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Research article

Results on homoclinic solutions of a partial difference equation involving the mean curvature operator

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Abstract: By variational technique coupled with the mountain pass lemma and fountain theorem, we investigate a second-order partial difference equation involving the mean curvature operator in the present paper. We establish a number of criteria to guarantee the existence of multiple nontrivial homoclinic solutions. Our results generalize and improve some known ones. Additionally, two examples are provided to demonstrate applications of our obtained results.

Keywords: variational technique; mountain pass lemma; fountain theorem; partial difference equation; homoclinic solution; mean curvature operator
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1. Introduction

In the present paper, our goal is to investigate the existence and multiplicity of nontrivial homoclinic solutions of the following second-order partial difference equation involving the mean curvature operator with a parameter $\lambda > 0$:

$$-\Delta_1\left[\phi_c\left(\Delta_1 u(k-1,l)\right)\right]-\Delta_2\left[\phi_c\left(\Delta_2 u(k,l-1)\right)\right]+b(k,l)u(k,l)=\lambda f\left((k,l),u(k,l)\right),\quad (k,l)\in\mathbb{Z}^2. \quad (1.1)$$

Here, forward difference operators are given by $\Delta_1 u(k,l) = u(k+1,l)-u(k,l)$ and $\Delta_2 u(k,l) = u(k,l+1)-u(k,l)$. For all $s \in \mathbb{R}$, let the mean curvature operator be denoted by $\phi_c(s) = \frac{s}{\sqrt{1+s^2}}$ and $\Phi_c(s) := \int_0^s \phi_c(t)dt = \sqrt{1+s^2}-1$. For all $(k,l) \in \mathbb{Z}^2$, we assume that the nonlinear term $f((k,l),u)$ is continuous in u with $F\left((k,l),u\right) = \int_0^u f\left((k,l),\tau\right)d\tau$ and fulfills the following basic hypotheses:
(F_1) $\lim\limits_{|u|\to 0}\frac{f((k,l),u)}{u} = 0$;

(F_2) $\lim\limits_{|u|\to+\infty} \frac{F((k,l),u)}{u^2} = \infty$;
(F_3) there exist $\alpha > 2$ and $d > 0$ such that $|f((k,l),u)| \leq d|u|^{\alpha-1}$ for all $u \in \mathbb{R}$;
(F_4) there exists a constant $\beta \geq 1$ such that $\beta G((k,l),u) \geq G((k,l),\nu u)$, here $u \in \mathbb{R}$, $\nu \in [0,1]$ and $G((k,l),u) = f((k,l),u)u - 2F((k,l),u)$;
(F_5) $\sup\limits_{|u|\leq T} |F((k,l),u)| \in l^1$ for all $T > 0$;
(F_6) $f((k,l),-u) = -f((k,l),u)$ for all $u \in \mathbb{R}$.
Further, we assume that the potential $b : \mathbb{Z}^2 \to \mathbb{R}$ satisfies
(B) $b(k,l) \geq b_* > 0$, $\lim\limits_{|k|+|l|\to\infty} b(k,l) = +\infty$, where $b_* = \min\{b(k,l) : (k,l) \in \mathbb{Z}^2\}$.

Difference equations, regarded as discrete analogues of differential equations, have been used extensively in a variety of fields, including biology, economics, computer science, machine learning, artificial intelligence, and other fields over the last few decades [1–3]. It has sparked a lot of interest and attention from academics. And abundant research results on various aspects have been acquired, such as periodic solutions [4, 5], boundary value problems [6, 7], homoclinic solutions [8, 9], heteroclinic solutions [10, 11], etc. It is worthy of pointing out that the theory of difference equations developed quickly as a result of the groundbreaking work of Guo and Yu [12], who were the first to study difference equations using the variational method and critical point theory.

It is widely acknowledged that difference equations play an important role in mathematical modeling for real-world problem solving. As our time continues to progress, more and more factors need to be taken into account in many aspects of our lives. Consequently, partial difference equations with two or more variables appear in a wide range of domains, including fluid dynamics, mechanical engineering analysis, population growth, quantum mechanics, and image processing [13–16]. In this situation, studying partial difference equations makes sense, and numerous significant works have been published recently. Here mention a couple; Long reviewed the discrete Kirchhoff-type problems and provided multiple results on nontrivial solutions [17, 18] and least energy sign-changing solutions [19]. Homoclinic solutions for partial difference equations with p-Laplacian were displayed in [20]. Very recently, the nonexistence and existence of periodic solutions for partial difference equations were presented in [21, 22].

The mean curvature operator, involved in Eq (1.1), has broad applicability and important theoretical significance. It has been widely used to describe a variety of problems. For example, the dynamic problem of combustible gas [23], the capillarity problem in hydrodynamics [24], and the flux-limited diffusion phenomenon [25]. We refer the reader to [26] for more detail. Much like partial difference equations with p-Laplacian, partial difference equations involving mean curvature operators and their various modified forms have drawn a lot of attention in recent decades. For instance, given parameter $\lambda > 0$, Wang and Zhou [27] considered Eq (1.1) with Dirichlet boundary conditions and obtained at least three solutions.

It is evident from the above-mentioned results that partial difference equations are widely applied and thoroughly researched. However, most given results are dependent on finite-dimensional space. Meanwhile, homoclinic solutions, without periodic condition assumptions, are studied in an infinite-dimensional space, which brings us an obstacle to overcome the lack of compactness of corresponding variational functional. There is, of course, comparatively less work. Moreover, Eq (1.1) involves both the parameter λ and the ϕ_c-Laplacian, which both make the study more challenging and complex. Therefore, it will be interesting and significant to deal with homoclinic solutions of Eq (1.1).

Inspired by the aforementioned reasons, we shall manage to investigate the existence and multiplicity of homoclinic solutions of Eq (1.1) by critical point theory. As usual, we say $u = \{u(k,l)\}_{(k,l)\in\mathbb{Z}^2} \neq 0$ is a nontrivial homoclinic solution of Eq (1.1) refers to u solving Eq (1.1) and satisfying

$$u(k,l) \to 0 \qquad \text{as} \quad |k|+|l| \to +\infty.$$

Now we are in a position to state our main results, which read as follows.

Theorem 1.1. *If (B) and (F_1)–(F_4) are valid, then Eq (1.1) admits at least one nontrivial homoclinic solution for all $\lambda > \frac{4+b_*}{2b_*}$.*

Theorem 1.2. *If (B) and (F_2)–(F_6) are satisfied, then Eq (1.1) has infinitely many homoclinic solutions for all $\lambda > 0$.*

We also display the following remarks to demonstrate that our obtained results are easier to verify and more widely applied.

Remark 1.1. *In [28], the authors studied*

$$-\Delta(\phi_c(\Delta u(k-1))) + b(k)u(k) = f(k,u(k)), \qquad k \in \mathbb{Z}, \tag{1.2}$$

and obtained some similar results to Theorems 1.1 and 1.2. Obviously, (1.2) is a very special case of Eq (1.1). And our results improve and generalize those in [28].

