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Research article

Surfaces of coordinate finite II-type

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Abstract: We study the class of surfaces of revolution in the 3-dimensional Euclidean space E^3 with nonvanishing Gauss curvature whose position vector $\boldsymbol{x}$ satisfies the condition $\Delta^{II}\boldsymbol{x} = A\boldsymbol{x}$, where A is a square matrix of order 3 and Δ^{II} denotes the Laplace operator of the second fundamental form II of the surface. We show that a surface of revolution satisfying the preceding relation is a catenoid or part of a sphere.

Keywords: surfaces in E^3; surfaces of revolution; surfaces of coordinate finite type; Beltrami operator
Mathematics Subject Classification: 53A05, 53A45

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1. Introduction

Surfaces of finite I-type is one of the main topics that attracted the interest of many differential geometers from the moment that B. Y. Chen introduced the notion of surfaces of finite I-type with respect to the first fundamental form I about four decades ago. Many results concerning this subject have been collected in [1].

Surfaces of revolution of finite Chen type have applications in various fields of science and engineering, such as domes and cooling towers, and are widely used in architecture and structural engineering. Finite Chen-type surfaces help in optimizing these structures for minimal energy configurations, ensuring stability and efficiency. Also, understanding Chen-type surfaces helps in designing aerodynamic shapes with minimal drag and structural stress distribution [2].

Let $\boldsymbol{x} : M^2 \rightarrow E^3$ be a parametric representation of a surface in the 3-dimensional Euclidean space E^3. Denote by Δ^I the second Laplace operator according to the first fundamental form I of M^2 and by $\boldsymbol{H}$ the mean curvature field of M^2. Then, it is well known that [3]

$$\Delta^I \boldsymbol{x} = -2\boldsymbol{H}.$$

Moreover, in [4], T.Takahashi showed that the position vector $\boldsymbol{x}$ of M^2 for which $\Delta^I \boldsymbol{x} = \lambda \boldsymbol{x}$, is either the minimals with eigenvalue $\lambda = 0$ or M^2 lies in an ordinary sphere S^2 with a fixed nonzero eigenvalue.

O. Garay, in his article [5], has made a generalization of T. Takahashi's condition. Actually, so far, he studied surfaces in $\mathbb{E}^3$ satisfying $\Delta^I r_i = \mu_i r_i, i = 1, 2, 3$, where (r_1, r_2, r_3) are the coordinate functions of $\mathbf{r}$. Another general problem was also studied in [6], for which the surfaces in $\mathbb{E}^3$ satisfy $\Delta^I \mathbf{r} = K\mathbf{r} + L(\S)$, where $K \in M(3 \times 3)$ and $L \in M(3 \times 1)$. It was proved that minimal surfaces, spheres, and circular cylinders are the only surfaces in $\mathbb{E}^3$ satisfying (§). Surfaces satisfying (§) are said to be of coordinate finite type.

In the framework of the theory of surfaces of finite I-type in E^3, a general Gauss map was studied within this context in [7]. So, one can ask which surfaces in E^3 are of finite I-type Gauss map. On the other hand, it is also interesting to study surfaces of finite I-type in E^3 whose Gauss map $\boldsymbol{n}$ satisfies a condition of the form $\Delta^I \boldsymbol{n} = A\boldsymbol{n}$, where $A \in \mathbb{R}^{3\times3}$. Surfaces in E^3 whose Gauss map is of coordinate finite type corresponding to the first fundamental form were investigated by many researchers. More precisely, the class of tubular surfaces was studied in [8], while in [9], authors studied the quadric surfaces in terms of their finite type Gauss map. In [10–12], Baikoussis and Verstraelen introduced the classes of spiral, translation, and helicoidal surfaces in the Euclidean 3-space. In [13], the authors proved that planes, circular cylinders, and spheres are the only surfaces of revolution whose Gauss map is of finite-coordinate type. In [14] the family of anchor rings was investigated in the Euclidean 3-space. Finally, in [15], researchers conducted a comprehensive study of all types of surfaces that have been studied and are still under investigation but have not yet concluded.

