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Research article

Finite-time flocking of a Cucker-Smale system with external perturbation and intermittent control

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Abstract: This study presents a modified Cucker-Smale system with external perturbation and intermittent control. First, a new finite-time stability lemma is developed, in which the settling time is determined only by the control interval's duration. Second, by imposing an assumption on external perturbation, the system can achieve collision-avoidance flocking. Third, finite-time flocking is achieved without the sign function, which effectively avoids the undesirable chattering phenomenon in the system. Finally, numerical simulations validate the theoretical analysis, and parameter sensitivity analysis offers important guidance for practical applications such as unmanned aerial vehicle (UAV) formations.

Keywords: Cucker-Smale system; finite-time flocking; collision avoidance; external perturbation; intermittent control
Mathematics Subject Classification: 93A16, 93D40, 91D30

1. Introduction

The phenomenon of flocking behavior, which is both mesmerizing and widespread in nature and society, has attracted considerable interest in recent years due to its efficiency and self-organizing features. This behavior exemplifies how individuals, by sharing local information and following simple coordination rules, can evolve from disorder to cohesive unity. From the harmonious flight patterns of birds [1] to the coordinated movements of fish schools [2] and the intricate formations of drones in modern applications [3], these collective behaviors highlight the unique charm and far-reaching implications of flocking.

In an effort to understand the underlying dynamics and evolutionary principles that drive this phenomenon, researchers have developed several classic mathematical models for detailed analysis [4–6]. For instance, tracing back to 2007, Cucker and Smale jointly introduced the celebrated Cucker-Smale (called C-S) model [6]. This model bears a resemblance to Newton's equations in classical mechanics, with its ingenuity rooted in modeling the force terms as a weighted average of velocity differences among the agents. This approach vividly replicates the attributes of interactions occurring within a flocking. The C-S model is formulated as the following dynamical system:

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = v_i(t), \quad i = 1, \cdots, N, \\ \dfrac{\mathrm{d}v_i(t)}{\mathrm{d}t} = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \hat{\theta}(\|x_j(t) - x_i(t)\|)(v_j(t) - v_i(t)), \end{cases}$$

where $x_i \in \mathbb{R}^d$ denotes the position of i-th particle and $v_i \in \mathbb{R}^d$ indicates velocity. $\|\cdot\|$ denotes the Euclidean norm. The communication rate $\hat{\theta}$ is described as

$$\hat{\theta}(r) = \frac{\hat{K}}{(\hat{\sigma}^2 + r^2)^{\hat{\beta}}}, \ \hat{K}, \ \hat{\sigma}, \ \hat{\beta} > 0,$$

which quantifies the strength of influence between two particles. It is crucial to note that the parameter $\hat{\beta}$ in the C-S model acts as a decisive threshold for flocking formation. When $\hat{\beta} \geq \frac{1}{2}$, flocking emerges under specific initial conditions $(x_i(0), v_i(0))$, otherwise known as conditional flocking (i.e., flocking that occurs under specific initial conditions). Strikingly, when $\hat{\beta} < \frac{1}{2}$, the occurrence of flocking becomes independent of the initial state, resulting in unconditional flocking (i.e., flocking that occurs independently of the initial conditions). Afterward, the C-S model was rapidly extended in other domains, such as multi-cluster flocking [7–9], time delay [10–12], pattern formation [13–15], nonlinear velocity couplings [16–18] and external disturbances [19, 20]. Currently, the model has advanced to become an indispensable and powerful tool for analyzing flocking behavior, demonstrating significant superiority both theoretically and practically.

Be it basic flocking or precise formation, convergence time is a key index for evaluating system performance. Previous studies have primarily focused on asymptotic flocking [21, 22], where systems eventually reached a flocking state at infinite time. However, recent scholarly attention has shifted toward the research of finite-time flocking [23–25], which implies that the system can achieve flocking within a finite time. For example, in 2016, Han et al. innovatively proposed a non-Lipschitz continuous C-S model [23], which achieves finite-time flocking while ensuring the existence of the lower bound of the communication weight. In particular, when the settling time is independent of the initial state, it is referred to as fixed-time flocking [26].

For analyzing flocking behavior, the impact of external disturbances is an indispensable factor. During the flocking process, the system is subject to various random disturbances [27, 28]. However, the role of deterministic disturbances [9,29]—such as external wind forces, terrain changes, and human control commands—remains equally important and deserves further attention. These disturbances have clear patterns and are predictable, but their specific effects on the formation and maintenance of the flock still require further investigation.

In recent research, many studies have adopted various control strategies. However, in the context of large-scale groups, continuous control methods often result in unnecessary resource waste, which highlights the advantages of intermittent control schemes in optimizing resource use and enhancing efficiency [30–32]. To better understand the intermittent control strategy, we will visually depict its fundamental principles in Figure 1. For a time sequence $t_0 < t_1 < \cdots < t_k < \cdots, k \in \mathbb{N}_+$, the control works at $[t_{2k}, t_{2k+1})$, which is called work interval, while the rest works at $t \in [t_{2k+1}, t_{2k+2})$. Specifically, if the intervals satisfy $t_{2k+1} - t_{2k} \equiv T_{a^*}$ and $t_{2k+2} - t_{2k+1} \equiv T_{b^*}$, the control strategy becomes a periodic intermittent control. Notably, when $T_{b^*} = 0$, the controller degenerates from intermittent control to continuous control. Recently, there has been a significant amount of research on finite-time flocking under intermittent control. For instance, fixed-time collision-avoidance flocking of C-S models with periodic intermittent control was investigated by Liu et al. [30].

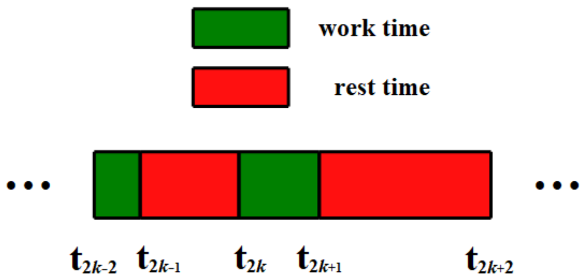


Figure 1. The intermittent control operating principle.

Unfortunately, to date, there has been no research on the finite-time flocking problem of the C-S model with external disturbances and intermittent control. Inspired by Liu et al. [30], a novel Cucker-Smale system with external perturbation and intermittent control is investigated here. The main contributions of this article are summarized below:

(1) Based on the Lyapunov function method, sufficient conditions for collision-avoidance flocking are derived, without the necessity of setting a lower bound on communication weights.

(2) A new finite-time stability lemma is proposed, which is applicable to both periodic intermittent control and aperiodic intermittent control. Different from [31], the settling time is related to the length of the control interval and independent of the length of the non-control interval.

(3) Compared to existing work [30], finite-time flocking can be successfully achieved without the sign function, which avoids the chattering phenomenon within the work time interval.

The rest of this article is structured as follows: Section 2 introduces some fundamental definitions and lemmas. Section 3.2 presents the proof of flocking estimation. Section 3.3 derives the sufficient conditions for ensuring collision avoidance. Following this, simulation examples and parameter sensitivity analysis are provided in Section 4. The article concludes with Section 5.

2. Preliminaries

In this section, we introduce a series of essential definitions and lemmas that form the foundation for the following discussions.

Definition 2.1. *The system reaches finite-time flocking if the solutions* $\{x_i, v_i\}_{i=1}^N$ *satisfy*

$$\lim_{t\to\infty} \|x_j(t) - x_i(t)\| < \infty \quad \text{and} \quad \lim_{t\to T} \|v_j(t) - v_i(t)\| = 0,$$

where T is the settling time. Moreover, if the minimum distance between particles meets

$$\|x_j(t) - x_i(t)\| > 0, \quad i \neq j, \quad t \geq 0,$$

then we claim that the system achieves finite-time collision-avoidance flocking. Furthermore, when T is independent of the initial data, it is called a fixed-time flocking.

