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Research article

The DH5 system and the Chazy and Ramamani equations

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Abstract: This article considers a class of nonlinear ordinary differential equations (ODEs) whose general solutions possess a natural boundary in the complex plane and admit special solutions that are automorphic under the action of subgroups of the modular group $\mathrm{PSL}_2(\mathbb{Z})$. Specifically, a 3×3 matrix valued system known as the Darboux-Halphen 9 (DH9) system and its fifth-order reduction to the DH5 system are discussed. These systems arise as reductions of the self-dual Yang-Mills equations of mathematical physics. It is shown that the equations discovered by J. Chazy (1908) and V. Ramamani (1970) are embedded in the DH5 system.

Keywords: Darboux-Halphen; Chazy; Ramamani; Schwarzian; automorphic functions; hypergeometric
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1. Introduction

Ramanujan [26,27] in 1916, introduced the functions

$$P(q) := 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \quad Q(q) := 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}, \quad R(q) := 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}, \tag{1.1}$$

and studied their properties. In particular, he showed that these functions satisfy the ODEs

$$q\frac{dP}{dq} = \frac{P^2 - Q}{12}, \qquad q\frac{dQ}{dq} = \frac{PQ - R}{3}, \qquad q\frac{dR}{dq} = \frac{PR - Q^2}{2}. \tag{1.2}$$

In modern terminology (see e.g., [29,30]), the functions P, Q, R are the well-known Eisenstein series E_2, E_4, and E_6, respectively, for the modular group $\mathrm{PSL}_2(\mathbb{Z})$, which is the group of fractional linear transformations with integer coefficients. If one defines

$$y(t) := -\pi i P(q), \qquad q = e^{2\pi i t}, \ \ \mathrm{Im}\, t > 0, \tag{1.3}$$

and expresses the system (1.2) as a single scalar, ODE for $y(t)$, then one obtains the eponymous Chazy equation

$$\dddot{y} + 2y\ddot{y} - 3\dot{y}^2 = 0, \tag{1.4}$$

where the "dot" denotes differentiation with respect to t. Equation (1.4) appears in a series of papers [11–13] by J. Chazy between 1909 and 1911 in his work on the classification of third-order ODEs. Chazy introduced 13 distinct classes of third-order equations, of which Chazy Class III contains Eq (1.4). Subsequently, (1.4) was shown to arise in several areas of mathematical physics involving the study of magnetic monopoles [3], self-dual Yang-Mills and Einstein equations [7, 21], and topological field theory [15], and special reductions of hydrodynamic-type equations [17].

In 1970, Ramamani [25] introduced an analogue of the Ramanujan nonlinear system (1.2). She considered the functions

$$\mathcal{P}(q) := 1 - 8\sum_{n=1}^{\infty}\frac{(-1)^n nq^n}{1-q^n}, \quad \widetilde{\mathcal{P}}(q) := 1 + 24\sum_{n=1}^{\infty}\frac{nq^n}{1+q^n}, \quad \mathcal{Q}(q) := 1 + 16\sum_{n=1}^{\infty}\frac{(-1)^n n^3 q^n}{1-q^n}.$$

These functions are the Eisenstein series associated with the congruence (congruent modulo 2) subgroup $\Gamma_0(2)$ of $\mathrm{PSL}_2(\mathbb{Z})$ (see, e.g., [30]). Here, $\Gamma_0(2)$ is defined by

$$\Gamma_0(2) := \left\{ T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod 2 \right\}.$$

In the same manner as (1.2), these functions $\mathcal{P}(q)$, $\widetilde{\mathcal{P}}(q)$, and $\mathcal{Q}(q)$ satisfy the differential equations [25]

$$q\frac{d\mathcal{P}}{dq} = \frac{\mathcal{P}^2 - \mathcal{Q}}{4}, \qquad q\frac{d\widetilde{\mathcal{P}}}{dq} = \frac{\widetilde{\mathcal{P}}\mathcal{P} - \mathcal{Q}}{2}, \qquad q\frac{d\mathcal{Q}}{dq} = \mathcal{P}\mathcal{Q} - \widetilde{\mathcal{P}}\mathcal{Q}. \tag{1.5}$$

If one defines $y(t) := \pi i\mathcal{P}(q)$, $q = e^{2\pi i t}$, then $y(t)$ satisfies the third-order nonlinear ODE

$$\dddot{y} + 2y\ddot{y} - \dot{y}^2 - 2\frac{(\ddot{y} + y\dot{y})^2}{2\dot{y} + y^2} = 0 \tag{1.6}$$

considered in [2, 4]. Both the Chazy Eq (1.4) and the Ramamani Eq (1.6) are examples of *natural barrier* equations. That is, given general initial conditions, the solutions of these equations evolve to form a natural boundary in the complex t-plane. The solutions are analytic inside (outside) the boundary but can *not* be continued analytically across the boundary that forms a dense set of essential singular points of the function $y(t)$. The barrier is movable in the sense that it can be conformally mapped to a circle whose center and radius depends on the prescribed initial conditions.

The system of nonlinear ODEs

$$\dot{\omega}_i = \omega_j\omega_k - \omega_i(\omega_j + \omega_k), \tag{1.7}$$

where the distinct indices $\{i, j, k\}$ are cyclic permutations of $\{1, 2, 3\}$ is referred to as the Darboux-Halphen (DH) system. This system first arose in Darboux's study of triply orthogonal surfaces [14] and was later solved by Halphen [19, 20]. The DH system also appeared in studies of self-dual Bianchi-IX metrics with Euclidean signature [3, 18] and in reductions of the associativity equations on a three-dimensional Frobenius manifold [15]. Furthermore, if $(\omega_1, \omega_2, \omega_3)$ is a solution of the classical Darboux-Halphen system, then

$$y := 2(\omega_1 + \omega_2 + \omega_3)$$

satisfies the Chazy Eq (1.4). Thus the general solution to the DH system also admits a movable natural boundary.

There is a strong connection between the integrability of a system of ODEs and the Painlevé property, which asserts that all solutions of the system are single-valued in their domain of existence where the movable singularities are either poles or isolated essential singular points [24]. A common feature of the above-mentioned Eqs (1.4), (1.6), and (1.7) is that they are integrable in the sense that their solutions can be expressed in terms of two linearly independent solutions of certain hypergeometric equations. However, they do not possess the Painlevé property in the usual sense because their solutions admit a movable natural barrier instead of isolated singular points. Another important distinction is that while solutions of Painlevé-type ODEs are usually given in terms of rational, single-, or double-periodic functions, or Painlevé transcendents, the solutions of (1.4), (1.6), and (1.7) are invariant under the subgroups of the automorphic group $\mathrm{SL}_2(\mathbb{Z})$. These solutions are given in terms of automorphic or modular functions, which form an entirely novel class of functions in the theory of integrable ODEs.

In this article, we further explore the relationships among automorphic ODEs by demonstrating how the Chazy and Ramamani equations can be derived as reductions of certain equations arising in mathematical physics. In particular, we consider the Darboux-Halphen-9 (DH9) system, which was first introduced in [6] as a reduction of the self-dual Yang-Mills equation. In Section 2, we will review the DH9 system and its solution in terms of a generalized version of (1.7) called the generalized Darboux-Halphen (gDH) system. Next, we will consider a special case of the DH9 system, which is a fifth-order system of ODEs referred here as the DH5 system. We will then establish the connection between the DH5 system and the Chazy (1.4) and Ramamani (1.6) equations in Section 3. The DH5 system is characterized by an important parameter γ, which takes the values from $\{0, \frac{1}{n}, n \in \mathbb{N}\}$ for single-valued solutions. We will delineate the values of γ for which the corresponding DH5 system is related to either the Chazy or the Ramamani equation. Finally, we provide explicit expressions for the solutions of these equations for the specified values of γ, in terms of hypergeometric functions.

2. The DH9 system and its reduction to DH5

Consider the following nonlinear ODE for a complex, 3×3 matrix valued function $M(t)$

$$\dot{M} = (\operatorname{adj} M)^T + M^T M - (\operatorname{Tr} M)M, \tag{2.1}$$

where M^T denotes the transpose of M, and $\operatorname{adj} M$ is the adjoint matrix of M satisfying $(\operatorname{adj} M)\, M = \det(M)I$. Equation (2.1) was obtained in [6] as a reduction of the self-dual Yang-Mills equations associated with the gauge group of diffeomorphisms $\mathrm{Diff}(S^3)$ of the 3-sphere. They were also derived from the Nahm equations for a triple of left-invariant vector fields on the 3-sphere S^3 and describe SU(2)-invariant hypercomplex 4-manifolds [22]. The matrix-valued function $M(t)$ on the level sets of t define a framing of left-invariant vector fields on the tangent space of S^3. That is, any triple (v_1, v_2, v_3) of left-invariant vector fields on S^3 is related to a fixed triple (e_1, e_2, e_3) via $v_i = \sum_j M_{ij} e_j$. Thus, M is a uniquely defined, invertible matrix.