Remark 1.2. *The nonlinearity F can be sign-changing, which is much weaker and more applicable than the nonnegative case ($F((k,l),u) \geq 0$ for all $(k,l) \in \mathbb{Z}^2$ and $u \in \mathbb{R}$) in many related articles, for example, [29–31].*

Remark 1.3. *As known to all, either a classical Ambrosetti-Rabinowitz condition*
(AR) there exist constants $\mu > 2$ and $d_0 > 0$ such that

$$0 < \mu F(k,u) \leq uf(k,u) \qquad \textit{for} \quad (k,u) \in \mathbb{Z}\times\mathbb{R} \quad \textit{and} \quad |u| \geq d_0,$$

or a generalization,
$(AR)'$ there exists $\mu > 2$ and $d_0 > 0$ such that

$$0 < \mu F((k,l),u) \leq uf((k,l),u) \qquad \textit{for} \quad ((k,l),u) \in \mathbb{Z}^2\times\mathbb{R} \quad \textit{and} \quad |u| \geq d_0, \tag{1.3}$$

plays a vital role in the application of critical point theory to ensure that any $(C)_c$ sequence of the corresponding energy functional is bounded. Meanwhile, (AR) or $(AR)'$ indicates that there exist r_1, $r_2 > 0$ such that

$$F((k,l),u) \geq r_1|u|^\mu - r_2 \qquad \textit{for} \quad ((k,l),u) \in \mathbb{Z}^2\times\mathbb{R}.$$

That is (F_2). However, it is not necessary for F to fulfill $(AR)'$ in many practical problems. For instance, for $((k,l),u) \in \mathbb{Z}^2\times\mathbb{R}$, let

$$F((k,l),u) = \frac{1}{2}u^2\ln(1+|u|) - \frac{1}{2}\left(\ln(1+|u|) + \frac{1}{2}u^2 - |u|\right).$$

Then

$$G((k,l),u) = f((k,l),u)u - 2F((k,l),u) = \ln(1+|u|) + \frac{1}{2}u^2 - |u|.$$

We depict them as Figure 1 for the reader's convenience.

It can be certified that $F((k,l),u)$ does not satisfy $(AR)'$. But it does satisfy (F_1)–(F_6). In this sense, our results release $(AR)'$ or (AR) somehow and improve some existing results.

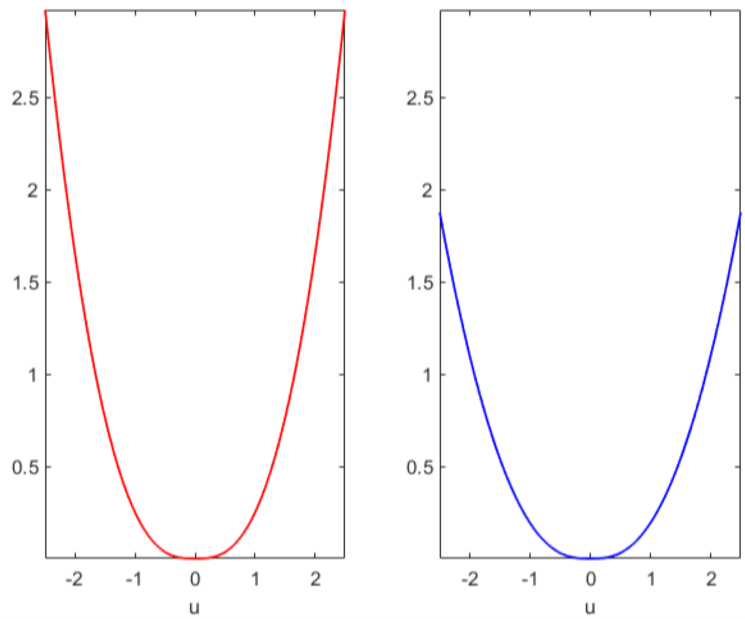

Figure 1. The images of $F((k,l),u)$ and $G((k,l),u)$ respectively.

The structure of this paper is developed as follows: In Section 2, we introduce some basic results and establish the variational structure related to Eq (1.1). Moreover, two essential lemmas, including the compactness of the $(C)_c$ sequence and the compact embedding of spaces X to space l^2, are provided. Detailed proofs of main results are presented in Section 3. Finally, we illustrate our results with two examples in Section 4 as an end.

2. Variational structure and some auxiliary results

To prove the main results, we construct the corresponding variational functional of Eq (1.1) and provide some essential lemmas.

For $1 \le r < +\infty$, denote the set of all functions $u : \mathbb{Z}^2 \to \mathbb{R}$ by

$$l^r \equiv l^r(\mathbb{Z}^2) = \{u = \{u(k,l)\} : (k,l) \in \mathbb{Z}^2, u(k,l) \in \mathbb{R}, \sum_{(k,l)\in\mathbb{Z}^2} |u(k,l)|^r < +\infty\}.$$

For any $u \in l^r$, we define

$$\|u\|_{l^r}^r = \sum_{(k,l)\in\mathbb{Z}^2} |u(k,l)|^r < +\infty,$$

and let l^∞ represent the set of all functions $u : \mathbb{Z}^2 \to \mathbb{R}$ such that

$$\|u\|_\infty = \sup_{(k,l)\in\mathbb{Z}^2} |u(k,l)| < +\infty.$$

Then $(l^r, \|\cdot\|_{l^r})$ is a reflexive Banach space. Moreover, for $1 \leq \kappa \leq \iota \leq +\infty$, there holds

$$l^\kappa \subseteq l^\iota \quad \text{and} \quad \|u\|_{l^\iota} \leq \|u\|_{l^\kappa}. \tag{2.1}$$

Denoted reflexive Banach space by $(X, \|\cdot\|_X)$, where

$$X = \{u \in l^2 : \sum_{(k,l)\in\mathbb{Z}^2} b(k,l)|u(k,l)|^2 < +\infty\},$$

and

$$\|u\|_X^2 = \sum_{(k,l)\in\mathbb{Z}^2} b(k,l)|u(k,l)|^2 \qquad \forall u \in X.$$

Then we have

$$\|u\|_\infty \leq \|u\|_{l^2} \leq \frac{1}{\sqrt{b_*}}\|u\|_X \qquad \forall u \in X. \tag{2.2}$$

For $\lambda > 0$, define an energy functional $I_\lambda : X \to \mathbb{R}$, associated to Eq (1.1), as

$$\begin{aligned} I_\lambda(u) =& \sum_{(k,l)\in\mathbb{Z}^2}[\Phi_c(\Delta_1 u(k-1,l)) + \Phi_c(\Delta_2 u(k,l-1))] + \frac{1}{2}\sum_{(k,l)\in\mathbb{Z}^2} b(k,l)|u(k,l)|^2 - \lambda\sum_{(k,l)\in\mathbb{Z}^2} F((k,l),u(k,l)) \\ :=& \Phi(u) - \lambda\Psi(u), \qquad \forall u \in X. \end{aligned} \tag{2.3}$$

