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Research article

Solving functional integrodifferential equations with Liouville-Caputo fractional derivatives by fixed point techniques

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Abstract: The existence and uniqueness of solutions to fractional-order functional and neutral functional integrodifferential equations with infinite delay and multi-term fractional integral boundary conditions are investigated in this paper. Rigorous mathematical frameworks for analyzing these hybrid equations are established utilizing fixed point theorems. Notably, the fractional derivative is defined in the Liouville-Caputo sense, allowing for a comprehensive examination of nonlocal dynamics. Illustrative examples are provided to complement the theoretical results and demonstrate the applicability and practicality of the main results.

Keywords: fractional derivative; fixed point technique; existence result; evolution metrics; boundary value problem; phase space
Mathematics Subject Classification: 26A33, 34B10, 34B15, 74H10

Abbreviations

FDE, fractional differential equation
LC, Liouville-Caputo
RL, Riemann-Liouville
BS, Banach space
FFIDE, fractional functional integrodifferential equation
BVP, boundary value problems

1. Introduction

The theory of fractional differential equations (FDEs) generalizes classical differential equations by introducing fractional derivatives, enabling the modeling of complex phenomena exhibiting non-locality, memory, and power-law behavior [1, 2]. FDEs have been extensively developed using various fractional derivatives, such as Riemann-Liouville (RL), Caputo, and Grünwald-Letnikov [3]. These equations describe anomalous diffusion, relaxation, and oscillation processes, making them suitable for modeling real-world problems in physics, engineering, and biology. FDEs have diverse applications across disciplines, including viscoelasticity [4], chaotic dynamics [5], image processing [6], model financial systems [7], population dynamics [8], and electrical circuits [9]. Recent studies have explored fractional-order controllers in robotics [10] and biomedical signal processing [11]. These applications demonstrate the versatility and potential of FDEs in describing complex systems.

Fractional calculus and fixed point theory have emerged as powerful tools in addressing optimization and inverse problems across various scientific and engineering disciplines. Fractional derivatives and integrals, with their inherent nonlocal properties, are particularly well-suited for modeling systems exhibiting memory effects and long-range dependencies, which are often encountered in optimization problems involving complex systems. Furthermore, fixed point (FP) methods, especially those tailored for non-smooth or set-valued mappings, provide robust frameworks for solving inverse problems, including those arising in image processing and signal reconstruction. The combination of fractional calculus and FP theory offers a synergistic approach, enabling the analysis and solution of challenging optimization and inverse problems that are often intractable using classical techniques. See [12–16] for more information.

Recent years have witnessed significant interest in boundary value problems (BVPs) of FDEs, encompassing various boundary conditions such as the existence and uniqueness of solutions to fractional boundary value problems [17–20], the existence and uniqueness of solutions to hybrid fractional systems under multi-point, periodic, and anti-periodic boundary conditions [21, 22], and the stability of mixed integral fractional delay dynamic system equations and pantograph differential equations under impulsive effects and nonlocal conditions [23, 24]. Integral boundary conditions, in particular, have far-reaching implications in applied fields like heat conduction, electric power networks, elastic stability, telecommunications and electric railway systems. Multi-point BVPs, arising from practical applications, also warrant attention. For example, the existence results for FDEs are established in [25–29]. Alos, the existence of solutions to fractional functional differential equations [30], semilinear fractional differential inclusions [31], Hadamard fractional integro-differential equations [32], systems of multi-point boundary value problems [33], fractional hybrid delay differential equations [34], and nonlinear Atangana-Baleanu-type fractional differential equations [35–38] have been established. The theory of fractional functional BVPs remains underdeveloped, necessitating further research in mathematical modeling, numerical methods, and computational simulations to address the unresolved aspects.

Benchohra et al. [26] proved the existence of a solution via Leray-Schauder nonlinear alternative and uniqueness via Banach's FP theorem for fractional functional differential equations with infinite delay. Chauhan et al. [20] explored existence solutions for fractional integro-differential equations with impulses, infinite delay, and integral boundary conditions. Dabas and Gautam [39] examined existence results for impulsive neutral fractional integrodifferential equations featuring state-dependent delays and integral boundary conditions.

Inspired by the contributions of [20, 26, 39], first, our study focuses on establishing existence and uniqueness results for a fractional functional integrodifferential equation (FFIDE) featuring infinite delay. It takes the form

$$\begin{cases} {}^{LC}D^{p}\varpi(\varsigma) = g\left(\varsigma, \varpi_{\varsigma}, \int_{0}^{\varsigma} \Omega\left(\varsigma, \vartheta, \varpi_{\vartheta}\right) d\vartheta, \int_{0}^{\sigma} \Upsilon\left(\varsigma, \vartheta, \varpi_{\vartheta}\right) d\vartheta\right), \ p \in (2,3], \ \varsigma \in U = [0,\sigma] \\ \varpi(\varsigma) = \psi\left(\varsigma\right), \ \varsigma \in (-\infty, 0] \\ \varpi(\sigma) = \sum_{j=1}^{u} b_{j}\left(I_{0^{+}}^{q_{j}}\varpi\right)\left(\lambda_{j}\right), \ 0 < \lambda_{1} < \lambda_{2} < \cdots < \lambda_{u} < \sigma, \end{cases} \tag{1.1}$$

where ${}^{LC}D^{p}$ is the Liouville-Caputo (LC) fractional derivative with order p. Assume that $\mho = \{(\varsigma, \vartheta) : 0 \le \vartheta \le \varsigma \le \sigma\}$, Ξ is a BS, and Θ is a phase space. Then $g : U \times \Theta \times \Xi \to \Xi$, $\Omega, \Upsilon : \mho \times \Theta \to \Xi$ are continuous functions and $\psi \in \Theta$. Furthermore, $I_{0^{+}}^{q_{j}}$ refers to the RL fractional integral of order $q_{j} > 0$, and b_{j} represents suitable real constants for $j = 1, 2, \ldots, u$.

Supposing that $\varpi : (-\infty, \sigma] \to \Theta$ and $\varsigma \in U$, we denote $\varpi_{\varsigma} \in \Theta$ as an element defined by

$$\varpi_{\varsigma}\left(\xi\right) = \varpi\left(\varsigma + \xi\right), \ \xi \in (-\infty, 0].$$

Throughout this manuscript, we suppose that $\varpi_{\varsigma}(.)$ is the historical state trajectory from time $-\infty$ to ς, and $\varpi_{\varsigma} \in \Theta$, where Θ is an abstract phase space.

The second main result here is to investigate the existence and uniqueness of solutions to the neutral FFIDE with BVPs. It takes the form

$$\begin{cases} {}^{LC}D_{\varsigma}^{p}\left[\varpi(\varsigma) - \int_{0}^{\varsigma} \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} h\left(\vartheta, \varpi_{\vartheta}, \int_{0}^{\vartheta} \Omega_{1}\left(\vartheta, \mu, \varpi_{\mu}\right) d\mu, \int_{0}^{\sigma} \Upsilon_{1}\left(\vartheta, \mu, \varpi_{\mu}\right) d\mu\right)\right] \\ = g\left(\varsigma, \varpi_{\varsigma}, \int_{0}^{\varsigma} \Omega_{2}\left(\varsigma, \vartheta, \varpi_{\vartheta}\right) d\vartheta, \int_{0}^{\sigma} \Upsilon_{2}\left(\varsigma, \vartheta, \varpi_{\vartheta}\right) d\vartheta\right) \ p \in (2,3], \ \varsigma, \vartheta \in U = [0,\sigma] \\ \varpi(\varsigma) = \psi\left(\varsigma\right), \ \varsigma \in (-\infty, 0] \\ \varpi(\sigma) = \sum_{j=1}^{u} b_{j}\left(I_{0^{+}}^{q_{j}}\varpi\right)\left(\lambda_{j}\right), \ 0 < \lambda_{1} < \lambda_{2} < \cdots < \lambda_{u} < \sigma, \end{cases} \tag{1.2}$$

where $h, g : U \times \Theta \times \Xi \to \Xi$, $\Omega_{1}, \Upsilon_{1}, \Omega_{2}$, and $\Upsilon_{2} : \mho \times \Theta \to \Xi$ are continuous functions.

- This paper provides a systematic exploration of fractional functional integrodifferential equations.
- Section 2 lays the groundwork by establishing the foundational definitions, notation, and preliminary results.
- Existence and uniqueness criteria for FFIDEs are developed in Section 3, employing both Krasnoselskii's FP and Banach's FP theorem.
- Building upon these findings, Section 4 extends the existence and uniqueness results to neutral FFIDEs with BVPs.
- The applicability and practicality of the theoretical framework are demonstrated through illustrative examples provided in Section 5.

