we obtain, $|z(\upsilon_2) - z(\upsilon_1)| \to 0$. Therefore, $\Psi_2(\Omega_{\gamma^*})$ is equicontinuous. Combining the results from Steps 1 and 2, and employing the theorem of Arzelà-Ascoli, we can confirm the completely continuity of Ψ_2.

Step 3. $\Psi_2(\varkappa)$ is convex for all $\varkappa \in \mathbb{E}$. Let $z_1, z_2 \in \Psi_2(\varkappa)$. Then $\sigma_1, \sigma_2 \in \mathcal{R}_{\daleth,\varkappa}$ exist such that for each $\upsilon \in \mathfrak{B}$

$$\begin{aligned} z_j(\upsilon) = &\left[\Lambda_3 \left(\varsigma_0(\tilde{b})\right)^{r_3+\delta-1} - \Lambda_1 \left(\varsigma_0(\upsilon)\right)^{r_3+\delta-1}\right] {}^{\mathrm{RL}}\mathfrak{I}^{r_1-\delta+1,\varsigma}\sigma_j(\tilde{b}) \\ &+ {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma}\sigma_j(\upsilon) - \Lambda_2\, {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma}\sigma_j(\tilde{b}), \qquad j = 1, 2. \end{aligned}$$

Let $\mu \in [0, 1]$. Then for any $\upsilon \in \mathfrak{B}$,

$$(\mu z_1(\upsilon) + (1-\mu)\, z_2(\upsilon)) = \Big[\Lambda_3 \left(\varsigma_0(\tilde{b})\right)^{r_3+\delta-1} - \Lambda_1 \left(\varsigma_0(\upsilon)\right)^{r_3+\delta-1}\Big] {}^{\mathrm{RL}}\mathfrak{I}^{r_1-\delta+1,\varsigma} \left(\mu\sigma_1(\tilde{b}) + (1-\mu)\,\sigma_2(\tilde{b})\right) + {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma} \left(\mu\sigma_1(\upsilon) + (1-\mu)\,\sigma_2(\upsilon)\right) - \Lambda_2\, {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma} \left(\mu\sigma_1(\tilde{b}) + (1-\mu)\,\sigma_2(\tilde{b})\right).$$

Since $\daleth$ has convex values, $\mathcal{R}_{\daleth,\varkappa}$ is convex, and for $\mu \in [0,1]$, $(\mu\sigma_1(\upsilon) + (1-\mu)\,\sigma_2(\upsilon)) \in \mathcal{R}_{\daleth,\varkappa}$. Therefore, $\mu z_1(\upsilon) + (1-\mu)\, z_2(\upsilon) \in \Psi_2(\varkappa)$, which shows that Ψ_2 is convex-valued. Moreover, Ψ_1 is compact and convex-valued.

Step 4. We prove that $\mathrm{Gr}(\Psi_2)$ is closed. Let $\varkappa_n \to \varkappa_*$, $z_n \in \Psi_2(\varkappa_n)$ and $z_n \to z_*$. We show that $z_* \in \Psi_2(\varkappa_*)$. Since $z_n \in \Psi_2(\varkappa_n)$, there is a $\sigma_n \in \mathcal{R}_{\daleth,\varkappa_n}$ such that,

$$z_n(\upsilon) = \left[\Lambda_3 \left(\varsigma_0(\tilde{b})\right)^{r_3+\delta-1} - \Lambda_1 \left(\varsigma_0(\upsilon)\right)^{r_3+\delta-1}\right] {}^{\mathrm{RL}}\mathfrak{I}^{r_1-\delta+1,\varsigma}\sigma_n(\tilde{b}) + {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma}\sigma_n(\upsilon) - \Lambda_2\, {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma}\sigma_n(\tilde{b}).$$

Therefore, we need to prove the existence of $\sigma_* \in \mathcal{R}_{\daleth,\varkappa_*}$ such that for each $\upsilon \in \mathfrak{B}$,

$$z_*(\upsilon) = \left[\Lambda_3 \left(\varsigma_0(\tilde{b})\right)^{r_3+\delta-1} - \Lambda_1 \left(\varsigma_0(\upsilon)\right)^{r_3+\delta-1}\right] {}^{\mathrm{RL}}\mathfrak{I}^{r_1-\delta+1,\varsigma}\sigma_*(\tilde{b}) + {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma}\sigma_*(\upsilon) - \Lambda_2\, {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma}\sigma_*(\tilde{b}), \qquad \upsilon \in \mathfrak{B}.$$

Let $\Upsilon : L^1(\mathfrak{B}, \mathbb{R}) \to C(\mathfrak{B}, \mathbb{R})$ be a continuous linear operator defined as follows:

$$\sigma \to \Upsilon(\sigma)(\upsilon) = \left[\Lambda_3 \left(\varsigma_0(\tilde{b})\right)^{r_3+\delta-1} - \Lambda_1 \left(\varsigma_0(\upsilon)\right)^{r_3+\delta-1}\right] I^{a_1-\delta+1,\varsigma}\sigma(\tilde{b}) + {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma}\sigma(\upsilon) - \Lambda_2\, {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma}\sigma(\tilde{b}), \qquad \upsilon \in \mathfrak{B}.$$

Notice that

$$\|z_n - z_*\| = \Big\|\Big[\Lambda_3 \left(\varsigma_0(\tilde{b})\right)^{r_3+\delta-1} - \Lambda_1 \left(\varsigma_0(\upsilon)\right)^{r_3+\delta-1}\Big] {}^{\mathrm{RL}}\mathfrak{I}^{r_1-\delta+1,\varsigma} \left(\sigma_n(\tilde{b}) - \sigma_*(\tilde{b})\right) + {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma} \left(\sigma_n(\upsilon) - \sigma_*(\upsilon)\right) - \Lambda_2\, {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma} \left(\sigma_n(\tilde{b}) - \sigma_*(\tilde{b})\right)\Big\| \to 0,$$

when $n \to \infty$. Therefore, by Lemma 2.11, $\Upsilon \circ \mathcal{R}_{\daleth,\varkappa}$ is a closed graph operator. Additionally, $z_n \in \Upsilon(\mathcal{R}_{\daleth,\varkappa_n})$. Since $\varkappa_n \to \varkappa_*$, Lemma 2.11 gives

$$z_*(\upsilon) = \left[\Lambda_3 \left(\varsigma_0(\tilde{b})\right)^{r_3+\delta-1} - \Lambda_1 \left(\varsigma_0(\upsilon)\right)^{r_3+\delta-1}\right] {}^{\mathrm{RL}}\mathfrak{I}^{r_1-\delta+1,\varsigma}\sigma_*(\tilde{b}) + {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma}\sigma_*(\upsilon) - \Lambda_2\, {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma}\sigma_*(\tilde{b}),$$

for some $\sigma_* \in \mathcal{R}_{\daleth,\varkappa_*}$. Thus, the graph of Ψ_2 is closed. As a result, Ψ_2 is compact and upper semi-continuous.