2.1. The DH9 system

In this section, we outline how to solve (2.1) by a factorization method, that first appeared in [1]. Note that since M is invertible, its characteristic equation can be expressed as

$$(\det M)M^{-1} = M^2 - (\mathrm{Tr}M)M + \tfrac{1}{2}((\mathrm{Tr}M)^2 - \mathrm{Tr}M^2)I,$$

where the last term on the right-hand side is $\mathrm{Tr}(\mathrm{adj}M)$. Inserting the above in (2.1), and decomposing the matrix M into its symmetric (M_s) and antisymmetric (M_a) parts, yields

$$\dot{M}_s = 2(M_s^2 - (\mathrm{Tr}M_s)M_s) + [M_s, M_a] + \tfrac{1}{2}((\mathrm{Tr}M_s)^2 - \mathrm{Tr}(M_s^2 - M_a^2))I, \tag{2.2a}$$
$$\dot{M}_a = -\{M_s, M_a\}, \tag{2.2b}$$

where $[\cdot,\cdot], \{\cdot,\cdot\}$ denote the matrix commutator and anti-commutator, respectively. Assuming that the eigenvalues of M_s are distinct, it can be diagonalized such that

$$M = M_s + M_a = P(D + A)P^{-1},$$

where $P \in \mathrm{SO}(3, \mathbf{C})$, $D = \mathrm{diag}(\omega_1, \omega_2, \omega_3)$ with $\omega_i \neq \omega_j$, $i \neq j$, and $A = P^{-1}M_aP$ is skew-symmetric with elements $A_{12} = \tau_3, A_{23} = \tau_1$ and $A_{31} = \tau_2$. After decomposing (2.2a) into diagonal, off-diagonal symmetric parts, and from (2.2b), one obtains the following ODEs for D, P, and A. Equation (2.2b) gives

$$\dot{\tau}_1 = -\tau_1(\omega_2 + \omega_3), \qquad \dot{\tau}_2 = -\tau_2(\omega_3 + \omega_1), \qquad \dot{\tau}_3 = -\tau_3(\omega_1 + \omega_2); \tag{2.3}$$

the off-diagonal symmetric part of (2.2a) gives a linear equation for P

$$\dot{P} = -PA; \tag{2.4}$$

and the diagonal part of (2.2a) leads to the generalized Darboux-Halphen (gDH) system

$$\dot{\omega}_1 = \omega_2\omega_3 - \omega_1(\omega_2 + \omega_3) + \tau^2, \tag{2.5}$$
$$\dot{\omega}_2 = \omega_3\omega_1 - \omega_2(\omega_3 + \omega_1) + \tau^2, \tag{2.6}$$
$$\dot{\omega}_3 = \omega_1\omega_2 - \omega_3(\omega_1 + \omega_2) + \tau^2, \tag{2.7}$$

with $\tau^2 = \tau_1^2 + \tau_2^2 + \tau_3^2$.

Equations (2.3), (2.4), and (2.6) constitute the DH9 system given by (2.1) in terms of the factorization $M = P(D + A)P^{-1}$. Taking differences of equations in system (2.6) yields $\dot{\omega}_1 - \dot{\omega}_2 = -2\omega_3(\omega_1 - \omega_2)$, and its cyclic permutations, which can be used to solve the τ_is in terms of the ω_is as

$$\tau_1^2 = \alpha^2(\omega_1 - \omega_2)(\omega_3 - \omega_1), \quad \tau_2^2 = \beta^2(\omega_2 - \omega_3)(\omega_1 - \omega_2), \quad \tau_3^2 = \gamma^2(\omega_3 - \omega_1)(\omega_2 - \omega_3), \tag{2.8}$$

where α, β, and γ are integration constants. Thus the solution of the DH9 Eq (2.1) is given via the solutions of the gDH system since (2.4) can be solved once the matrix elements of A are determined via (2.8). Next, we outline how to linearize the gDH system. If we define the cross ratio

$$s := \frac{\omega_1 - \omega_2}{\omega_3 - \omega_2},$$

then it follows from Eq (2.6) that the ω_is can be parametrized as

$$\omega_1 = -\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}}{s-1}, \quad \omega_2 = -\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}}{s}, \quad \omega_3 = -\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}}{s(s-1)}. \tag{2.9}$$

Substituting (2.9) in either of the three equations in (2.6) shows that $s(t)$ satisfies the Schwarz differential equation

$$\frac{d}{dt}\left(\frac{\ddot{s}}{\dot{s}}\right) - \frac{1}{2}\left(\frac{\ddot{s}}{\dot{s}}\right)^2 + \frac{\dot{s}^2}{2}V(s) = 0, \qquad V(s) = \frac{1-\alpha^2}{s^2} + \frac{1-\beta^2}{(s-1)^2} + \frac{\alpha^2+\beta^2-\gamma^2-1}{s(s-1)}. \tag{2.10}$$

The general solution of Eq (2.10) is given implicitly by the inverse function

$$t(s) = \frac{u_1(s)}{u_2(s)}, \tag{2.11}$$

where $u_1(s)$ and $u_2(s)$ are linearly independent solutions to the Fuchsian equation with three regular singular points at $s = 0, 1, \infty$

$$\frac{d^2u}{ds^2} + \frac{1}{4}V(s)u = 0, \tag{2.12}$$

which is also equivalent to the hypergeometric equation,

$$s(s-1)\frac{d^2F}{ds^2} + [(a+b+1)s - c]\frac{dF}{ds} + abF = 0, \tag{2.13}$$

where $a = (1-\alpha-\beta-\gamma)/2$, $b = (1-\alpha-\beta+\gamma)/2$, $c = 1-\alpha$, and

$$u(s) = s^{(1-\alpha)/2}(s-1)^{(1-\beta)/2}F(s).$$

If $V(s)$ in (2.10) is to be further restricted such that α, β, γ are real and non-negative with $\alpha+\beta+\gamma < 1$, then a branch of $t(s)$ defined by (2.11) maps the upper-half s-plane (Im $s \geq 0$) onto a hyperbolic triangle T whose sides are three circular arcs subtending interior angles $\alpha\pi$, $\beta\pi$, and $\gamma\pi$ at the vertices $t(0)$, $t(1)$, and $t(\infty)$, respectively, in the extended complex t-plane. By the Schwarz reflection principle, the analytic extension of this branch maps the lower-half plane to an adjacent triangle T′ that is the image of T under reflection across the circular arc that forms their common boundary. This process can be continued indefinitely to obtain a Riemann surface that is the union of an infinite number of circular triangles obtained by inversions across the boundaries of T and its images. The necessary and sufficient condition that this Riemann surface is a plane region D (being the uniform covering of non-overlapping triangles) is that the parameters α, β, γ be either zero or reciprocals of positive integers. Then the inverse $s(t)$ is a single-valued, meromorphic function on T and on each of its reflected images. Further details of this conformal map can be found in [23].

All possible analytic extensions of the pair (u_1, u_2) of solutions of the Fuchsian Eq (2.12) are determined by the monodromy group $G \subset \mathrm{GL}_2(\mathbb{C})$, which depends on the parameters α, β, γ in $V(s)$. G acts projectively on the ratio $t(s)$ via fractional linear transformations

$$t \to g(t) := \frac{au_1(s) + bu_2(s)}{cu_1(s) + du_2(s)} = \frac{at+b}{ct+d}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,$$

and the projectivized monodromy group is the quotient $\Gamma \cong G/\lambda I \subseteq \mathrm{PSL}_2(\mathbb{C})$, $\lambda \in \mathbb{C}$ where I is the 2×2 identity matrix. Note that Eq (2.10) is invariant under the projective transformation:

$$t \to g(t), \qquad s(g(t)) = s(t), \qquad g \in \Gamma.$$

When α, β, γ is either zero or reciprocals of positive integers greater than 1, Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ and is generated by an even number of reflections across the boundaries of the circular triangles. Thus, the Riemann surface in the t-plane is tiled by the pair of triangles T and T′ together with their images under Γ. In this case, the single-valued inverse $s(t)$ satisfying $s(g(t)) = s(t),\ g \in \Gamma$, is called a Γ-invariant automorphic function. The boundary of D in the t-plane is a Γ-invariant circle, which is orthogonal to (all three sides of) the triangle T and its reflected images. This orthogonal circle is the set of limit points for the automorphic group Γ. It turns out to be a set of essential singularities forming a *natural boundary* for the function $s(t)$. Inside its domain of existence D, $s(t)$ is meromorphic with simple poles at those vertices of the triangles tessellating D, where $s(t) = \infty$.