2. Preliminary work

This section presents fundamental definitions, notation, and lemmas essential for the subsequent analysis. Let Ξ denote a BS equipped with the norm $\|.\|$. Furthermore, $C(U, \Xi)$ represents the BS of continuous functions from the interval U to Ξ, endowed with a uniform convergence topology and the norm $\|.\|_{C}$.

Definition 2.1. [2] For the function $g \in L^1(\mathbb{R}_+)$

(i) The RL fractional integral of order $p > 0$ is given by

$$I^p_{0^+}g(\varsigma) = \frac{1}{\Gamma(p)} \int_0^\varsigma (\varsigma - \vartheta)^{p-1} g(\vartheta) d\vartheta,$$

whenever the integral exists.

(ii) The LC fractional derivative of order $p \in (v-1, v]$ is described as

$${}^{LC}D^p_\varsigma g(\varsigma) = \frac{1}{\Gamma(v-p)} \int_0^\varsigma (\varsigma - \vartheta)^{v-p-1} g^{(v)}(\vartheta) d\vartheta,$$

where g has absolutely continuous derivatives up to order $(v-1)$.

Remark 2.2. It should be noted that, if we take $v = 1$ in Definition 2.1 (ii), we have $0 < p \leq 1$ and

$${}^{LC}D^p_\varsigma g(\varsigma) = \frac{1}{\Gamma(1-p)} \int_0^\varsigma (\varsigma - \vartheta)^p g'(\vartheta) d\vartheta,$$

where $g'(\vartheta) = \frac{dg(\vartheta)}{d\vartheta}$.

Now, for simplicity, we denote ${}^{LC}D^p_\varsigma$ and $I^p_{0^+}$ by ${}^{LC}D^p$ and I^p, respectively.

Lemma 2.3. *[2] Assume that $p, q \geq 0$, and $g \in L^1[b, c]$. Then, for all $\varsigma \in [b, c]$, we have*

(i) $I^q I^p g(\varsigma) = I^{q+p} g(\varsigma) = I^p I^q g(\varsigma)$;

(ii) ${}^{LC}D^p_\varsigma I^q g(\varsigma) = g(\varsigma)$.

Theorem 2.4. *[40] (Krasnoselskii's theorem) Assume that $\Lambda \neq \emptyset$ is a closed and convex subset of a BS Ξ and that $\mathfrak{I}, \mathfrak{R}$ are two operators satisfying*

(i) for $\varpi, \varrho \in \Lambda$, $\mathfrak{I}\varpi + \mathfrak{R}\varrho \in \Lambda$,

(ii) $\mathfrak{I}$ is continuous and compact,

(iii) $\mathfrak{R}$ is a contraction.

Then, $w \in \Lambda$ exists such that $w = \mathfrak{I}w + \mathfrak{R}w$.

This paper considers a seminormed linear state space $(\Theta, \|.\|_\Theta)$ of functions from $(-\infty, 0]$ to Ξ satisfying the following hypotheses of Hale and Kato [41]:

(H$_1$) On the interval $(-\infty, \sigma]$, if $\varpi : (-\infty, \sigma] \to \Xi$ is continuous and $\varpi_0 \in \Theta$, then for $\varsigma \in U$, we have the following stipulations:

(1) $\varpi_\varsigma \in \Theta$,

(2) $\|\varpi(\varsigma)\|_\Theta \leq \kappa \left\|\varpi_\varsigma\right\|_\Theta$, where κ is a non-negative constant and is independent of $\varpi(.)$,

(3) There is a continuous function $N_1 : [0, \infty) \to [0, \infty)$ and a locally bounded function $N_2 : [0, \infty) \to [0, \infty)$ in order that

$$\left\|\varpi_\varsigma\right\|_\Theta \leq N_1(\varsigma) \sup\{\|\varpi(\vartheta)\| : 0 \leq \vartheta \leq \varsigma\} + N_2(\varsigma) \|\varpi(.)\|_\Theta,$$

where N_1 and N_2 are independent of $\varpi(.)$.

(H$_2$) The space Θ is complete.

(H$_3$) On the interval U, ϖ_ς is a B-valued continuous function, where $\varpi(.)$ is described in **(H$_1$)**.

Here, we consider $N_1^* = \sup_{\varsigma \in U} N_1(\varsigma)$ and $N_2^* = \sup_{\varsigma \in U} N_2(\varsigma)$.

3. Solving the FFIDE

This section is devoted to investigating the existence and uniqueness of solution to the considered problem (1.1) by applying Krasnoselskii's and Banach's FP theorems.

Assume the space

$$\widetilde{\mho} = \{\varpi : (-\infty, \sigma] \to \Xi : \varpi_{(-\infty,0]} \in \Theta \text{ and } \varpi_U \text{ is continuous}\},$$

and select $P\varpi(\varsigma) = \int_0^\varsigma \Omega(\varsigma, \vartheta, \varpi_\vartheta)\, d\vartheta$, and $Q\varpi(\varsigma) = \int_0^\sigma \Upsilon(\varsigma, \vartheta, \varpi_\vartheta)\, d\vartheta$.

Definition 3.1. We say that the function $\varpi \in \widetilde{\mho}$ is a solution to the FFIDE (1.1) if it fulfills the problem

$$\begin{cases} {}^{LC}D^p \varpi(\varsigma) = g(\varsigma, \varpi_\varsigma, P\varpi(\varsigma), Q\varpi(\varsigma)), \\ \varpi(\varsigma) = \psi(\varsigma), \ \varsigma \in (-\infty, 0], \\ \varpi(\sigma) = \sum\limits_{j=1}^{u} b_j (I^{q_j}\varpi)(\lambda_j), \ 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_u < \sigma. \end{cases}$$

We initiate our analysis of the nonlinear problem (1.1) by examining its linear counterpart, thereby obtaining a foundational solution.

Lemma 3.2. *Assume that $\varpi(\varsigma) \in C(U, \Xi)$ satisfies the following problem:*

$$\begin{cases} {}^{LC}D^p \varpi(\varsigma) = g(\varsigma), \ p \in (2, 3], \ \varsigma \in U, \\ \varpi(\varsigma) = \psi(\varsigma), \ \varsigma \in (-\infty, 0], \\ \varpi(\sigma) = \sum\limits_{j=1}^{u} b_j (I^{q_j}\varpi)(\lambda_j), \ 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_u < \sigma. \end{cases} \tag{3.1}$$

Then the unique solution of the fractional BVP (3.1) can be written as

$$\varpi(\varsigma) = \begin{cases} \psi(\varsigma), \quad \varsigma \in (-\infty, 0], \\ I^p g(\varsigma) + \frac{\varsigma}{B}\left(\sum\limits_{j=1}^{u} b_j I^{q_j+p} g(\lambda_j) - I^p g(\sigma)\right) \\ +\psi(0)\left(1 + \frac{\varsigma}{B}\left(\sum\limits_{j=1}^{u} \frac{b_j \lambda_j^{q_i}}{\Gamma(q_i+1)} - 1\right)\right), \ \varsigma \in U, \end{cases}$$

where $B = \sigma - \sum\limits_{j=1}^{u} \frac{b_j \lambda_j^{q_i+1}}{\Gamma(q_i+2)} \neq 0$, provided that $\sum\limits_{j=1}^{u} \frac{b_j \lambda_j^{q_i}}{\Gamma(q_i+1)} > 1$.

Proof. Suppose that $\rho_0, \rho_1 \in \Xi$ are vector constants. Based on [2], the solution of (3.1) takes the form

$$\varpi(\varsigma) = I^p g(\varsigma) + \rho_0 + \rho_1 \varsigma. \tag{3.2}$$

Applying the condition $\varpi(\varsigma) = \psi(\varsigma)$, we get

$$\rho_0 = \psi(0). \tag{3.3}$$

Using the condition $\varpi(\sigma) = \sum\limits_{j=1}^{u} b_j (I^{q_j}\varpi)(\lambda_j)$, we have

$$\rho_1 = \frac{1}{\left(\sigma - \sum\limits_{j=1}^{u} \frac{b_j \lambda_j^{q_i+1}}{\Gamma(q_i+2)}\right)} \left\{\sum_{j=1}^{u} b_j I^{q_j+p} g(\lambda_j) + \psi(0)\left(\sum_{j=1}^{u} \frac{b_j \lambda_j^{q_i}}{\Gamma(q_i+1)} - 1\right) - I^p g(\sigma)\right\}. \tag{3.4}$$