In 2003, S. Stamatakis and H. Al-Zoubi in [16] followed the ideas of B. Y. Chen. They introduced the notion of surfaces of finite type regarding the second or third fundamental forms, and since then much work has been done in this context.

Ruled surfaces [17], tubes [18], quadrics [19], and a special case of surfaces of revolution [20] are the classes of surfaces studied in terms of finite type classification with respect to the third fundamental form. Tubes [21] and ruled surfaces [22] are the only classes of surfaces investigated in terms of finite type classification with respect to the second fundamental form.

Another generalization can be made by studying surfaces in E^3 whose position vector $\boldsymbol{x}$ satisfies the following condition:

$$\Delta^J \boldsymbol{x} = A\boldsymbol{x}, \quad J = II, III, \tag{1.1}$$

where $A \in \mathbb{R}^{3\times3}$.

Regarding the third fundamental form in 3-dimensional Euclidean space, it was proved that spheres and catenoids are the only surfaces of revolution satisfying condition (1.1) [23]. Next, in [24], the authors found that helicoids are the only ruled surfaces that satisfy (1.1), and spheres are the only quadric surfaces that satisfy (1.1). Finally, in [25], it was shown that Scherk's surface is the only translation surface that satisfies (1.1).

2. Main theorem

Let M^2 be a smooth surface in E^3 parametrized by $\boldsymbol{x} = \boldsymbol{x}(u^1, u^2)$ on a region $U := (a, b) \times \mathbb{R}$ whose Gaussian curvature never vanishes. The standard unit normal vector field $\boldsymbol{n}$ on M^2 is defined by

$$\boldsymbol{n} = \frac{\boldsymbol{x}_{u^1} \times \boldsymbol{x}_{u^2}}{||\boldsymbol{x}_{u^1} \times \boldsymbol{x}_{u^2}||}, \tag{2.1}$$

where $\boldsymbol{x}_{u^1} := \frac{\partial x(u^1,u^2)}{\partial u^1}$ and " $\times$ " denotes the Euclidean vector product. We denote by

$$I = g_{ij}du^i du^j, \quad II = b_{ij}du^i du^j, \tag{2.2}$$

the first and second fundamental forms of M^2, respectively, where we put

$$g_{11} =< \boldsymbol{x}_{u^1}, \boldsymbol{x}_{u^1} >, \quad g_{12} =< \boldsymbol{x}_{u^1}, \boldsymbol{x}_{u^2} >, \quad g_{22} =< \boldsymbol{x}_{u^2}, \boldsymbol{x}_{u^2} >,$$

$$b_{11} =< \boldsymbol{x}_{u^1u^1}, \boldsymbol{n} >, \quad b_{12} =< \boldsymbol{x}_{u^1u^2}, \boldsymbol{n} >, \quad b_{22} =< \boldsymbol{x}_{u^2u^2}, \boldsymbol{n} >,$$

and $<,>$ is the Euclidean inner product. For two sufficiently differentiable functions $p(u^1, u^2)$ and $q(u^1, u^2)$ on M^2, the first differential parameter of Beltrami with respect to the second fundamental form II is defined by [26]

$$\nabla^{II}(p, q) = b^{ij} p_{/i} q_{/j}, \tag{2.3}$$

where $p_{/i} := \frac{\partial p}{\partial u^i}$ and b^{ij} are the components of the inverse tensor of b_{ij}. The second Beltrami operator, according to the fundamental form II of M^2, is defined by

$$\Delta^{II} p = -b^{ij} \nabla^{II}_i p_j = -\frac{1}{\sqrt{|b|}} \frac{\partial}{\partial u^i}(\sqrt{|b|} b^{ij} \frac{\partial p}{\partial u^j}), \tag{2.4}$$

where p is a sufficiently differentiable function, ∇^{II}_i is the covariant derivative in the u^i direction and $b = \det(b_{ij})$ [19].