2.2. The DH5 reduction of DH9

Let us now consider the case in which M has the special form

$$M = \begin{pmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ 0 & 0 & M_{33} \end{pmatrix}. \tag{2.14}$$

In this case, the DH9 system (2.1) reduces to the DH5 system, which was considered in [5] and analyzed using an associated evolving monodromy problem. Here, we will solve the DH5 system explicitly utilizing the factorization method described in Section 2.1 (see also [10]).

We decompose the special form of M in (2.14) into symmetric and skew-symmetric parts as before. Due to the special block structure of M, its symmetric part M_s can be diagonalized by an orthogonal matrix of the form

$$P = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.15}$$

where θ is a (generally complex) function of t to be determined. That is, $M_s = PDP^{-1}$, where $D = \mathrm{diag}(\omega_1, \omega_2, \omega_3)$. Furthermore, the skew-symmetric part of M is unchanged by the adjoint action of P. That is,

$$A = P^{-1} M_a P = M_a = \frac{1}{2}(M - M^T) = \begin{pmatrix} 0 & \tau_3(t) & 0 \\ -\tau_3(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.16}$$

where $\tau_3(t) = \frac{1}{2}(M_{12} - M_{21})$. Since $\tau_1 = \tau_2 = 0$ in this case, the ω_is are given by Eq (2.9), where $s(t)$ solves Eq (2.10) with $\alpha = \beta = 0$. From Eqs (2.8) and (2.9), we have

$$\tau_3(t) = \frac{i\gamma}{2} \frac{\dot{s}}{\sqrt{s}(s-1)},$$

where the constant γ is the same as in (2.8). With the $\tau_3(t)$ given above, Eq (2.4) with P as in (2.15) and A as in (2.16) can be integrated to give

$$\theta(s(t)) = i\gamma \log(\sqrt{s} - \sqrt{s-1}) + \theta_0,$$

where θ_0 is a constant. Finally, the matrix $M = P(D+A)P^{-1}$ in (2.14) is reconstructed from the various components P, D, and A. Note that in order to obtain any solution of Eq (2.1) where M is given by (2.14), we must fix the two constants γ and θ_0 and choose a solution $s(t)$ to the Schwarz Eq (2.10) with $\alpha = \beta = 0$ and the fixed value of γ.

3. The DH5 system and special third-order ODEs

As demonstrated in the previous section, the DH5 system can effectively be solved in terms of a solution $s(t)$ of the Schwarz Eq (2.10) with parameters $(0, 0, \gamma)$. The solution $s(t)$ is often referred to as the Schwarz function $S(0, 0, \gamma; t)$. In this section, we will show that for certain values of the parameter γ, the solutions to the DH5 system given by $S(0, 0, \gamma; t)$ are related to the Chazy and Ramamani Eqs (1.4), (1.6) introduced in Section 1. More specifically, we will establish that the gDH system (2.6) with $\alpha = \beta = 0$, associated with the DH5 equations can be reduced to third-order, scalar ODEs depending on the choice of the parameter γ. Here, we will consider only those choices of γ for which the resulting third-order ODE is either the Chazy or the Ramamani equation. The reduction procedure is outlined below in the general case first, i.e., for the gDH system (2.6) with three arbitrary parameters α, β, γ. The restriction $\alpha = \beta = 0$ is imposed later.

Starting from (2.6), let us define the function

$$y(t) = a_1\omega_1 + a_2\omega_2 + a_3\omega_3, \tag{3.1}$$

where the coefficients a_i are real constants. Then one obtains from (2.9)

$$y(t) = -\frac{p}{2}\frac{d}{dt}\log g(t), \qquad g(t) = \frac{\dot{s}(t)}{s^{1-\alpha_1}(s-1)^{1-\beta_1}}, \tag{3.2}$$

with $\alpha_1 = a_1/p$, $\beta_1 = a_2/p$, and $p = a_1 + a_2 + a_3$. As a result, all the higher derivatives of $y(t)$ are also expressed via $s(t)$, $\dot{s}(t)$, and $\ddot{s}(t)$ from (3.2) by repeated use of the Schwarz differential Eq (2.10). Next we proceed to derive a third-order, nonlinear ODE in $y(t)$ that is either a polynomial or rational in y and its derivatives. To that end, we introduce a sequence of special differential polynomials by

$$f_{n+1} = \dot{f}_n + \frac{2n}{p}yf_n, \quad n \geq 2, \qquad f_2 := \dot{y} + \frac{y^2}{p}. \tag{3.3}$$

The first few differential polynomials are given by

$$f_2 := \dot{y} + \frac{y^2}{p}, \qquad f_3 = \ddot{y} + \frac{6y\dot{y}}{p} + \frac{4y^3}{p^2}, \qquad f_4 = \dddot{y} + \frac{6}{p}(2y\ddot{y} - 3\dot{y}^2 + 4f_2^2).$$

A direct calculation using (3.2) and (2.10) then shows that the differential polynomials f_n are given by

$$f_n = \frac{p}{2}V_n(s)\dot{s}(t)^n, \qquad V_{n+1}(s) = V_n'(s) + nq(s)V_n(s), \quad n \geq 2, \tag{3.4}$$

where $V_n(s)$ are rational functions defined recursively as

$$V_2(s) = \frac{V(s)}{2} + q'(s) + \frac{q(s)^2}{2}, \qquad q(s) = \frac{1-\alpha_1}{s} + \frac{1-\beta_1}{(s-1)}, \tag{3.5}$$

with $V(s)$ given in (2.10) and $'$ above denoting differentiation with respect to s. Then from (3.4) for $n = 2, 3, 4$, and assuming $f_2 \neq 0$, one can construct the following rational expressions in $y(t), \dot{y}(t), \ddot{y}(t)$, and $\dddot{y}(t)$

$$I_1 := \frac{p}{2}\frac{f_3^2}{f_2^3} = \frac{V_3(s)^2}{V_2(s)^3}, \qquad I_2 := \frac{p}{2}\frac{f_4}{f_2^2} = \frac{V_4(s)}{V_2(s)^2}. \tag{3.6}$$

The right-hand side of (3.6) implies that the quantities I_1 and I_2 are rational functions of s only, depending explicitly on the parameters p, α_1, β_1 (equivalently, a_1, a_2, a_3) and α, β, γ. The pair (I_1, I_2) can be regarded as defining a complex curve in $\mathbb{C}^2$ parametrized by s. Eliminating s from the two equations in (3.6) leads to a third-order, nonlinear ODE for $y(t)$, which depends explicitly on the parameters $(a_1, a_2, a_3; \alpha, \beta, \gamma)$. In general, such an ODE is highly nonlinear and can be expressed in the form of a quasi-homogeneous polynomial in f_2, f_3, f_4, as $P(f_2, f_3, f_4) = 0$ where each monomial $f_2^a f_3^b f_4^c$ in P has weight $4a + 6b + 8c = 48$. However, for special choices of the parameter set $(a_1, a_2, a_3; \alpha, \beta, \gamma)$, one can obtain a variety of quasilinear, third-order ODEs [9]. In particular, by simply taking a linear combination

$$aI_1 + bI_2 + c = 0 \tag{3.7}$$

of the quantities I_1, I_2 defined in (3.6) with constants $a, b \neq 0, c$, it is possible to derive a class of interesting third-order ODEs, which were studied in [9], although this class was not completely classified. We show below that both the Chazy and Ramamani ODEs belong to the class (3.7).