Lemma 2.1. *[33] Let $\xi_1, \xi_2, \ldots, \xi_m \geq 0$, then*

$$\left(\sum_{i=1}^{m} \xi_i\right)^{\lambda} \leq \sum_{i=1}^{m} \xi_i^{\lambda} \leq m^{1-\lambda} \left(\sum_{i=1}^{m} \xi_i\right)^{\lambda}, \quad 0 < \lambda \leq 1,$$

and

$$m^{1-\lambda} \left(\sum_{i=1}^{m} \xi_i\right)^{\lambda} \leq \sum_{i=1}^{m} \xi_i^{\lambda} \leq \left(\sum_{i=1}^{m} \xi_i\right)^{\lambda}, \quad \lambda > 1.$$

Lemma 2.2. *Assume that a continuous, non-negative function $V(t)$ is defined on $t \in [0, +\infty)$, and the following inequality is met:*

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leq \begin{cases} -\rho V(t) - \alpha V^{p}(t), & t \in [t_{2k}, t_{2k+1}), \\ 0, & t \in [t_{2k+1}, t_{2k+2}), \end{cases} \tag{2.1}$$

in which $\alpha > 0, \rho > 0, 0 < p < 1$. The settling time T is estimated by

$$T = t_{2k^*} + \frac{1}{\rho(1-p)} \ln\left(1 + \frac{\rho}{\alpha} V^{1-p}(0)\right) - \sum_{i=0}^{k^*-1} (t_{2i+1} - t_{2i}),$$

where

$$k^* = \max\{k \in \mathbb{N}_+ : \ln\left(1 + \frac{\rho}{\alpha} V^{1-p}(0)\right) - \rho(1-p) \sum_{i=0}^{k-1} (t_{2i+1} - t_{2i}) > 0\}.$$

Proof. We consider the following comparison system:

$$\begin{cases} \frac{\mathrm{d}v(t)}{\mathrm{d}t} = \begin{cases} -\rho v(t) - \alpha v^{p}(t), & t \in [t_{2k}, t_{2k+1}), \\ 0, & t \in [t_{2k+1}, t_{2k+2}), \end{cases} \\ v(0) = V(0), \end{cases} \tag{2.2}$$

from (2.1) and (2.2), it is clear that $0 \leq V(t) \leq v(t)$. This indicates that if there exists a constant $T > 0$ such that $v(T) = 0$ and $v(t) \equiv 0$ for $t > T$, it follows that $V(t) \equiv 0$ for $t > T$. Thus, to achieve the desired conclusion, we only need to show that the solution of (2.2) is finitely stable.

Note that Eq (2.2) is a standard Bernoulli equation, therefore we set

$$G(t) = v^{1-p}(t),$$

then system (2.2) reduces to

$$\begin{cases} \frac{\mathrm{d}G(t)}{\mathrm{d}t} = \begin{cases} -\rho(1-p)G(t) - \alpha(1-p), & t \in [t_{2k}, t_{2k+1}), \\ 0, & t \in [t_{2k+1}, t_{2k+2}), \end{cases} \\ G(0) = V^{1-p}(0). \end{cases}$$

After a straightforward calculation, we obtain

$$G(t) = \left(G(t_{2k}) + \frac{\alpha}{\rho}\right)e^{-\rho(1-p)(t-t_{2k})} - \frac{\alpha}{\rho}, \quad t \in [t_{2k}, t_{2k+1}), \tag{2.3}$$

and

$$G(t) = G(t_{2k+1}), \quad t \in [t_{2k+1}, t_{2k+2}). \tag{2.4}$$

When $t \in [0, t_1)$, it follows from (2.3) that

$$G(t) = \left(G(0) + \frac{\alpha}{\rho}\right)e^{-\rho(1-p)t} - \frac{\alpha}{\rho},$$

and when $t \to t_1^-$, we have

$$G(t_1) = \left(G(0) + \frac{\alpha}{\rho}\right)e^{-\rho(1-p)t_1} - \frac{\alpha}{\rho}, \tag{2.5}$$

when $t \in [t_1, t_2)$, one derives from (2.4) to (2.5) that

$$G(t) = G(t_1) = \left(G(0) + \frac{\alpha}{\rho}\right)e^{-\rho(1-p)t_1} - \frac{\alpha}{\rho} = G(t_2), \tag{2.6}$$

when $t \in [t_2, t_3)$, from (2.3) to (2.6) one has

$$\begin{aligned} G(t) &= \left(G(t_2) + \frac{\alpha}{\rho}\right)e^{-\rho(1-p)(t-t_2)} - \frac{\alpha}{\rho} \\ &= \left(G(0) + \frac{\alpha}{\rho}\right)e^{-\rho(1-p)t_1-\rho(1-p)(t-t_2)} - \frac{\alpha}{\rho}, \end{aligned}$$

and similar to (2.5), we obtain

$$G(t_3) = \left(G(0) + \frac{\alpha}{\rho}\right)e^{-\rho(1-p)t_1-\rho(1-p)(t_3-t_2)} - \frac{\alpha}{\rho}.$$

Now, we use a similar induction to obtain

$$G(t) = \begin{cases} \left(G(0) + \frac{\alpha}{\rho}\right)e^{-\rho(1-p)\sum_{i=0}^{k-1}(t_{2i+1}-t_{2i})-\rho(1-p)(t-t_{2k})} - \frac{\alpha}{\rho}, & t \in [t_{2k}, t_{2k+1}), \\ \left(G(0) + \frac{\alpha}{\rho}\right)e^{-\rho(1-p)\sum_{i=0}^{k-1}(t_{2i+1}-t_{2i})} - \frac{\alpha}{\rho}, & t \in [t_{2k+1}, t_{2k+2}). \end{cases} \tag{2.7}$$

From the definition of k^*, it follows that $G(t_{2k^*}) > 0$ and $G(t_{2(k^*+1)}) = 0$. According to the second equation in (2.7), $G(t)$ is non-decreasing on $[t_{2k^*+1}, t_{2k^*+2})$, leading to $G(t_{2k^*+1}) = 0$. Moreover, since $\dot{G}(t) < 0$ on $[t_{2k^*}, t_{2^*+1})$, it implies that there is only one $T \in [t_{2k^*}, t_{2k^*+1})$ such that $G(T) = 0$. Thus, we obtain

$$T = t_{2k^*} + \frac{1}{\rho(1-p)}\ln\left(1 + \frac{\rho}{\alpha}V^{1-p}(0)\right) - \sum_{i=0}^{k^*-1}(t_{2i+1} - t_{2i}).$$

This completes the proof.

Corollary 2.1. *Note that for $\rho \to 0$, the system corresponds to a special case of system (2.1). Under this condition, we obtain*

$$\lim_{\rho\to 0}\left[\frac{1}{\rho(1-p)}\ln\left(1+\frac{\rho}{\alpha}V^{1-p}(0)\right)\right]=\frac{1}{\alpha(1-p)}V^{1-p}(0),$$

thus, the settling time T is estimated by

$$T = t_{2k^*} + \frac{1}{\alpha(1-p)}V^{1-p}(0) - \sum_{i=0}^{k^*-1}(t_{2i+1}-t_{2i}),$$

where

$$k^* = \max\{k\in\mathbb{N}_+ : \frac{1}{\alpha(1-p)}V^{1-p}(0) - \sum_{i=0}^{k-1}(t_{2i+1}-t_{2i}) > 0\}.$$

Remark 2.1. *Unlike the approach in a previous study [31], our proposed lemmas simplify the settling time, making it dependent solely on the length of the control interval rather than the non-control interval. The expression for the settling time T shows that a wider control interval results in a shorter convergence time under given parameters, which is consistent with practical engineering applications. Furthermore, our lemmas can be applied to both periodic and aperiodic intermittent controllers, thereby enhancing their flexibility.*

3. Finite-time flocking

3.1. Model description

Motivated by previous findings [16, 30], we further explore the impact of external perturbation and intermittent control on system dynamics. In this context, we consider the following dynamical system:

$$\begin{cases}\dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = v_i(t), \quad i=1,\cdots,N,\\ \dfrac{\mathrm{d}v_i(t)}{\mathrm{d}t} = \dfrac{K}{N}\displaystyle\sum_{j=1}^{N}\psi(\|x_j(t)-x_i(t)\|)(v_j(t)-v_i(t)) + g(t,v_i(t)) + u_1(t),\end{cases} \tag{3.1}$$

where

$$u_1(t) = \begin{cases}\dfrac{K_0}{N}\displaystyle\sum_{j=1}^{N}\varphi(\|x_j(t)-x_i(t)\|)\Gamma_\lambda(v_j(t)-v_i(t)), & t\in[t_{2k},t_{2k+1}),\\ 0, & t\in[t_{2k+1},t_{2k+2}).\end{cases} \tag{3.2}$$

where $K > 0$ and $K_0 > 0$ represent the coupling strengths, and $\Gamma_\lambda(v)$ denotes the nonlinear velocity coupling, which is given by

$$\Gamma_\lambda(v) = v\|v\|^{2(\lambda-1)}, \quad \frac{1}{2}<\lambda<1. \tag{3.3}$$

The communication weights $\psi(r)$ and $\varphi(r)$ are positive and non-increasing functions. $g(t, v_i(t))$ describes the external deterministic perturbation of agent i at time t, and we assume that g satisfies the following condition:

Assumption 3.1. *The external perturbation function $g(t, v_i(t))$ satisfies*

$$\|g(t, v_i(t)) - g(t, v_j(t))\| \leq L\|v_i(t) - v_j(t)\|, \ L > 0. \tag{3.4}$$

Remark 3.1. *When $u_1(t) = 0$ for $t \in [t_{2k}, t_{2k+1})$ and $g(t, v_i(t)) = 0$, system (3.1) reduces to the classical C-S model [6], leading to asymptotic flocking behavior. Inspired by Ha et al. [16], we introduce a intermittent controller $u_1(t)$ based on the C-S model. This allows the system to transition from asymptotic flocking to finite-time flocking without inducing chattering in work interval.*

Set the center of the mass system $(x_c(t), v_c(t))$ as follows:

$$x_c(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t), \quad v_c(t) = \frac{1}{N}\sum_{i=1}^{N} v_i(t). \tag{3.5}$$

When $t \in [t_{2k}, t_{2k+1})$, a straightforward calculation yields

$$\frac{\mathrm{d}v_c(t)}{\mathrm{d}t} = \frac{1}{N}\sum_{i=1}^{N} g(t, v_i(t)), \tag{3.6}$$

obviously, when $t \in [t_{2k+1}, t_{2k+2})$, the relation (3.6) remains valid.