In the present paper, we mainly focus on surfaces of finite II-type by studying surfaces of revolution in E^3 which are connected, complete, and of that, their position vector $\boldsymbol{x}$ satisfies the following relation:

$$\Delta^{II} \boldsymbol{x} = A\boldsymbol{x}, \tag{2.5}$$

Our main result is

Theorem 2.1. *Spheres and catenoids are the only surfaces of revolution in E^3 whose position vector $\boldsymbol{x}$ satisfies condition (2.5).*

3. Proof of the main theorem

Let C be a smooth curve lying on the xz-plane parametrized by

$$\boldsymbol{r}(u) = (p(u), 0, q(u)), \quad u \in (a, b),$$

where p, q are smooth functions and p is a positive function. When C revolves around the z-axis, the resulting point set S is called the surface of revolution generated by the curve C. In this case, the

z-axis is called the axis of revolution of S, and C is called the profile curve of S. On the other hand, a subgroup of the rotation group that fixes the vector $(0, 0, 1)$ is generated by

$$\begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the position vector of S is given by

$$\boldsymbol{x}(u, v) = \big(p(u)\cos v, p(u)\sin v, q(u)\big), \quad u \in (a, b), \quad v \in [0, 2\pi). \tag{3.1}$$

(For the parametric representation of surfaces of revolution, see [13, 27, 28]).

Here, we shall assume that C has the arc-length parametrization, i.e.,

$$(p\prime)^2 + (q\prime)^2 = 1, \tag{3.2}$$

where $\prime := \frac{d}{du}$. On the other hand, $p\prime q\prime \neq 0$, because if $p =$ const. or $q =$ const. then S is a circular cylinder or part of a plane. Hence the Gaussian curvature of S vanishes; a case that has been excluded.

Using the natural frame $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ of S defined by

$$\boldsymbol{x_u} = \left(p\prime(u)\cos v, p\prime(u)\sin v, q\prime(u)\right),$$

and

$$\boldsymbol{x_v} = \left(-p(u)\sin v, p(u)\cos v, 0\right),$$

the components g_{ij} of the first fundamental form in (local) coordinates are the following

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = p^2.$$

Denoting by R_1, R_2 the principal radii of curvature of S and κ the curvature of the curve C, we have

$$R_1 = \kappa, \quad R_2 = \frac{q\prime}{p}.$$

The mean and the Gaussian curvature of S are, respectively,

$$2H = R_1 + R_2 = \kappa + \frac{q\prime}{p}, \quad K = R_1 R_2 = \frac{\kappa q\prime}{p} = -\frac{p\prime\prime}{p}.$$

The components b_{ij} of the second fundamental form in (local) coordinates are the following:

$$b_{11} = \kappa, \quad b_{12} = 0, \quad b_{22} = pq\prime.$$

The Beltrami operator Δ^{II} in terms of local coordinates (u, v) of S can be expressed as follows:

$$\Delta^{II} = -\frac{1}{\kappa}\frac{\partial^2}{\partial u^2} - \frac{1}{pq\prime}\frac{\partial^2}{\partial v^2} + \frac{1}{2}\left(\frac{\kappa\prime}{\kappa^2} - \frac{p\prime q\prime + \kappa p p\prime}{\kappa p q\prime}\right)\frac{\partial}{\partial u}. \tag{3.3}$$

On account of (3.2), we can put

$$p\prime = \cos\varphi, \quad q\prime = \sin\varphi, \tag{3.4}$$

where $\varphi = \varphi(u)$. Then $\kappa = \varphi\prime$ and the relation (3.3) become

$$\Delta^{II} = -\frac{1}{\varphi\prime}\frac{\partial^2}{\partial u^2} - \frac{1}{p\sin\varphi}\frac{\partial^2}{\partial v^2} + \frac{1}{2}\left(\frac{\varphi\prime\prime}{(\varphi\prime)^2} - \frac{\cos\varphi\sin\varphi + p\varphi\prime\cos\varphi}{p\varphi\prime\sin\varphi}\right)\frac{\partial}{\partial u}, \tag{3.5}$$

while the mean and the Gaussian curvature of S become

$$2H = \varphi\prime + \frac{\sin\varphi}{p}, \tag{3.6}$$

$$K = \frac{\varphi\prime\sin\varphi}{p}. \tag{3.7}$$

Let (x_1, x_2, x_3) be the coordinate functions of $\boldsymbol{x}$ of (3.1). Then

$$\Delta^{II}\boldsymbol{x} = (\Delta^{II}x_1, \Delta^{II}x_2, \Delta^{II}x_3). \tag{3.8}$$