From (3.3) and (3.4) in (3.2), we can write

$$\varpi(\varsigma) = I^p g(\varsigma) + \frac{\varsigma}{B}\left(\sum_{j=1}^{u} b_j I^{q_j+p} g\left(\lambda_j\right) - I^p g(\sigma)\right) + \psi(0)\left(1 + \frac{\varsigma}{B}\left(\sum_{j=1}^{u} \frac{b_j \lambda_j^{q_i}}{\Gamma(q_i+1)} - 1\right)\right).$$

□

After that, we need the following assertions:

(A$_1$) For all $\varsigma, \vartheta \in U$, $\psi_1, \psi_2 \in \Theta$ and $\varpi_1, \varpi_2, \widetilde{\varpi}_1, \widetilde{\varpi}_2 \in \Xi$, ℓ_g, ℓ_P, ℓ_Q exist in order that

$$\begin{cases} \|g(\varsigma, \psi_1, \varpi_1, \widetilde{\varpi}_1) - g(\varsigma, \psi_2, \varpi_2, \widetilde{\varpi}_2)\|_\Xi \le \ell_g \left(\|\psi_1 - \psi_2\|_\Theta + \|\varpi_1 - \varpi_2\|_\Xi + \|\widetilde{\varpi}_1 - \widetilde{\varpi}_2\|_\Xi\right), \\ \|P(\varsigma, \vartheta, \psi_1) - P(\varsigma, \vartheta, \psi_2)\|_\Xi \le \ell_P \|\psi_1 - \psi_2\|_\Theta, \\ \|Q(\varsigma, \vartheta, \psi_1) - Q(\varsigma, \vartheta, \psi_2)\|_\Xi \le \ell_Q \|\psi_1 - \psi_2\|_\Theta. \end{cases}$$

(A$_2$) For all $(\varsigma, \psi, \varpi_1, \varpi_2) \in U \times \Theta \times \Xi \times \Xi$ and $(\varsigma, \vartheta, \psi) \in \mho \times \Theta$, $V_j \in L^1(U, \mathbb{R}_+)$ $(j = 1, 2, 3, 4, 5)$ exists such that

$$\begin{cases} \|g(\varsigma, \psi, \varpi_1, \varpi_2)\|_\Xi \le V_1(\varsigma)\|\psi\|_\Theta + V_2(\varsigma)\|\varpi_1\|_\Xi + V_3(\varsigma)\|\varpi_2\|_\Xi, \\ \|P(\varsigma, \vartheta, \psi)\|_\Xi \le V_4(\varsigma)\|\psi\|_\Theta, \\ \|Q(\varsigma, \vartheta, \psi)\|_\Xi \le V_5(\varsigma)\|\psi\|_\Theta. \end{cases}$$

(A$_3$) We consider $S = \ell_g N_1^* \{\xi_1 + \xi_2(\ell_P + \ell_Q)\} < 1$, where

$$\begin{cases} \xi_1 = \left(1 + \frac{\sigma}{|B|}\right)\nu_1 + \frac{\sigma}{|B|}\nu_3, \\ \xi_2 = \left(1 + \frac{\sigma}{|B|}\right)\nu_2 + \frac{\sigma}{|B|}\nu_4, \\ \nu_1 = \frac{\sigma^p}{\Gamma(1+p)},\ \nu_2 = \frac{\sigma^{p+1}}{\Gamma(2+p)}, \\ \nu_3 = \sum_{j=1}^{u} \left|b_j\right| \frac{\lambda_j^{q_i+p}}{\Gamma(q_i+p+1)},\ \nu_4 = \sum_{j=1}^{u} \left|b_j\right| \frac{\lambda_j^{q_i+p+1}}{\Gamma(q_i+p+2)}. \end{cases}$$

Now, the first main result in this part is as follows:

Theorem 3.3. *Under Assertions (**A**$_1$) and (**A**$_2$), the BVP (1.1) has at least one solution on* $(-\infty, \sigma]$, *provided that*

$$\ell = \frac{\sigma}{|B|}\ell_g N_1^* \{(\nu_1 + \nu_3) + (\nu_2 + \nu_4)(\ell_P + \ell_Q)\} < 1.$$

Proof. The FP technique involves equating a given operator to the problem at hand and seeking a unique FP, which corresponds to the problem's unique solution. Therefore, we convert the BVP (1.1) to an FP problem. Define the operator $M : \widetilde{\mho} \to \widetilde{\mho}$ as

$$(M\varpi)(\varsigma) = \begin{cases} \psi(\varsigma), \quad \varsigma \in (-\infty, 0], \\ \int_0^\varsigma \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} g(\vartheta, \varpi_\vartheta, P\varpi(\vartheta), Q\varpi(\vartheta))\, d\vartheta \\ +\frac{\varsigma}{B}\left(\sum_{j=1}^{u} b_j \int_0^{\lambda_j} \frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)} g(\vartheta, \varpi_\vartheta, P\varpi(\vartheta), Q\varpi(\vartheta))\, d\vartheta \right. \\ \left. - \int_0^\sigma \frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)} g(\vartheta, \varpi_\vartheta, P\varpi(\vartheta), Q\varpi(\vartheta))\, d\vartheta\right) + \psi(0)\left(1 + \frac{\varsigma}{B}\left(\sum_{j=1}^{u} \frac{b_j \lambda_j^{q_i}}{\Gamma(q_i+1)} - 1\right)\right),\ \varsigma \in U. \end{cases} \tag{3.5}$$

Assume that $\varrho(.) : (-\infty, \sigma] \to \Xi$ is a function described as

$$\varrho(\varsigma) = \begin{cases} \psi(\varsigma), & \varsigma \in (-\infty, 0], \\ 0, & \varsigma \in U. \end{cases}$$

It is clear that $\varrho_0 = 0$. For every $\omega \in C(U, \Xi)$ with $\omega(0) = 0$, we select

$$\widetilde{\omega}(\varsigma) = \begin{cases} 0, & \varsigma \in (-\infty, 0], \\ \omega(\varsigma), & \varsigma \in U. \end{cases}$$

If $\varpi(.)$ fulfills (3.5), then we decompose $\varpi(.)$ as $\varpi(\varsigma) = \varrho(\varsigma) + \widetilde{\omega}(\varsigma)$, which leads to $\varpi_\varsigma = \varrho_\varsigma + \widetilde{\omega}_\varsigma$ for all $\varsigma \in U$, and $\omega(.)$ satisfies

$$\begin{aligned} \omega(\varsigma) &= \int_0^\varsigma \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} g\left(\vartheta, \varrho_\vartheta + \widetilde{\omega}_\vartheta, P\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right], Q\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right) d\vartheta \\ &\quad + \frac{\varsigma}{B}\left(\sum_{j=1}^{u} b_j \int_0^{\lambda_j} \frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)} g\left(\vartheta, \varrho_\vartheta + \widetilde{\omega}_\vartheta, P\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right], Q\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right) d\vartheta \right.\\ &\quad \left. - \int_0^\sigma \frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)} g\left(\vartheta, \varrho_\vartheta + \widetilde{\omega}_\vartheta, P\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right], Q\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right) d\vartheta \right) \\ &\quad + \psi(0)\left(1 + \frac{\varsigma}{B}\left(\sum_{j=1}^{u} \frac{b_j \lambda_j^{q_i}}{\Gamma(q_i+1)} - 1\right)\right). \end{aligned}$$

Put $G_0 = \{\omega \in C(U, \Xi) : \omega_0 = 0\}$ and consider $\|.\|_{G_0}$ to be the seminorm in G_0 given by

$$\|\omega\|_{G_0} = \sup_{\varsigma \in U} \|\omega(\varsigma)\|_\Xi + \|\omega_0\|_\Theta = \sup_{\varsigma \in U} \|\omega(\varsigma)\|_\Xi, \ \omega \in G_0.$$

Hence, $\left(G_0, \|.\|_{G_0}\right)$ is a BS. Describe the operator $\Phi : G_0 \to G_0$ as

$$\begin{aligned} (\Phi\omega)(\varsigma) &= \int_0^\varsigma \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} g\left(\vartheta, \varrho_\vartheta + \widetilde{\omega}_\vartheta, P\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right], Q\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right) d\vartheta \\ &\quad + \frac{\varsigma}{B}\left(\sum_{j=1}^{u} b_j \int_0^{\lambda_j} \frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)} g\left(\vartheta, \varrho_\vartheta + \widetilde{\omega}_\vartheta, P\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right], Q\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right) d\vartheta \right.\\ &\quad \left. - \int_0^\sigma \frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)} g\left(\vartheta, \varrho_\vartheta + \widetilde{\omega}_\vartheta, P\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right], Q\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right) d\vartheta \right) \\ &\quad + \psi(0)\left(1 + \frac{\varsigma}{B}\left(\sum_{j=1}^{u} \frac{b_j \lambda_j^{q_i}}{\Gamma(q_i+1)} - 1\right)\right). \end{aligned}$$

The existence of an FP for an operator M is equivalent to the operator Φ having an FP. Hence, we focus on establishing the existence of an FP for Φ.