Step 5. We prove that Ψ_1 is a contraction in $\mathbb{E}$. Let $\varkappa_1, \varkappa_2 \in \mathbb{E}$. By using the assumption (P3), we get,

$$|\Psi_1\varkappa_1(\upsilon) - \Psi_1\varkappa_2(\upsilon)| \le l_y \left(\varsigma_0(\tilde{b})\right)^{r_3} \left(\frac{|\Lambda_3|+|\Lambda_1|}{\Gamma(2-\delta)} + \frac{1+|\Lambda_2|}{\Gamma(r_3+1)}\right) \|\varkappa_1 - \varkappa_2\|.$$

Thus, $\|\Psi_1 \varkappa_1 - \Psi_1 \varkappa_2\| \leq l_{\mathrm{y}} \lambda_2 \|\varphi - \overline{\varphi}\|$. As $l_{\mathrm{y}} \lambda_2 < 1$, we conclude that Ψ_1 is a contraction. Thus, the operators Ψ_1 and Ψ_2 meet the theorem [27] hypotheses. As a result, we conclude that either of the two following conditions holds, (a) Θ has an FP in $\overline{\mathbb{E}}$, (b) we have $\varkappa \in \partial \mathbb{E}$ and $\xi \in (0,1)$ with $\varkappa \in \xi F(\varkappa)$. We show that conclusion (b) is not possible. If $\varkappa \in \xi \Psi_1(\varkappa) + \xi \Psi_2(\varkappa)$ for $\xi \in (0,1)$. Then, $\sigma \in \mathcal{R}_{\beth,\varkappa}$ exists such that

$$\begin{aligned}|\varkappa(\upsilon)| = \Big| & \xi\, {}^{\mathrm{RL}}\mathfrak{I}^{r_3,\varsigma} \mathrm{y}(\upsilon, \varkappa(\upsilon)) + \xi\, {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma} \sigma(\upsilon) + \xi \Big[\Lambda_3 \left(\varsigma_0(\tilde{b}) \right)^{r_3+\delta-1} \\ & - \Lambda_1 \left(\varsigma_0(\upsilon) \right)^{r_3+\delta-1} \Big] \left({}^{\mathrm{RL}}\mathfrak{I}^{1-\delta,\varsigma} \mathrm{y}\left(\tilde{b}, \varkappa(\tilde{b})\right) + {}^{\mathrm{RL}}\mathfrak{I}^{r_1-\delta+1,\varsigma} \sigma(\tilde{b}) \right) \\ & - \xi \Lambda_2 \left({}^{\mathrm{RL}}\mathfrak{I}^{r_3,\varsigma} \mathrm{y}\left(\tilde{b}, \varkappa(\tilde{b})\right) + {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma} \sigma(\tilde{b}) \right) \Big| \leq \lambda_1 \|\widetilde{\varpi}_1\| \,\widetilde{\varpi}_2(\varkappa) + \lambda_2 \left\| \vartheta_{\mathrm{y}} \right\|,\end{aligned}$$

which implies that $|\varkappa(\upsilon)| \leq \lambda_1 \|\widetilde{\varpi}_1\| \,\widetilde{\varpi}_2(\varkappa) + \lambda_2 \left\|\vartheta_{\mathrm{y}}\right\|$, for each $\upsilon \in \mathfrak{B}$. If criterion of [27, Theorem-(b)] is true, then $\xi \in (0,1)$ and $\varkappa \in \partial \mathbb{E}$ with $\varkappa = \xi \Theta(\varkappa)$ exist. Therefore, $\varkappa$ is a solution of (1.3) with $\|\varkappa\| = \mathcal{N}$. Now, thanks to $|\varkappa(\upsilon)| \leq \lambda_1 \|\widetilde{\varpi}_1\| \,\widetilde{\varpi}_2(\varkappa) + \lambda_2 \left\|\vartheta_{\mathrm{y}}\right\|$, we get

$$\frac{\mathcal{N}}{\lambda_1 \|\widetilde{\varpi}_1\| \widetilde{\varpi}_2(\mathcal{N}) + \lambda_2 \left\|\vartheta_{\mathrm{y}}\right\|} \leq 1,$$

which contradicts (3.10). Thus, it follows from the theorem [27] that Θ admits an FP, and it is a solution of (1.3). □

3.2. The Lipschitz case

We try to establish a more general existence criterion for the FDI (1.3) under new hypotheses. Specifically, we demonstrate the desired existence result for a nonconvex-valued right-hand side using the theorem of Covitz and Nadler [28]. For a metric space $(\mathbb{E}, \varrho)$, we define

$$\begin{cases} \mathscr{H}^{\varrho} : \mathcal{P}(\mathbb{E}) \times \mathcal{P}(\mathbb{E}) \to \mathbb{R}^{+} \cup \{\infty\}, \\ \mathscr{H}^{\varrho}(\widetilde{R}_1, \widetilde{R}_2) = \max \left\{ \sup\limits_{\widetilde{r}_1 \in \widetilde{R}_1} \varrho(\widetilde{r}_1, \widetilde{R}_2),\ \sup\limits_{\widetilde{r}_2 \in \widetilde{R}_2} \varrho(\widetilde{R}_1, \widetilde{r}_2) \right\}, \end{cases}$$

where $\varrho(\widetilde{R}_1, \widetilde{r}_2) = \inf_{\widetilde{r}_1 \in \widetilde{R}_1} \varrho(\widetilde{r}_1, \varrho_2)$ and $\varrho(\widetilde{r}_1, \widetilde{R}_2) = \inf_{\widetilde{r}_2 \in \widetilde{R}_2} \varrho(\widetilde{r}_1, \widetilde{r}_2)$. Then $(\mathcal{P}_{\mathrm{b,cl}}(\mathbb{E}), \mathscr{H}^{\varrho})$ forms a metric space [29].

Definition 3.4. *An SVM $\Omega : \mathbb{E} \to \mathcal{P}_{\mathrm{cl}}(\mathbb{E})$ is a $\widetilde{\eta}$-Lipschitz if and only if $\widetilde{\eta} > 0$ exists such that*

$$\mathscr{H}^{\varrho}(\Omega(\varkappa_1), \Omega(\varkappa_2)) \leq \widetilde{\eta} \varrho(\varkappa_1, \varkappa_2), \qquad \forall \varkappa_1, \varkappa_2 \in \mathbb{E}.$$

In particular, Ω is a contraction whenever $\widetilde{\eta} < 1$.

Theorem 3.5. *Assume that* (P3) *and the following conditions hold:*

- P6) *The map $\beth : \mathfrak{B} \times \mathbb{R} \to \mathcal{P}_{\mathrm{cp}}(\mathbb{R})$ is such that $\beth(\cdot, \varphi) : \mathfrak{B} \to \mathcal{P}_{\mathrm{cp}}(\mathbb{R})$ is measurable for any $\varkappa \in \mathbb{R}$;*
- P7) *The condition $\mathscr{H}^{\varrho}(\beth(\upsilon, \varkappa_1), \beth(\upsilon, \varkappa_2)) \leq \mathrm{n}(\upsilon) |\varkappa_1 - \varkappa_2|$ holds for a.e. $\upsilon \in \mathfrak{B}$ and $\varkappa_1, \varkappa_2 \in \mathbb{R}$ with $\mathrm{n} \in C(\mathfrak{B}, \mathbb{R}^{+})$ and $\varrho(0, \beth(\upsilon, 0)) \leq \mathrm{n}(\upsilon)$ for a.e. $\upsilon \in \mathfrak{B}$.*

Then FDI (1.3) *has at least one solution for $\mathfrak{B}$ whenever $\|\mathrm{n}\| \lambda_1 + l_{\mathrm{y}} \lambda_2 < 1$, where λ_1, λ_2 are given in* (3.11).