Clearly

$$\Phi(u) = \sum_{(k,l)\in\mathbb{Z}^2}[\Phi_c(\Delta_1 u(k-1,l)) + \Phi_c(\Delta_2 u(k,l-1))] + \frac{1}{2}\sum_{(k,l)\in\mathbb{Z}^2} b(k,l)|u(k,l)|^2,$$

$$\Psi(u) = \sum_{(k,l)\in\mathbb{Z}^2} F((k,l),u(k,l)).$$

Then definitions of Φ and Ψ guarantee that Φ and Ψ are continuously Gateaux differentiable. Thus, $I_\lambda \in C^1(X,\mathbb{R})$. Moreover, for any $u, v \in X$, direct computation yields that

$$\begin{aligned} \Phi'(u)(v) =& \lim_{\tau\to 0^+}\frac{\Phi(u+\tau v) - \Phi(u)}{\tau} \\ =& \sum_{(k,l)\in\mathbb{Z}^2}\phi_c(\Delta_1 u(k-1,l))\Delta_1 v(k-1,l) + \sum_{(k,l)\in\mathbb{Z}^2}\phi_c(\Delta_2 u(k,l-1))\Delta_2 v(k,l-1) + \sum_{(k,l)\in\mathbb{Z}^2} b(k,l)u(k,l)v(k,l) \\ =& \sum_{(k,l)\in\mathbb{Z}^2}\phi_c(\Delta_1 u(k-1,l))v(k,l) - \sum_{(k,l)\in\mathbb{Z}^2}\phi_c(\Delta_1 u(k-1,l))v(k-1,l) + \sum_{(k,l)\in\mathbb{Z}^2}\phi_c(\Delta_2 u(k,l-1))v(k,l) \\ & - \sum_{(k,l)\in\mathbb{Z}^2}\phi_c(\Delta_2 u(k,l-1))v(k,l-1) + \sum_{(k,l)\in\mathbb{Z}^2} b(k,l)u(k,l)v(k,l) \\ =& -\sum_{(k,l)\in\mathbb{Z}^2}[\Delta_1\phi_c(\Delta_1 u(k-1,l)) + \Delta_2\phi_c(\Delta_2 u(k,l-1))]v(k,l) + \sum_{(k,l)\in\mathbb{Z}^2} b(k,l)u(k,l)v(k,l), \end{aligned}$$

and

$$\Psi'(u)(v) = \lim_{\tau\to 0^+}\frac{\Psi(u+\tau v) - \Psi(u)}{\tau} = \sum_{(k,l)\in\mathbb{Z}^2} f((k,l),u(k,l))v(k,l).$$

Namely,

$$\begin{aligned}\langle I'_\lambda(u), v\rangle = &-\sum_{(k,l)\in\mathbb{Z}^2}[\Delta_1\phi_c(\Delta_1 u(k-1,l)) + \Delta_2\phi_c(\Delta_2 u(k,l-1))]v(k,l)\\ &+\sum_{(k,l)\in\mathbb{Z}^2} b(k,l)u(k,l)v(k,l) - \lambda\sum_{(k,l)\in\mathbb{Z}^2} f((k,l),u(k,l))v(k,l).\end{aligned} \tag{2.4}$$

Since $v \in X$ is arbitrary, (2.4) implies $\langle I'_\lambda(u), v\rangle = 0$ if and only if

$$-\Delta_1[\phi_c(\Delta_1 u(k-1,l))] - \Delta_2[\phi_c(\Delta_2 u(k,l-1))] + b(k,l)u(k,l) = \lambda f((k,l),u(k,l)),$$

which is exactly Eq (1.1). Consequently, it is sufficient to seek nonzero critical points of I_λ (defined by (2.3)) to obtain nontrivial homoclinic solutions of Eq (1.1).

For $I_\lambda \in C^1(X,\mathbb{R})$, we list the definition of the $(C)_c$ condition and two famous theorems that are essential tools to verify our main results.

Definition 2.1. *Let I be a C^1-functional defined on a real reflexive Banach space X. A Cerami sequence ($(C)_c$ sequence for short) refers to a sequence $\{u_n\} \subset X$ that fulfills*

$$I(u_n) \to c \quad \textit{and} \quad (1 + \|u_n\|)\|I'(u_n)\| \to 0 \qquad \textit{as} \quad n \to \infty.$$

We say that I satisfies the $(C)_c$ condition in X if any $(C)_c$ sequence for I has a convergent subsequence for some $c \in \mathbb{R}$.

Lemma 2.1. *(Mountain Pass Lemma [32]) Let I be a C^1-functional defined on a real reflexive Banach space X. Suppose that*
(J_1) there exist $\gamma, \widetilde{a} > 0$ such that $I(u) \geq \widetilde{a}$ for $\|u\| = \gamma$;
(J_2) there is an $e \in X$ with $\|e\| \geq \gamma$ such that $I(e) \leq 0$.
Then I possesses a $(C)_c$ sequence with $c = \inf_{g\in\Gamma}\max_{s\in[0,1]} I(g(s)) \geq \widetilde{a}$, where

$$\Gamma = \{g \in C([0,1], X) : g(0) = 0, g(1) = e\}.$$

Further, I attains a critical value c if the $(C)_c$ condition is met.

Before introducing the fountain theorem, we give some notations to help illustrate the content. Let reflexive Banach space X be separable. Then for every $k \in \mathbb{N}$, there is a finite dimensional space $X_k \subset X$ such that $X = \overline{\bigoplus_{k\in\mathbb{N}} X_k}$. Set

$$S_n = \bigoplus_{k=0}^{n} X_k \quad \text{and} \quad T_n = \overline{\bigoplus_{k=n}^{\infty} X_k}.$$

Lemma 2.2. *(Fountain theorem [33]) Suppose that $I \in C^1(X,\mathbb{R})$ fulfills the $(C)_c$ condition for some $c \in \mathbb{R}$ and $I(-u) = I(u)$ for $u \in X$. For every $n \in \mathbb{N}$, if there exist $\rho_n > r_n > 0$ satisfying*
(i) $H_n := \inf_{u\in T_n, \|u\|_X = r_n} I(u) \to \infty$ as $n \to \infty$;
(ii) $I_n := \max_{u\in S_n, \|u\|_X = \rho_n} I(u) \leq 0$.
Then I has an unbounded sequence of critical values.