Consider the set $H_s = \left\{\omega \in G_0 : \|\omega\|_{G_0} \leq s\right\}$. Hence, H_s is a bounded, closed, and convex subset of G_0. Assume that there is a positive constant ε such that $\varepsilon < s$, where

$$\varepsilon = \|q\|_{L^1} s^* \left[\left(1 + \frac{\sigma}{|B|}\right)(\nu_1 + \nu_2) + \frac{\sigma}{|B|}(\nu_3 + \nu_4)\right]$$

$$+\|\psi(0)\|\left(1+\frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}\frac{b_j\,\lambda_j^{q_i}}{\Gamma(q_i+1)}-1\right)\right).$$

Now, we decompose Φ as $\Phi_1+\Phi_2$ on H_s, where

$$(\Phi_1\omega)(\varsigma)=\int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)}g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)d\vartheta,$$

and

$$\begin{aligned}(\Phi_2\omega)(\varsigma) &= \frac{\varsigma}{B}\left(\sum_{j=1}^{u}b_j\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)}g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)d\vartheta\right.\\&\quad\left.-\int_0^{\sigma}\frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)}g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)d\vartheta\right)\\&\quad+\psi(0)\left(1+\frac{\varsigma}{B}\left(\sum_{j=1}^{u}\frac{b_j\,\lambda_j^{q_i}}{\Gamma(q_i+1)}-1\right)\right).\end{aligned}$$

Now, if we let $\omega,\omega^*\in H_s$ and $\varsigma\in U$, we get

$$\begin{aligned}&\|(\Phi_1\omega)(\varsigma)+(\Phi_2\omega^*)(\varsigma)\|_\Xi\\ \le\;&\int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)}\left\|g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right\|_\Xi d\vartheta\\&+\frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}\left|b_j\right|\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)}\left\|g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}^*_\vartheta,P\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right],Q\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right)\right\|_\Xi d\vartheta\right.\\&\left.+\int_0^{\sigma}\frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)}\left\|g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}^*_\vartheta,P\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right],Q\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right)\right\|_\Xi d\vartheta\right)\\&+\|\psi(0)\|\left(1+\frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}\frac{\left|b_j\right|\,\lambda_j^{q_i}}{\Gamma(q_i+1)}-1\right)\right)\\ \le\;&\int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)}\left[V_1(\vartheta)\|\varrho_\vartheta+\widetilde{\omega}_\vartheta\|_\Theta+V_2(\vartheta)\left\|P\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_\Xi+V_3(\vartheta)\left\|Q\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_\Xi\right]\\&+\frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}\left|b_j\right|\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)}\left(\begin{array}{c}V_1(\vartheta)\left\|\varrho_\vartheta+\widetilde{\omega}^*_\vartheta\right\|_\Theta+V_2(\vartheta)\left\|P\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right\|_\Xi\\+V_3(\vartheta)\left\|Q\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right\|_\Xi\end{array}\right)d\vartheta\right.\\&\left.+\int_0^{\sigma}\frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)}\left(\begin{array}{c}V_1(\vartheta)\left\|\varrho_\vartheta+\widetilde{\omega}^*_\vartheta\right\|_\Theta+V_2(\vartheta)\left\|P\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right\|_\Xi\\+V_3(\vartheta)\left\|Q\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right\|_\Xi\end{array}\right)d\vartheta\right)\\&+\|\psi(0)\|\left(1+\frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}\frac{\left|b_j\right|\,\lambda_j^{q_i}}{\Gamma(q_i+1)}-1\right)\right)\\ \le\;&\|V\|_{L^1}\,s^*\left[\left(1+\frac{\sigma}{|B|}\right)(\nu_1+\nu_2)+\frac{\sigma}{|B|}(\nu_3+\nu_4)\right]+\|\psi(0)\|\left(1+\frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}\frac{b_j\,\lambda_j^{q_i}}{\Gamma(q_i+1)}-1\right)\right)\\ =\;&\varepsilon.\end{aligned}$$

Hence,

$$\|\Phi_1\omega + \Phi_2\omega^*\|_{\Xi} \le \varepsilon, \tag{3.6}$$

where $V(\varsigma) = \max\{V_1(\varsigma), V_2(\varsigma), V_3(\varsigma), V_4(\varsigma)\}$ and

$$\begin{aligned}
\|\varrho_\vartheta + \widetilde{\omega}_\vartheta\|_\Theta &\le \|\varrho_\vartheta\|_\Theta + \|\widetilde{\omega}_\vartheta\|_\Theta \\
&\le N_1(\vartheta)\sup_{0\le\mu\le\vartheta}\|\varrho(\mu)\| + N_2(\vartheta)\|\varrho(0)\| + N_1(\vartheta)\sup_{0\le\mu\le\vartheta}\|\widetilde{\omega}(\mu)\| + N_2(\vartheta)\|\widetilde{\omega}(0)\| \\
&\le N_1^* s + N_2^*\|\psi\|_\Theta \le s^*.
\end{aligned}$$

It follows from (3.6) that $\Phi_1\omega + \Phi_2\omega^* \in H_s$. Now, we show that Φ_2 is a contraction. For this, assume that $\omega, \omega^* \in H_s$, and $\varsigma \in U$. We then

$$\begin{aligned}
&\|(\Phi_2\omega)(\varsigma) - (\Phi_2\omega^*)(\varsigma)\|_\Xi \\
\le\ & \frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}|b_j|\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)}\left\|g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right. \\
&\left. - g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}^*_\vartheta,P\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right],Q\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right)\right\|d\vartheta + \int_0^{\sigma}\frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)} \\
&\times\left\|g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right. \\
&\left.\left. - g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}^*_\vartheta,P\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right],Q\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right)\right\|d\vartheta\right) \\
\le\ & \frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}|b_j|\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)}\ell_g\left(\left\|(\varrho_\vartheta+\widetilde{\omega}_\vartheta)-(\varrho_\vartheta+\widetilde{\omega}^*_\vartheta)\right\|_\Theta\right.\right. \\
&\left. + \left\|P\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]-P\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right\|_\Xi + \left\|Q\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]-Q\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right\|_\Xi\right)d\vartheta \\
&+ \int_0^{\sigma}\frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)}\ell_g\left(\left\|(\varrho_\vartheta+\widetilde{\omega}_\vartheta)-(\varrho_\vartheta+\widetilde{\omega}^*_\vartheta)\right\|_\Theta\right. \\
&\left.\left. + \left\|P\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]-P\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right\|_\Xi + \left\|Q\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]-Q\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right\|_\Xi\right)d\vartheta\right) \\
\le\ & \frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}|b_j|\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)}\ell_g\left(\left\|\widetilde{\omega}_\vartheta-\widetilde{\omega}^*_\vartheta\right\|_\Theta + \ell_P\left\|\widetilde{\omega}_\mu-\widetilde{\omega}^*_\mu\right\|_\Theta\vartheta + \ell_Q\left\|\widetilde{\omega}_\mu-\widetilde{\omega}^*_\mu\right\|_\Theta\vartheta\right)d\vartheta\right. \\
&\left. + \int_0^{\sigma}\frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)}\ell_g\left(\left\|\widetilde{\omega}_\vartheta-\widetilde{\omega}^*_\vartheta\right\|_\Theta + \ell_P\left\|\widetilde{\omega}_\mu-\widetilde{\omega}^*_\mu\right\|_\Theta\vartheta + \ell_Q\left\|\widetilde{\omega}_\mu-\widetilde{\omega}^*_\mu\right\|_\Theta\vartheta\right)d\vartheta\right) \\
\le\ & \frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}|b_j|\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)}\ell_g\left(N_1^*\sup_{\vartheta\in[0,\varsigma]}\|\omega(\vartheta)-\omega^*(\vartheta)\| + \ell_P N_1^*\sup_{\mu\in[0,\vartheta]}\|\omega(\mu)-\omega^*(\mu)\|\vartheta\right.\right. \\
&\left. + \ell_Q N_1^*\sup_{\mu\in[0,\vartheta]}\|\omega(\mu)-\omega^*(\mu)\|\vartheta\right)d\vartheta + \int_0^{\sigma}\frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)}\ell_g\left(N_1^*\sup_{\vartheta\in[0,\varsigma]}\|\omega(\vartheta)-\omega^*(\vartheta)\|\right. \\
&\left.\left. + \ell_P N_1^*\sup_{\mu\in[0,\vartheta]}\|\omega(\mu)-\omega^*(\mu)\|\vartheta + \ell_Q N_1^*\sup_{\mu\in[0,\vartheta]}\|\omega(\mu)-\omega^*(\mu)\|\vartheta\right)d\vartheta\right) \\
\le\ & \frac{\varsigma}{|B|}N_1^*\ell_g\left(\sum_{j=1}^{u}|b_j|\frac{(\lambda_j)^{q_j+p}}{\Gamma(q_j+p+1)} + (\ell_P+\ell_Q)\sum_{j=1}^{u}|b_j|\frac{(\lambda_j)^{q_j+p+1}}{\Gamma(q_j+p+2)}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sigma^{p}}{\Gamma(p+1)}+(\ell_P+\ell_Q)\,\frac{\sigma^{p}+1}{\Gamma(p+2)}\bigg)\|\omega-\omega^*\|_{G_0}\\
\leq\;& \frac{\sigma}{|B|}\ell_g N_1^*\left\{(\nu_1+\nu_3)+(\nu_2+\nu_4)\,(\ell_P+\ell_Q)\right\}\|\omega-\omega^*\|_{G_0}\\
=\;& \ell\,\|\omega-\omega^*\|_{G_0}\,.
\end{aligned}
$$