Proof. By assumption (P6) and [30, Theorem III.6], $\gimel$ has a measurable selection $\sigma : \mathfrak{B} \to \mathbb{R}$, with $\sigma \in L^1(\mathfrak{B})$, which implies that $\gimel$ is integrability bounded. Therefore, $\mathcal{R}_{\gimel,\varkappa} \neq \emptyset$. We demonstrate that the operator $\Omega : \mathbb{E} \to \mathcal{P}(\mathbb{E})$ described in (3.12) meets the conditions required by Nadler and Covitz's FPT. Specifically, we prove that $\Omega(\varkappa)$ is closed for each $\varkappa \in \mathbb{E}$. Assume a sequence such that $\{u_n\}_{n\geq 0} \in \Omega(\varkappa)$ and $u_n \to u$ $(n \to \infty)$ in $\mathbb{E}$. Then $u \in \mathbb{E}$ and $\sigma_n \in \mathcal{R}_{\mathcal{G},\varkappa_n}$ exists such that

$$\begin{aligned} u_n(\upsilon) = {}& {}^{\mathrm{RL}}\mathfrak{J}^{r_3,\varsigma}\mathrm{y}(\upsilon,\varkappa(\upsilon)) + {}^{\mathrm{RL}}\mathfrak{J}^{r_1+r_3,\varsigma}\sigma_n(\upsilon) + \Big[\Lambda_3\left(\varsigma_0(\tilde{b})\right)^{r_3+\delta-1} \\ & - \Lambda_1\left(\varsigma_0(\upsilon)\right)^{r_3+\delta-1}\Big]\left({}^{\mathrm{RL}}\mathfrak{J}^{1-\delta,\varsigma}\mathrm{y}\left(\tilde{b},\varkappa(\tilde{b})\right) + {}^{\mathrm{RL}}\mathfrak{J}^{r_1-\delta+1,\varsigma}\sigma_n(\tilde{b})\right) \\ & - \Lambda_2\left({}^{\mathrm{RL}}\mathfrak{J}^{r_3,\varsigma}\mathrm{y}\left(\tilde{b},\varkappa(\tilde{b})\right) + {}^{\mathrm{RL}}\mathfrak{J}^{r_1+r_3,\varsigma}\sigma_n(\tilde{b})\right). \end{aligned}$$

So there is a subsequence σ_n that converges to σ in $L^1(\mathfrak{B})$, because $\gimel$ has compact values. As a result, $\sigma \in \mathcal{R}_{\gimel,\varkappa}$, and we get

$$\begin{aligned} u_n(\upsilon) \to u(\upsilon) = {}& {}^{\mathrm{RL}}\mathfrak{J}^{r_3,\varsigma}\mathrm{y}(\upsilon,\varkappa(\upsilon)) + {}^{\mathrm{RL}}\mathfrak{J}^{r_1+r_3,\varsigma}\sigma(\upsilon) + \Big[\Lambda_3\left(\varsigma_0(\tilde{b})\right)^{r_3+\delta-1} \\ & - \Lambda_1\left(\varsigma_0(\upsilon)\right)^{r_3+\delta-1}\Big]\left({}^{\mathrm{RL}}\mathfrak{J}^{1-\delta,\varsigma}\mathrm{y}\left(\tilde{b},\varkappa(\tilde{b})\right) + {}^{\mathrm{RL}}\mathfrak{J}^{r_1-\delta+1,\varsigma}\sigma(\tilde{b})\right) \\ & - \Lambda_2\left({}^{\mathrm{RL}}\mathfrak{J}^{r_3,\varsigma}\mathrm{y}\left(\tilde{b},\varkappa(\tilde{b})\right) + {}^{\mathrm{RL}}\mathfrak{J}^{r_1+r_3,\varsigma}\sigma(\tilde{b})\right). \end{aligned}$$

Hence $u \in \Omega(\varkappa)$. Next, we show that a $\Delta \in (0,1)$, $(\Delta = \|\mathfrak{n}\|\,\lambda_1 + l_{\mathrm{y}}\lambda_2)$ exists such that

$$\mathscr{H}^{\varrho}\left(\Omega(\varkappa_1), \Omega(\varkappa_2)\right) \leq \Delta\,\|\varkappa_1 - \varkappa_2\|, \qquad \forall \varkappa_1, \varkappa_2 \in \mathbb{E}.$$

Let $\varkappa_1, \varkappa_2 \in \mathbb{E}$ and $v_1 \in \Omega(\varkappa_1)$. Then $\sigma_1(\upsilon) \in \gimel(\upsilon, \varkappa_1(\upsilon))$ exists such that for all $\upsilon \in \mathfrak{B}$ and

$$\begin{aligned} v_1(\upsilon) = {}& {}^{\mathrm{RL}}\mathfrak{J}^{r_3,\varsigma}\mathrm{y}(\upsilon,\varkappa_1(\upsilon)) + {}^{\mathrm{RL}}\mathfrak{J}^{r_1+r_3,\varsigma}\sigma_1(\upsilon) + \Big[\Lambda_3\left(\varsigma_0(\tilde{b})\right)^{r_3+\delta-1} \\ & - \Lambda_1\left(\varsigma_0(\upsilon)\right)^{r_3+\delta-1}\Big]\left({}^{\mathrm{RL}}\mathfrak{J}^{1-\delta,\varsigma}\mathrm{y}\left(\tilde{b},\varkappa_1(\tilde{b})\right) + {}^{\mathrm{RL}}\mathfrak{J}^{r_1-\delta+1,\varsigma}\sigma_1(\tilde{b})\right) \\ & - \Lambda_2\left({}^{\mathrm{RL}}\mathfrak{J}^{r_3,\varsigma}\mathrm{y}\left(\tilde{b},\varkappa_1(\tilde{b})\right) + {}^{\mathrm{RL}}\mathfrak{J}^{r_1+r_3,\varsigma}\sigma_1(\tilde{b})\right). \end{aligned}$$

By (P7), we have

$$\mathscr{H}^{\varrho}\left(\gimel(\upsilon,\varkappa_1(\upsilon)), \gimel(\upsilon,\varkappa_2(\upsilon))\right) \leq \mathfrak{n}(\upsilon)\,|\varkappa_1(\upsilon) - \varkappa_2(\upsilon)|.$$

Thus, $\chi(\upsilon) \in \gimel(\upsilon,\varkappa_2)$ exists such that $|\sigma_1(\upsilon) - \chi| \leq \mathfrak{n}(\upsilon)\,|\varkappa_1(\upsilon) - \varkappa_2(\upsilon)|$, for each $\upsilon \in \mathfrak{B}$. We build an SVM, $O : \mathfrak{B} \to \mathcal{P}(\mathbb{R})$ as follows:

$$O(\upsilon) = \Big\{\chi \in \mathbb{R} : |\sigma_1(\upsilon) - \chi| \leq \mathfrak{n}(\upsilon)\,|\varkappa_1(\upsilon) - \varkappa_2(\upsilon)|\Big\}.$$

Notice that σ_1 and $\omega = \mathfrak{n}\,|\varkappa_1 - \varkappa_2|$ are measurable, so it follows that $O(\upsilon) \cap \gimel(\upsilon,\varkappa_2)$ is measurable. Next, we select the function $\sigma_2(\upsilon) \in \gimel(\upsilon,\varkappa_2)$ such that,

$$|\sigma_1(\upsilon) - \sigma_2(\upsilon)| \leq \mathfrak{n}(\upsilon)\,|\varkappa_1(\upsilon) - \varkappa_2(\upsilon)|, \qquad \forall \upsilon \in \mathfrak{B}.$$

Define

$$v_2(\upsilon) = {}^{\mathrm{RL}}\mathfrak{J}^{r_3,\varsigma}\mathrm{y}(\upsilon,\varkappa_2(\upsilon)) + I\,{}^{\mathrm{RL}}\mathfrak{J}^{r_1+r_3,\varsigma}\sigma_2(\upsilon) + \Big[\Lambda_3\left(\varsigma_0(\tilde{b})\right)^{r_3+\delta-1}$$

$$- \Lambda_1 (\varsigma_0(\upsilon))^{r_3+\delta-1} \Big] \left({}^{\mathrm{RL}}\mathfrak{I}^{1-\delta,\varsigma} \mathrm{y}\left(\tilde{b}, \varkappa_2(\tilde{b})\right) + {}^{\mathrm{RL}}\mathfrak{I}^{r_1-\delta+1,\varsigma} \sigma_2(\tilde{b}) \right)$$
$$- \Lambda_2 \left({}^{\mathrm{RL}}\mathfrak{I}^{r_3,\varsigma} \mathrm{y}\left(\tilde{b}, \varkappa_2(\tilde{b})\right) + {}^{\mathrm{RL}}\mathfrak{I}^{r_1+r_3,\varsigma} \sigma_2(\tilde{b}) \right).$$