We define

$$\hat{x}_i(t) = x_i(t) - x_c(t), \quad \hat{v}_i(t) = v_i(t) - v_c(t),$$

when $t \in [t_{2k}, t_{2k+1})$, then the fluctuations $(\hat{x}_i(t), \hat{v}_i(t))$ satisfy the following system:

$$\begin{cases} \dfrac{\mathrm{d}\hat{x}_i(t)}{\mathrm{d}t} = \hat{v}_i(t), \quad i = 1, \cdots, N, \\ \dfrac{\mathrm{d}\hat{v}_i(t)}{\mathrm{d}t} = \dfrac{K}{N}\displaystyle\sum_{j=1}^{N} \psi(\|\hat{x}_j(t) - \hat{x}_i(t)\|)(\hat{v}_j(t) - \hat{v}_i(t)) \\ + \dfrac{K_0}{N}\displaystyle\sum_{j=1}^{N} \varphi(\|\hat{x}_j(t) - \hat{x}_i(t)\|)\Gamma_\lambda(\hat{v}_j(t) - \hat{v}_i(t)) + \dfrac{1}{N}\displaystyle\sum_{j=1}^{N}(g(t, v_i(t)) - g(t, v_j(t))), \end{cases} \tag{3.7}$$

when $t \in [t_{2k+1}, t_{2k+2})$, similarly we have

$$\begin{cases} \dfrac{\mathrm{d}\hat{x}_i(t)}{\mathrm{d}t} = \hat{v}_i(t), \quad i = 1, \cdots, N, \\ \dfrac{\mathrm{d}\hat{v}_i(t)}{\mathrm{d}t} = \dfrac{K}{N}\displaystyle\sum_{j=1}^{N} \psi(\|\hat{x}_j(t) - \hat{x}_i(t)\|)(\hat{v}_j(t) - \hat{v}_i(t)) + \dfrac{1}{N}\displaystyle\sum_{j=1}^{N}(g(t, v_i(t)) - g(t, v_j(t))). \end{cases} \tag{3.8}$$

To simplify the following analysis, we let

$$\mathcal{X}(t) = \left(\frac{1}{N}\sum_{i=1}^{N} \|\hat{x}_i(t)\|^2\right)^{\frac{1}{2}}, \quad \mathcal{V}(t) = \left(\frac{1}{N}\sum_{i=1}^{N} \|\hat{v}_i(t)\|^2\right)^{\frac{1}{2}}.$$

3.2. Flocking estimation

Lemma 3.1. *Let $\{(\hat{x}_i(t), \hat{v}_i(t))\}_{i=1}^N$ be the solution of systems (3.7) and (3.8). Assume that Assumption 3.1 holds, then $\mathcal{X}(t)$ and $\mathcal{V}(t)$ satisfy the following relations:*

$$\left|\frac{\mathrm{d}\mathcal{X}(t)}{\mathrm{d}t}\right| \le \mathcal{V}(t), \tag{3.9}$$

and

$$\frac{\mathrm{d}\mathcal{V}(t)}{\mathrm{d}t} \le \begin{cases} -C_1\mathcal{V}(t) - 2^{\lambda-1}K_0N^{2\lambda-2}\varphi\left(2\sqrt{N}\mathcal{X}(t)\right)\mathcal{V}^{2\lambda-1}(t), \ t \in [t_{2k}, t_{2k+1}), \\ -C_1\mathcal{V}(t), \ t \in [t_{2k+1}, t_{2k+2}). \end{cases}$$

where

$$C_1 = K\psi\left(2\sqrt{N}\mathcal{X}(t)\right) - \sqrt{2}L. \tag{3.10}$$

Proof. Based on the definition of $\mathcal{X}(t)$, for $t \ge 0$, we have

$$\begin{aligned}\left|\frac{\mathrm{d}\mathcal{X}^2(t)}{\mathrm{d}t}\right| &= \left|\frac{1}{N}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^N \|\hat{x}_i(t)\|^2\right| \le 2\left(\frac{1}{N}\sum_{i=1}^N \|\hat{x}_i(t)\|^2\right)^{\frac{1}{2}}\left(\frac{1}{N}\sum_{i=1}^N \|\hat{v}_i(t)\|^2\right)^{\frac{1}{2}} \\ &= 2\mathcal{X}(t)\mathcal{V}(t),\end{aligned}$$

this yields the desired estimate (3.9). Next, we will discuss the two cases of $t \in [t_{2k}, t_{2k+1})$ and $t \in [t_{2k+1}, t_{2k+2})$, $k \in \mathbb{N}_+$.

For $t \in [t_{2k}, t_{2k+1})$, the following holds:

$$\begin{aligned}\frac{\mathrm{d}\mathcal{V}^2(t)}{\mathrm{d}t} &= \frac{1}{N}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^N \|\hat{v}_i(t)\|^2 \\ &= \frac{2}{N}\sum_{i=1}^N \langle \hat{v}_i(t), \frac{K}{N}\sum_{j=1}^N \psi(\|\hat{x}_j(t) - \hat{x}_i(t)\|)(\hat{v}_j(t) - \hat{v}_i(t)) + u_1(t)\rangle \\ &= \frac{2K}{N^2}\sum_{i,j=1}^N \langle \hat{v}_i(t), \psi(\|\hat{x}_j(t) - \hat{x}_i(t)\|)(\hat{v}_j(t) - \hat{v}_i(t))\rangle \\ &+ \frac{2K_0}{N^2}\sum_{i,j=1}^N \langle \hat{v}_i(t), \varphi(\|\hat{x}_j(t) - \hat{x}_i(t)\|)\Gamma_\lambda(\hat{v}_j(t) - \hat{v}_i(t))\rangle \\ &+ \frac{2}{N}\sum_{i=1}^N \langle \hat{v}_i(t), \frac{1}{N}\sum_{j=1}^N (g(t, v_i(t)) - g(t, v_j(t)))\rangle \\ &= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.\end{aligned}$$

For $\mathcal{J}_1$, we obtain

$$\mathcal{J}_1 = -\frac{K}{N^2}\sum_{i,j=1}^N \psi(\|\hat{x}_j(t) - \hat{x}_i(t)\|)\|\hat{v}_j(t) - \hat{v}_i(t)\|^2,$$

note that

$$\|\hat{x}_j(t) - \hat{x}_i(t)\| \le \|\hat{x}_j(t)\| + \|\hat{x}_i(t)\| \le \sqrt{N}\mathcal{X}(t) + \sqrt{N}\mathcal{X}(t) = 2\sqrt{N}\mathcal{X}(t), \tag{3.11}$$

and from the property of ψ, it follows that

$$\mathcal{J}_1 \leq -2K\psi\left(2\sqrt{N}\mathcal{X}(t)\right)\mathcal{V}^2(t), \tag{3.12}$$

where we use

$$\sum_{i,j=1}^{N} \|\hat{v}_j(t) - \hat{v}_i(t)\|^2 = 2N^2\mathcal{V}^2(t).$$

For $\mathcal{J}_2$, we derive that

$$\begin{aligned}\mathcal{J}_2 &\leq -\frac{K_0}{N^2}\varphi\left(2\sqrt{N}\mathcal{X}(t)\right)\sum_{i,j=1}^{N}\langle \hat{v}_j(t) - \hat{v}_i(t), \Gamma_\lambda(\hat{v}_j(t) - \hat{v}_i(t))\rangle \\ &= -\frac{K_0}{N^2}\varphi\left(2\sqrt{N}\mathcal{X}(t)\right)\sum_{i,j=1}^{N}\|\hat{v}_j(t) - \hat{v}_i(t)\|^{2\lambda},\end{aligned} \tag{3.13}$$

and we apply the property of Γ_λ being an odd function to arrive at

$$\begin{aligned}&\sum_{i,j=1}^{N}\varphi(\|\hat{x}_j(t) - \hat{x}_i(t)\|)\langle \hat{v}_i(t), \Gamma_\lambda(\hat{v}_j(t) - \hat{v}_i(t))\rangle \\ =& -\frac{1}{2}\sum_{i,j=1}^{N}\varphi(\|\hat{x}_j(t) - \hat{x}_i(t)\|)\langle \hat{v}_j(t) - \hat{v}_i(t), \Gamma_\lambda(\hat{v}_j(t) - \hat{v}_i(t))\rangle.\end{aligned} \tag{3.14}$$