Lemma 2.3. *Assume that condition (B) holds. Then the embedding $X \hookrightarrow l^2$ is compact.*

Proof. Let $\{u_n\} \subset X$ be bounded. By the Banach-Steinhaus theorem, we obtain that $\sup_{n\in\mathbb{N}} \|u_n\|_X < \infty$. Then there is a constant $M_0 > 0$ such that

$$\|u_n\|_X^2 = \sum_{(k,l)\in\mathbb{Z}^2} b(k,l)|u_n(k,l)|^2 \leq M_0 \qquad \text{for all} \quad n \in \mathbb{N}.$$

Taking a subsequence, still denoted by $\{u_n\}$, we have

$$u_n \rightharpoonup u \qquad \text{in} \quad X.$$

Without loss of generality, we assume $u = 0$, in particular $u_n(k,l) \to 0$ as $n \to +\infty$ for all $(k,l) \in \mathbb{Z}^2$. In view of (B), for any $\varepsilon > 0$, there is $N_0 \in \mathbb{N}$ such that

$$b(k,l) \geq \frac{1 + M_0}{\varepsilon} \qquad \text{for all} \quad |k| + |l| > N_0.$$

Then

$$\sum_{|k|+|l|>N_0} |u_n(k,l)|^2 \leq \frac{\varepsilon}{1 + M_0} \sum_{|k|+|l|>N_0} b(k,l)|u_n(k,l)|^2 \leq \frac{M_0}{1 + M_0}\varepsilon. \tag{2.5}$$

Additionally, the continuity of the finite sum implies that there exists $N_1 \in \mathbb{N}$ such that

$$\sum_{|k|+|l|\leq N_0} |u_n(k,l)|^2 \leq \frac{\varepsilon}{1 + M_0} \qquad \text{for all} \quad n > N_1. \tag{2.6}$$

Choosing $N = \max\{N_0, N_1\}$ and jointing (2.5) with (2.6), it follows that

$$\sum_{(k,l)\in\mathbb{Z}^2} |u_n(k,l)|^2 = \sum_{|k|+|l|>N} |u_n(k,l)|^2 + \sum_{|k|+|l|\leq N} |u_n(k,l)|^2 \leq \frac{M_0}{1 + M_0}\varepsilon + \frac{\varepsilon}{1 + M_0} = \varepsilon,$$

which means that $u_n \to 0$ in l^2. Therefore, the embedding $X \hookrightarrow l^2$ is compact. □

Lemma 2.4. *Assume (B) and (F_2)–(F_4) hold. Then for $c \in \mathbb{R}$, I_λ satisfies the $(C)_c$ condition.*

Proof. Set $\{u_n\}$ be a $(C)_c$ sequence of I_λ, that is,

$$I_\lambda(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|_X)\|I'_\lambda(u_n)\|_X \to 0 \qquad \text{as} \quad n \to \infty. \tag{2.7}$$

First, for any fixed $\lambda > 0$, we are to show that $\{u_n\}$ is bounded in X. Arguing indirectly, we assume $\|u_n\|_X \to +\infty$ as $n \to \infty$. Let $w_n = u_n/\|u_n\|_X$, then $\|w_n\|_X = 1$, which means that $\{w_n\}$ is bounded in X and has a weakly convergence subsequence. Suppose that there is $w \in X$ such that $w_n \rightharpoonup w$ in X. In view of Lemma 2.3, we have

$$w_n \to w \qquad \text{in} \quad l^2.$$

By (F_3), for all $(k,l) \in \mathbb{Z}^2$ and $u \in \mathbb{R}$, simple calculation gives that

$$|F((k,l),u)| = \left|\int_0^u f((k,l),\tau)d\tau\right| = \left|\int_0^1 uf((k,l),u\tau)d\tau\right| \leq d\int_0^1 |u||u\tau|^{\alpha-1}d\tau \leq \frac{d}{\alpha}|u|^\alpha. \tag{2.8}$$

Case 1. We show $w \neq 0$ is impossible. Let $\Lambda = \{(k,l) \in \mathbb{Z}^2 : w(k,l) \neq 0\}$. Due to $I_\lambda(u_n) \to c$ as $n \to \infty$, there exists a constant C such that $I_\lambda(u_n) \geq C$. Moreover, $\Phi_c(s) := \sqrt{1+s^2} - 1$ means that $\Phi_c(u) \leq \frac{1}{2}u^2$. Combining with (2.2), we obtain

$$\begin{aligned} C \leq& I_\lambda(u_n) \\ \leq& \frac{1}{2}\left(\sum_{(k,l)\in\mathbb{Z}^2} |\Delta_1 u_n(k-1,l)|^2 + \sum_{(k,l)\in\mathbb{Z}^2} |\Delta_2 u_n(k,l-1)|^2\right) + \frac{1}{2}\sum_{(k,l)\in\mathbb{Z}^2} b(k,l)|u_n(k,l)|^2 - \lambda \sum_{(k,l)\in\mathbb{Z}^2} F((k,l),u_n(k,l)) \\ \leq& \left(\frac{4}{b_*} + \frac{1}{2}\right)\|u_n\|_X^2 - \lambda \sum_{(k,l)\in\mathbb{Z}^2} F((k,l),u_n(k,l)). \end{aligned} \tag{2.9}$$

Dividing both sides of (2.9) by $\|u_n\|_X^2$, it follows that

$$\lambda \sum_{(k,l)\in\mathbb{Z}^2} \frac{F((k,l),u_n(k,l))}{\|u_n\|_X^2} \leq \frac{4}{b_*} + \frac{1}{2} - \frac{C}{\|u_n\|_X^2} < +\infty. \tag{2.10}$$

However, fixing $(k_0,l_0) \in \Lambda$, we have

$$u_n(k_0,l_0) = w_n(k_0,l_0)\|u_n\|_X \to +\infty \qquad \text{as} \quad n \to \infty.$$

Thus, (F_2) means that

$$\frac{F((k_0,l_0),u_n(k_0,l_0))}{\|u_n\|_X^2} = \frac{F((k_0,l_0),u_n(k_0,l_0))}{u_n^2(k_0,l_0)} w_n^2(k_0,l_0) \to +\infty,$$

which is a contradiction of (2.10). Subsequently, $w \neq 0$ is absurd.

Case 2. We verify $w = 0$ is false. Define

$$I_\lambda(\nu_n u_n) := \max_{\nu\in[0,1]} I_\lambda(\nu u_n).$$

Recall $\|u_n\|_X \to +\infty$ as $n \to \infty$. Then, for any $M > 1$, there is an $n \in \mathbb{N}$ sufficiently large such that $\|u_n\|_X \geq 2M^{\frac{1}{2}}$. Write

$$\widetilde{w_n} = 2M^{\frac{1}{2}} w_n = 2M^{\frac{1}{2}} \frac{u_n}{\|u_n\|_X}.$$

Then there holds

$$\begin{aligned} I_\lambda(\nu_n u_n) =& \max_{\nu\in[0,1]} I_\lambda(\nu u_n) \geq I_\lambda\left(\frac{2M^{\frac{1}{2}}}{\|u_n\|_X} u_n\right) \\ =& \sum_{(k,l)\in\mathbb{Z}^2} [\Phi_c(\Delta_1 \widetilde{w_n}(k-1,l)) + \Phi_c(\Delta_2 \widetilde{w_n}(k,l-1))] \\ &+ \frac{1}{2}\sum_{(k,l)\in\mathbb{Z}^2} b(k,l)|\widetilde{w_n}(k,l)|^2 - \lambda \sum_{(k,l)\in\mathbb{Z}^2} F((k,l),\widetilde{w_n}(k,l)) \\ \geq& 2M - \lambda \sum_{(k,l)\in\mathbb{Z}^2} F((k,l),\widetilde{w_n}(k,l)). \end{aligned} \tag{2.11}$$

Moreover, (2.8) and $w_n \to w$ in l^2 ensure that

$$\sum_{(k,l)\in\mathbb{Z}^2} F((k,l),\widetilde{w_n}(k,l)) \leq \frac{d}{\alpha}\|\widetilde{w_n}\|_{l^2}^{\alpha} \to 0 \qquad \text{as} \quad n\to\infty. \tag{2.12}$$