As a results, we arrive at,

$$|v_1(\upsilon) - v_2(\upsilon)| \leq \left(\|\mathrm{n}\| \, \lambda_1 + l_{\mathrm{y}} \lambda_2 \right) \|\varkappa_1 - \varkappa_2\|,$$

which implies $\|v_1 - v_2\| \leq \left(\|\mathrm{n}\| \, \lambda_1 + l_{\mathrm{y}} \lambda_2 \right) \|\varkappa_1 - \varkappa_2\|$. Now, by interchanging the roles of $\varkappa_1$ and $\varkappa_2$, we obtain,

$$\mathscr{H}^{\varrho} \left(\Omega(\varkappa_1), \Omega(\varkappa_2) \right) \leq \left(\|\mathrm{n}\| \, \lambda_1 + l_{\mathrm{y}} \lambda_2 \right) \|\varkappa_1 - \varkappa_2\|.$$

Since Ω is a contraction, it follows that the Covitz and Nadler theorem that Ω has an FP, which is a solution of the FDI (1.3). □

4. Examples

In order to validate the theoretical findings, we provide specific cases of FDIs in this section. In fact, we focus on the FDI with the following form:

$$\begin{cases} {}^{\mathrm{H}}\mathfrak{D}^{r_1,r_2,\varsigma} \left({}^{\mathrm{C}}\mathfrak{D}^{r_3,\varsigma} \varkappa(\upsilon) - \mathrm{y}\left(\upsilon, \varkappa(\upsilon)\right) \right) \in \daleth\left(\upsilon, \varkappa(\upsilon)\right), \quad \upsilon \in \mathfrak{B}, \\ \varkappa(0) + \eta_1 \varkappa(\tilde{b}) = 0, \\ {}^{\mathrm{C}}\mathfrak{D}^{\delta+r_3-1,\varsigma} \varkappa(0) + \eta_2 \, {}^{\mathrm{C}}\mathfrak{D}^{\delta+r_3-1,\varsigma} \varkappa(\tilde{b}) = 0. \end{cases} \tag{4.1}$$

The examples below are special cases of FDIs given by (4.1).

Example 4.1. *Using the FDIs defined by* (4.1) *and taking* $r_1 \in \{\frac{1}{2}, \frac{2}{3}, \frac{5}{6}\}$, $r_2 = \frac{1}{3}$, $r_3 = \frac{1}{5}$, $\varsigma(\upsilon) = \upsilon^2$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{6}$, $\delta = 0.666, 0.777, 0.888$, *and* $\tilde{b} = 1$, *the problem* (4.1) *is reduced to*

$$\begin{cases} {}^{\mathrm{H}}\mathfrak{D}^{1/2,1/3,\upsilon^2} \left({}^{\mathrm{C}}\mathfrak{D}^{1/5,\upsilon^2} \varkappa(\upsilon) - \mathrm{y}\left(\upsilon, \varkappa(\upsilon)\right) \right) \in \daleth\left(\upsilon, \varkappa(\upsilon)\right), \\ \varkappa(0) + \frac{1}{4} \varkappa(1) = 0, \\ {}^{\mathrm{C}}\mathfrak{D}^{-2/15,\upsilon^2} \varkappa(0) + \frac{1}{6} \, {}^{\mathrm{C}}\mathfrak{D}^{-2/15,\upsilon^2} \varkappa(1) = 0, \end{cases} \tag{4.2}$$

for $\upsilon \in \mathfrak{B}$. *With these data, it follows from* (3.3)*, that we have*

$$\Lambda_1 = \frac{\eta_2}{(\eta_2+1)\Gamma(r_3+\delta)} \simeq \begin{cases} 0.1302, & r_1 = 1/2, \\ 0.1409, & r_1 = 2/3, \\ 0.1494, & r_1 = 5/6, \end{cases}$$

$$\Lambda_2 = \frac{\eta_1}{\eta_1+1} \simeq \begin{cases} 0.2000, & r_1 = 1/2, \\ 0.2000, & r_1 = 2/3, \\ 0.2000, & r_1 = 5/6, \end{cases}$$

$$\Lambda_3 = \frac{\eta_1 \eta_2}{(\eta_2+1)\Gamma(r_3+\delta)} \simeq \begin{cases} 0.0325, & r_1 = 1/2, \\ 0.0352, & r_1 = 2/3, \\ 0.0373, & r_1 = 5/6. \end{cases}$$

We define the function $\mathbb{y}$ *and the SVM* $\daleth : \mathfrak{B} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *as follows:*

$$\mathbb{y}(\upsilon, \varkappa) = \frac{\cos(\upsilon)}{\upsilon^2+2}\left(\frac{|\varkappa|}{|\varkappa|+1}\right), \qquad \forall (\upsilon, \varkappa) \in \mathfrak{B} \times \mathbb{R}, \tag{4.3}$$

and

$$\daleth(\upsilon, \varkappa) = \left[\frac{1}{\left(5\upsilon^2+7\exp(\upsilon)\right)}\frac{\varkappa}{5(\varkappa+3)},\ \frac{1}{\sqrt{\upsilon^2+16}}\frac{|\varkappa|}{|\varkappa|+1}\right]. \tag{4.4}$$

For $\varkappa, \overline{\varkappa} \in \mathbb{R}$, *we have*

$$\left|\mathbb{y}(\upsilon, \varkappa) - \mathbb{y}(\upsilon, \overline{\varkappa})\right| = \left|\frac{\cos(\upsilon)}{\upsilon^2+2}\left(\frac{|\varkappa|}{|\varkappa|+1} - \frac{|\overline{\varkappa}|}{|\overline{\varkappa}|+1}\right)\right| \leq \frac{1}{\upsilon^2+2}\left(\frac{|\varkappa-\overline{\varkappa}|}{(1+|\varkappa|)\left(1+|\overline{\varkappa}|\right)}\right) \leq l_{\mathbb{y}}\left|\varkappa - \overline{\varkappa}\right|, \tag{4.5}$$

with $l_{\mathbb{y}} = \frac{1}{2}$ *and also,*

$$\mathbb{y}(\upsilon, \varkappa) \leq \frac{1}{\exp\left(\upsilon^2\right)+1} = \vartheta_{\mathbb{y}}(\upsilon), \qquad \forall(\upsilon, \varkappa) \in \mathfrak{B} \times \mathbb{R}.$$

Thus, the assumptions (P3) *and* (P4) *hold. It is also clear that the SVM* $\daleth$ *satisfies the assumption* (P1) *and*

$$\|\daleth(\upsilon, \varkappa)\|_{\mathcal{P}} = \sup\left\{|\eta| : \eta \in \daleth(\upsilon, \varkappa)\right\} \leq \frac{1}{\sqrt{\upsilon^2+16}} = \widetilde{\varpi}_1(\upsilon)\widetilde{\varpi}_2(\|\varkappa\|),$$

where $\|\widetilde{\varpi}_1\| = \frac{1}{4}$ *and* $\widetilde{\varpi}_2(\|\varkappa\|) = 1$. *Thus,* (P2) *holds, and by* (P5),