3.1. The Chazy reduction of the DH5 system

In order to derive the Chazy equation, we take $a = 0$ in (3.7) and set

$$I_2 = J \quad \Longleftrightarrow \quad f_4 - \frac{2J}{p} f_2^2 = 0, \tag{3.8}$$

where J is a constant to be determined later. Note that the second equality above follows from the definition of I_2 in (3.6), which also implies a simple algebraic relation between the rational functions $V_2(s)$ and $V_4(s)$, namely,

$$V_4(s) = J V_2^2(s). \tag{3.9}$$

If (3.9) were to to hold for *all* s, the parameter set $(a_1, a_2, a_3; \alpha, \beta, \gamma)$ appearing in the functions $V_2(s)$ and $V_4(s)$ must satisfy certain conditions. Using the expression for $V(s)$ in (2.10), the rational function $V_2(s)$ in (3.5) can be written as

$$\begin{gathered} V_2(s) = \frac{1}{2}\left[\frac{A}{s^2} + \frac{B}{(s-1)^2} + \frac{C}{s(s-1)}\right], \\ A = \alpha_1^2 - \alpha^2, \qquad B = \beta_1^2 - \beta^2, \qquad C = \gamma_1^2 - A - B - \gamma^2, \end{gathered} \tag{3.10}$$

where $\gamma_1 = 1 - \alpha_1 - \beta_1$ and recall (from below (3.2)) that $\alpha_1 = a_1/p, \beta_1 = a_2/p$. Then $V_4(s)$ is readily computed from the recurrence relation in (3.4). Upon substituting the expressions for $V_2(s)$ and $V_4(s)$ into (3.9) and rationalizing the resulting expression, one finds that (3.9) is satisfied if and only if

$$u_1 s^4 + u_2 s^3(s-1) + u_3 s^2(s-1)^2 + u_4 s(s-1)^3 + u_5(s-1)^4 = 0,$$

for all values of s. The last identity is equivalent to the vanishing of the coefficients u_i, i.e.,

$$u_1 := JB^2 - 12B\beta_1^2 = 0, \tag{3.11a}$$
$$u_2 := 2JBC - 4[(1-\alpha_1)(1-6\beta_1)B + \tfrac{1}{2}(1-2\beta_1)(1-3\beta_1)C] = 0, \tag{3.11b}$$
$$\begin{aligned} u_3 := {} & J(2AB + C^2) - 4[(2-3\beta_1)(1-\beta_1)A + (2-3\alpha_1)(1-\alpha_1)B \\ & + \tfrac{1}{2}[(2-3\beta_1)(1-2\alpha_1) + (2-3\alpha_1)(1-2\beta_1)]C] = 0, \end{aligned} \tag{3.11c}$$
$$u_4 := 2JAC - 4[(1-\beta_1)(1-6\alpha_1)A + \tfrac{1}{2}(1-2\alpha_1)(1-3\alpha_1)C] = 0, \tag{3.11d}$$
$$u_5 := JA^2 - 12A\alpha_1^2 = 0. \tag{3.11e}$$

System (3.11) represents a set of coupled, algebraic equations for the parameters $(\alpha_1, \beta_1; \alpha, \beta, \gamma)$ and J. Next, we impose the conditions $\alpha = \beta = 0$ since we consider the DH5 system (2.14) and choose $\gamma \in \{\frac{1}{m},\ m \in \mathbb{N},\ m > 1\} \cup \{0\}$ so that the underlying Schwarz function $s(t) = S(0, 0, \gamma; t)$ is single-valued. Moreover, it is easy to verify that if either $\alpha_1 = 0$ or $\beta_1 = 0$, then the system (3.11) does not admit a consistent solution. Henceforth, we choose $\alpha_1 \neq 0, \beta_1 \neq 0$. Then it follows immediately from (3.11a) amd (3.11e) that $J = 12$. Substituting this value of J in (3.8) and setting $p = 6$ yields

$$f_4 - 4f_2^2 = 0,$$

which is the Chazy Eq (1.4) as can be verified from the expressions of the differential polynomials listed below (3.3). Our remaining task is to find the parameter values $(\alpha_1, \beta_1; \gamma)$ that satisfy the rest of the Eqs (3.11b)–(3.11d) above.

Substituting $J = 12, A = \alpha_1^2, B = \beta_1^2$ in (3.11b) and (3.11d) yields

$$(1-6\alpha_1)\left[(1-\beta_1)\alpha_1^2 + \tfrac{1}{2}(1+\alpha_1)\,C\right] = 0,$$
$$(1-6\beta_1)\left[(1-\alpha_1)\beta_1^2 + \tfrac{1}{2}(1+\beta_1)\,C\right] = 0,$$

which leads to 4 cases: (a) $\alpha_1 = \beta_1 = \frac{1}{6}$, (b) $\alpha_1 = \frac{1}{6}, \beta_1 \neq \frac{1}{6}$, (c) $\alpha_1 \neq \frac{1}{6}, \beta_1 = \frac{1}{6}$, and (d) $\alpha_1 \neq \frac{1}{6}, \beta_1 \neq \frac{1}{6}$. Note that since (3.11b) and (3.11d) are symmetric with respect to interchanging α_1 and β_1, the cases (b) and (c) can be treated in a similar fashion. For each of these 4 cases, one can solve for α_1, β_1, and C from Eqs (3.11b)–(3.11d), and obtain the values of γ that correspond to the Schwarz function $S(0, 0, \gamma; t)$ that parametrizes the Chazy solution $y(t)$ via (3.2) with $p = 6$. In other words, $y(t)$ given by (3.2) solves the Chazy Eq (1.4) if and only if $S(0, 0, \gamma; t)$ satisfies (2.10) with $\alpha = 0$, $\beta = 0$ and the obtained values of γ. The results are summarized in Table 1 below.

The first two columns of Table 1 list the parameters $(\alpha_1, \beta_1, \gamma_1)$ and the value γ in the Schwarz function $S(0, 0, \gamma; t)$. The third column lists $y(t)$ as a linear combination of the gDH variables ω_i as in (3.1). Recall that the coefficients $(a_1, a_2, a_3) = p(\alpha_1, \beta_1, \gamma_1)$ with $p = 6$ for the Chazy reduction. The first row consists of three distinct cyclic permutations of $(\alpha_1, \beta_1, \gamma_1)$ for the same value $\gamma = 0$. This is due to the fact that $s(t) = S(0, 0, 0; t)$ is defined on the fundamental hyperbolic triangle T whose vertices corresponding to $s = 0, 1, \infty$ subtend the same angle 0π. As a result, the repeated action of the homographic transformation

$$\sigma_1 : s \to \frac{1}{1-s}, \qquad \sigma_1^3 = id,$$

cyclically permutes these vertices but leave the Schwarz Eq (2.10) for $S(0, 0, 0; t)$ invariant. On the other hand, the gDH-triple $(\omega_1, \omega_2, \omega_3)$ is cyclically permuted under the transformations σ_1, σ_1^2 as can be easily verified from (2.9). Hence, we have three distinct linear combinations of the ω_i leading to the same parametrization of the Chazy solution $y(t)$ in terms of the Schwarz function $S(0,0,0;t)$. Alternatively, one can also verify this fact directly from the expression (3.2) of $y(t)$ in terms of $g(t)$ by computing $g(\sigma_1(s(t)))$ and $g(\sigma_1^2(s(t)))$. The two distinct choices of $(\alpha_1,\beta_1,\gamma_1)$ in the second and third rows corresponding to $\gamma = \frac{1}{2}$ and $\gamma = \frac{1}{3}$, respectively, are also explained similarly. In this case, the transformation $\sigma_2 : s \to 1-s$ switches the vertices corresponding to $s = 0$ and $s = 1$ of the fundamental triangle T for the Schwarz functions $S(0,0,\frac{1}{2};t)$ and $S(0,0,\frac{1}{3};t)$ but leaves the vertex corresponding to $s = \infty$ fixed. This results in the permutation $(\omega_1,\omega_2,\omega_3) \to (\omega_2,\omega_1,\omega_3)$ of the gDH variables as shown in the third column of Table 1.

Table 1. Admissible sets of parameters and automorphic groups for the Chazy solution $y(t)$ associated with DH5.