It follows that

$$
\|\Phi_2\omega-\Phi_2\omega^*\|_{G_0}\leq \ell\,\|\omega-\omega^*\|_{G_0}\,.
$$

Since $\ell < 1$, then, Φ_2 is contraction. Because g, P and Q are continuous, and thus Φ_1 is continuous. Consider

$$
\begin{aligned}
&\|(\Phi_1\omega)\,(\varsigma)\|_{\Xi}\\
\leq\;& \int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)}\left\|g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right\|_{\Xi}d\vartheta\\
\leq\;& \int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)}\left[V_1(\vartheta)\,\|\varrho_\vartheta+\widetilde{\omega}_\vartheta\|_{\Theta}+V_2(\vartheta)\left\|P\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_{\Xi}+V_3(\vartheta)\left\|Q\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_{\Xi}\right]d\vartheta\\
\leq\;& \|V\|_{L^1}\,s^*\,(\nu_1+\nu_2)\,.
\end{aligned}
$$

This proves that Φ_1 is uniformly bounded on H_s. Finally, we prove that Φ_1 is compact. Indeed, we claim that Φ_1 is equicontinuous. For $\varsigma_1,\varsigma_2\in U$, with $\varsigma_1<\varsigma_2$ and $\omega\in H_s$, one can write

$$
\begin{aligned}
&\|(\Phi_1\omega)\,(\varsigma_2)-(\Phi_1\omega)\,(\varsigma_1)\|_{\Xi}\\
\leq\;& \int_0^{\varsigma_1}\frac{(\varsigma_2-\vartheta)^{p-1}-(\varsigma_1-\vartheta)^{p-1}}{\Gamma(p)}\left\|g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right\|_{\Xi}d\vartheta\\
&+\int_{\varsigma_1}^{\varsigma_2}\frac{(\varsigma_2-\vartheta)^{p-1}}{\Gamma(p)}\left\|g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right\|_{\Xi}d\vartheta\\
\leq\;& \int_0^{\varsigma_1}\frac{(\varsigma_2-\vartheta)^{p-1}-(\varsigma_1-\vartheta)^{p-1}}{\Gamma(p)}\Big[V_1(\vartheta)\,\|\varrho_\vartheta+\widetilde{\omega}_\vartheta\|_{\Theta}+V_2(\vartheta)\left\|P\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_{\Xi}\\
&+V_3(\vartheta)\left\|Q\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_{\Xi}\Big]d\vartheta+\int_{\varsigma_1}^{\varsigma_2}\frac{(\varsigma_2-\vartheta)^{p-1}}{\Gamma(p)}\Big[V_1(\vartheta)\,\|\varrho_\vartheta+\widetilde{\omega}_\vartheta\|_{\Theta}\\
&+V_2(\vartheta)\left\|P\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_{\Xi}+V_3(\vartheta)\left\|Q\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_{\Xi}\Big]d\vartheta\\
\leq\;& \|V\|_{L^1}\,s^*\left(\int_0^{\varsigma_1}\frac{(\varsigma_2-\vartheta)^{p-1}-(\varsigma_1-\vartheta)^{p-1}}{\Gamma(p)}(1+\vartheta)d\vartheta+\int_{\varsigma_1}^{\varsigma_2}\frac{(\varsigma_2-\vartheta)^{p-1}}{\Gamma(p)}(1+\vartheta)d\vartheta\right).
\end{aligned}
$$

Clearly, $\|(\Phi_1\omega)\,(\varsigma_2)-(\Phi_1\omega)\,(\varsigma_2)\|_{\Xi}\to 0$ as $\varsigma_1\to\varsigma_2$. Consequently, Φ_1 is equicontinuous. Applying the Arzelà-Ascoli theorem, we establish that Φ_1 is compact on H_s. Consequently, invoking Krasnoselskii's FP theorem, we prove the existence of an FP $\omega\in G_0$, satisfying $\Phi\omega=\omega$, thereby yielding a solution to the fractional BVP (1.1). □

Now, for the uniqueness, we apply Banach's FP theorem as follows:

Theorem 3.4. *Via Assertions* (A_1) *and* (A_3)*, the BVP (1.1) owns a unique solution on* $(-\infty,\sigma]$.

Proof. Recall the set $H_s = \left\{\omega \in G_0 : \|\omega\|_{G_0} \leq s\right\}$ and assume that $\omega, \omega^* \in G_0$. For $\varsigma \in U$, one has

$$
\begin{aligned}
& \|(\Phi\omega)(\varsigma) - (\Phi\omega^*)(\varsigma)\|_{\Xi} \\
\leq\ & \int_0^{\varsigma} \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} \left\| g\left(\vartheta, \varrho_\vartheta + \widetilde{\omega}_\vartheta, P\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right], Q\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right) \right. \\
& \left. - g\left(\vartheta, \varrho_\vartheta + \widetilde{\omega}^*_\vartheta, P\left[\varrho(\vartheta) + \widetilde{\omega}^*(\vartheta)\right], Q\left[\varrho(\vartheta) + \widetilde{\omega}^*(\vartheta)\right]\right) \right\|_{\Xi} d\vartheta \\
& + \frac{\varsigma}{|B|} \left(\sum_{j=1}^{u} \left|b_j\right| \int_0^{\lambda_j} \frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)} \left\| g\left(\vartheta, \varrho_\vartheta + \widetilde{\omega}_\vartheta, P\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right], Q\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right) \right. \right. \\
& \left. - g\left(\vartheta, \varrho_\vartheta + \widetilde{\omega}^*_\vartheta, P\left[\varrho(\vartheta) + \widetilde{\omega}^*(\vartheta)\right], Q\left[\varrho(\vartheta) + \widetilde{\omega}^*(\vartheta)\right]\right) \right\|_{\Xi} d\vartheta \\
& + \int_0^{\sigma} \frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)} \left\| g\left(\vartheta, \varrho_\vartheta + \widetilde{\omega}_\vartheta, P\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right], Q\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right) \right. \\
& \left. \left. - g\left(\vartheta, \varrho_\vartheta + \widetilde{\omega}^*_\vartheta, P\left[\varrho(\vartheta) + \widetilde{\omega}^*(\vartheta)\right], Q\left[\varrho(\vartheta) + \widetilde{\omega}^*(\vartheta)\right]\right) \right\|_{\Xi} d\vartheta \right) \\
\leq\ & \int_0^{\varsigma} \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} \ell_g \left(\left\|\widetilde{\omega}_\vartheta - \widetilde{\omega}^*_\vartheta\right\|_{\Theta} + \ell_P \left\|\widetilde{\omega}_\mu - \widetilde{\omega}^*_\mu\right\|_{\Theta} \vartheta + \ell_Q \left\|\widetilde{\omega}_\mu - \widetilde{\omega}^*_\mu\right\|_{\Theta} \vartheta \right) d\vartheta \\
& + \frac{\varsigma}{|B|} \left(\sum_{j=1}^{u} \left|b_j\right| \int_0^{\lambda_j} \frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)} \ell_g \left(\left\|\widetilde{\omega}_\vartheta - \widetilde{\omega}^*_\vartheta\right\|_{\Theta} + \ell_P \left\|\widetilde{\omega}_\mu - \widetilde{\omega}^*_\mu\right\|_{\Theta} \vartheta + \ell_Q \left\|\widetilde{\omega}_\mu - \widetilde{\omega}^*_\mu\right\|_{\Theta} \vartheta \right) d\vartheta \right. \\
& \left. + \int_0^{\sigma} \frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)} \ell_g \left(\left\|\widetilde{\omega}_\vartheta - \widetilde{\omega}^*_\vartheta\right\|_{\Theta} + \ell_P \left\|\widetilde{\omega}_\mu - \widetilde{\omega}^*_\mu\right\|_{\Theta} \vartheta + \ell_Q \left\|\widetilde{\omega}_\mu - \widetilde{\omega}^*_\mu\right\|_{\Theta} \vartheta \right) d\vartheta \right) \\
\leq\ & \int_0^{\varsigma} \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} \ell_g \left(N_1^* \|\omega - \omega^*\|_{G_0} + \ell_P \|\omega - \omega^*\|_{G_0} \vartheta + \ell_Q \|\omega - \omega^*\|_{G_0} \vartheta \right) \\
& + \frac{\varsigma}{|B|} \left(\sum_{j=1}^{u} \left|b_j\right| \int_0^{\lambda_j} \frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)} \ell_g \left(N_1^* \|\omega - \omega^*\|_{G_0} + \ell_P \|\omega - \omega^*\|_{G_0} \vartheta + \ell_Q \|\omega - \omega^*\|_{G_0} \vartheta \right) d\vartheta \right. \\
& \left. + \int_0^{\sigma} \frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)} \ell_g \left(N_1^* \|\omega - \omega^*\|_{G_0} + \ell_P \|\omega - \omega^*\|_{G_0} \vartheta + \ell_Q \|\omega - \omega^*\|_{G_0} \vartheta \right) d\vartheta \right) \\
\leq\ & \ell_g N_1^* \left\{ \left[\frac{\sigma^p}{\Gamma(p+1)} + \frac{\sigma}{|B|} \sum_{j=1}^{u} \left|b_j\right| \frac{(\lambda_j)^{q_j+p}}{\Gamma(q_j+p+1)} + \frac{\sigma^{p+1}}{|B|\,\Gamma(p+1)} \right] \right. \\
& \left. + (\ell_P + \ell_Q) \left[\frac{\sigma^{p+1}}{\Gamma(p+2)} + \frac{\sigma}{|B|} \sum_{j=1}^{u} \left|b_j\right| \frac{(\lambda_j)^{q_j+p+1}}{\Gamma(q_j+p+2)} + \frac{\sigma^{p+2}}{|B|\,\Gamma(p+2)} \right] \right\} \|\omega - \omega^*\|_{G_0} \\
\leq\ & \ell_g N_1^* \left\{ \xi_1 + \xi_2 (\ell_P + \ell_Q) \right\} \|\omega - \omega^*\|_{G_0} \\
=\ & S \|\omega - \omega^*\|_{G_0}.
\end{aligned}
$$