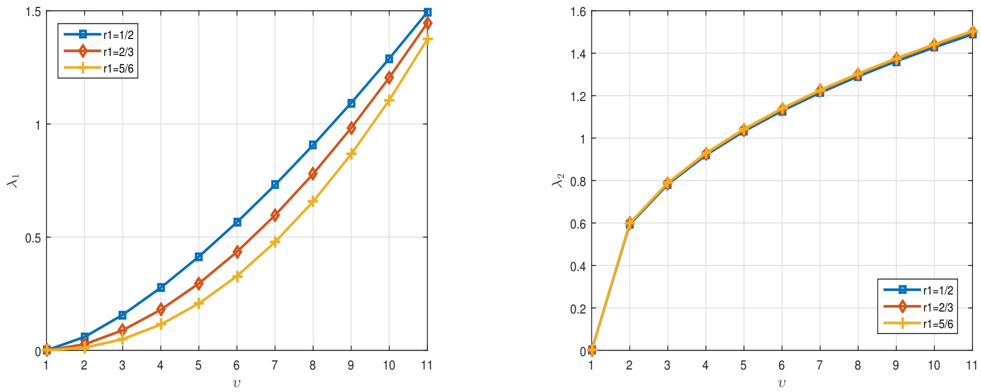
$$\lambda_1 = \left(\varsigma_0(\tilde{b})\right)^{r_3+r_1}\left[\frac{|\Lambda_3|+|\Lambda_1|}{\Gamma(r_1-\delta+2)} + \frac{1+|\Lambda_2|}{\Gamma(r_1+r_3+1)}\right] \simeq \begin{cases} 1.494, & r_1 = 1/2, \\ 1.446, & r_1 = 2/3, \\ 1.374, & r_1 = 5/6, \end{cases}$$

$$\lambda_2 = \left(\varsigma_0(\tilde{b})\right)^{r_3}\left[\frac{|\Lambda_3|+|\Lambda_1|}{\Gamma(2-\delta)} + \frac{1+|\Lambda_2|}{\Gamma(r_3+1)}\right] \simeq \begin{cases} 1.489, & r_1 = 1/2, \\ 1.500, & r_1 = 2/3, \\ 1.504, & r_1 = 5/6, \end{cases}$$

for which the curves are shown in Figure 1, Moreover,

$$\mathcal{N} > \lambda_1\|\widetilde{\varpi}_1\|\,\widetilde{\varpi}_2(\mathcal{N}) + \lambda_2\left\|\vartheta_{\mathbb{y}}\right\| \simeq \begin{cases} 1.140, & r_1 = 1/2, \\ 1.117, & r_1 = 2/3, \\ 1.086, & r_1 = 5/6, \end{cases}$$

whenever $\mathcal{N} = 1.15$, *which it is shown in Figure 2. As seen in Table 1, the effect of the order of the derivative* r_1 *is very insignificant. So all assumptions of Theorem 3.3 are valid. Hence the FDI* (4.2) *has a solution for* $\mathfrak{B}$.



(a) λ_1 **(b)** λ_2

Figure 1. Graphical representation of the λ_i, $i = 1, 2$ of the FDI (4.2) with three different values of $r1$.

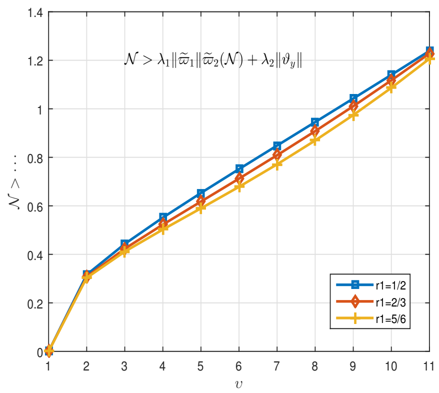


Figure 2. Graphical representation of the $\mathcal{N}$ of the FDI (4.2) for $r_1 \in \{\frac{1}{2}, \frac{2}{3}, \frac{5}{6}\}$.

Table 1. The data obtained for the FDI (4.2) with three different values of r_1.

υ	$r_1 = \frac{1}{2}$			$r_1 = \frac{2}{3}$			$r_1 = \frac{5}{6}$		
	λ_1	λ_2	$\mathcal{N} > ...$	λ_1	λ_2	$\mathcal{N} > ...$	λ_1	λ_2	$\mathcal{N} > ...$
0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.10	0.059	0.593	0.316	0.027	0.597	0.307	0.012	0.599	0.303
0.20	0.157	0.782	0.443	0.089	0.788	0.423	0.049	0.790	0.411
0.30	0.277	0.920	0.552	0.179	0.927	0.523	0.114	0.929	0.502
0.40	0.414	1.032	0.653	0.295	1.040	0.618	0.207	1.043	0.590
0.50	0.566	1.129	0.751	0.435	1.137	0.712	0.328	1.140	0.678
0.60	0.731	1.214	0.849	0.597	1.223	0.809	0.478	1.226	0.771
0.70	0.907	1.291	0.945	0.779	1.301	0.908	0.657	1.304	0.869
0.80	1.093	1.362	1.042	0.982	1.372	1.011	0.866	1.376	0.974
0.90	1.289	1.428	1.140	1.205	1.438	1.117	1.105	1.442	1.086
1.00	1.494	1.489	1.238	1.446	1.500	1.228	1.374	1.504	1.206

In the next example, we check the changes in the derivative order r_2.

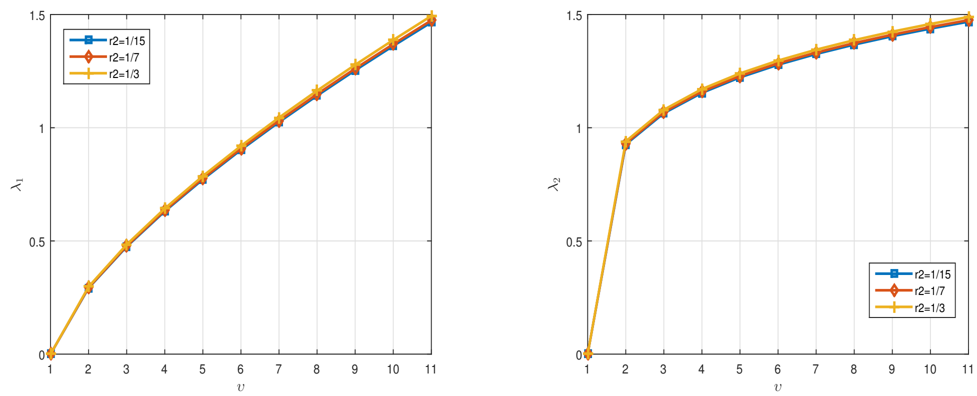
Example 4.2. *Using the FDI defined by* (4.1) *and taking* $r_1 = \frac{1}{2}$, $r_2 \in \{\frac{1}{15}, \frac{1}{7}, \frac{1}{3}\}$, $r_3 = \frac{1}{5}$, $\varsigma(\upsilon) = \upsilon$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{6}$, $\delta = 0.533, 0.571, 0.666$, *and* $\tilde{b} = 1$, *4.1 is reduced to*

$$\begin{cases} {}^{\mathrm{H}}\mathfrak{D}^{1/2,1/3,\upsilon}\left({}^{\mathrm{C}}\mathfrak{D}^{1/5,\upsilon}\varkappa(\upsilon) - \mathrm{y}(\upsilon, \varkappa(\upsilon))\right) \in \gimel(\upsilon, \varkappa(\upsilon)), \quad \upsilon \in \mathfrak{B}, \\ \varkappa(0) + \frac{1}{4}\varkappa(1) = 0, \\ {}^{\mathrm{C}}\mathfrak{D}^{-2/15,\upsilon}\varkappa(0) + \frac{1}{6}\,{}^{\mathrm{C}}\mathfrak{D}^{-2/15,\upsilon}\varkappa(1) = 0. \end{cases} \tag{4.6}$$

With these data, we find

$$\Lambda_1 \simeq \begin{cases} 0.114, & r_2 = 1/15, \\ 0.119, & r_2 = 1/7, \\ 0.130, & r_2 = 1/3, \end{cases} \qquad \Lambda_2 \simeq \begin{cases} 0.200, & r_2 = 1/15, \\ 0.200, & r_2 = 1/7, \\ 0.200, & r_2 = 1/3, \end{cases} \qquad \Lambda_3 \simeq \begin{cases} 0.028, & r_2 = 1/15, \\ 0.029, & r_2 = 1/7, \\ 0.032, & r_2 = 1/3. \end{cases}$$