From (3.5) and (3.8)

$$\Delta^{II}x_1 = \Delta^{II}(p\cos v) = \left(\sin\varphi + \frac{1}{\sin\varphi} + \frac{\varphi\prime\prime\cos\varphi}{2\varphi\prime^2} - \frac{H\cos^2\varphi}{\varphi\prime\sin\varphi}\right)\cos v, \tag{3.9}$$

$$\Delta^{II}x_2 = \Delta^{II}(p\sin v) = \left(\sin\varphi + \frac{1}{\sin\varphi} + \frac{\varphi\prime\prime\cos\varphi}{2\varphi\prime^2} - \frac{H\cos^2\varphi}{\varphi\prime\sin\varphi}\right)\sin v, \tag{3.10}$$

$$\Delta^{II}x_3 = \Delta^{III}(q) = -\frac{3}{2}\cos\varphi + \frac{\varphi\prime\prime\sin\varphi}{2\varphi\prime^2} - \frac{\sin\varphi\cos\varphi}{2p\varphi\prime}. \tag{3.11}$$

We denote by $a_{ij}, i, j = 1, 2, 3$, the entries of the matrix A. By using (3.9), (3.10), and (3.11), condition (2.5) is found to be equivalent to the following system:

$$\left(\sin\varphi + \frac{1}{\sin\varphi} + \frac{\varphi\prime\prime\cos\varphi}{2\varphi\prime^2} - \frac{H\cos^2\varphi}{\varphi\prime\sin\varphi}\right)\cos v = a_{11}p\cos v + a_{12}p\sin v + a_{13}q, \tag{3.12}$$

$$\left(\sin\varphi + \frac{1}{\sin\varphi} + \frac{\varphi\prime\prime\cos\varphi}{2\varphi\prime^2} - \frac{H\cos^2\varphi}{\varphi\prime\sin\varphi}\right)\sin v = a_{21}p\cos v + a_{22}p\sin v + a_{23}q, \tag{3.13}$$

$$-\frac{3}{2}\cos\varphi + \frac{\varphi\prime\prime\sin\varphi}{2\varphi\prime^2} - \frac{\sin\varphi\cos\varphi}{2p\varphi\prime} = a_{31}p\cos v + a_{32}p\sin v + a_{33}q. \tag{3.14}$$

From (3.14) it can be easily verified that $a_{31} = a_{32} = 0$. On differentiating (3.12) and (3.13) twice with respect to v we have

$$\left(\sin\varphi + \frac{1}{\sin\varphi} + \frac{\varphi\prime\prime\cos\varphi}{2\varphi\prime^2} - \frac{H\cos^2\varphi}{\varphi\prime\sin\varphi}\right)\cos v = a_{11}p\cos v + a_{12}p\sin v, \tag{3.15}$$

$$\left(\sin\varphi + \frac{1}{\sin\varphi} + \frac{\varphi\prime\prime\cos\varphi}{2\varphi\prime^2} - \frac{H\cos^2\varphi}{\varphi\prime\sin\varphi}\right)\sin v = a_{21}p\cos v + a_{22}p\sin v. \tag{3.16}$$

Thus, $a_{13}q = a_{23}q = 0$, so that a_{13}, and a_{23} vanish. Equations (3.12)–(3.14) are equivalent to

$$\left(\sin\varphi + \frac{1}{\sin\varphi} + \frac{\varphi'' \cos\varphi}{2\varphi'^2} - \frac{H\cos^2\varphi}{\varphi' \sin\varphi}\right)\cos v = a_{11}p\cos v + a_{12}p\sin v, \tag{3.17}$$

$$\left(\sin\varphi + \frac{1}{\sin\varphi} + \frac{\varphi'' \cos\varphi}{2\varphi'^2} - \frac{H\cos^2\varphi}{\varphi' \sin\varphi}\right)\sin v = a_{21}p\cos v + a_{22}p\sin v, \tag{3.18}$$

$$-\frac{3}{2}\cos\varphi + \frac{\varphi'' \sin\varphi}{2\varphi'^2} - \frac{\sin\varphi\cos\varphi}{2p\varphi'} = a_{33}q. \tag{3.19}$$