Hence, combining (2.11) with (2.12), we obtain

$$I_\lambda(\nu_n u_n) \geq M, \qquad n \quad \text{large enough}. \tag{2.13}$$

Consequently, by the arbitrariness of M, (2.13) means

$$I_\lambda(\nu_n u_n) \to \infty \qquad \text{as} \quad n\to\infty. \tag{2.14}$$

Since $I_\lambda \in C^1(X,\mathbb{R})$ and $I_\lambda(\nu_n u_n) = \max_{\nu\in[0,1]} I_\lambda(\nu u_n)$, $I_\lambda(\nu u_n)$ reaches its maximum at $\nu_n \in (0,1)$ as n is large enough. Thus,

$$\langle I'_\lambda(\nu_n u_n), u_n\rangle = 0. \tag{2.15}$$

On the other side, based on the fact that $\Phi_c(u) - \frac{1}{2}\phi_c(u)u = \sqrt{1+u^2} - 1 - \frac{1}{2}\frac{u^2}{\sqrt{1+u^2}}$ is an even function and increases in $[0,\infty)$, (2.15), and (F_4) indicate that

$$\begin{aligned}
I_\lambda(\nu_n u_n) =& I_\lambda(\nu_n u_n) - \frac{1}{2}\langle I'_\lambda(\nu_n u_n), \nu_n u_n\rangle\\
=& \sum_{(k,l)\in\mathbb{Z}^2}\left[\Phi_c(\nu_n\Delta_1 u_n(k-1,l)) - \frac{1}{2}\phi_c(\nu_n\Delta_1 u_n(k-1,l))\nu_n\Delta_1 u_n(k-1,l)\right]\\
&+ \sum_{(k,l)\in\mathbb{Z}^2}\left[\Phi_c(\nu_n\Delta_2 u_n(k,l-1)) - \frac{1}{2}\phi_c(\nu_n\Delta_2 u_n(k,l-1))\nu_n\Delta_2 u_n(k,l-1)\right]\\
&+ \lambda\sum_{(k,l)\in\mathbb{Z}^2}\left[\frac{1}{2}f((k,l),\nu_n u_n(k,l))\nu_n u_n(k,l) - F((k,l),\nu_n u_n(k,l))\right]\\
\leq& \sum_{(k,l)\in\mathbb{Z}^2}\left[\Phi_c(\Delta_1 u_n(k-1,l)) - \frac{1}{2}\phi_c(\Delta_1 u_n(k-1,l))\Delta_1 u_n(k-1,l)\right]\\
&+ \sum_{(k,l)\in\mathbb{Z}^2}\left[\Phi_c(\Delta_2 u_n(k,l-1)) - \frac{1}{2}\phi_c(\Delta_2 u_n(k,l-1))\Delta_2 u_n(k,l-1)\right]\\
&+ \beta\lambda\sum_{(k,l)\in\mathbb{Z}^2}\left[\frac{1}{2}f((k,l),u_n(k,l))u_n(k,l) - F((k,l),u_n(k,l))\right]\\
\leq& \beta\left(I_\lambda(u_n) - \frac{1}{2}\langle I'_\lambda(u_n),u_n\rangle\right).
\end{aligned}$$

Then by (2.7), we deduce that

$$I_\lambda(\nu_n u_n) \leq \beta(I_\lambda(u_n) - \frac{1}{2}\langle I'_\lambda(u_n),u_n\rangle) \to \beta c \qquad \text{as} \quad n\to\infty,$$

which is a contradiction with (2.14). Hence, $w=0$ is also invalid. Therefore, $\|u_n\|_X \to +\infty$ as $n\to\infty$ is impossible. And $\{u_n\}$ is bounded.

Next, we intend to verify that the bounded sequence $\{u_n\}$ has a convergent subsequence. Notice that $\{u_n\}$ has a weakly convergent subsequence, which might still be denoted by $\{u_n\}$. Suppose $u_n \rightharpoonup u$ in X; by Lemma 2.3, we have

$$u_n \to u \qquad \text{in} \quad l^2.$$

Making use of (F_3) and the Hölder inequality, we have that

$$\begin{aligned}
&\left|\sum_{(k,l)\in\mathbb{Z}^2}[f((k,l),u_n(k,l))-f((k,l),u(k,l))][u_n(k,l)-u(k,l)]\right| \\
\le& d\sum_{(k,l)\in\mathbb{Z}^2}|u_n(k,l)|^{\alpha-1}|u_n(k,l)-u(k,l)| + d\sum_{(k,l)\in\mathbb{Z}^2}|u(k,l)|^{\alpha-1}|u_n(k,l)-u(k,l)| \\
\le& d\left(\sum_{(k,l)\in\mathbb{Z}^2}|u_n(k,l)|^{\alpha}\right)^{(\alpha-1)/\alpha}\left(\sum_{(k,l)\in\mathbb{Z}^2}|u_n(k,l)-u(k,l)|^{\alpha}\right)^{1/\alpha} \\
&+ d\left(\sum_{(k,l)\in\mathbb{Z}^2}|u(k,l)|^{\alpha}\right)^{(\alpha-1)/\alpha}\left(\sum_{(k,l)\in\mathbb{Z}^2}|u_n(k,l)-u(k,l)|^{\alpha}\right)^{1/\alpha} \\
\le& d\left(\|u_n\|_{l^\alpha}^{\alpha-1}+\|u\|_{l^\alpha}^{\alpha-1}\right)\|u_n-u\|_{l^2}.
\end{aligned} \tag{2.16}$$

Since $\{u_n\}$ is bounded in X and $u_n \to u$ in l^2, (2.1), (2.2), and (2.16) ensure that

$$\sum_{(k,l)\in\mathbb{Z}^2}[f((k,l),u_n(k,l))-f((k,l),u(k,l))][u_n(k,l)-u(k,l)] \to 0 \qquad \text{as} \quad n\to\infty. \tag{2.17}$$

Notice the fact that $\phi_c(u)$ is strictly increasing for all $u \in \mathbb{R}$, we obtain

$$(\phi_c(x)-\phi_c(y))(x-y) \ge 0 \qquad \text{for any} \quad x,y\in\mathbb{R}.$$

Hence,

$$\begin{aligned}
&\langle I'_\lambda(u_n)-I'_\lambda(u),u_n-u\rangle = \langle I'_\lambda(u_n),u_n-u\rangle - \langle I'_\lambda(u),u_n-u\rangle \\
=& \sum_{(k,l)\in\mathbb{Z}^2}[\phi_c(\Delta_1u_n(k-1,l))-\phi_c(\Delta_1u(k-1,l))][\Delta_1u_n(k-1,l)-\Delta_1u(k-1,l)] \\
&+ \sum_{(k,l)\in\mathbb{Z}^2}[\phi_c(\Delta_2u_n(k,l-1))-\phi_c(\Delta_2u(k,l-1))][\Delta_2u_n(k,l-1)-\Delta_2u(k,l-1)] \\
&+ \sum_{(k,l)\in\mathbb{Z}^2}b(k,l)|u_n(k,l)-u(k,l)|^2 \\
&- \lambda\sum_{(k,l)\in\mathbb{Z}^2}[f((k,l),u_n(k,l))-f((k,l),u(k,l))][u_n(k,l)-u(k,l)] \\
\ge& \sum_{(k,l)\in\mathbb{Z}^2}b(k,l)|u_n(k,l)-u(k,l)|^2 \\
&- \lambda\sum_{(k,l)\in\mathbb{Z}^2}[f((k,l),u_n(k,l))-f((k,l),u(k,l))][u_n(k,l)-u(k,l)].
\end{aligned}$$