Consider the SVM $\gimel : \mathfrak{B} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is defined by,* $\varphi \to \gimel(\upsilon, \varkappa) = \left[0, \frac{\sin(\varkappa)}{5\sqrt{\upsilon^2+4}} + \frac{1}{12}\right]$, *and the function* y *defined in* (4.3). *From* (4.5), *we see that the assumption* (P3) *is satisfied with* $l_{\mathrm{y}} = \frac{1}{2}$. *Next, we have* $\mathscr{H}^{\varrho}(\gimel(\upsilon, \varkappa), \gimel(\upsilon, \overline{\varkappa})) \leq \pi(\upsilon)\left|\varkappa - \overline{\varkappa}\right|$, *where* $\pi(\upsilon) = \frac{1}{5\sqrt{\upsilon^2+4}}$ *and* $\varrho(0, \gimel(\upsilon, 0)) = \frac{1}{12} \leq \pi(\upsilon)$ *for a.e.* $\upsilon \in \mathfrak{B}$. *Figure 3 shows the curves of* λ_i, $i = 1, 2$, *whenever* r_2 *varies in the interval* $\mathfrak{B}$. *By comparing the curves and data in Table 2, it can be clearly seen that as* r_2 *approaches zero,* λ_i *decreases.*



(a) λ_1 **(b)** λ_2

Figure 3. Graphical representation of the λ_i, $i = 1, 2$ of the FDI (4.6) with three different values of r_2.

Furthermore, we obtain $\|\pi\| = \frac{1}{10}$, *resulting in*

$$\|\pi\|\,\lambda_1 + l_{\mathrm{y}}\lambda_2 \simeq \begin{cases} 0.881, & r_2 = 1/15, \\ 0.885, & r_2 = 1/7, \\ 0.894, & r_2 = 1/3. \end{cases} < 1. \tag{4.7}$$

These results are shown in Table 2. Furthermore, the curves of Eq (4.7) *for three cases of* r_2 *are shown in Figure 4.*

Table 2. The data obtained for the FDI (4.6) with three different values of r_2.

υ	$r_2=\frac{1}{15}$			$r_2=\frac{1}{7}$			$r_2=\frac{1}{3}$		
	λ_1	λ_2	$\|\mathfrak{n}\|\,\lambda_1+l_y\lambda_2$	λ_1	λ_2	$\|\mathfrak{n}\|\,\lambda_1+l_y\lambda_2$	λ_1	λ_2	$\|\mathfrak{n}\|\,\lambda_1+l_y\lambda_2$
0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.10	0.292	0.927	0.493	0.294	0.931	0.495	0.298	0.940	0.500
0.20	0.475	1.064	0.580	0.478	1.069	0.582	0.484	1.079	0.588
0.30	0.631	1.154	0.640	0.635	1.159	0.643	0.643	1.171	0.650
0.40	0.772	1.223	0.688	0.776	1.228	0.692	0.787	1.240	0.699
0.50	0.902	1.278	0.729	0.907	1.284	0.733	0.919	1.296	0.740
0.60	1.025	1.326	0.765	1.031	1.332	0.769	1.045	1.345	0.777
0.70	1.142	1.367	0.798	1.148	1.374	0.802	1.164	1.387	0.810
0.80	1.254	1.404	0.828	1.261	1.411	0.831	1.278	1.424	0.840
0.90	1.361	1.438	0.855	1.369	1.444	0.859	1.388	1.458	0.868
1.00	1.466	1.468	0.881	1.474	1.475	0.885	1.494	1.489	0.894

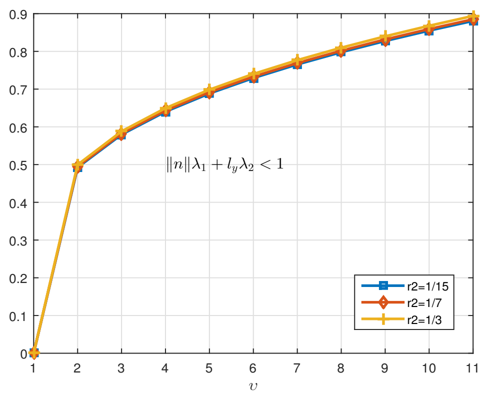


Figure 4. Graphical representation of $\|\mathfrak{n}\|\,\lambda_1 + l_y\lambda_2$ in Eq (4.7) of the FDI (4.6) for $r_2 \in \{\frac{1}{15}, \frac{1}{7}, \frac{1}{3}\}$.

Therefore, all the assumptions of Theorem 3.5 are satisfied, which implies that at least one solution to the problem (4.6) *for* $\mathfrak{B}$.

In Example 4.3, we examine our proven theorems for changes of function $\varsigma(\upsilon)$.

Example 4.3. *Using the FDIs defined by* (4.1) *and taking* $r_1 \in \frac{2}{3}$, $r_2 = \frac{1}{3}$, $r_3 = \frac{1}{5}$,

$$\varsigma_1(\upsilon) = \upsilon^2, \quad \varsigma_2(\upsilon) = \upsilon, \quad \varsigma_3(\upsilon) = \sqrt{\upsilon}, \quad \varsigma_4(\upsilon) = \ln(\upsilon + 0.01), \tag{4.8}$$

$\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{6}$, $\delta = 0.777$, $\tilde{b} = 1$, *the problem* (4.1) *is reduced to*

$$\begin{cases} {}^{\mathrm{H}}\mathfrak{D}^{2/3,1/3,\varsigma_j(\upsilon)}\left({}^{\mathrm{C}}\mathfrak{D}^{1/5,\varsigma_j(\upsilon)}\varkappa(\upsilon) - \mathrm{y}\left(\upsilon,\varkappa(\upsilon)\right)\right) \in \daleth\left(\upsilon,\varkappa(\upsilon)\right), \\ \varkappa(0) + \frac{1}{4}\varkappa(1) = 0, \\ {}^{\mathrm{C}}\mathfrak{D}^{-2/15,\varsigma_j(\upsilon)}\varkappa(0) + \frac{1}{6}\,{}^{\mathrm{C}}\mathfrak{D}^{-2/15,\varsigma_j(\upsilon)}\varkappa(1) = 0, \end{cases} \tag{4.9}$$

for $\upsilon \in \mathfrak{B}$. With these data, it follows from (3.3) *that*

$$\Lambda_1 = \frac{\eta_2}{(\eta_2+1)\Gamma(r_3+\delta)} \simeq 0.1409, \quad \Lambda_2 = \frac{\eta_1}{\eta_1+1} \simeq 0.2000, \quad \Lambda_3 = \frac{\eta_1\eta_2}{(\eta_2+1)\Gamma(r_3+\delta)} \simeq 0.0352.$$

We define the function y and the SVM $\daleth : \mathfrak{B} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ as follows:

$$\mathrm{y}(\upsilon, \varkappa) = \frac{\cos(\upsilon)}{\upsilon^2+2}\left(\frac{|\varkappa|}{|\varkappa|+1}\right), \qquad \forall (\upsilon, \varkappa) \in \mathfrak{B} \times \mathbb{R},$$

and

$$\daleth(\upsilon, \varkappa) = \left[\frac{1}{\left(5\upsilon^2+7\exp(\upsilon)\right)}\frac{\varkappa}{5(\varkappa+3)}, \ \frac{1}{\sqrt{\upsilon^2+16}}\frac{|\varkappa|}{|\varkappa|+1}\right].$$