But since $\sin v$ and $\cos v$ are linearly independent functions of v, we finally obtain $a_{12} = a_{21} = 0, a_{11} = a_{22}$. Putting $a_{11} = a_{22} = \lambda$, and $a_{33} = \mu$, we see that the system of Eqs (3.17), (3.18) and (3.19) reduces to the following two equations:

$$\sin\varphi + \frac{1}{\sin\varphi} + \frac{\varphi'' \cos\varphi}{2\varphi'^2} - \frac{H\cos^2\varphi}{\varphi' \sin\varphi} = \lambda p, \tag{3.20}$$

$$-\frac{3}{2}\cos\varphi + \frac{\varphi'' \sin\varphi}{2\varphi'^2} - \frac{\sin\varphi\cos\varphi}{2p\varphi'} = \mu q. \tag{3.21}$$

Hence the matrix A for which relation (2.5) is satisfied becomes

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

We distinguish the following cases:

Case I. $\lambda = \mu = 0$. Equations (3.20) and (3.21) become

$$\sin\varphi + \frac{1}{\sin\varphi} + \frac{\varphi'' \cos\varphi}{2\varphi'^2} - \frac{H\cos^2\varphi}{\varphi' \sin\varphi} = 0, \tag{3.22}$$

$$-\frac{3}{2}\cos\varphi + \frac{\varphi'' \sin\varphi}{2\varphi'^2} - \frac{\sin\varphi\cos\varphi}{2p\varphi'} = 0. \tag{3.23}$$

From (3.6), relation (3.22) becomes

$$\sin\varphi + \frac{1}{\sin\varphi} + \frac{\varphi'' \cos\varphi}{2\varphi'^2} - \frac{\cos^2\varphi}{2\sin\varphi} - \frac{\cos^2\varphi}{2p\varphi'} = 0, \tag{3.24}$$

Multiplying (3.23) by $-\cos\varphi$, (3.24) by $\sin\varphi$, and adding the result of these two equations, it follows that $\cos^2\varphi + \sin^2\varphi + 1 = 0$, a contradiction.

Case II. $\lambda = \mu \neq 0$. Equations (3.20) and (3.21) become

$$\sin\varphi + \frac{1}{\sin\varphi} + \frac{\varphi'' \cos\varphi}{2\varphi'^2} - \frac{H\cos^2\varphi}{\varphi' \sin\varphi} = \lambda p, \tag{3.25}$$

$$-\frac{3}{2}\cos\varphi + \frac{\varphi'' \sin\varphi}{2\varphi'^2} - \frac{\sin\varphi\cos\varphi}{2p\varphi'} = \lambda q. \tag{3.26}$$

From (3.6), relation (3.25) becomes

$$\sin\varphi + \frac{1}{\sin\varphi} + \frac{\varphi'' \cos\varphi}{2\varphi'^2} - \frac{\cos^2\varphi}{2\sin\varphi} - \frac{\cos^2\varphi}{2p\varphi'} = \lambda p. \tag{3.27}$$

Similarly, multiplying (3.26) by $-\cos\varphi$, (3.27) by $\sin\varphi$, and adding the result of these two equations, it follows that

$$\lambda p \sin\varphi - \lambda q \cos\varphi = 2. \tag{3.28}$$

On differentiating the last equation with respect to u, we find

$$\lambda p' \sin\varphi + \lambda p\varphi' \cos\varphi - \lambda q' \cos\varphi + \lambda q\varphi' \sin\varphi = 0,$$

which can be written

$$\lambda(p' \sin\varphi - q' \cos\varphi) + \lambda\varphi'(p\cos\varphi + q\sin\varphi) = 0. \tag{3.29}$$

Since $p' = \cos\varphi$ and $q' = \sin\varphi$. Thus, (3.29) reduces to

$$\lambda\varphi'(pp' + qq') = 0. \tag{3.30}$$

$\lambda\varphi'$ cannot be 0; otherwise, from (3.7), the Gaussian curvature vanishes. Hence, $pp' + qq' = 0$, i.e., $(p^2 + q^2)' = 0$. Therefore, $p^2 + q^2 = const$. Thus, C is part of a circle, and S is obviously part of a sphere.