Hence,

$$
\|\Phi(\omega) - \Phi(\omega^*)\|_{G_0} \leq S \|\omega - \omega^*\|_{G_0}.
$$

By **(A_3)**, $S < 1$, Φ is a contraction. By Banach's FP theorem, Φ possesses a unique FP, which is a unique solution to the problem (1.1) on the interval $(-\infty, \sigma]$. □

4. Solving the neutral FFIDE

In this section, we discuss the existence and uniqueness of solution to the considered problem (1.2) by applying Krasnoselskii's and Banach's FP theorems.

Assume that the space $\widetilde{\mho}$ is defined as in the section above and choose

$$\begin{cases} P_1\varpi(\varsigma) = \int_0^{\varsigma} \Omega_1(\varsigma, \vartheta, \varpi_\vartheta)\, d\vartheta, \\ P_2\varpi(\varsigma) = \int_0^{\varsigma} \Omega_2(\varsigma, \vartheta, \varpi_\vartheta)\, d\vartheta, \\ Q_1\varpi(\varsigma) = \int_0^{\sigma} \Upsilon_1(\varsigma, \vartheta, \varpi_\vartheta)\, d\vartheta, \\ Q_2\varpi(\varsigma) = \int_0^{\sigma} \Upsilon_2(\varsigma, \vartheta, \varpi_\vartheta)\, d\vartheta. \end{cases}$$

Definition 4.1. We say that the function $\varpi \in \widetilde{\mho}$ is a solution to the FFIDE (1.1) if it fulfills the problem

$$\begin{cases} {}^{LC}D^p\left[\varpi(\varsigma) - \int_0^{\varsigma} \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} h(\vartheta, \varpi_\vartheta, P_1\varpi(\vartheta), Q_1\varpi(\vartheta))\right] = g(\varsigma, \varpi_\varsigma, P_2\varpi(\varsigma), Q_2\varpi(\varsigma)), \ \varsigma \in U, \\ \varpi(\varsigma) = \psi(\varsigma), \ \varsigma \in (-\infty, 0], \\ \varpi(\sigma) = \sum\limits_{j=1}^{u} b_j (I^{q_j}\varpi)\left(\lambda_j\right), \ 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_u < \sigma. \end{cases}$$

With the aid of Lemma 3.2, the solution of the neutral FFIDE (1.2) takes the form

$$\varpi(\varsigma) = \begin{cases} \psi(\varsigma), \quad \varsigma \in (-\infty, 0], \\ \int_0^{\varsigma} \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} g(\vartheta, \varpi_\vartheta, P_2\varpi(\vartheta), Q_2\varpi(\vartheta)) + \int_0^{\varsigma} \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} h(\vartheta, \varpi_\vartheta, P_1\varpi(\vartheta), Q_1\varpi(\vartheta)) \\ +\frac{\varsigma}{B}\left(\sum\limits_{j=1}^{u} b_j \int_0^{\lambda_j} \frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)} g(\vartheta, \varpi_\vartheta, P_2\varpi(\vartheta), Q_2\varpi(\vartheta))\right. \\ + \sum\limits_{j=1}^{u} b_j \int_0^{\lambda_j} \frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)} h(\vartheta, \varpi_\vartheta, P_2\varpi(\vartheta), Q_2\varpi(\vartheta)) - \int_0^{\sigma} \frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)} g(\vartheta, \varpi_\vartheta, P_2\varpi(\vartheta), Q_2\varpi(\vartheta)) \\ \left. - \int_0^{\sigma} \frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)} h(\vartheta, \varpi_\vartheta, P_1\varpi(\vartheta), Q_1\varpi(\vartheta))\right) \\ +\psi(0)\left(1 + \frac{\varsigma}{B}\left(\sum\limits_{j=1}^{u} \frac{b_j\, \lambda_j^{q_i}}{\Gamma(q_i+1)} - 1\right)\right), \quad \varsigma \in U, \end{cases}$$

where $B = \sigma - \sum\limits_{j=1}^{u} \frac{b_j\, \lambda_j^{q_i+1}}{\Gamma(q_i+2)} \neq 0$.

To accomplish our main task here, we needs the following assertions:

(A$_4$) For all $\varsigma, \vartheta \in U$, $\psi_1, \psi_2 \in \Theta$ and $\varpi_1, \varpi_2, \widetilde{\varpi}_1, \widetilde{\varpi}_2 \in \Xi$, $\ell_g, \ell_h, \ell_{P_1}, \ell_{Q_1}, \ell_{P_2}, \ell_{Q_2}$ exist such that

$$\begin{cases} \|g(\varsigma, \psi_1, \varpi_1, \widetilde{\varpi}_1) - g(\varsigma, \psi_2, \varpi_2, \widetilde{\varpi}_2)\|_\Xi \leq \ell_g\left(\|\psi_1 - \psi_2\|_\Theta + \|\varpi_1 - \varpi_2\|_\Xi + \|\widetilde{\varpi}_1 - \widetilde{\varpi}_2\|_\Xi\right), \\ \|h(\varsigma, \psi_1, \varpi_1, \widetilde{\varpi}_1) - h(\varsigma, \psi_2, \varpi_2, \widetilde{\varpi}_2)\|_\Xi \leq \ell_h\left(\|\psi_1 - \psi_2\|_\Theta + \|\varpi_1 - \varpi_2\|_\Xi + \|\widetilde{\varpi}_1 - \widetilde{\varpi}_2\|_\Xi\right), \\ \|P_1(\varsigma, \vartheta, \psi_1) - P_1(\varsigma, \vartheta, \psi_2)\|_\Xi \leq \ell_{P_1}\|\psi_1 - \psi_2\|_\Theta, \\ \|P_2(\varsigma, \vartheta, \psi_1) - P_2(\varsigma, \vartheta, \psi_2)\|_\Xi \leq \ell_{P_2}\|\psi_1 - \psi_2\|_\Theta, \\ \|Q_1(\varsigma, \vartheta, \psi_1) - Q_1(\varsigma, \vartheta, \psi_2)\|_\Xi \leq \ell_{Q_1}\|\psi_1 - \psi_2\|_\Theta, \\ \|Q_2(\varsigma, \vartheta, \psi_1) - Q_2(\varsigma, \vartheta, \psi_2)\|_\Xi \leq \ell_{Q_2}\|\psi_1 - \psi_2\|_\Theta. \end{cases}$$