When $\frac{1}{2} < \lambda < 1$, Lemma 2.1 directly implies

$$\sum_{i,j=1}^{N}\|\hat{v}_j(t) - \hat{v}_i(t)\|^{2\lambda} \geq \left(\sum_{i,j=1}^{N}\|\hat{v}_j(t) - \hat{v}_i(t)\|^2\right)^{\lambda},$$

then Eq (3.13) now reads

$$\mathcal{J}_2 \leq -2^\lambda K_0 N^{2\lambda-2}\varphi\left(2\sqrt{N}\mathcal{X}(t)\right)\mathcal{V}^{2\lambda}(t). \tag{3.15}$$

For $\mathcal{J}_3$, we have

$$\begin{aligned}\mathcal{J}_3 &= \frac{2}{N}\sum_{i=1}^{N}\langle \hat{v}_i(t), \frac{1}{N}\sum_{j=1}^{N}(g(t, v_i(t)) - g(t, v_j(t)))\rangle \\ &\leq \frac{2}{N^2}\sum_{i,j}^{N}\|\hat{v}_i(t)\|\|g(t, v_i(t)) - g(t, v_j(t))\| \\ &\leq \frac{2L}{N^2}\sum_{i,j}^{N}\|\hat{v}_i(t)\|\|\hat{v}_i(t) - \hat{v}_j(t)\| \\ &\leq \frac{2L}{N^2}\left(\sum_{i,j}^{N}\|\hat{v}_i(t)\|^2\right)^{\frac{1}{2}}\left(\sum_{i,j}^{N}\|\hat{v}_i(t) - \hat{v}_j(t)\|^2\right)^{\frac{1}{2}} \\ &\leq 2\sqrt{2}L\mathcal{V}^2(t),\end{aligned} \tag{3.16}$$

where the second inequality is based on Assumption 3.1.

Combining (3.12), (3.15), and (3.16), we obtain

$$\frac{\mathrm{d}\mathcal{V}(t)}{\mathrm{d}t} \leq -C_1\mathcal{V}(t) - 2^{\lambda-1}K_0N^{2\lambda-2}\varphi\Big(2\sqrt{N}\mathcal{X}(t)\Big)\mathcal{V}^{2\lambda-1}(t), t \in [t_{2k}, t_{2k+1}), \tag{3.17}$$

where C_1 is defined in (3.10).

For $t \in [t_{2k+1}, t_{2k+2})$, the following holds:

$$\frac{\mathrm{d}\mathcal{V}(t)}{\mathrm{d}t} \leq -C_1\mathcal{V}(t). \tag{3.18}$$

In summary, combining (3.17) and (3.18), we obtain the desired result. This completes the proof.

Next, we will present the main theorem of this paper.

Theorem 3.1. *Let Assumption 3.1 hold and $\{(\hat{x}_i(t), \hat{v}_i(t))\}_{i=1}^N$ be the solution of systems (3.7) and (3.8). If the initial state $\{(\hat{x}_i(0), \hat{v}_i(0))\}_{i=1}^N$ satisfies*

(H1): $K\psi\Big(2\sqrt{N}\mathcal{X}(0)\Big) - \sqrt{2}L > 0$,

(H2): there exists a positive constant $r^* > \mathcal{X}(0)$ such that

$$\mathcal{V}(0) < \int_{\mathcal{X}(0)}^{r^*}\Big[K\psi\Big(2\sqrt{N}\omega\Big) - \sqrt{2}L\Big]\mathrm{d}\omega, K\psi\Big(2\sqrt{N}r^*\Big) - \sqrt{2}L > 0. \tag{3.19}$$

Then, there exists a positive constant $\mathcal{X}_{M_0}$ such that

$$\mathcal{X}(t) \leq \mathcal{X}_{M_0}, \quad t \geq 0, \tag{3.20}$$

where $\mathcal{X}_{M_0}$ is given by

$$\mathcal{V}(0) = \int_{\mathcal{X}(0)}^{\mathcal{X}_{M_0}}\Big[K\psi\Big(2\sqrt{N}\omega\Big) - \sqrt{2}L\Big]\mathrm{d}\omega. \tag{3.21}$$

Proof. From Lemma 3.1, we obtain

$$\frac{\mathrm{d}\mathcal{V}(t)}{\mathrm{d}t} \leq -\Big[K\psi\Big(2\sqrt{N}\mathcal{X}(t)\Big) - \sqrt{2}L\Big]\mathcal{V}(t), \quad \forall t \geq 0,$$

inspired by the method in Ru and Xue's research [34], we introduce the following Lyapunov function:

$$\Xi(t) := \mathcal{V}(t) + \int_0^{\mathcal{X}(t)}\Big[K\psi\Big(2\sqrt{N}\omega\Big) - \sqrt{2}L\Big]\mathrm{d}\omega,$$

now, we compute the derivative of the above equation, yielding

$$\begin{aligned}\frac{\mathrm{d}\Xi(t)}{\mathrm{d}t} &= \frac{\mathrm{d}\mathcal{V}(t)}{\mathrm{d}t} + \Big[K\psi\Big(2\sqrt{N}\mathcal{X}(t)\Big) - \sqrt{2}L\Big]\frac{\mathrm{d}\mathcal{X}(t)}{\mathrm{d}t} \\ &= \Big[K\psi\Big(2\sqrt{N}\mathcal{X}(t)\Big) - \sqrt{2}L\Big]\left(-\mathcal{V}(t) + \left|\frac{\mathrm{d}\mathcal{X}(t)}{\mathrm{d}t}\right|\right).\end{aligned}$$

To show that $\Xi(t)$ is non-increasing, since inequality (3.9) holds (i.e.,$-\mathcal{V}(t) + \left|\frac{\mathrm{d}\mathcal{X}(t)}{\mathrm{d}t}\right| \leq 0$), we only need to verify that $K\psi\Big(2\sqrt{N}\mathcal{X}(t)\Big) - \sqrt{2}L > 0$ for $t \geq 0$. For this purpose, we define

$$\mathcal{T} = \Big\{s > 0 : \forall t \in [0, s), K\psi\Big(2\sqrt{N}\mathcal{X}(t)\Big) - \sqrt{2}L > 0\Big\}, \tilde{T} = \sup\mathcal{T}, \tag{3.22}$$

due to $K\psi\left(2\sqrt{N}\mathcal{X}(0)\right) - \sqrt{2}L > 0$, $r^* > \mathcal{X}(0)$ and the strictly deceasing property of ψ, we have $K\psi\left(2\sqrt{N}\mathcal{X}(0)\right) - \sqrt{2}L > 0$. Since $\mathcal{X}(t)$ is a continuous function, there exists a constant t_1 such that $K\psi\left(2\sqrt{N}\mathcal{X}(t)\right) - \sqrt{2}L > 0$ for all $t \in [0, t_1)$. This implies $\mathcal{T} \neq \emptyset$. Now we claim $\tilde{T} = +\infty$. Suppose now, we obtain $\tilde{T} < \infty$, it follows that

$$K\psi\left(2\sqrt{N}\mathcal{X}(t)\right) - \sqrt{2}L > 0,\ t \in [0, \tilde{T}), K\psi\left(2\sqrt{N}\mathcal{X}(\tilde{T})\right) - \sqrt{2}L = 0, \tag{3.23}$$

which implies $\mathcal{X}(\tilde{T}) > r^*$ and

$$\frac{\mathrm{d}\Xi(t)}{\mathrm{d}t} \le 0,\ t \in [0, \tilde{T}), \tag{3.24}$$

where we use (3.9). Since $K\psi\left(2\sqrt{N}r^*\right) - \sqrt{2}L \ge 0$, it follows that $K\psi\left(2\sqrt{N}\omega\right) - \sqrt{2}L > 0, \omega \in (\mathcal{X}(0), r^*)$. According to (3.19), there exists a constant $\mathcal{X}_{M_0} \in [\mathcal{X}(0), r^*)$ such that

$$\mathcal{V}(0) = \int_{\mathcal{X}(0)}^{\mathcal{X}_{M_0}} \left[K\psi\left(2\sqrt{N}\omega\right) - \sqrt{2}L\right] \mathrm{d}\omega, K\psi\left(2\sqrt{N}\mathcal{X}_{M_0}\right) - \sqrt{2}L > 0, \tag{3.25}$$

together with (3.24), we have

$$\begin{aligned} &\mathcal{V}(\tilde{T}) + \int_0^{\mathcal{X}(\tilde{T})} \left[K\psi\left(2\sqrt{N}\omega\right) - \sqrt{2}L\right] \mathrm{d}\omega \\ &\le \mathcal{V}(0) + \int_0^{\mathcal{X}(0)} \left[K\psi\left(2\sqrt{N}\omega\right) - \sqrt{2}L\right] \mathrm{d}\omega \\ &= \int_0^{\mathcal{X}_{M_0}} \left[K\psi\left(2\sqrt{N}\omega\right) - \sqrt{2}L\right] \mathrm{d}\omega. \end{aligned} \tag{3.26}$$