$(\alpha_1,\beta_1,\gamma_1)$	γ	$y = a_1w_1 + a_2w_2 + a_3w_3$	Γ
$(\frac{1}{6},\frac{1}{6},\frac{2}{3})$		$y = \omega_1 + \omega_2 + 4\omega_3$	
$(\frac{2}{3},\frac{1}{6},\frac{1}{6})$	0	$y = 4\omega_1 + \omega_2 + \omega_3$	$\Gamma_0(4)$
$(\frac{1}{6},\frac{2}{3},\frac{1}{6})$		$y = \omega_1 + 4\omega_2 + \omega_3$	
$(\frac{1}{6},\frac{1}{3},\frac{1}{2})$	$\frac{1}{2}$	$y = \omega_1 + 2\omega_2 + 3\omega_3$	$\Gamma_0(2)$
$(\frac{1}{3},\frac{1}{6},\frac{1}{2})$		$y = 2\omega_1 + \omega_2 + 3\omega_3$	
$(\frac{1}{6},\frac{1}{2},\frac{1}{3})$	$\frac{1}{3}$	$y = \omega_1 + 3\omega_2 + 2\omega_3$	$\Gamma_0(3)$
$(\frac{1}{2},\frac{1}{6},\frac{1}{3})$		$y = 3\omega_1 + \omega_2 + 2\omega_3$	
$(\frac{1}{6},\frac{1}{6},\frac{2}{3})$	$\frac{2}{3}$	$y = \omega_1 + \omega_2 + 4\omega_3$	Γ^*
$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	0	$y = 2\omega_1 + 2\omega_2 + 2\omega_3$	$\Gamma(2)$

The last column of Table 1 lists the automorphism groups Γ of the various Schwarz functions $S(0,0,\gamma;t)$. Recall from Section 2.1 that these correspond to the transformation

$$t \to g(t) = \frac{at+b}{ct+d}, \quad S(\alpha,\beta,\gamma;g(t)) = S(\alpha,\beta,\gamma;t), \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

The automorphic group Γ is a subgroup of the modular group $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ when the vertex parameters (α,β,γ) of the fundamental triangle T are either 0 or reciprocal of a positive integer greater than 1. Except for Γ^*, the automorphic groups listed in Table 1 are level-N congruence subgroups of $\Gamma(1)$ that are defined by

$$\Gamma_0(N) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod N \right\}, \tag{3.12}$$

for $N = 2,3,4$. Both the first and last row of Table 1 correspond to $\gamma = 0$ but with different combinations of the ω_i in the expression for $y(t)$. These two cases correspond to the automorphic groups $\Gamma(2)$ and $\Gamma_0(4)$ that are related via the conjugation: $\Gamma_0(4) = g\Gamma(2)g^{-1}$, $g = \mathrm{diag}(1,2)$. The automorphic function for $\Gamma(2)$ is called the elliptic modular function $\lambda(t)$, which is related to the triangle function of $\Gamma_0(4)$ as

$$S(0,0,0;t) = \frac{\lambda(2t)}{\lambda(2t)-1}.$$

Finally, the automorphic group Γ^* corresponding to the case $\gamma = \frac{2}{3}$ in the penultimate row of Table 1 is not a subgroup of the modular group $\Gamma(1)$ since $1/\gamma$ is not a positive integer. Nonetheless, the single-valued function $S(0, \frac{1}{2}, \frac{1}{3}; t)$ is expressible as a degree-2 rational function of $S(0, \frac{2}{3}, 0; t)$ [23], namely

$$S(0, \tfrac{1}{2}, \tfrac{1}{3}; \epsilon t) = -\frac{4S(0, \frac{2}{3}, 0; t)}{(S(0, \frac{2}{3}, 0; z) - 1)^2}, \quad \epsilon = \sqrt[3]{-\tfrac{1}{4}}.$$

Using this relation to express $g(t)$ in (3.2), it can be shown that $y(t)$ obtained from $S(0, \frac{2}{3}, 0; t)$ is a single-valued function of t. We remark here that the results in Table 1 are consistent with the more general parametrization of the Chazy equation in [8] in terms of Schwarz functions $S(\alpha, \beta, \gamma; t)$ where (α, β) are not necessarily $(0, 0)$, unlike the situation associated with the DH5 system that is considered in this article.

Next, we discuss the analytic behavior of the Chazy solution $y(t)$ in its domain of definition D. Since D is tessellated by the fundamental triangle T and its images, it suffices to examine the analyticity only on T. It is evident from (3.2) that $y(t)$ depends on the Schwarz function $s(t)$ and its derivatives. The analytic properties of $s(t)$ can be deduced from its inverse $t(s)$ defined via (2.11), which was carried out in [8]. It was shown that $s(t)$ is analytic inside the fundamental triangle T except at the vertex $t(\infty)$ where it is meromorphic. Moreover, $s(t)$ can be represented as a Laurent series in a neighborhood of each vertex in terms of a suitable local uniformizing variable depending on the angle $\mu\pi$ subtended at the vertex. We omit the details that can be found in [8].

For the DH5 system, $\mu = (0, 0, \gamma)$ corresponding to the singular points at $s = (0, 1, \infty)$, in that order. Then $s(t)$ can be expressed as a power series near the vertex $t_0 \in \{t(0), t(1)$, given by

$$s(t) = s(t_0) + \sum_{j=1}^{\infty} a_j q^j,$$

where the uniformizing variable $q := e^{\frac{2\pi i}{k(t-t_0)}}$ if $t_0 \neq \infty$ and $q = e^{\frac{2\pi i t}{k}}$ if $t_0 = \infty$, and $k \in \mathbb{N}$. Near the vertex $t_0 = t(\infty)$, one has $s(t) = (t - t_0)^{-n}\phi(t)$ if $\gamma = 1/n$, $1 < n \in \mathbb{N}$, where $\phi(t)$ is analytic with $\phi(t_0) \neq 0$. If $\gamma = 0$, then $s(t)$ admits a Laurent expansion

$$s(t) = \frac{b_{-1}}{q} + \sum_{j=0}^{\infty} b_j q^j,$$

where q is defined above.

Since the gDH variables ω_i are also expressed in terms of $s(t)$ and its derivatives as in (2.9), their analytic properties can also be determined from those of $s(t)$ on each fundamental triangle T. If $t_0 \in \{t(0), t(1)\}$ where $\alpha = \beta = 0$, then a direct calculation using Eq (2.9) and the analytic behavior of $s(t)$ outlined above shows that each ω_i approaches a finite value as $t \to t_0$ from the interior of the triangle T. In particular, if $t_0 = t(0)$, then $\omega_1 = O(q)$, $\omega_2 = \frac{1}{2} + O(q)$, $\omega_3 = O(q)$; and if $t_0 = t(1)$, then $\omega_1 = \frac{1}{2} + O(q)$, $\omega_2 = O(q)$, $\omega_3 = O(q)$, where q is the local uniformizing variable in the neighborhood of t_0, and $q \to 0$ as $t \to t_0$. If $t_0 = t(\infty)$ then there are two possibilities: (i) if $\gamma = 0$, then $\omega_1 = O(q)$, $\omega_2 = O(q)$, $\omega_3 = \frac{1}{2} + O(q)$; (ii) if $\gamma = \frac{1}{m}$, $m \in \mathbb{N}, m > 1$, then all ω_i have first-order poles at $t(\infty)$ and $\mathrm{Res}_{t=t(\infty)}\{\omega_1, \omega_2, \omega_3\} = \{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}(m-1)\}$. Consequently, the Chazy solution given by (3.1) is at most meromorphic in the fundamental triangle T as well as in its entire domain of definition with possibly a simple pole at the vertex $t(\infty)$ and its images, obtained by reflection across the sides of T. The residue at the pole is given by

$$\text{Res}_{t=t(\infty)}y(t) = \frac{a_3 m - p}{2} = 3\left(\frac{\gamma_1}{\gamma} - 1\right),$$

where we have used $p = 6$ and $\gamma = \frac{1}{m}$. The residue $\text{Res}_{t=t(\infty)}y(t)$ vanishes when $\gamma = \gamma_1$, which occurs in *all* the cases in Table 1 where $\gamma = \frac{1}{m}$. Thus we conclude that the Chazy solution $y(t)$ associated with the DH5 system is *analytic* in its domain of definition D. However, as discussed earlier in Section 2.1, the boundary of D is a natural boundary of essential singularities formed by the circle orthogonal to T and its images. In fact, it can be shown (see, e.g., [23]) that the vertices of the fundamental triangle T with angle 0π lie on this orthogonal circle. This is precisely the case for the vertices $t(0), t(1)$ (and their images) where $\alpha = \beta = 0$. In addition, when $\gamma = 0$, the vertex $t(\infty)$ and its images also lie on the boundary of the domain D.