That is,

$$\begin{aligned}&\sum_{(k,l)\in\mathbb{Z}^2} b(k,l)|u_n(k,l)-u(k,l)|^2\\ \leq&\langle I'_\lambda(u_n)-I'_\lambda(u),u_n-u\rangle+\lambda\sum_{(k,l)\in\mathbb{Z}^2}[f((k,l),u_n(k,l))-f((k,l),u(k,l))][u_n(k,l)-u(k,l)].\end{aligned}\tag{2.18}$$

Further, weak convergence and boundedness of the $(C)_c$ sequence in X guarantee that

$$\langle I'_\lambda(u_n)-I'_\lambda(u),u_n-u\rangle\to 0 \qquad \text{as} \quad n\to\infty. \tag{2.19}$$

Thanks to (2.17)–(2.19), we draw a conclusion that

$$\sum_{(k,l)\in\mathbb{Z}^2} b(k,l)|u_n(k,l)-u(k,l)|^2\to 0 \qquad \text{as} \quad n\to\infty,$$

which just is

$$||u_n-u||_X\to 0 \qquad \text{as} \quad n\to\infty.$$

Consequently, the above procedure of proof manifests that $I_\lambda : X\to\mathbb{R}$ satisfies the $(C)_c$ condition. And this completes the proof. □

3. Proofs of main results

In this section, we exhibit associated proofs of Theorems 1.1 and 1.2 at length.

Proof of Theorem 1.1. Owing to (F_1), we know that for any $\varepsilon>0$, there is $\delta>0$ such that

$$|F((k,l),u)|\leq\varepsilon|u|^2 \qquad \text{for all} \quad (k,l)\in\mathbb{Z}^2 \quad \text{and} \quad |u|\leq\delta. \tag{3.1}$$

Let ∂B_γ be the boundary of B_γ, where $B_\gamma(0)$ is an open ball with center 0 and radius $\gamma>0$.

Select $\gamma=||u||_X=\sqrt{b_*}\delta$. By (2.2), it follows that

$$||u||_\infty\leq(\sqrt{b_*})^{-1}||u||_X=\delta.$$

For all $u\in B_\gamma(0)\setminus\{0\}$, (3.1) implies that

$$\sum_{(k,l)\in\mathbb{Z}^2} F((k,l),u(k,l))\leq\varepsilon\sum_{(k,l)\in\mathbb{Z}^2}|u(k,l)|^2\leq\frac{\varepsilon}{b_*}\sum_{(k,l)\in\mathbb{Z}^2} b(k,l)|u(k,l)|^2=\frac{\varepsilon}{b_*}||u||_X^2.$$

Then by the definitions of I_λ and Φ_c, there holds

$$\begin{aligned}I_\lambda(u)\geq&\frac{1}{2}\sum_{(k,l)\in\mathbb{Z}^2} b(k,l)|u(k,l)|^2-\lambda\sum_{(k,l)\in\mathbb{Z}^2} F((k,l),u(k,l))\\ \geq&\frac{1}{2}||u||_X^2-\frac{\lambda\varepsilon}{b_*}||u||_X^2=(\frac{1}{2}-\frac{\lambda\varepsilon}{b_*})||u||_X^2.\end{aligned}\tag{3.2}$$

Take $\varepsilon = \frac{b_*}{4\lambda}$ in (3.2), we find that

$$I_\lambda(u) \geq \frac{1}{4}\gamma^2 = \widetilde{a} > 0 \qquad \text{for} \quad u \in \partial B_\gamma \triangleq \{u \in X : \|u\|_X = \gamma\},$$

which manifests that I_λ satisfies (J_1) of Lemma 2.1.

Next, we verify (J_2) is fulfilled. Since $b_* = \min\{b(k,l) : (k,l) \in \mathbb{Z}^2\}$, there is $(k_*, l_*) \in \mathbb{Z}^2$ satisfying $b(k_*, l_*) = b_*$. Define $e = \{e(k,l)\}$ by

$$e(k,l) = \begin{cases} 1, & \text{if} \quad (k,l) = (k_*, l_*), \\ 0, & \text{if} \quad (k,l) \neq (k_*, l_*). \end{cases}$$

By (F_2), there is $\xi > 0$ such that

$$F((k,l), u) \geq b_* u^2 \qquad \text{for} \quad |u| \geq \xi.$$

Let $u = te \in X$, for sufficiently large $|t|$ such that $|t| > \xi$, we obtain that

$$\begin{aligned} I_\lambda(te) =& \sum_{(k,l)\in\mathbb{Z}^2} [\Phi_c(\Delta_1 te(k-1,l)) + \Phi_c(\Delta_2 te(k,l-1)] + \sum_{(k,l)\in\mathbb{Z}^2} \frac{1}{2} t^2 b(k,l) e^2(k,l) - \lambda \sum_{(k,l)\in\mathbb{Z}^2} F((k,l), te(k,l)) \\ \leq& \sum_{(k,l)\in\mathbb{Z}^2} \left[\frac{1}{2}|\Delta_1 te(k-1,l)|^2 + \frac{1}{2}|\Delta_2 te(k,l-1)|^2\right] + \sum_{(k,l)\in\mathbb{Z}^2} \frac{1}{2} t^2 b(k,l) e^2(k,l) - \lambda \sum_{(k,l)\in\mathbb{Z}^2} F((k,l), te(k,l)) \\ \leq& 2t^2 + \frac{1}{2} b_* t^2 - \lambda b_* t^2 = (2 + \frac{1}{2} b_* - \lambda b_*) t^2. \end{aligned} \tag{3.3}$$

Remind $\lambda > \frac{4+b_*}{2b_*}$. Then (3.3) arrives that $I_\lambda(te) \to -\infty$, as $|t| \to \infty$. Thus, there exists $t_0 \in \mathbb{R}$ such that

$$\|t_0 e\|_X > \gamma \quad \text{and} \quad I_\lambda(t_0 e) < 0,$$

which means that I_λ satisfies (J_2). Subsequently, Lemma 2.1 ensures that I_λ admits a $(C)_c$ sequence with

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I_\lambda(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], X) : g(0) = 0, g(1) = t_0 e\}.$$

Moreover, Lemma 2.4 verifies that I_λ satisfies the $(C)_c$ condition. Using Lemma 2.1 once more, we know that I_λ gets a critical value $c \geq \widetilde{a} > 0$. Thereby, I_λ possesses a nontrivial critical point u with $I_\lambda(u) = c$. Subsequently, Eq (1.1) has at least one nontrivial homoclinic solution in X. □

The proof of Theorem 1.2 is done by fountain theorem. We display the verification at length as follow.