Case III. $\lambda \neq 0, \mu = 0$. Following the same procedure as in *CaseI* and *CaseII*, we can obtain

$$2 - \lambda p \sin\varphi = 0. \tag{3.31}$$

Differentiating (3.31) with respect to u, we have

$$(\sin\varphi + p\varphi')\cos\varphi = 0. \tag{3.32}$$

Taking into account relation (3.6), Eq (3.32) becomes

$$2Hp\cos\varphi = 0, \tag{3.33}$$

which implies mean curvature H vanishes identically. Therefore, the surface is minimal; that is, it is a catenoid. Furthermore, a catenoid satisfies the condition (2.5).

Case IV. $\lambda = 0, \mu \neq 0$. In this case, (3.20) and (3.21) are given, respectively, by

$$\sin\varphi + \frac{1}{\sin\varphi} + \frac{\varphi''\cos\varphi}{2\varphi'^2} - \frac{H\cos^2\varphi}{\varphi'\sin\varphi} = 0, \tag{3.34}$$

$$-\frac{3}{2}\cos\varphi + \frac{\varphi''\sin\varphi}{2\varphi'^2} - \frac{\sin\varphi\cos\varphi}{2p\varphi'} = \mu q. \tag{3.35}$$

On account of (3.6), relation (3.34) becomes

$$\sin\varphi + \frac{1}{\sin\varphi} + \frac{\varphi''\cos\varphi}{2\varphi'^2} - \frac{\cos^2\varphi}{2\sin\varphi} - \frac{\cos^2\varphi}{2p\varphi'} = 0. \tag{3.36}$$

Multiplying (3.35) by $-\cos\varphi$, (3.36) by $\sin\varphi$, and adding the result of these two equations, we find

$$2 + \mu q \cos\varphi = 0. \tag{3.37}$$

Differentiating this equation we have

$$q\varphi\prime - \cos\varphi = 0, \tag{3.38}$$

from which

$$\varphi\prime = \frac{\cos\varphi}{q}. \tag{3.39}$$

Another derivative of (3.38), gives

$$2\varphi\prime \sin\varphi + q\varphi\prime\prime = 0. \tag{3.40}$$

From (3.39) and (3.40), we have

$$\varphi\prime\prime = -\frac{2\sin\varphi\cos\varphi}{q^2}. \tag{3.41}$$

Equation (3.34) can be written

$$1 + \sin^2\varphi + \frac{\varphi\prime\prime\cos\varphi\sin\varphi}{2\varphi\prime^2} - \frac{1}{2}\cos^2\varphi - \frac{\cos^2\varphi\sin\varphi}{2p\varphi\prime} = 0. \tag{3.42}$$

Consequently, from (3.37), (3.39), and (3.41), Eq (3.42) becomes

$$2 - \cos^2\varphi + \frac{2\sin\varphi}{\mu p} = 0, \tag{3.43}$$

from which

$$\mu p = \frac{2\sin\varphi}{\cos^2\varphi - 2}. \tag{3.44}$$

Differentiating (3.44), we obtain

$$\mu = \frac{2\varphi\prime}{\cos^2\varphi - 2} + \frac{4\varphi\prime\sin^2\varphi}{(\cos^2\varphi - 2)^2}. \tag{3.45}$$

Using (3.39) and (3.37) after some computation, we obtain $\sin\varphi = 0$, that is, $q = const.$, which implies that the Gauss curvature vanishes. A case that was excluded. Thus, there are no surfaces of revolution that satisfy this case.