3.2. The Ramamani reduction of the DH5 system

Next, we outline the reduction of the DH5 system to the Ramamani Eq (1.6). We will keep our discussion brief since the reduction procedure is similar to the Chazy case. First, we define $y(t)$ as in (3.1), then choose $a = -1, b = 1$ in (3.7) and set

$$I_2 - I_1 = R \quad \Longleftrightarrow \quad f_4 f_2 - f_3^2 = \frac{2R}{p} f_2^3, \tag{3.13}$$

where R is a constant to be determined later. Consequently, from (3.6), we obtain an algebraic relation among the rational functions $V_2(s)$, $V_3(s)$, and $V_4(s)$ given by

$$V_4(s)V_2(s) - V_3^2 = RV_2^3(s).$$

Using (3.10) and (3.4) to obtain $V_3(s)$, $V_4(s)$, substituting these in the above equation, and rationalizing the resulting expression, yields a polynomial equation

$$\sum_{i=i}^{7} v_i s^{i-1}(s-1)^{7-i} = 0,$$

that should hold identically for all s. This implies that the coefficients v_i, $i = 1, \ldots, 7$ depending on the parameters $(\alpha_1, \beta_1; \gamma)$ and R must vanish and leads to the following overdetermined system of equations

$$v_1 := A^2(4\alpha_1^2 - RA) = 0, \tag{3.14a}$$

$$v_2 := A[4A(1-\beta_1)(1-2\alpha_1) + 2C(1-\alpha_1+4\alpha_1^2) - 3RAC] = 0, \tag{3.14b}$$

$$\begin{aligned} v_3 :=& 4[A\{3B\alpha_1^2 + M + (1-\beta_1)(1-6\alpha_1)C\} + \tfrac{1}{2}(1-2\alpha_1)(1-3\alpha_1)C^2] - 3RA(AB+C^2) \\ &- 8[(1-\beta_1)A + \tfrac{1}{2}(1-2\alpha_1)C]^2 + 8A\alpha_1[2B(1-\alpha_1) + C(1-2\beta_1)] = 0, \end{aligned} \tag{3.14c}$$

$$\begin{aligned} v_4 :=& 4[\{(1-\alpha_1)(1-6\beta_1) + (1-\beta_1)(1-6\alpha_1)\}AB + \tfrac{1}{2}\{(1-2\beta_1)(1-3\beta_1)A + \\ &(1-2\alpha_1)(1-3\alpha_1)B + 2M\}C] - 16\alpha_1\beta_1 AB - 16[(1-\alpha_1)B + \tfrac{1}{2}(1-2\beta_1)C] \times \\ &[(1-\beta_1)A + \tfrac{1}{2}(1-2\alpha_1)C] - RC(C^2 + 6AB) = 0, \end{aligned} \tag{3.14d}$$

where A, B, C are given by (3.10) with $\alpha = \beta = 0$, and

$$M = (2 - 3\beta_1)(1 - \beta_1)A + (2 - 3\alpha_1)(1 - \alpha_1)B + \tfrac{1}{2}[(2 - 3\beta_1)(1 - 2\alpha_1) + (2 - 3\alpha_1)(1 - 2\beta_1)]C.$$

The system (3.14) is symmetric with respect to α_1 and β_1. Hence the remaining coefficients v_5, v_6, v_7 are obtained from v_3, v_2, v_1, respectively, by switching α_1 and β_1. We do not explicitly include them here.

We next proceed to determine the values (if any) of the constant R and parameters $(\alpha_1, \beta_1; \gamma)$ that would yield a consistent solution for the system (3.11). The equations $v_1 = v_7 = 0$ lead to 4 possible cases that need to be further examined, namely (a) $\alpha_1 \neq 0$, $\beta_1 \neq 0$, (b) $\alpha_1 = 0$, $\beta_1 \neq 0$, (c) $\alpha_1 \neq 0$, $\beta_1 = 0$, and (d) $\alpha_1 = 0$, $\beta_1 = 0$. The first 3 cases all lead to the choice $R = 4$. Substituting this value of R in (3.13) and setting $p = 2$, we obtain

$$f_4 f_2 - f_3^2 = 4 f_2^3,$$

which is the Ramamani Eq (1.6), as can be verified by using the expressions of the differential polynomials listed below (3.3). For case (d), all the equations in (3.14) are identically satisfied except (3.14d), which gives rise to 2 subcases: $C = 0$ or $C = 4/R$. For $C = 0$, using its definition in (3.10) and the fact that $\gamma_1 = 1$, we get $\gamma = 1$, which is ruled out since $\gamma \in \{\frac{1}{m}, m \in \mathbb{N}, m > 1\} \cup \{0\}$ as in the Chazy case. If $C = 4/R$, then it turns out that $\gamma^2 = 1 - 4/R$. Setting $\gamma = 0$ again leads to $R = 4$, which corresponds to the Ramamani Eq (1.6). However, if we choose $\gamma = \frac{1}{m}$, $m \in \mathbb{N}, m > 1$, then $R = 4m^2/(m^2 - 1)$. This case leads to a generalized version of Ramamani equation found in [9] but we do not pursue it here. Upon examining the remaining equations from (3.14) corresponding to the first 3 cases mentioned earlier, we find a consistent set of parameter values for $\{\alpha_1, \beta_1; \gamma\}$ that solve (3.14). These are listed in Table 2.

Table 2. Admissible sets of parameters and automorphic groups for the Ramamani solution $y(t)$ associated with DH5.

$(\alpha_1, \beta_1, \gamma_1)$	γ	$y = a_1 w_1 + a_2 w_2 + a_3 w_3$	Γ
$(\frac{1}{2}, \frac{1}{2}, 0)$		$y = \omega_1 + \omega_2$	
$(0, \frac{1}{2}, \frac{1}{2})$	0	$y = \omega_2 + \omega_3$	$\Gamma_0(4)$
$(\frac{1}{2}, 0, \frac{1}{2})$		$y = \omega_1 + \omega_3$	
$(\frac{1}{2}, 0, \frac{1}{2})$	$\frac{1}{2}$	$y = \omega_1 + \omega_3$	$\Gamma_0(2)$
$(0, \frac{1}{2}, \frac{1}{2})$		$y = \omega_2 + \omega_3$	
$(1, 0, 0)$		$y = 2\omega_1$	
$(0, 1, 0)$	0	$y = 2\omega_2$	$\Gamma(2)$
$(0, 0, 1)$		$y = 2\omega_3$	

From Table 2, one finds that the gDH system associated with the DH5 equations, i.e., (2.6) with $\alpha = \beta = 0$, can be reduced to the Ramamani equation only if $\gamma = 0$ or $\frac{1}{2}$. For $\gamma = 0$, there are two cases corresponding to modular subgroup $\Gamma_0(4)$ and its conjugate $\Gamma(2)$, whereas $\gamma = \frac{1}{2}$ corresponds to the level-2 congruence subgroup $\Gamma_0(2)$ of the modular group. These were already described in Section 3.1. Furthermore, like the Chazy case, the transformation $s \to (1-s)^{-1}$ that cyclically permutes $\{\omega_1, \omega_2, \omega_3\}$ but leaves (2.10) invariant for the Schwarz function $S(0, 0, 0; t)$ is the reason why there are 3 sets of

$\{\alpha_1, \beta_1, \gamma_1\}$ parameters when $\gamma = 0$. Similarly, the transformation $s \to 1 - s$ accounts for the 2 sets of parameters when $\gamma = \frac{1}{2}$.

When $\gamma = 0$, the gDH system (2.6) reduces simply to the classical Darboux-Halphen system given by (1.7) in Section 1. Then, one can verify by a direct calculation using system (1.7) that $y(t) = 2\omega_i(t)$ or $y(t) = \omega_i(t) + \omega_j(t)$, $i \neq j$ satisfy the Ramamani Eq (1.6), as indicated in the first and third rows of Table 2. One can also verify (1.6) directly for $y(t)$ given in the second row corresponding to $\gamma = \frac{1}{2}$, although the calculations are a bit more involved since one needs to use (2.6) instead of (1.7). Of course, these remarks also hold for the Chazy case with $y(t)$ given in Table 1. In fact, the case where $y = 2(\omega_1 + \omega_2 + \omega_3)$ was already discussed in Section 1.

Like the Chazy case, the Ramamani solution $y(t)$ can have at most a simple pole at the vertex corresponding to $s = \infty$ of the fundamental triangle T when $\gamma = \frac{1}{2}$. However, in this case, $\gamma_1 = \gamma$ as well, thus the residue vanishes at the pole. Therefore, $y(t)$ is analytic inside T as well as inside its entire domain of definition D but cannot be analytically continued across the boundary of D, as explained before.