Proof of Theorem 1.2. First, by (F_6) and Lemma 2.4, we obtain that $I_\lambda \in C^1(X, \mathbb{R})$ is an even function and satisfies the $(C)_c$ condition. According to Lemma 2.2, it is shown that both (i) and (ii) are valid. As a matter of convenience, we list some notations and basic results needed for later use. For $(i, j) \in \mathbb{Z}^2$, define $\eta_{(i,j)} = \{\eta_{(i,j)}(k, l)\}$ as follows:

$$\eta_{(i,j)}(k,l) = \begin{cases} 1, & \text{if} \quad (k,l) = (i,j), \\ 0, & \text{if} \quad (k,l) \neq (i,j). \end{cases}$$

Then $X = \overline{span\{\eta_{(i,j)} : (i,j) \in \mathbb{Z}^2\}}$. Denote

$$E_{(i,j)} = span\{\eta_{(i,j)} : (i,j) \in \mathbb{Z}^2\}, \quad S_n = \bigoplus_{|i|+|j|=0}^{n} E_{(i,j)}, \quad T_n = \overline{\bigoplus_{|i|+|j|=n}^{\infty} E_{(i,j)}}.$$

Choose $\vartheta_n = \sup_{u \in T_n, \|u\|_X = 1} \|u\|_{l^2}$, then

$$\lim_{n\to\infty} \vartheta_n = 0 \quad \text{and} \quad \|u\|_{l^\alpha} \leq \|u\|_{l^2} \leq \vartheta_n \|u\|_X, \tag{3.4}$$

where α is given by (F_3).

Let $r_n = \left(\frac{4\lambda d \vartheta_n^\alpha}{\alpha}\right)^{\frac{1}{2-\alpha}}$. Then we obtain that $r_n > 0$ and $\lim_{n\to\infty} r_n = +\infty$. In view of (2.8), (3.4) with $\|u\|_X = r_n$ for $u \in T_n$, we have

$$\begin{aligned} I_\lambda(u) &= \sum_{(k,l)\in\mathbb{Z}^2} \Phi_c(\Delta_1 u(k-1,l)) + \Phi_c(\Delta_2 u(k,l-1)) + \frac{1}{2} \sum_{(k,l)\in\mathbb{Z}^2} b(k,l)|u(k,l)|^2 - \lambda \sum_{(k,l)\in\mathbb{Z}^2} F((k,l),u(k,l)) \\ &\geq \frac{1}{2}\|u\|_X^2 - \frac{\lambda d}{\alpha}\|u\|_{l^\alpha}^\alpha \geq \frac{1}{2}\|u\|_X^2 - \frac{\lambda d \vartheta_n^\alpha}{\alpha}\|u\|_X^\alpha = \frac{1}{4}\left(\frac{4\lambda d \vartheta_n^\alpha}{\alpha}\right)^{\frac{2}{2-\alpha}}. \end{aligned} \tag{3.5}$$

For $\alpha > 2$, it follows that

$$H_n = \inf_{u \in T_n, \|u\|_X = r_n} I_\lambda(u) \to +\infty \qquad \text{as} \quad n \to \infty.$$

Thus, (i) of Lemma 2.2 holds.

Since S_n is finite-dimensional, which implies that all forms of S_n are equivalent. Hence, there is $B_n > 0$ such that

$$\left(\frac{4}{b_*} + \frac{1}{2}\right)\|u\|_X^2 \leq \lambda B_n \|u\|_\infty^2. \tag{3.6}$$

In view of (F_2), we can obtain that there is a $T > 0$ such that

$$F((k,l),u) \geq 2B_n u^2 \qquad \text{for} \quad ((k,l),u) \in \mathbb{Z}^2 \times \mathbb{R} \quad \text{with} \quad |u| > T. \tag{3.7}$$

From (F_5), it follows that there is a non-negative $\theta \in l^1$ satisfying

$$F((k,l),u) \geq -\theta(k,l) \qquad \text{for} \quad ((k,l),u) \in \mathbb{Z}^2 \times \mathbb{R} \quad \text{with} \quad |u| \leq T. \tag{3.8}$$

For any $u \in S_n$ with $\|u\|_\infty > T$, write

$$L_{u,1} = \{(k,l) \in \mathbb{Z}^2 : |u(k,l)| > T\} \quad \text{and} \quad L_{u,2} = \{(k,l) \in \mathbb{Z}^2 : |u(k,l)| \leq T\}.$$

Similar to (2.2), one has

$$\sup_{(k,l)\in L_{u,1}} |u(k,l)| \le \left(\sum_{(k,l)\in L_{u,1}} |u(k,l)|^2 \right)^{\frac{1}{2}}.$$

Noting that $\sup_{(k,l)\in L_{u,1}} |u(k,l)| = \sup_{(k,l)\in \mathbb{Z}^2} |u(k,l)| = \|u\|_\infty$, it can be shown that

$$\|u\|_\infty^2 \le \sum_{(k,l)\in L_{u,1}} |u(k,l)|^2. \tag{3.9}$$

Therefore, for any fixed $\lambda > 0$, from (3.6)–(3.9), we obtain that

$$\begin{aligned} I_\lambda(u) \le & \left(\frac{4}{b_*} + \frac{1}{2}\right)\|u\|_X^2 - \lambda \sum_{(k,l)\in L_{u,2}} F((k,l),u(k,l)) - \lambda \sum_{(k,l)\in L_{u,1}} F((k,l),u(k,l)) \\ \le & \left(\frac{4}{b_*} + \frac{1}{2}\right)\|u\|_X^2 - \lambda \sum_{(k,l)\in L_{u,1}} F((k,l),u(k,l)) + \lambda \sum_{(k,l)\in L_{u,2}} \theta(k,l) \\ \le & \lambda B_n \|u\|_\infty^2 - 2\lambda B_n \sum_{(k,l)\in L_{u,1}} |u(k,l)|^2 + \lambda \|\theta\|_{l^1} \\ \le & \lambda B_n \|u\|_\infty^2 - 2\lambda B_n \|u\|_\infty^2 + \lambda \|\theta\|_{l^1} \\ \le & -\left(\frac{4}{b_*} + \frac{1}{2}\right)\|u\|_X^2 + \lambda \|\theta\|_{l^1}, \end{aligned}$$

which means that we can choose ρ_n with $\rho_n > r_n > 0$ large enough such that

$$I_n = \max_{u\in S_n, \|u\|_X = \rho_n} I_\lambda(u) \le 0.$$

Thus, (*ii*) of Lemma 2.2 holds as well. Consequently, Lemma 2.2 is satisfied, and I_λ has a critical point sequence $\{u_n\} \subset X$ satisfying $I_\lambda(u_n) \to \infty$ as $n \to \infty$. This completes the proof of Theorem 1.2. □

4. Examples

In sequence, we state two examples to illustrate the applicability of our conclusions.