Case V. $\lambda \neq 0, \mu \neq 0$. If we multiply (3.25) by $\sin\varphi$, and (3.26) by $-\cos\varphi$ then adding the resulting equations, we easily obtain

$$\lambda p \sin\varphi - \mu q \cos\varphi = 2. \tag{3.46}$$

Let

$$\Omega := \lambda p \sin\varphi + \mu q \cos\varphi. \tag{3.47}$$

By using (3.46), the derivative of Ω is the following:

$$\Omega\prime = \lambda\cos^2\varphi + \mu\sin^2\varphi - 2\varphi\prime. \tag{3.48}$$

Differentiating the Eq (3.46) and using (3.47), we find

$$\Omega\varphi\prime = (\mu - \lambda)\cos\varphi\sin\varphi. \tag{3.49}$$

It is easily verified $\Omega \neq 0$; hence, (3.49) can be written

$$\varphi\prime = \frac{(\mu - \lambda)\cos\varphi\sin\varphi}{\Omega}. \tag{3.50}$$

Differentiating the last equation and using (3.48) and (3.49), we obtain

$$\varphi\prime\prime = \frac{((\lambda - 2\mu)\sin^2\varphi + (\mu - 2\lambda)\cos^2\varphi)\varphi\prime + 2\varphi\prime^2}{\Omega}. \tag{3.51}$$

In view of (3.50), and (3.51) relation (3.21) takes the following form:

$$\frac{\Omega\cos\varphi}{p} + \frac{2(\lambda - \mu)\cos\varphi\sin\varphi}{\Omega} - 2\mu(\lambda - \mu)q\cos\varphi - (\lambda - 2\mu) = 0. \tag{3.52}$$

Multiplying the last equation by $\Omega p\cos\varphi$, we have

$$2(\lambda - \mu)p\cos^2\varphi\sin\varphi - 2\mu(\lambda - \mu)pq\Omega\cos^2\varphi - (\lambda - 2\mu)\Omega p\cos\varphi + \Omega^2\cos^2\varphi = 0. \tag{3.53}$$

From (3.46) and (3.47), it can be is easily verified that

$$\Omega\cos\varphi = \lambda p - 2\sin\varphi. \tag{3.54}$$

Therefore, on using (3.46) and (3.54), relation (3.53) becomes

$$a_1p^3 + a_2p^2 + a_3p + a_4 = 0, \tag{3.55}$$

where

$$a_1 = \lambda^2(\mu - \lambda)\sin\varphi, \qquad a_2 = \lambda[(2\lambda - \mu) - 2(\mu - \lambda)\sin^2\varphi],$$
$$a_3 = [(\mu - \lambda)\sin^2\varphi - (\mu - 4\lambda)]\sin\varphi, \qquad a_4 = 2\sin^2\varphi.$$

Taking the derivative of (3.55), and then by using (3.47), (3.50) and (3.54), we obtain

$$b_1p^3 + b_2p^2 + b_3p + b_4 = 0, \tag{3.56}$$

where

$$b_1 = \lambda^2(\mu - \lambda)\sin\varphi[(2\lambda + \mu) - (\mu - \lambda)\sin^2\varphi],$$
$$b_2 = 2\lambda[\lambda(2\lambda - \mu) - (\mu - \lambda)(3\lambda + 2\mu)\sin^2\varphi + 2(\mu - \lambda)^2\sin^4\varphi],$$
$$b_3 = [(8\lambda\mu - 8\lambda^2 - \mu^2) + 2(\mu - \lambda)(2\mu + \lambda)\sin^2\varphi - 3(\mu - \lambda)^2\sin^4\varphi]\sin\varphi,$$
$$b_4 = 6(\mu - 2\lambda)\sin^2\varphi - 6(\mu - \lambda)\sin^4\varphi.$$