For $\forall r \in [0, \mathcal{X}(\tilde{T}))$, we have $K\psi\left(2\sqrt{N}r\right) - \sqrt{2}L > 0$. Combining (3.19) and (3.24), $\mathcal{X}_{M_0} < r^* < \mathcal{X}(\tilde{T})$. Therefore, we have

$$\mathcal{V}(\tilde{T}) + \int_0^{\mathcal{X}(\tilde{T})} \left[K\psi\left(2\sqrt{N}\omega\right) - \sqrt{2}L\right] \mathrm{d}\omega > \int_0^{\mathcal{X}_{M_0}} \left[K\psi\left(2\sqrt{N}\omega\right) - \sqrt{2}L\right] \mathrm{d}\omega, \tag{3.27}$$

this finding contradicts (3.26). Consequently, $\tilde{T} = \infty$, ensuring that $\Xi(t)$ is non-increasing on $[0, \infty)$. For $t \ge 0$, we derive

$$\begin{aligned} &\mathcal{V}(t) + \int_0^{\mathcal{X}(t)} \left[K\psi\left(2\sqrt{N}\omega\right) - \sqrt{2}L\right] \mathrm{d}\omega \\ &\le \mathcal{V}(0) + \int_0^{\mathcal{X}(0)} \left[K\psi\left(2\sqrt{N}\omega\right) - \sqrt{2}L\right] \mathrm{d}\omega \\ &\le \int_0^{\mathcal{X}_{M_0}} \left[K\psi\left(2\sqrt{N}\omega\right) - \sqrt{2}L\right] \mathrm{d}\omega, \end{aligned} \tag{3.28}$$

this implies $\mathcal{X}(t) \le \mathcal{X}_{M_0}$. Together with (3.11), we obtain

$$\|x_j(t) - x_i(t)\| = \|\hat{x}_j(t) - \hat{x}_i(t)\| \le 2\sqrt{N}\mathcal{X}(t) \le 2\sqrt{N}\mathcal{X}_{M_0}. \tag{3.29}$$

This indicates that the distance between any two agents in the group always has a well-defined upper bound. This concludes the proof.

Theorem 3.2. *If Assumption 3.1 holds and system (3.1) satisfies conditions (H1) and (H2), then system (3.1) achieves finite-time flocking, and the settling time is estimated by*

$$T = t_{2k^*} + \frac{1}{(2-2\lambda)C_M}\ln\left(Q_{M^*}+1\right) - \sum_{i=0}^{k^*-1}(t_{2i+1}-t_{2i}), \tag{3.30}$$

where

$$k^* = \max\{k\in\mathbb{N}_+ : \frac{1}{(2-2\lambda)C_M}\ln\left(Q_{M^*}+1\right) - \sum_{i=0}^{k-1}(t_{2i+1}-t_{2i}) > 0\},$$

and

$$C_M = K\psi\left(2\sqrt{N}\mathcal{X}_{M_0}\right) - \sqrt{2}L,\, Q_{M^*} = \frac{C_M\mathcal{V}^{2-2\lambda}(0)}{2^{\lambda-1}K_0N^{2\lambda-2}\varphi\left(2\sqrt{N}\mathcal{X}_{M_0}\right)}.$$

Proof. From Theorem 3.1, we obtain $K\psi\left(2\sqrt{N}\mathcal{X}_{M_0}\right) - \sqrt{2}L > 0$. From Lemma 3.1 and (3.29), we have

$$\frac{\mathrm{d}\mathcal{V}(t)}{\mathrm{d}t} \le \begin{cases} -C_M\mathcal{V}(t) - 2^{\lambda-1}K_0N^{2\lambda-2}\varphi\left(2\sqrt{N}\mathcal{X}_{M_0}\right)\mathcal{V}^{2\lambda-1}(t), t\in[t_{2k},t_{2k+1}), \\ -C_M\mathcal{V}(t) \le 0, t\in[t_{2k+1},t_{2k+2}), \end{cases}$$

where

$$C_M = K\psi\left(2\sqrt{N}\mathcal{X}_{M_0}\right) - \sqrt{2}L > 0. \tag{3.31}$$

By applying Lemma 2.2 and Definition 2.1, flocking occurs in a finite time, and the settling time T is defined in (3.30). This leads to

$$\mathcal{V}(t) \equiv 0, \quad \forall t \ge T.$$

This completes the proof.

Remark 3.2. *When $t_{2k+1} - t_{2k} \equiv T_a > 0$ and $t_{2k+2} - t_{2k+1} \equiv T_b > 0$, then the control strategy becomes periodic intermittent control. Further, the settling time is determined by*

$$\begin{aligned} T &= k^*(T_a+T_b) + \frac{1}{\rho(1-p)}\ln\left(1+\frac{\rho}{\alpha}V^{1-p}(0)\right) - k^*T_a \\ &= k^*T_b + \frac{1}{\rho(1-p)}\ln\left(1+\frac{\rho}{\alpha}V^{1-p}(0)\right), \end{aligned}$$

where

$$k^* = \max\{k\in\mathbb{N}_+ : \frac{1}{\rho(1-p)}\ln\left(1+\frac{\rho}{\alpha}V^{1-p}(0)\right) - kT_a > 0\}.$$

Remark 3.3. *When $t_{2k+2} - t_{2k+1} = 0$, the control system switches to continuous control. Furthermore, system (3.1) can still achieve finite-time flocking, and the settling time is estimated by*

$$T = \frac{1}{(2-2\lambda)C_M}\ln\left(Q_{M^*}+1\right),$$

where C_M and Q_{M^*} are defined in Theorem 3.2.

When $g(t, v_i(t)) = 0$, we obtain the following corollary.

Corollary 3.1. *If there exists $r_1^* > \mathcal{X}(0)$ such that $\mathcal{V}(0) < \int_{\mathcal{X}(0)}^{r_1^*} K\psi\left(2\sqrt{N}\omega\right)\mathrm{d}\omega$, then there exists a positive constant $\mathcal{X}_{M_1}$ such that*

$$\mathcal{X}(t) \leq \mathcal{X}_{M_1},$$

where $\mathcal{X}_{M_1}$ is defined by $\mathcal{V}(0) = \int_{\mathcal{X}(0)}^{\mathcal{X}_{M_1}} K\psi\left(2\sqrt{N}\omega\right)\mathrm{d}\omega$. Then, system (3.1) can achieve finite-time flocking, and the settling time is given by

$$T = t_{2k^*} + \frac{1}{(2-2\lambda)K\psi\left(2\sqrt{N}\mathcal{X}_{M_1}\right)}\ln(Q_1+1) - \sum_{k=0}^{k^*-1}(t_{2i+1}-t_{2i}),$$

where

$$k^* = \max\{k\in\mathbb{N}_+ : \frac{1}{(2-2\lambda)K\psi\left(2\sqrt{N}\mathcal{X}_{M_1}\right)}\ln(Q_1+1) - \sum_{i=0}^{k-1}(t_{2i+1}-t_{2i}) > 0\},$$

and $Q_1 = K\psi\left(2\sqrt{N}\mathcal{X}_{M_1}\right)\mathcal{V}^{2-2\lambda}(0)/2^{\lambda-1}K_0N^{2\lambda-2}\phi\left(2\sqrt{N}\mathcal{X}_{M_1}\right)$.

3.3. Collision avoidance

Many studies have primarily focused on velocity synchronization and position boundedness, with relatively little attention given to collision avoidance. However, collision avoidance plays a central role in drone formations and tactical operations. It is the foundation for ensuring drones operate safely and efficiently in complex and dynamic environments, and it is also a critical means to mitigate potential collision risks, prevent mission interruptions, and reduce unnecessary resource consumption.

Theorem 3.3. *Assuming Assumption 3.1 holds and system (3.1) satisfies conditions (H1) and (H2), the initial condition $\{x_i(0), v_i(0)\}_{i=1}^N$ satisfies*

$$\min_{i\neq j}\|X_{ij}(0)\| > \sqrt{2N}T\mathcal{V}(0), \quad \frac{1}{2} < \lambda < 1, \tag{3.32}$$

then for any solution of (3.1), the agent trajectory remains non-collisional for $t \geq 0$.