We close this section by pointing out that the parametrizations of the Chazy or Ramamani solutions $y(t)$ in terms of different Schwarz functions lead to interesting rational transformations among themselves. Suppose $s(t)$ and $s'(t)$ are two Schwarz functions that parametrize the *same* solution $y(t)$. Then from (3.2), we obtain the differential relation

$$\frac{\dot{s}(t)}{s^{1-\alpha_1}(s-1)^{1-\beta_1}} = C\frac{\dot{s}'(t)}{s'^{1-\alpha'_1}(s'-1)^{1-\beta'_1}},$$

for some constant C, which can be solved to obtain a relation of the form $f(s, s') = 0$. We illustrate this by considering an explicit example using information from the second and third rows of Table 2. Let us choose $\gamma = \frac{1}{2}$ with parameters $\{\alpha_1, \beta_1, \gamma_1\} = \{\frac{1}{2}, 0, \frac{1}{2}\}$, which yield via (3.2) an expression for $y(t)$ in terms of the Schwarz function $s(t) = S(0, 0, \frac{1}{2}; t)$ and its derivatives. Similarly, a second differential expression for $y(t)$ in terms of the elliptic modular function $\lambda(t)$ and its derivatives is obtained by choosing $\gamma = 0$ and $\{\alpha_1, \beta_1, \gamma_1\} = \{1, 0, 0\}$ from the third row. Recall from Section 3.1 that $\lambda(t)$ is the canonical Schwarz function associated with $\Gamma(2)$. Equating these two expressions from (3.2) yields

$$\frac{\dot{s}}{\sqrt{s}(s-1)} = \frac{\dot{\lambda}}{\lambda - 1},$$

where we have set $C = 1$ for convenience. The above equation can be integrated by elementary means, and after appropriately choosing the integration constant, one obtains the rational relation

$$s(\lambda) = \left(\frac{2}{\lambda} - 1\right)^2,$$

which was also found in [2] via a different method. This process can also be carried out in the Chazy case by considering the parameters corresponding to 2 distinct values of γ.

4. Solutions $y(t)$ and hypergeometric functions

In Section 2.1, it was shown that the Schwarz Eq (2.10) for $s(t)$ can be linearized in the sense that its inverse function $t(s)$ can be expressed as a ratio of two linearly independent solutions of the hypergeometric Eq (2.13). As a result, it is possible to express the solution $y(t)$ of both the Chazy and Ramamani equations directly in terms of appropriate hypergeometric functions. This yields another interesting connection between the nonlinear ODEs (1.4) and (1.6) with the classical hypergeometric theory.

Let $\{F_1, F_2\}$ be a pair of linearly independent solutions of (2.13) such that $t(s) = F_2(s)/F_1(s)$ in (2.11). Then

$$\dot{s}(t) = \frac{1}{t'(s)} = \frac{F_1^2}{W(F_1, F_2)},$$

where $W(F_1, F_2)$ denotes the Wronskian that, via Abel's formula, is given by $W(F_1, F_2) = W_0 s^{\alpha-1}(s-1)^{\beta-1}$. The constant W_0 depends on the pair of hypergeometric functions chosen. Inserting the above expression for $\dot{s}$ in (3.2) yields an explicit expression for $y(s(t))$ in terms of the hypergeometric function $F_1(s)$, namely

$$y(s(t)) = -\frac{p s^{-\alpha}(s-1)^{-\beta}}{2W_0}\Big(2s(s-1)F_1F_1' - [(\alpha+\beta-\alpha_1-\beta_1)s + (\alpha_1-\alpha)]F_1^2\Big). \tag{4.1}$$

In what follows, we provide a hypergeometric parametrization for $y(t)$ for both the Chazy and Ramamani equations associated with the automorphic group $\Gamma_0(N)$, $N = 2, 3, 4$ given in Tables 1 and 2, as well as the group Γ^* in Table 1, although the latter is not a subgroup of the modular group $\Gamma(1)$. Recall that in these cases, the parameters $(\alpha,\beta,\gamma) = (0, 0, \frac{1}{N})$ for $N = 2, 3$, $(\alpha,\beta,\gamma) = (0, 0, 0)$ when $N = 4$, and $(\alpha,\beta,\gamma) = (0, 0, \frac{2}{3})$ for Γ^*. From the expressions given below (2.13), the parameters $(a, b, c) = ((1-\gamma)/2,\ (1+\gamma)/2,\ 1)$ so that $b = 1-a$. The resulting hypergeometric equation associated with $\Gamma_0(N)$ and Γ^* depends on only one parameter a and is given by

$$s(s-1)\frac{d^2F}{ds^2} + [2s-1]\frac{dF}{ds} + a(1-a)F = 0, \tag{4.2}$$

where $a = \frac{1}{4}, \frac{1}{3}$ and $\frac{1}{2}$ for $N = 2, 3$ and 4, respectively, while $a = \frac{1}{6}$ corresponding to $\gamma = \frac{2}{3}$ for Γ^*. The regular solutions of (4.2) near $s = 0$ are spanned by the hypergeometric function

$$F(s) = {}_2F_1(a, 1-a; 1; s) := \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{n!^2} s^n, \quad (a)_n := a(a+1)(a+2)\cdots(a+n-1),$$

that is normalized as $F(0) = 1$. We will first construct a canonical triangle function $s = S(0, 0, \gamma; t)$ associated with the group $\Gamma_0(N)$, $N = 2, 3, 4$ and Γ^* from its inverse $t(s)$ by choosing an appropriate pair $\{F_1, F_2\}$ of linearly independent hypergeometric solutions of (4.2). It is customary to choose the fundamental triangle T associated with the Schwarz function $S(0, 0, \gamma; t)$ such that one of its vertices $t(0) = i\infty$ where $\alpha = 0$. To that end, we define $t(s)$ in (2.11) as follows:

$$t(s) = \frac{F_2}{F_1}, \qquad F_1 = {}_2F_1(a, 1-a; 1; s), \quad F_2 = K\, {}_2F_1(a, 1-a; 1; 1-s), \tag{4.3}$$

where the constant K will be determined by the vertex condition $t(0) = i\infty$. The analytic continuation of ${}_2F_1(a, 1-a; 1; 1-s)$ to the neighborhood of $s = 0$ is given by (see, e.g., [16]),

$${}_2F_1(a, 1-a; 1; 1-s) = \frac{1}{\Gamma(a)\Gamma(1-a)}\left[-\log(s)\,{}_2F_1(a, 1-a; 1; s) + \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(n!)^2} h_n s^n\right].$$

Here, $\Gamma(\cdot)$ are Gamma functions that satisfy $\Gamma(a)\Gamma(1-a) = \pi\csc(\pi a)$, and the coefficients $h_n := 2\psi(1+n) - \psi(a+n) - \psi(1-a+n)$, where $\psi(\cdot) := \Gamma'(\cdot)/\Gamma(\cdot)$ are the Gauss Digamma functions. Hence, (4.3) implies that

$$t(s) = K\,\frac{{}_2F_1(a, 1-a; 1; 1-s)}{{}_2F_1(a, 1-a; 1; s)} = K\frac{\sin(\pi a)}{\pi}(-\log(s) + h(s))$$

in a neighborhood of $s = 0$, and $h(s)$ is analytic there. The constant K is determined by the requirements $t(s) \to i\infty$ as $s \to 0^+$, and $t \to t+1$ onto the next branch as s makes a circuit around $s = 0$. The translation: $t \to t+1$ is an automorphism $s(t+1) = s(t)$, which stabilizes the vertex $t(0) = i\infty$. These conditions are satisfied if we set $2K = i\csc(\pi a)$, which leads to $q(s) = e^{2\pi i t(s)} = se^{-h(s)}$. Finally, inverting this relation gives the desired q-expansion of the Schwarz function near the vertex $t(0) = i\infty$, i.e.,

$$s(t) = e^{h_0}q\Big(1 + b_1 q + b_2 q^2 + \cdots\Big), \qquad q = e^{2\pi i t}, \qquad h_0 = 2\psi(1) - \psi(a) - \psi(1-a).$$

Using the formulas for Digamma function $\psi(\cdot)$ at rational arguments (see, e.g., [16]), the constant $e^{h_0} = 432, 64, 27$, and 16, when $a = \frac{1}{6}, \frac{1}{4}, \frac{1}{3}$, and $\frac{1}{2}$, respectively. We note here that the Schwarz triangle functions corresponding to $\Gamma_0(N)$ are usually expressed in terms of null theta or Dedekind's eta functions, which admit such q-expansions derived above. Here, we demonstrated that such representations can be directly deduced from the classical hypergeometric theory.