For $\varkappa, \overline{\varkappa} \in \mathbb{R}$, we have

$$\left|\mathrm{y}(\upsilon, \varkappa) - \mathrm{y}(\upsilon, \overline{\varkappa})\right| = \left|\frac{\cos(\upsilon)}{\upsilon^2+2}\left(\frac{|\varkappa|}{|\varkappa|+1} - \frac{|\overline{\varkappa}|}{|\overline{\varkappa}|+1}\right)\right| \leq \frac{1}{\upsilon^2+2}\left(\frac{|\varkappa-\overline{\varkappa}|}{(1+|\varkappa|)(1+|\overline{\varkappa}|)}\right) \leq l_{\mathrm{y}}\left|\varkappa - \overline{\varkappa}\right|,$$

with $l_{\mathrm{y}} = \frac{1}{2}$, as well as $\mathrm{y}(\upsilon, \varkappa) \leq \frac{1}{\exp(\upsilon^2)+1} = \vartheta_{\mathrm{y}}(\upsilon)$, for each $(\upsilon, \varkappa) \in \mathfrak{B} \times \mathbb{R}$. Thus, the assumptions (P3) *and* (P4) *hold. It is also clear that the SVM $\daleth$ satisfies the assumption* (P1) *and*

$$\|\daleth(\upsilon, \varkappa)\|_{\mathcal{P}} = \sup\left\{|\eta| \ : \ \eta \in \daleth(\upsilon, \varkappa)\right\} \leq \frac{1}{\sqrt{\upsilon^2+16}} = \widetilde{\varpi}_1(\upsilon)\widetilde{\varpi}_2(\|\varkappa\|),$$

where $\|\widetilde{\varpi}_1\| = \frac{1}{4}$ and $\widetilde{\varpi}_2(\|\varkappa\|) = 1$. Thus, (P2) *holds, and by* (P5)

$$\lambda_1 = \left(\varsigma_0(\tilde{b})\right)^{r_3+r_1}\left[\frac{|\Lambda_3|+|\Lambda_1|}{\Gamma(r_1-\delta+2)} + \frac{1+|\Lambda_2|}{\Gamma(r_1+r_3+1)}\right] \simeq \begin{cases} 1.494, & \varsigma_1(\upsilon) = \upsilon^2, \\ 1.446, & \varsigma_2(\upsilon) = \upsilon, \\ 1.374, & \varsigma_3(\upsilon) = \sqrt{\upsilon}, \\ 1.374, & \varsigma_4(\upsilon) = \ln(\upsilon + 0.01), \end{cases}$$

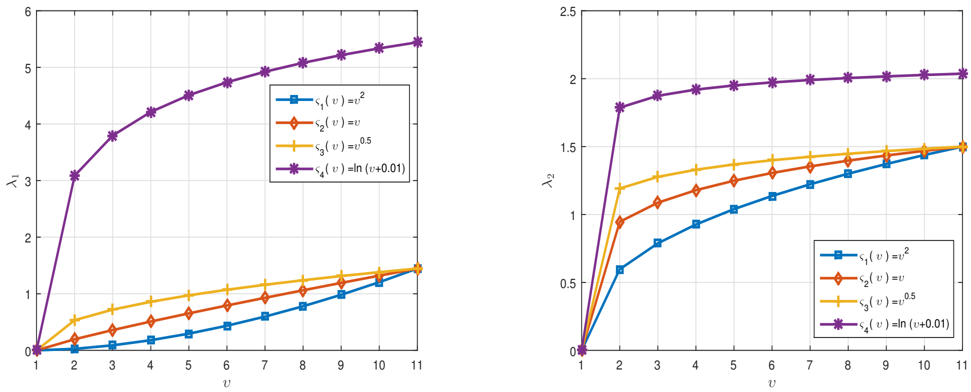
$$\lambda_2 = \left(\varsigma_0(\tilde{b})\right)^{r_3}\left[\frac{|\Lambda_3|+|\Lambda_1|}{\Gamma(2-\delta)} + \frac{1+|\Lambda_2|}{\Gamma(r_3+1)}\right] \simeq \begin{cases} 1.494, & \varsigma_1(\upsilon) = \upsilon^2, \\ 1.446, & \varsigma_2(\upsilon) = \upsilon, \\ 1.374, & \varsigma_3(\upsilon) = \sqrt{\upsilon}, \\ 1.374, & \varsigma_4(\upsilon) = \ln(\upsilon + 0.01), \end{cases}$$

for which the curves are shown in Figure 5. Moreover

$$\mathcal{N} > \lambda_1\|\widetilde{\varpi}_1\|\widetilde{\varpi}_2(\mathcal{N}) + \lambda_2\left\|\vartheta_{\mathrm{y}}\right\| \simeq \begin{cases} 1.117, & \varsigma_1(\upsilon) = \upsilon^2, \\ 1.114, & \varsigma_2(\upsilon) = \upsilon, \\ 1.138, & \varsigma_3(\upsilon) = \sqrt{\upsilon}, \\ 1.142, & \varsigma_4(\upsilon) = \ln(\upsilon + 0.01), \end{cases}$$

whenever $\mathcal{N} = 1.15$, which is shown in Figure 6. As seen in Table 3, the effect of $\varsigma(\upsilon)$ is very remarkable.

So all the assumptions of Theorem 3.3 are valid. Hence the FDI (4.9) *has a solution for $\mathfrak{B}$.*



(a) λ_1 **(b)** λ_2

Figure 5. Graphical representation of the λ_i, $i = 1, 2$ of the FDI (4.9) with four cases of $\varsigma(\upsilon)$.

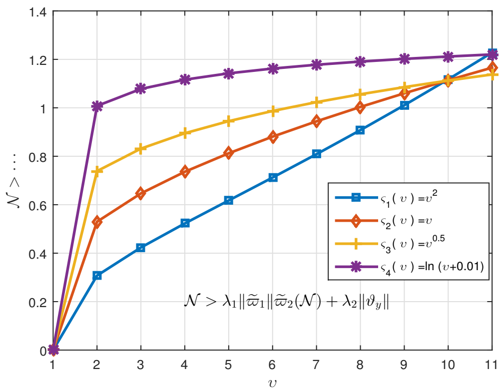


Figure 6. Graphical representation of the $\mathcal{N}$ of the FDI (4.9) with four cases of $\varsigma(\upsilon)$ as defined in (4.8).

Table 3. The data obtained for the FDI (4.2) with four cases of $\varsigma(\upsilon)$.

υ	$\varsigma_1(\upsilon) = \upsilon^2$			$\varsigma_2(\upsilon) = \upsilon$			$\varsigma_3(\upsilon) = \sqrt{\upsilon}$			$\varsigma_4(\upsilon) = \ln(\upsilon + 0.01)$		
	λ_1	λ_2	$\mathcal{N} > \ldots$	λ_1	λ_2	$\mathcal{N} > \ldots$	λ_1	λ_2	$\mathcal{N} > \ldots$	λ_1	λ_2	$\mathcal{N} > \ldots$
0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.10	0.027	0.597	0.307	0.197	0.946	0.530	0.533	1.192	0.739	3.086	1.787	1.008
0.20	0.089	0.788	0.423	0.358	1.087	0.647	0.720	1.277	0.832	3.795	1.874	1.078
0.30	0.179	0.927	0.523	0.509	1.179	0.736	0.858	1.330	0.895	4.213	1.920	1.116
0.40	0.295	1.040	0.618	0.654	1.249	0.812	0.972	1.369	0.945	4.508	1.950	1.142
0.50	0.435	1.137	0.712	0.793	1.306	0.881	1.071	1.400	0.987	4.737	1.973	1.162
0.60	0.597	1.223	0.809	0.929	1.354	0.944	1.159	1.425	1.023	4.923	1.990	1.178
0.70	0.779	1.301	0.908	1.062	1.397	1.004	1.239	1.447	1.056	5.081	2.005	1.191
0.80	0.982	1.372	1.011	1.192	1.435	1.060	1.313	1.467	1.085	5.216	2.017	1.202
0.90	1.205	1.438	1.117	1.320	1.469	1.114	1.382	1.484	1.113	5.336	2.027	1.212
1.00	1.446	1.500	1.228	1.446	1.500	1.166	1.446	1.500	1.138	5.443	2.037	1.220

5. Conclusions

In the investigation of FDEs and FDIs that contain Hilfer fractional derivative operators, a zero initial condition is typically required. To address this limitation, we proposed a novel approach that combines Hilfer and Caputo fractional derivatives. In this research, we applied this method to study a class of FDEs for FDIs with non-separated BCs, incorporating both Hilfer and Caputo fractional derivative operators. The existence results are established by examining cases where the set-valued map has either convex or nonconvex values. For convex SVMs, the Leray-Schauder FPT was applied, whereas Nadler's and Covitz's FPTs are used for nonconvex SVMs. The findings are well demonstrated with two relevant illustrative examples. The findings of this study contribute significantly to the emerging field of FDIs. In future work, we aim to apply this method to study other types of FDEs with nonzero initial conditions, as well as coupled systems of FDEs that incorporate both Hilfer and Caputo FDs.