Proof. Let

$$X_{ij}(t) = \|x_j(t) - x_i(t)\|, \quad V_{ij}(t) = \|v_j(t) - v_i(t)\|, \quad i \neq j,$$

based on the definition of $\mathcal{V}(t)$, we obtain

$$V_{ij}(t) \leq \sqrt{2N}\mathcal{V}(t).$$

In the case of $X_{ij}(t)$, it is easy to see that

$$\frac{\mathrm{d}X_{ij}^2(t)}{\mathrm{d}t} \leq 2\|x_j(t)-x_i(t)\|\|v_j(t)-v_i(t)\| = 2X_{ij}(t)V_{ij}(t),$$

which implies

$$\left|\frac{\mathrm{d}X_{ij}(t)}{\mathrm{d}t}\right| \leq V_{ij}(t),$$

by integrating both sides from 0 to t, it follows that

$$\|X_{ij}(t) - X_{ij}(0)\| \leq \int_0^t |V_{ij}(\tau)| d\tau \leq \sqrt{2N} \int_0^t |\mathcal{V}(\tau)| d\tau. \tag{3.33}$$

From Theorem 3.2, we have

$$\mathcal{V}(t) \leq \mathcal{V}(0), \tag{3.34}$$

combining (3.33) and (3.34), we obtain

$$\|X_{ij}(t) - X_{ij}(0)\| \leq \sqrt{2N} \int_0^T |\mathcal{V}(\tau)| d\tau \leq \sqrt{2N} T \mathcal{V}(0),$$

this gives

$$\|X_{ij}(t)\| > \min_{i \neq j} \|X_{ij}(0)\| - \sqrt{2N} T \mathcal{V}(0).$$

It is easy to see

$$\|x_j(t) - x_i(t)\| > 0, \quad t \geq 0,$$

from Definition 2.1, system (3.1) reaches finite-time collision-free flocking. Thus, during the flocking process, no collisions between any two agents will occur, ensuring the agent's safe and efficient operation. This conclusion will be further validated in Section 4.

4. Simulations

To validate our theoretical results, we use the fourth-order Runge-Kutta algorithm in MATLAB for numerical simulations. The initial position and velocity are listed in Table 1. Following this, we will then present numerical examples to illustrate these results. To more accurately characterize the flocking behavior, we define the following two indicators:

$$\delta_x(t) = \frac{1}{N} \sqrt{\sum_{i=1}^{N} [x_i(t) - x_c(t)]^2}, \quad \delta_v(t) = \frac{1}{N} \sqrt{\sum_{i=1}^{N} [v_i(t) - v_c(t)]^2},$$

where $x_c(t)$ and $v_c(t)$ are defined as in Section 3.1. It is worth noting that the system can be said to have successfully achieved flocking at time T^* if, for $t \geq T^*$, $\delta_v(t) < 10^{-7}$ and $\delta_x(t) < \infty$.

Table 1. The initial data.

Agents	Initial position	Initial velocity
1	(0.0100,0.0101)	(0.0043, 0.0046)
2	(0.0102,4.0110)	(0.0014, 0.0088)
3	(4.0100, 0.0400)	(0.0095, 0.0149)
4	(0.0130, 7.9902)	(0.0145, 0.0043)
5	(8.0012, 0.0111)	(0.0050, 0.0122)
6	(7.9923, 4.0130)	(0.0038, 0.0066)
7	(8.0011, 8.0003)	(0.0040, 0.0022)
8	(4.0052, 8.0104)	(0.0077, 0.0163)
9	(4.0032, 4.0021)	(0.0030, 0.0010)
10	(12.0005, 2.0052)	(0.0012, 0.0034)

Example 4.1. *Let* $N = 10, K = 5, K_0 = 7, \lambda = 0.88, \psi(r) = \frac{1}{(1+r)^{0.81}}, \phi(r) = \frac{1}{(1+r)^{1.5}}, g(t, v_i(t)) = \frac{l}{N}(v_i(t) - v_c(t)), l = 1, L = 0.1, r^* = 5.1, T = 10.$

$$u_1(t) = \begin{cases} \frac{K_0}{N}\sum_{j=1}^{N}\varphi(\|x_j(t) - x_i(t)\|)\Gamma_\lambda(v_j(t) - v_i(t)), & t \in [2k, 2k + 0.3), \\ 0, & t \in [2k + 0.3, 2k + 1). \end{cases}$$

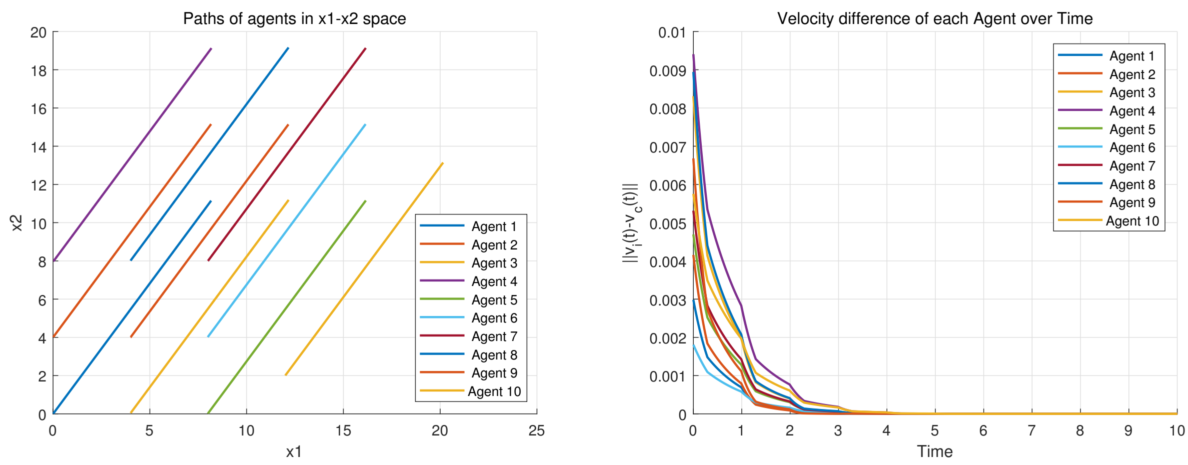
Through simple calculation, we have

$$\mathcal{V}(0) = 0.0064 < \int_{\mathcal{X}(0)}^{r^*}\left[K\psi\left(2\sqrt{N}\omega\right) - \sqrt{2}L\right]\mathrm{d}\omega = 0.0117, \mathcal{X}_{M_0} = 5.0653,$$

and

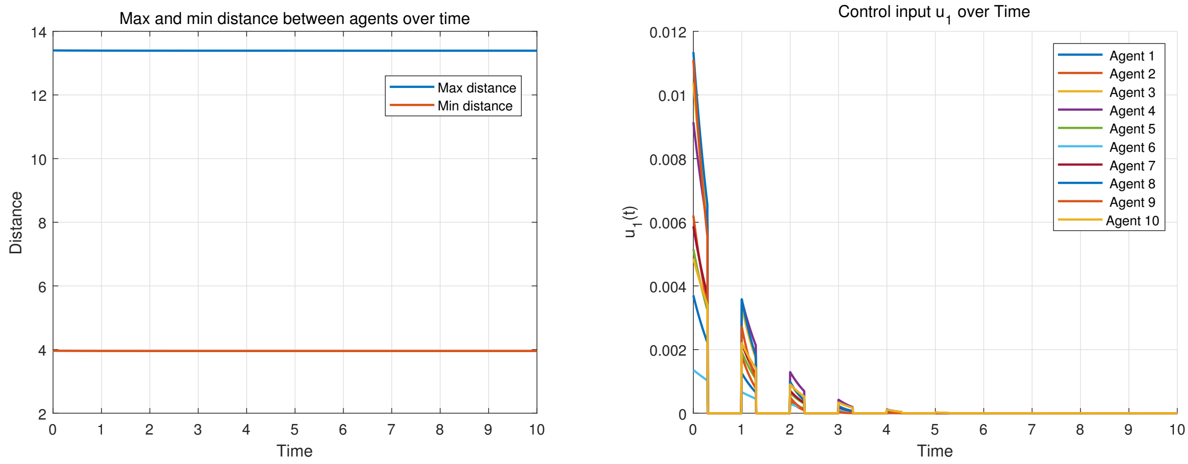
$$\min_{i\neq j}\|X_{ij}(0)\| = 3.9621 > \sqrt{2NT\mathcal{V}(0)} = 3.0786.$$

Figure 2(a) shows the position trajectories of ten agents. The curve trajectories of $\|v_i(t) - v_c(t)\|$ over time are displayed in Figure 2(b). We observe significant velocity fluctuations within the control intervals, while outside these regions, changes are relatively slower. From Figure 2(b), despite the external interference $g(t, v_i(t))$, system (3.1) still achieves flocking. Combining the expression for $g(t, v_i(t))$ and (3.6), we easily obtain $\frac{\mathrm{d}v_c(t)}{\mathrm{d}t} = 0$, i.e., $v_c(t) = v_c(0)$. Further, according to Theorem 3.2, the final velocity of flocking is the average of the initial velocities. Therefore, under this disturbance condition, we can control the final flocking velocity by adjusting the initial velocities. Figure 3(a) illustrates the curve of the maximum and minimum distances between any two agents during the flocking process, confirming that no collisions occur. The time trajectory of the control output $u_1(t)$ is depicted in Figure 3(b). According to Theorem 3.2, a few basic computations yield $k^* = 107$, leading to $T = 107.1526$. From Figure 4(a),(b), it is evident that system (3.1) achieves collision-free finite-time flocking at approximately $t = 6.03 < T$.