(A$_5$) For all $(\varsigma, \psi, \varpi_1, \varpi_2) \in U \times \Theta \times \Xi \times \Xi$ and $(\varsigma, \vartheta, \psi) \in \mho \times \Theta$, $V_j \in L^1(U, \mathbb{R}_+)$ $(j = 1, 2, 3, 4, 5)$

exists in order that

$$\begin{cases} \|g(\varsigma,\psi,\varpi_1,\varpi_2)\|_\Xi \le V_1(\varsigma)\|\psi\|_\Theta + V_2(\varsigma)\|\varpi_1\|_\Xi + V_3(\varsigma)\|\varpi_2\|_\Xi\,,\\ \|h(\varsigma,\psi,\varpi_1,\varpi_2)\|_\Xi \le V_4(\varsigma)\|\psi\|_\Theta + V_5(\varsigma)\|\varpi_1\|_\Xi + V_6(\varsigma)\|\varpi_2\|_\Xi\,,\\ \|P_1(\varsigma,\vartheta,\psi)\|_\Xi \le V_7(\varsigma)\|\psi\|_\Theta\,,\\ \|P_2(\varsigma,\vartheta,\psi)\|_\Xi \le V_8(\varsigma)\|\psi\|_\Theta\,,\\ \|Q_1(\varsigma,\vartheta,\psi)\|_\Xi \le V_9(\varsigma)\|\psi\|_\Theta\,,\\ \|Q_2(\varsigma,\vartheta,\psi)\|_\Xi \le V_{10}(\varsigma)\|\psi\|_\Theta\,. \end{cases}$$

(A_6) Assume that $S^* = \ell_g N_1^* \{\xi_1 + \xi_2 (\ell_{P_2} + \ell_{Q_2})\} + \ell_h N_1^* \{\xi_1 + \xi_2 (\ell_{P_1} + \ell_{Q_1})\} < 1$, where

$$\begin{cases} \xi_1 = \left(1+\frac{\sigma}{|B|}\right)\nu_1 + \frac{\sigma}{|B|}\nu_3,\\ \xi_2 = \left(1+\frac{\sigma}{|B|}\right)\nu_2 + \frac{\sigma}{|B|}\nu_4,\\ \nu_1 = \frac{\sigma^p}{\Gamma(1+p)},\ \nu_2 = \frac{\sigma^{p+1}}{\Gamma(2+p)},\\ \nu_3 = \sum\limits_{j=1}^{u}\left|b_j\right|\frac{\lambda_j^{q_i+p}}{\Gamma(q_i+p+1)},\ \nu_4 = \sum\limits_{j=1}^{u}\left|b_j\right|\frac{\lambda_j^{q_i+p+1}}{\Gamma(q_i+p+2)}. \end{cases}$$

Theorem 4.2. *Under Assertions (A_4) and (A_5), the neutral BVP (1.2) has at least one solution on $(-\infty,\sigma]$, provided that*

$$\ell^* = \frac{\sigma N_1^*}{|B|}\Big(\ell_g\left[(\nu_1+\nu_3) + (\ell_{P_2}+\ell_{Q_2})(\nu_2+\nu_4)\right] + \ell_h\left[(\nu_1+\nu_3) + (\ell_{P_1}+\ell_{Q_1})(\nu_2+\nu_4)\right]\Big) < 1.$$

Proof. Define the operator $\eth : \widetilde{\mho} \to \widetilde{\mho}$ as

$$(\eth\varpi)(\varsigma) = \begin{cases} \psi(\varsigma),\quad \varsigma\in(-\infty,0],\\ \int_0^\varsigma \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} g(\varsigma,\varpi_\varsigma,P_2\varpi(\varsigma),Q_2\varpi(\varsigma))\,d\varsigma\\ +\int_0^\varsigma \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} h(\vartheta,\varpi_\vartheta,P_1\varpi(\vartheta),Q_1\varpi(\vartheta))\,d\varsigma\\ +\frac{\varsigma}{B}\Bigg(\sum\limits_{j=1}^{u} b_j\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)} g(\varsigma,\varpi_\varsigma,P_2\varpi(\varsigma),Q_2\varpi(\varsigma))\,d\varsigma\\ +\sum\limits_{j=1}^{u} b_j\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)} h(\vartheta,\varpi_\vartheta,P_1\varpi(\vartheta),Q_1\varpi(\vartheta))\,d\varsigma\\ -\int_0^\sigma \frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)} g(\varsigma,\varpi_\varsigma,P_2\varpi(\varsigma),Q_2\varpi(\varsigma))\,d\varsigma\\ -\int_0^\sigma \frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)} h(\vartheta,\varpi_\vartheta,P_1\varpi(\vartheta),Q_1\varpi(\vartheta))\,d\varsigma\Bigg)\\ +\psi(0)\left(1+\frac{\varsigma}{B}\left(\sum\limits_{j=1}^{u}\frac{b_j\,\lambda_j^{q_i}}{\Gamma(q_i+1)}-1\right)\right),\ \varsigma\in U. \end{cases} \tag{4.1}$$

Analogous to Theorem 3.3, define the operator $\Re : G_0 \to G_0$ as

$$(\Re\varpi)(\varsigma)=\begin{cases}\int_0^\varsigma \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\\ +\int_0^\varsigma \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} h\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\\ +\frac{\varsigma}{B}\Bigg(\sum_{j=1}^{u} b_j\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)} g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\\ +\sum_{j=1}^{u} b_j\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)} h\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\\ -\int_0^\sigma \frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)} g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\\ -\int_0^\sigma \frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)} h\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\Bigg)\\ +\psi(0)\left(1+\frac{\varsigma}{B}\left(\sum_{j=1}^{u}\frac{b_j\lambda_j^{q_i}}{\Gamma(q_i+1)}-1\right)\right),\ \varsigma\in U.\end{cases}$$

Describe the set $H_{\widehat{s}}$ as $H_{\widehat{s}}=\left\{\omega\in G_0:\|\omega\|_{G_0}\leq\widehat{s}\right\}$. Let there be a positive constant ε^* such that $\varepsilon^*<\widehat{s}$, where

$$\varepsilon^*=2\|V^*\|_{L^1}\widehat{s}^*\left[\xi_1+\xi_2\right]+\|\psi(0)\|\left(1+\frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}\frac{b_j\lambda_j^{q_i}}{\Gamma(q_i+1)}-1\right)\right),$$

where

$$V^*(\varsigma)=\max\left\{V_1(\varsigma),V_2(\varsigma),V_3(\varsigma),V_4(\varsigma),V_5(\varsigma),V_6(\varsigma),V_7(\varsigma),V_8(\varsigma),V_9(\varsigma),V_{10}(\varsigma)\right\}.$$

Now, we decompose $\Re$ as $\Re_1+\Re_2$ on $H_{\widehat{s}}$, where

$$(\Re_1\varpi)(\varsigma)=\begin{cases}\int_0^\varsigma \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\\ +\int_0^\varsigma \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} h\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right),\end{cases}$$

and

$$(\Re_2\varpi)(\varsigma)=\begin{cases}\frac{\varsigma}{B}\Bigg(\sum_{j=1}^{u} b_j\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)} g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\\ +\sum_{j=1}^{u} b_j\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)} h\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\\ -\int_0^\sigma \frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)} g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\\ -\int_0^\sigma \frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)} h\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\Bigg)\\ +\psi(0)\left(1+\frac{\varsigma}{B}\left(\sum_{j=1}^{u}\frac{b_j\lambda_j^{q_i}}{\Gamma(q_i+1)}-1\right)\right),\ \varsigma\in U.\end{cases}$$