Combining (3.55) and (3.56), we conclude that

$$c_1p^2 + c_2p + c_3 = 0, \tag{3.57}$$

where

$$c_1 = a_1 b_2 - a_2 b_1 = \lambda[2(\mu-\lambda)^2 \sin^4\varphi - 3\mu(\mu-\lambda)\sin^2\varphi - \mu(2\lambda-\mu)], \tag{3.58}$$

$$\begin{aligned} c_2 &= a_1 b_3 - a_3 b_1 = 2[-(\mu-\lambda)^2 \sin^4\varphi \\ &\quad +(\mu+2\lambda)(\mu-\lambda)\sin^2\varphi + \lambda(3\mu-8\lambda)]\sin\varphi, \end{aligned} \tag{3.59}$$

$$c_3 = a_1 b_4 - a_4 b_1 = [-4(\mu-\lambda)\sin^4\varphi + 4(\mu-4\lambda)\sin^2\varphi]. \tag{3.60}$$

Taking the derivative of (3.57) and then by using (3.47), (3.50), and (3.54), we obtain

$$d_1 p^2 + d_2 p + d_3 = 0, \tag{3.61}$$

where

$$d_1 = -4\lambda(\mu-\lambda)^3 \sin^6\varphi + \sum_{i=0}^{2} D_{1i}(\lambda,\mu)\sin^{2i}\varphi,$$

$$d_2 = 5(\mu-\lambda)^3 \sin^7\varphi + \sum_{i=0}^{2} D_{2i}(\lambda,\mu)\sin^{2i+1}\varphi,$$

$$d_3 = 10(\mu-\lambda)^2 \sin^6\varphi + \sum_{i=0}^{2} D_{3i}(\lambda,\mu)\sin^{2i}\varphi,$$

and $D_{ji}(\lambda,\mu),\ (j=1,2,3)$ are polynomials in λ and μ. Combining (3.57) and (3.61), we find that

$$e_1 p + e_2 = 0, \tag{3.62}$$

where

$$e_1 = c_1 d_2 - c_2 d_1 = 2(\mu-\lambda)^5 \sin^{10}\varphi + \sum_{i=0}^{4} E_{1i}(\lambda,\mu)\sin^{2i}\varphi, \tag{3.63}$$

$$e_2 = c_1 d_3 - c_3 d_1 = 20(\mu-\lambda)^4 \sin^9\varphi + \sum_{i=0}^{3} E_{2i}(\lambda,\mu)\sin^{2i+1}\varphi, \tag{3.64}$$

and $E_{ji}(\lambda,\mu),\ (j=1,2)$ are some polynomials in λ and μ. Following the same procedure by taking the derivative of (3.62) and taking into account (3.47), (3.50), and (3.54), we find

$$h_1 p + h_2 = 0, \tag{3.65}$$

where

$$h_1 = -20(\mu-\lambda)^6 \sin^{12}\varphi + \sum_{i=0}^{5} H_{1i}(\lambda,\mu)\sin^{2i}\varphi, \tag{3.66}$$

$$h_2 = -184(\mu-\lambda)^5 \sin^{11}\varphi + \sum_{i=0}^{4} H_{2i}(\lambda,\mu)\sin^{2i+1}\varphi, \tag{3.67}$$

and $H_{ji}(\lambda,\mu),\ (j=1,2)$ are polynomials in λ and μ. Combining (3.62) and (3.65), we finally find

$$32(\mu-\lambda)^{10}\sin^{20}\varphi + \sum_{i=0}^{9} P_i(\lambda,\mu)\sin^{2i}\varphi = 0. \tag{3.68}$$

where $P_i(\lambda,\mu),\ (i=0,1,...,9)$ are the known polynomials in λ and μ. Since this polynomial is equal to zero for every φ, all its coefficients must be zero. Therefore, we conclude that $\mu-\lambda=0$, which is a contradiction. Consequently, there are no surfaces of revolution in this case. This completes our proof.

4. Conclusions

This research article was divided into three sections, where, after the introduction, the needed definitions and relations regarding this interesting field of study were given. Then a formula for the Laplace operator corresponding to the second fundamental form was defined for the position vector of a surface. Finally, we classify the surfaces of revolutions of coordinate finite Chen type regarding the second fundamental form. An interesting study can be drawn, if this type of study can be applied to other classes of surfaces that have not been investigated yet, such as spiral surfaces, or tubular surfaces.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that he has no conflict of interest.

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