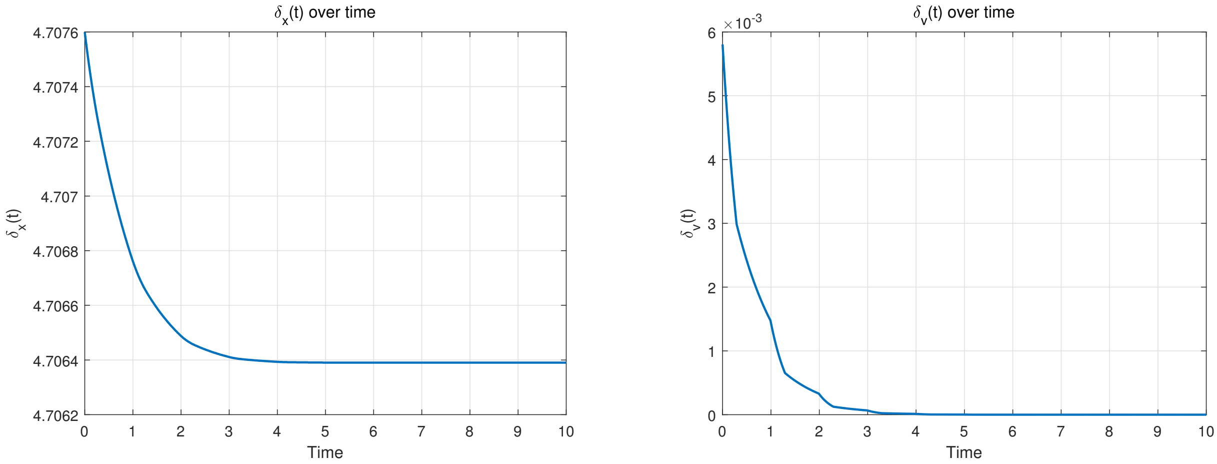
(a) (b)

Figure 2. (a) The position trajectory of agents. (b) The trajectory of $\|v_i(t) - v_c(t)\|$.



(a) (b)

Figure 3. (a) The evolution of the maximum and minimum distances. (b) The trajectory of $u_1(t)$.



(a) (b)

Figure 4. (a) The trajectory of $\delta_x(t)$. (b) The trajectory of $\delta_v(t)$.

Example 4.2. *Inspired by numerical simulations [34], we let*

$$g(t, v_i(t)) = 2(sin(8t)sin(v_i^1/30), cos(12t)cos(v_i^2/20)), L = \frac{\sqrt{2}}{10}, T = 20.$$

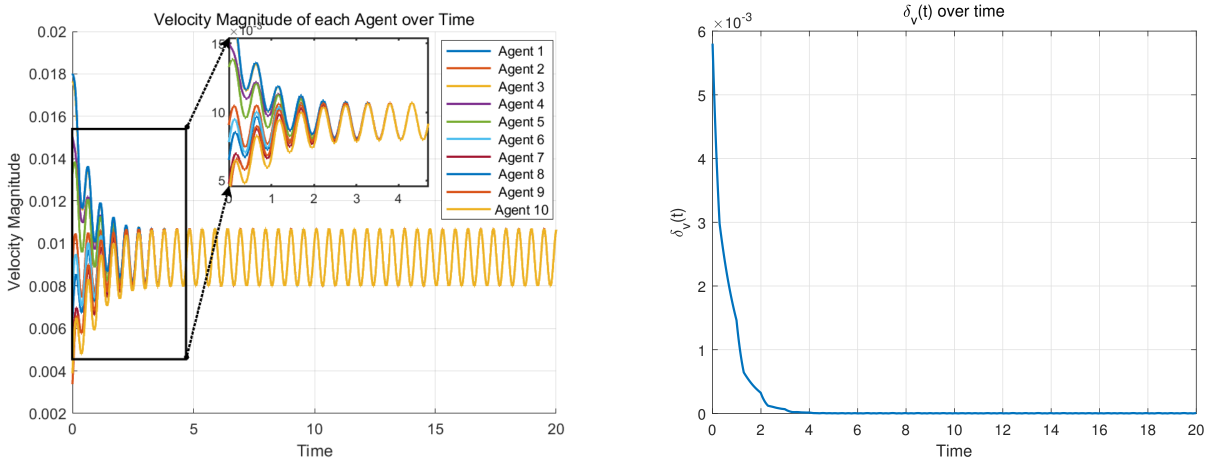
Other parameters are consistent with Example 4.1. Through similar verification, we obtain

$$\mathcal{V}(0) = 0.0064 < \int_{X(0)}^{r^*} \left[K\psi\left(2\sqrt{N}\omega\right) - \sqrt{2}L\right] \mathrm{d}\omega = 0.0074, X_{M_0} = 5.0901,$$

and

$$\min_{i \neq j} \|X_{ij}(0)\| = 3.9621 > \sqrt{2NT\mathcal{V}(0)} = 2.9922.$$

From Figure 5(a), it can be observed that under this disturbance condition, the system's flocking velocity is neither the average of the initial velocities nor remains constant, but eventually all agents achieve a consistent speed. According to Theorem 3.2, the calculation yields $k^* = 104$, and consequently, $T = 104.1457$. As shown in Figure 5(b), system (3.1) achieves finite-time flocking at approximately $t = 17.77 < T$.



(a)

(b)

Figure 5. (a) The trajectory of $\|v_i(t)\|$. (b) The trajectory of $\delta_v(t)$.

Example 4.3. *We provide a traditional finite-time control protocol:*

$$u_2(t) = \frac{K_0}{N} \sum_{j=1}^{N} \varphi(\|x_j(t) - x_i(t)\|)\text{sig}(v_j(t) - v_i(t))^{0.7}, t \in [t_{2k}, t_{2k+1}).$$

Other parameters are consistent with Example 4.1. For the purpose of comparison, we focus solely on the results of the finite-time control protocol within the control interval.

In Figure 6(a),(b), the variation trajectories of the finite-time control protocols $u_1(t)$ and $u_2(t)$ are presented, respectively. We observed that the traditional finite-time control protocol $u_2(t)$ causes chattering, whereas the protocol $u_1(t)$ proposed in this paper effectively avoids this phenomenon and is both continuous and smooth.

To further explore the impact of control parameters on control performance, we present the following comparisons under the disturbance conditions described in Example 4.1. First, as can be seen from Figure 7(a),(b), the larger the values of K and K_0, the faster the convergence speed of the flock. Moreover, changes in K have a relatively more noticeable impact on the convergence speed. Additionally, from Figure 8(a), it is evident that as λ decreases, the convergence speed of the flock increases. Finally, Figure 8(b) demonstrates that the longer the length of the control interval, the faster the convergence speed of the flock, further validating Remark 2.1. Based on the above analysis, we can adjust the relevant parameters according to practical application requirements.

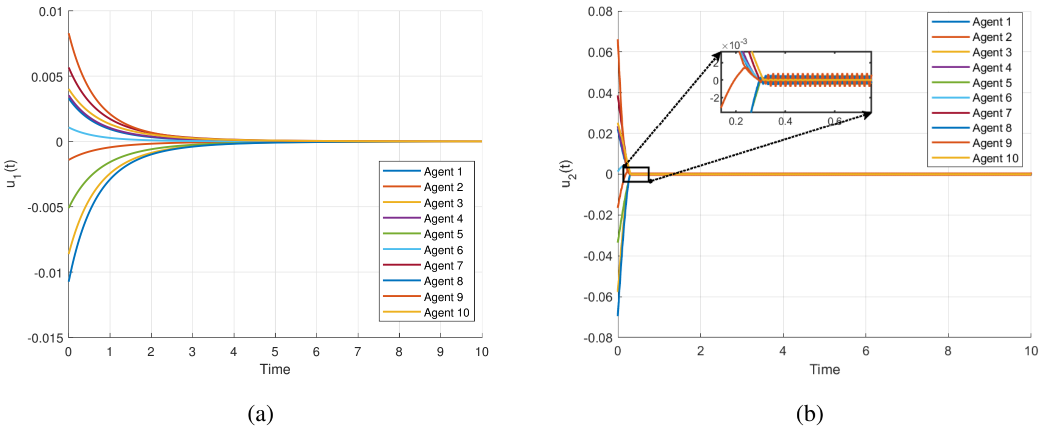


Figure 6. (a) The trajectory of $u_1(t)$. (b) The trajectory of $u_2(t)$.