Example 4.1. *Given* $b(k,l) = k^2 + l^2 + 1$. *For any* $(k,l) \in \mathbb{Z}^2$, *consider Eq (1.1) with*

$$f((k,l),u) = f(u) = \begin{cases} u^2 \ln(1+u^2), & u > 0, \\ 0, & u = 0, \\ u^2 \ln(1+u^2) + u^4, & u < 0. \end{cases} \tag{4.1}$$

By (4.1), it follows that

$$F((k,l),u) = \begin{cases} \frac{1}{3}u^3 \ln(1+u^2) - \frac{1}{3}\left(\frac{2}{3}u^3 - 2u + 2\arctan u\right), & u > 0, \\ 0, & u = 0, \\ \frac{1}{3}u^3 \ln(1+u^2) - \frac{1}{3}\left(\frac{2}{3}u^3 - 2u + 2\arctan u\right) + \frac{1}{5}u^5, & u < 0. \end{cases}$$

And F is a sign-changing function; see Figure 2. Moreover,

$$\frac{f((k,l),u)}{u} \to 0 \qquad \text{as} \quad |u| \to 0.$$

So (F_1) is fulfilled. If $|u| \to +\infty$, there holds

$$\lim_{u\to+\infty} \frac{F((k,l),u)}{u^2} = \lim_{u\to+\infty} \left[\frac{1}{3}u\ln(1+u^2) - \frac{1}{3}\left(\frac{2}{3}u - \frac{2}{u} + \frac{2\arctan u}{u^2}\right)\right] = +\infty,$$

$$\lim_{u\to-\infty} \frac{F((k,l),u)}{u^2} = \lim_{u\to-\infty} \left[\frac{1}{3}u\ln(1+u^2) - \frac{1}{3}\left(\frac{2}{3}u - \frac{2}{u} + \frac{2\arctan u}{u^2}\right) + \frac{1}{5}u^3\right] = -\infty.$$

Thus, (F_2) holds.

Choose $\alpha = 5$ and $d = 2$, we obtain

$$|f((k,l),u)| \le 2|u|^4 \qquad \text{for all} \quad u \in \mathbb{R} \quad \text{and} \quad (k,l) \in \mathbb{Z}^2.$$

Then (F_3) is true.

Further, direct computation yields that

$$f((k,l),u)u - 2F((k,l),u) = \begin{cases} \frac{1}{3}u^3\ln(1+u^2) + \frac{4}{3}\left(\frac{1}{3}u^3 - u + \arctan u\right), & u > 0, \\ 0, & u = 0, \\ \frac{1}{3}u^3\ln(1+u^2) + \frac{4}{3}\left(\frac{1}{3}u^3 - u + \arctan u\right) - \frac{2}{5}u^5, & u < 0, \end{cases}$$

which ensures that

$$G((k,l),u) = f((k,l),u)u - 2F((k,l),u) \ge 0. \tag{4.2}$$

Therefore, taking $\beta = 3$, (4.2) implies that (F_4) is true. In addition, it can be verified that $b(k,l)$ satisfies (B).

Consequently, f defined by (4.1) satisfies all assumptions of Theorem 1.1, and Eq (1.1) possesses at least one nontrivial homoclinic solution for all $\lambda > \frac{5}{2}$.

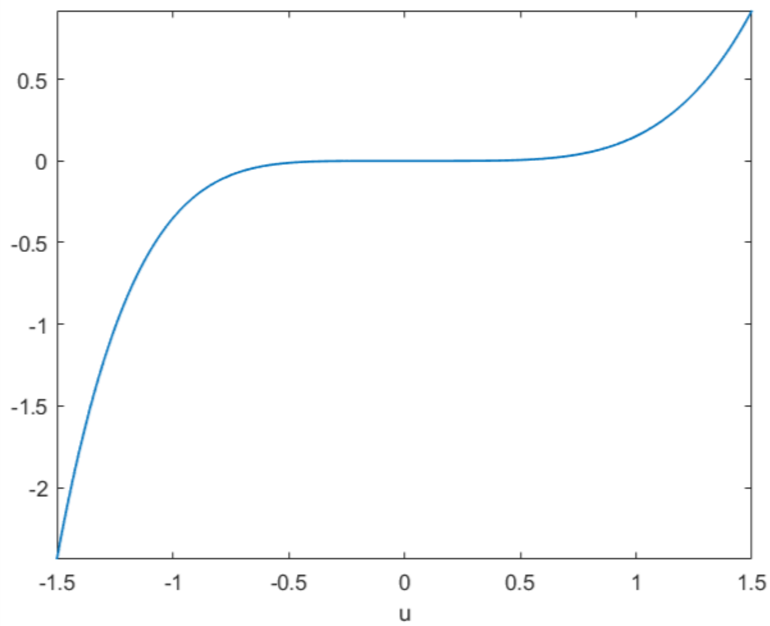

Figure 2. The image of $F((k,l),u)$.

Example 4.2. *For any* $(k,l) \in \mathbb{Z}^2$*, take*

$$b(k,l) = k^2 + l^2 + 2, \qquad f((k,l),u) = f(u) = u^2 \arctan u, \qquad \forall (k,l) \in \mathbb{Z}^2, \quad u \in \mathbb{R}. \tag{4.3}$$

Consider Eq (1.1) with (4.3).

From (4.3), we know that $b(k,l)$ satisfies (B), and $f((k,l),u)$ is odd in u that fulfills (F_6). Moreover, direct computations yield that

$$F((k,l),u) = \frac{1}{3}u^3 \arctan u - \frac{1}{6}u^2 + \frac{1}{6}\ln(1+u^2) \geq 0 \qquad \text{for all} \quad u \in \mathbb{R},$$

and (F_5) holds. Then

$$\frac{F((k,l),u)}{u^2} = \frac{1}{3}u \arctan u - \frac{1}{6} + \frac{1}{6}\frac{\ln(1+u^2)}{u^2} \to +\infty \qquad \text{as} \quad |u| \to +\infty,$$

which shows that (F_2) is true.

Choose $\alpha = 3$ and $d = \frac{\pi}{2}$, we obtain

$$|f((k,l),u)| \leq \frac{\pi}{2}u^2 \qquad \text{for all} \quad u \in \mathbb{R}.$$

Then (F_3) holds.

Further, for any $(k,l) \in \mathbb{Z}^2$, there holds

$$G((k,l),u) = f((k,l),u)u - 2F((k,l),u) = \frac{1}{3}\left(u^3 \arctan u + u^2 - \ln(1+u^2)\right) \geq \frac{1}{3}u^3 \arctan u. \tag{4.4}$$

Namely, $G((k,l),u) \geq 0$. Take $\beta = 2$. By (4.4), we get (F_4) is met.

Then f, defined by (4.3), satisfies all assumptions of Theorem 1.2. As a result, Eq (1.1) admits infinitely many solutions for $\lambda > 0$.

Author contributions

Yuhua Long: Conceptualization, Funding acquisition, Visualization, Writing-original draft, Writing-review & editing; Sha Li: Conceptualization, Visualization, Writing-original draft.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Authors state no conflict of interest.

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