Availability of data and material

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Author contributions

Adel Lachouri: Actualization, methodology, formal analysis, validation, investigation, initial draft and a major contribution to writing the manuscript. Naas Adjimi: Actualization, methodology, formal analysis, validation, investigation and review. Mohammad Esmael Samei: Actualization, methodology, formal analysis, validation, investigation, software, simulation, review and a major contribution to writing the manuscript. Manuel De la Sen: Validation, review, funding. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. S. Rogosin, M. Karapiyenya, Fractional models for analysis of economic risks, *Fract. Calc. Appl. Anal.*, **26** (2023), 2602–2617. https://doi.org/10.1007/s13540-023-00202-y

2. B. Acay, E. Bas, T. Abdeljawad, Fractional economic models based on market equilibrium in the frame of different type kernels, *Chaos Soliton. Fract.*, **130** (2020), 109438. https://doi.org/10.1016/j.chaos.2019.109438
3. R. P. Chaouhan, S. Kumar, B. S. T. Alkahtani, S. S. Alzaid, A study on fractional order financial model by using Caputo-Fabrizio derivative, *Results Phys.*, **57** (2024), 107335. https://doi.org/10.1016/j.rinp.2024.107335
4. L. Debnath, Recent applications of fractional calculus to science and engineering, *Int. J. Math. Math. Sci.*, **2003** (2003), 3413–3442. https://doi.org/10.1155/S0161171203301486
5. J. Alzabut, S. R. Grace, S. S. Santra, M. E. Samei, Oscillation of even-order nonlinear dynamic equations with sublinear and superlinear neutral terms on time scales, *Qual. Theory Dyn. Syst.*, **23** (2024), 103. https://doi.org/10.1007/s12346-024-00961-w
6. S. T. M. Thabet, M. Vivas-Cortez, I. Kedim, M. E. Samei, M. I. Ayari, Solvability of ϱ-Hilfer fractional snap dynamic system on unbounded domains, *Fractals Fract.*, **7** (2023), 607. https://doi.org/10.3390/fractalfract7080607
7. R. Magin, Fractional calculus in bioengineering, part 1, *Criti. Rev. Biomed. Eng.*, **32** (2004), 1–104. https://doi.org/10.1615/CritRevBiomedEng.v32.i1.10
8. A. S. Alshehry, S. Mukhtar, A. M. Mahnashi, Analytical methods in fractional biological population modeling: unveiling solitary wave solutions, *AIMS Math.*, **9** (2024), 15,966–15,987. https://doi.org/10.3934/math.2024773
9. B. Mohammadaliee, V. Roomi, M. E. Samei, SEIARS model for analyzing COVID-19 pandemic process via ψ-Caputo fractional derivative and numerical simulation, *Sci. Rep.*, **14** (2024), 723. https://doi.org/10.1038/s41598-024-51415-x
10. R. Hilfer, *Applications of fractional calculus in physics*, Singapore: World Scientific, 2000. https://doi.org/10.1142/3779
11. C. Thaiprayoon, W. Sudsutad, J. Alzabut, S. Etemad, S. Rezapour, On the qualitative analysis of the fractional boundary value problem describing thermostat control model via ψ-Hilfer fractional operator, *Adv. Differ. Equ.*, **2021** (2021), 201. https://doi.org/10.1186/s13662-021-03359-z
12. D. Baleanu, J. A. T. Machado, A. C. J. Luo, *Fractional dynamics and control*, Springer, 2002.
13. J. V. C. Sousa, E. C. D. Oliveira, On the ϑ-Hilfer fractional derivative, *Commun. Nonlinear Sci. Numer. Simul.*, **60** (2018), 72–91. https://doi.org/10.1016/j.cnsns.2018.01.005
14. S. Asawasamrit, A. Kijjathanakorn, S. K. Ntouyas, J. Tariboon, Nonlocal boundary value problems for Hilfer fractional differential equations, *Bull. Korean Math. Soc.*, **55** (2018), 1639–1657. https://doi.org/10.4134/BKMS.b170887
15. A. Samadi, S. K. Ntouyas, J. Tariboon, Mixed Hilfer and caputo fractional Riemann-Stieltjes integro-differential equations with non-separated boundary conditions, *Mathematics*, **12** (2024), 1361. https://doi.org/10.3390/math12091361
16. I. Ahmed, S. K. Ntouyas, J. Tariboon, Separated boundary value problems via quantum Hilfer and Caputo operators. *AIMS Math.*, **9** (2024), 19473–19494. https://doi.org/10.3934/math.2024949

17. A. Lachouri, M. S. Abdo, A. Ardjouni, S. Etemad, S. Rezapou, A generalized neutral-type inclusion problem in the frame of the generalized Caputo fractional derivatives, *Adv. Differ. Equ.*, **2021** (2021), 404. https://doi.org/10.1186/s13662-021-03559-7
18. S. Surang, S. K. Ntouyasm, S. Ayub, T. Jessada, Boundary value problems for ψ-Hilfer type sequential fractional differential equations and inclusions with integral multi-point boundary conditions, *Mathematics*, **9** (2021), 1001. https://doi.org/10.3390/math9091001
19. S. Etemad, I. Iqbal, M. E. Samei, S. Rezapour, J. Alzabut, W. Sudsutad, et al., Some inequalities on multi-functions for applying fractional caputo-hadamard jerk inclusion system, *J. Inequal. Appl.*, **2022** (2022), 84. https://doi.org/10.1186/s13660-022-02819-8
20. F. Jarad, T. Abdeljawad, Generalized fractional derivatives and Laplace transform, *Discrete Contin. Dyn. Syst.-Ser. S*, **13** (2019), 709–722. https://doi.org/10.3934/dcdss.2020039
21. R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci. Numer. Simul.*, **44** (2017), 460–481. https://doi.org/10.1016/j.cnsns.2016.09.006
22. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach Science Publishers, 1993.
23. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Vol. 204, Elsevier, 2006,
24. M. Aitalioubrahim, S. Sajid, Higher-order boundary value problems for Carathéodory differential inclusions, *Miskolc Math. Notes*, **9** (2008), 7–15. https://doi.org/ 10.18514/MMN.2008.180
25. K. Deimling, *Multivalued differential equations*, De Gruyter, 1992. https://doi.org/10.1515/9783110874228
26. A. Lasota, Z. Opial, An application of the Kakutani–Ky fan theorem in the theory of ordinary differential equations, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astoronom. Phys.*, **13** (1955), 781–786.
27. W. V. Petryshyn, P. M. Fitzpatrick, A degree theory, fixed point theorems, and mapping theorems for multivalued noncompact maps, *Trans. Amer. Math. Soc.*, **194** (1974), 1–25. https://doi.org/10.2307/1996791
28. H. Covitz, S. B. Nadler, Multivalued contraction mappings in generalized metric spaces, *Israel J. Math.*, **8** (1970), 5–11. https://doi.org/10.1007/BF02771543
29. M. Kisielewicz, *Differential inclusions and optimal control*, Vol. 44, Springer Dordrecht, 1991.
30. C. Castaing, M. Valadier, *Convex analysis and measurable multifunction*, Lecture Notes in Mathematics, Springer, 1977.