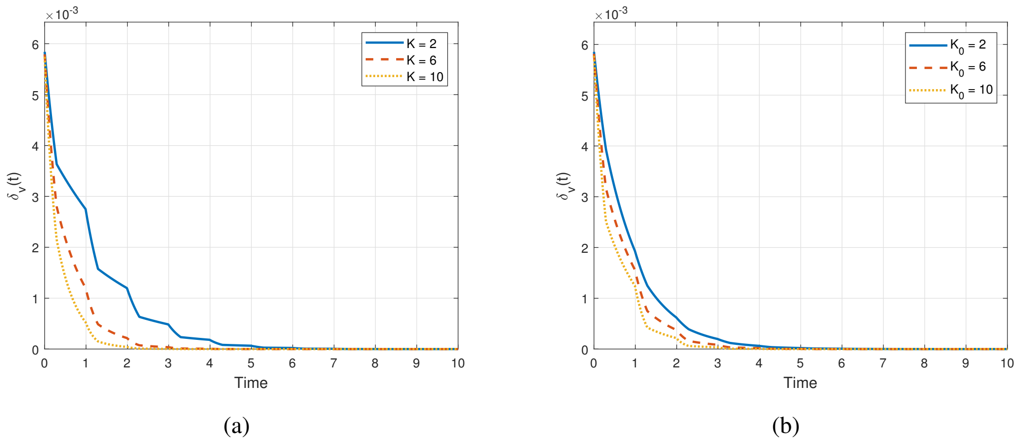


Figure 7. (a) The trajectory of $\delta_v(t)$ with different K. (b) The trajectory of $\delta_v(t)$ with different K_0.

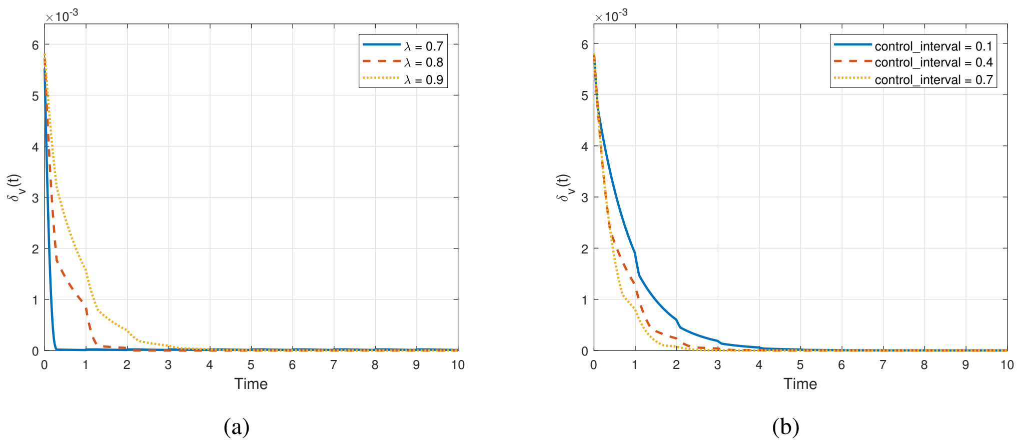
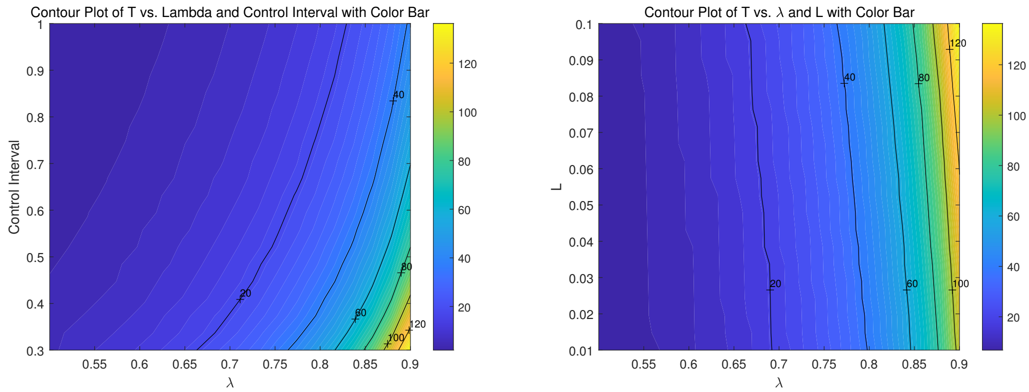


Figure 8. (a) The trajectory of $\delta_v(t)$ with different λ. (b) The trajectory of $\delta_v(t)$ with different control interval.

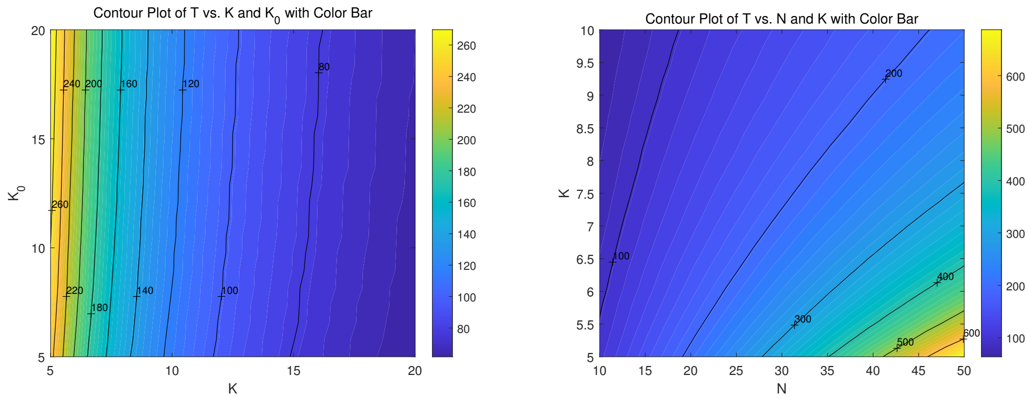
In practical applications, it is critical to examine how model parameters influence the settling time of flocking. This not only provides insight into how subtle or significant parameter changes affect the time required for the system to reach stability but also lays a solid foundation for optimizing design, thereby enhancing system efficiency and reliability. Next, we will conduct parameter sensitivity analysis with Example 4.1.

As shown in Figures 9(a), the settling time decreases significantly as the values of λ and the length of control interval increase. However, from Figure 9(b), the impact of L on the settling time is minimal and can be almost negligible. Figure 10(a) illustrates that an increase in K markedly shortens the settling time, whereas the impact of K_0 on the settling time is negligible. Additionally, as shown in Figure 10(b), the settling time increases as the number of individuals N in the group grows, which is consistent with previous findings [35]. From the analysis presented, it is evident that parameters such as K, λ, N, and control interval significantly influence the settling time. Therefore, we can focus on optimizing these most sensitive parameters to achieve the best balance between cost and performance, thereby avoiding unnecessary resource wastage.



(a) (b)

Figure 9. (a) The effect of λ and control interval $(t_{2k+1} - t_{2k})$ on T. (b) The effect of λ and L on T.



(a) (b)

Figure 10. (a) The effect of K and K_0 on T. (b) The effect of N and K on T.

5. Conclusions

This article mainly investigated the finite-time flocking of the C-S model with external perturbation and intermittent control. By constructing the appropriate Lyapunov function, system (3.1) can achieve collision-avoidance flocking without using the lower bound on communication weight. A new finite-time stability lemma is proposed, and the settling time is related to the length of the control interval and is independent of the length of the non-control interval. The finite-time flocking can be successfully achieved without using the sign function, and an upper bound on the settling time is obtained. In contrast to traditional finite-time flocking controllers, we effectively avoid the chattering phenomenon caused by the sign function during the working time intervals. Furthermore, sensitivity analysis helps us identify the parameters that critically affect the settling time, offering key insights for optimizing the system design. This process highlights the practical engineering significance of the main research findings presented in this paper.

As observed by Yin et al. [18], the precise convergence rate does not rely on the number N of agents. Deriving an upper bound on the settling time that is also independent of N is of significant interest, and we intend to explore this in our future work.

Author contributions

Qiming Liu: Writing-original draft, conceptualization, methodology; Jianlong Ren: Software, visualization; Shihua Zhang: Writing-review & editing, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors have no conflicts to disclose.

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