We next proceed to obtain expressions for the solution $y(t)$ given in (4.1). First, we need to find an expression for the constant W_0 arising from the Wronskian. If we use the expressions for the hypergeometric function $F_1(s)$ and the analytic continuation of $F_2(s)$ near $s = 0$ as given above, then a direct calculation shows that

$$W(F_1, F_2) = K\frac{\sin(\pi a)}{\pi}[W(F_1, h(s)) - \frac{F_1^2}{s}] = \frac{W_0}{s(s-1)},$$

where we have used the fact that $\alpha = \beta = 0$. Letting $s \to 0^+$ in above, and noting that $h(s)$ is analytic near $s = 0$ and $K = i\csc(\pi a)/2$, yields $W_0 = i/2\pi$. Inserting this expression in (4.1) along with $\alpha = \beta = 0$ leads to the following expression of $y(t)$

$$y(s(t)) = p\pi i\Big(2s(s-1)F_1F_1' - [\alpha_1 - (\alpha_1 + \beta_1)s]F_1^2\Big), \tag{4.4}$$

where $F_1(s) = {}_2F_1(a, 1-a; 1; s)$ and the parameters (α_1, β_1) are given in Table 1 for the Chazy equation with $p = 6$, and in Table 2 for the Ramamani equation with $p = 2$. Recall from (1.3) in Section 1 that the Ramanujan function $P(q) = iy(t)/\pi$, where $y(t)$ is the solution of the Chazy Eq (1.4). Similarly, the Ramamani function is also given by the same expression $\mathcal{P}(q) = iy(t)/\pi$, $y(t)$ being the solution of the Ramamani Eq (1.6). In the third column of Tables 3 and 4, we list the Chazy and Ramamani solutions that are obtained from (4.4) in terms of the hypergeometric function ${}_2F_1(a, 1-a; 1; s)$.

Note that the 3 different expressions for P and $\mathcal{P}$ associated with the group $\Gamma_0(4)$ correspond to the transformations $\sigma_1(s)$ and $\sigma_1^2(s)$ as mentioned in Section 3.1. For example, substituting $(\alpha_1, \beta_1) = (\frac{1}{6}, \frac{1}{6})$ from the first row of Table 3 into the expression $g(t)$ in (3.2) yields $g(t) = \dot{s}/s^{5/6}(s-1)^{5/6}$, which leads to the top expression for $P(s(t))$ in the first row. Replacing s by $\sigma_1(s) = (1-s)^{-1}$ in the above expression for $g(t)$ then gives $\tilde{g}(t) = \dot{s}/s^{5/6}(s-1)^{1/3}$, which leads to the function $g(t)$ in (3.2) but with a *new* set of coefficients $(\alpha_1, \beta_1) = (\frac{1}{6}, \frac{2}{3})$. This set of coefficients and the corresponding expression for $P(s(t))$ appear on the bottom of the first row in Table 3. Applying the transformation σ_1 once again and proceeding similarly yields the middle set of coefficients $(\alpha_1,\beta_1) = (\frac{2}{3},\frac{1}{6})$ and the corresponding $P(s(t))$. Similarly, the involution $\sigma_2(s) = 1 - s$ gives rise to the 2 different expressions for P and $\mathcal{P}$ associated with the groups $\Gamma_0(2)$ and $\Gamma_0(3)$.

Table 3. Ramanujan functions P in terms of $F_1(s) = {}_2F_1(a, 1-a; 1; s)$ with $p = 6$ in (4.4).

Γ	a	(α_1,β_1)	$P = iy/\pi$
$\Gamma_0(4)$	$\frac{1}{2}$	$(\frac{1}{6},\frac{1}{6})$	$P = -12s(s-1)F_1F_1' + (1-2s)F_1^2$
		$(\frac{2}{3},\frac{1}{6})$	$P = -12s(s-1)F_1F_1' + (4-5s)F_1^2$
		$(\frac{1}{6},\frac{2}{3})$	$P = -12s(s-1)F_1F_1' + (1-5s)F_1^2$
$\Gamma_0(3)$	$\frac{1}{3}$	$(\frac{1}{6},\frac{1}{2})$	$P = -12s(s-1)F_1F_1' + (1-4s)F_1^2$
		$(\frac{1}{2},\frac{1}{6})$	$P = -12s(s-1)F_1F_1' + (3-4s)F_1^2$
$\Gamma_0(2)$	$\frac{1}{4}$	$(\frac{1}{6},\frac{1}{3})$	$P = -12s(s-1)F_1F_1' + (1-3s)F_1^2$
		$(\frac{1}{3},\frac{1}{6})$	$P = -12s(s-1)F_1F_1' + (2-3s)F_1^2$
Γ^*	$\frac{1}{6}$	$(\frac{1}{6},\frac{1}{6})$	$P = -12s(s-1)F_1F_1' + (1-2s)F_1^2$

Table 4. Ramamani functions $\mathcal{P}$ in terms of $F_1(s) = {}_2F_1(a, 1-a; 1; s)$ with $p = 2$ in (4.4).

Γ	a	(α_1,β_1)	$\mathcal{P} = iy/\pi$
$\Gamma_0(4)$	$\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2})$	$\mathcal{P} = -4s(s-1)F_1F_1' + (1-2s)F_1^2$
		$(0,\frac{1}{2})$	$\mathcal{P} = -4s(s-1)F_1F_1' - sF_1^2$
		$(\frac{1}{2},0)$	$\mathcal{P} = -4s(s-1)F_1F_1' + (1-s)F_1^2$
$\Gamma_0(2)$	$\frac{1}{4}$	$(\frac{1}{2},0)$	$\mathcal{P} = -4s(s-1)F_1F_1' + (1-s)F_1^2$
		$(0,\frac{1}{2})$	$\mathcal{P} = -4s(s-1)F_1F_1' - sF_1^2$

We conclude this section by pointing out that the hypergeometric parametrizations for $P(t(s))$ in Table 3 were known to Ramanujan. In his second notebook [28], he gave an explicit parametrization of the functions P, Q, R in (1.2) corresponding to the case $a = \frac{1}{2}$, in terms of complete elliptic integral of the first kind, which is related to the hypergeometric function ${}_2F_1(\frac{1}{2},\frac{1}{2};1;s)$ via

$$K(s) := \int_0^{\pi/2} \frac{dt}{\sqrt{1 - s\sin^2 t}} = \frac{\pi}{2}\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;s).$$

He also stated the results for the remaining cases $a = \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ but without proof.

5. Conclusions

We have studied a special class of nonlinear ODEs with novel singularity structure of their general solution consisting of a natural barrier in the complex plane. In particular, this class includes the third-order equations discovered in the last century by Chazy, Ramanujan, and Ramamani. A special subclass of solutions of these ODEs is given in terms of Schwarz functions that are invariant under subgroups of the modular group. In this paper, we have established the connection between the Chazy and Ramamani equations and an important system of ODEs in mathematical physics, namely the DH5 system given by (2.1) together with (2.14). Our main results can be summarized as follows:

(a) Suppose $\omega_i(t)$ are the gDH variables associated with the DH5 system, satisfying (2.6) with $\tau^2 = \gamma^2(\omega_3 - \omega_1)(\omega_2 - \omega_3)$. Then there exist certain choices of the set of coefficients $\{a_1, a_2, a_3\}$ in the linear combination defined in (3.1) and corresponding choices of the parameter γ such that the function $y(t)$ in (3.1) satisfies either the Chazy equation (1.4) or the Ramamani equation (1.6).

(b) The corresponding solutions $y(t)$ are expressed via (3.2) in terms of the Schwarz function $S(0, 0, \gamma; t)$ associated with certain subgroups of the modular group $SL_2(\mathbb{Z})$. Both the set of admissible parameters and the type of subgroups corresponding to the Chazy and Ramamani equations are listed in Tables 1 and 2.

(c) An alternative description of the Chazy and Ramamani solutions are given in terms of the classical hypergeometric functions, and an explicit formula (4.4) for the solutions is derived. These results are tabulated in Tables 3 and 4.

To the best of our knowledge, the above results are new and we believe that they provide novel insights into the inter-relationships among these special class of ODEs. Furthermore, the method described in Section 3.1 to obtain these results can be extended to investigate reductions of the full gDH system (2.6) to other third (or higher)-order, nonlinear ODEs whose general solutions admit a natural barrier in the complex plane.

Author contributions

S. Chakravarty and P. Guha: Conceptualization, Methodology, Validation, Writing-original draft, Writing-review & editing. The authors contributed equally to the work and preparation of this article.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflict of interest in this article.

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