Now, for $\omega,\omega^*\in H_{\widehat{s}}$ and $\varsigma\in U$, we have

$$\begin{aligned}&\left\|(\Re_1\omega)(\varsigma)+(\Re_2\omega^*)(\varsigma)\right\|_\Xi\\ \leq\ &\int_0^\varsigma\frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)}\left\|g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right\|_\Xi d\vartheta\\ &+\int_0^\varsigma\frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)}\left\|h\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right\|_\Xi d\vartheta\end{aligned}$$

$$
\begin{aligned}
&+\frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}\left|b_j\right|\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)}\left\|g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}^*_\vartheta,P_2\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right],Q_2\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right)\right\|_\Xi d\vartheta\right.\\
&+\sum_{j=1}^{u}\left|b_j\right|\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)}\left\|h\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}^*_\vartheta,P_1\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right],Q_1\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right)\right\|_\Xi d\vartheta\\
&+\int_0^{\sigma}\frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)}\left\|g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}^*_\vartheta,P_2\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right],Q_2\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right)\right\|_\Xi d\vartheta\\
&\left.+\int_0^{\sigma}\frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)}\left\|h\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}^*_\vartheta,P_1\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right],Q_1\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right)\right\|_\Xi d\vartheta\right)\\
&+\|\psi(0)\|\left(1+\frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}\frac{\left|b_j\right|\lambda_j^{q_i}}{\Gamma(q_i+1)}-1\right)\right)\\
\leq\;&\int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)}\Big[V_1(\vartheta)\|\varrho_\vartheta+\widetilde{\omega}_\vartheta\|_\Theta+V_2(\vartheta)\left\|P_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_\Xi+V_3(\vartheta)\left\|Q_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_\Xi\\
&+V_4(\vartheta)\|\varrho_\vartheta+\widetilde{\omega}_\vartheta\|_\Theta+V_5(\vartheta)\left\|P_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_\Xi+V_6(\vartheta)\left\|Q_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_\Xi\Big]d\vartheta\\
&+\frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}\left|b_j\right|\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)}\Big[V_1(\vartheta)\|\varrho_\vartheta+\widetilde{\omega}_\vartheta\|_\Theta+V_2(\vartheta)\left\|P_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_\Xi\right.\\
&+V_3(\vartheta)\left\|Q_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_\Xi+V_4(\vartheta)\|\varrho_\vartheta+\widetilde{\omega}_\vartheta\|_\Theta+V_5(\vartheta)\left\|P_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_\Xi\\
&+V_6(\vartheta)\left\|Q_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_\Xi\Big]d\theta+\int_0^{\sigma}\frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)}\left[V_1(\vartheta)\|\varrho_\vartheta+\widetilde{\omega}_\vartheta\|_\Theta\right.\\
&+q_2(\vartheta)\left\|P_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_\Xi+q_3(\vartheta)\left\|Q_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_\Xi+V_4(\vartheta)\|\varrho_\vartheta+\widetilde{\omega}_\vartheta\|_\Theta\\
&\left.+V_5(\vartheta)\left\|P_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_\Xi+V_6(\vartheta)\left\|Q_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right\|_\Xi\right]d\theta\\
&+\|\psi(0)\|\left(1+\frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}\frac{\left|b_j\right|\lambda_j^{q_i}}{\Gamma(q_i+1)}-1\right)\right)\\
\leq\;&2\|V^*\|_{L^1}\widehat{s}^*\left[\xi_1+\xi_2\right]+\|\psi(0)\|\left(1+\frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}\frac{b_j\lambda_j^{q_i}}{\Gamma(q_i+1)}-1\right)\right)\\
=\;&\varepsilon^*,
\end{aligned}
$$

where $V^*(\varsigma)=\max\{V_1(\varsigma),V_2(\varsigma),V_3(\varsigma),V_4(\varsigma),V_5(\varsigma),V_6(\varsigma)\}$ and

$$\|\varrho_\vartheta+\widetilde{\omega}_\vartheta\|_\Theta\leq N_1^*\widehat{s}+N_2^*\widehat{s}\|\psi(0)\|_\Theta\leq\widehat{s}^*.$$

Hence,

$$\left\|\mathfrak{R}_1\omega+\mathfrak{R}_2\omega^*\right\|_{G_0}\leq\varepsilon^*.$$

Thus, $\mathfrak{R}_1\omega+\mathfrak{R}_2\omega^*\in H_{\widehat{s}}$. Now, we prove that $\mathfrak{R}_2$ is a contraction. Let $\omega,\omega^*\in H_{\widehat{s}}$ and $\varsigma\in U$. We then

$$
\begin{aligned}
&\left\|(\mathfrak{R}_2\omega)(\varsigma)-(\mathfrak{R}_2\omega^*)(\varsigma)\right\|_\Xi\\
\leq\;&\frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}\left|b_j\right|\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)}\left[\left\|g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right.\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
& - g\left(\vartheta, \varrho_{\vartheta} + \widetilde{\omega}^{*}_{\vartheta}, P_2\left[\varrho(\vartheta) + \widetilde{\omega}^{*}(\vartheta)\right], Q_2\left[\varrho(\vartheta) + \widetilde{\omega}^{*}(\vartheta)\right]\right)\Big\|_{\Xi} \\
& + \Big\|h\left(\vartheta, \varrho_{\vartheta} + \widetilde{\omega}_{\vartheta}, P_1\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right], Q_1\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right) \\
& - h\left(\vartheta, \varrho_{\vartheta} + \widetilde{\omega}^{*}_{\vartheta}, P_1\left[\varrho(\vartheta) + \widetilde{\omega}^{*}(\vartheta)\right], Q_1\left[\varrho(\vartheta) + \widetilde{\omega}^{*}(\vartheta)\right]\right)\Big\|_{\Xi}\Big] d\vartheta \\
& + \int_0^{\sigma} \frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)} \Big[\Big\|g\left(\vartheta, \varrho_{\vartheta} + \widetilde{\omega}_{\vartheta}, P_2\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right], Q_2\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right) \\
& - g\left(\vartheta, \varrho_{\vartheta} + \widetilde{\omega}^{*}_{\vartheta}, P_2\left[\varrho(\vartheta) + \widetilde{\omega}^{*}(\vartheta)\right], Q_2\left[\varrho(\vartheta) + \widetilde{\omega}^{*}(\vartheta)\right]\right)\Big\|_{\Xi} \\
& + \Big\|h\left(\vartheta, \varrho_{\vartheta} + \widetilde{\omega}_{\vartheta}, P_1\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right], Q_1\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right) \\
& - g\left(\vartheta, \varrho_{\vartheta} + \widetilde{\omega}^{*}_{\vartheta}, P_1\left[\varrho(\vartheta) + \widetilde{\omega}^{*}(\vartheta)\right], Q_1\left[\varrho(\vartheta) + \widetilde{\omega}^{*}(\vartheta)\right]\right)\Big\|_{\Xi}\Big] d\vartheta \\
\leq\ & \frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}\left|b_j\right| \int_0^{\lambda_j} \frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)} \left[\ell_g\left(\left\|\widetilde{\omega}_{\vartheta} - \widetilde{\omega}^{*}_{\vartheta}\right\|_{\Theta} + \ell_{P_2}\left\|\widetilde{\omega}_{\mu} - \widetilde{\omega}^{*}_{\mu}\right\|_{\Theta}\vartheta + \ell_{Q_2}\left\|\widetilde{\omega}_{\mu} - \widetilde{\omega}^{*}_{\mu}\right\|_{\Theta}\vartheta\right)\right.\right. \\
& \left. + \ell_h\left(\left\|\widetilde{\omega}_{\vartheta} - \widetilde{\omega}^{*}_{\vartheta}\right\|_{\Theta} + \ell_{P_1}\left\|\widetilde{\omega}_{\mu} - \widetilde{\omega}^{*}_{\mu}\right\|_{\Theta}\vartheta + \ell_{Q_1}\left\|\widetilde{\omega}_{\mu} - \widetilde{\omega}^{*}_{\mu}\right\|_{\Theta}\vartheta\right)\right] d\vartheta \\
& + \int_0^{\sigma} \frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)} \left[\ell_g\left(\left\|\widetilde{\omega}_{\vartheta} - \widetilde{\omega}^{*}_{\vartheta}\right\|_{\Theta} + \ell_{P_2}\left\|\widetilde{\omega}_{\mu} - \widetilde{\omega}^{*}_{\mu}\right\|_{\Theta}\vartheta + \ell_{Q_2}\left\|\widetilde{\omega}_{\mu} - \widetilde{\omega}^{*}_{\mu}\right\|_{\Theta}\vartheta\right)\right. \\
& \left.\left. + \ell_h\left(\left\|\widetilde{\omega}_{\vartheta} - \widetilde{\omega}^{*}_{\vartheta}\right\|_{\Theta} + \ell_{P_1}\left\|\widetilde{\omega}_{\mu} - \widetilde{\omega}^{*}_{\mu}\right\|_{\Theta}\vartheta + \ell_{Q_1}\left\|\widetilde{\omega}_{\mu} - \widetilde{\omega}^{*}_{\mu}\right\|_{\Theta}\vartheta\right)\right] d\vartheta\right) \\
\leq\ & \frac{\sigma N_1^{*}}{|B|}\left[\begin{array}{c} \ell_g\left\{(\nu_1+\nu_3) + (\nu_2+\nu_4)\left(\ell_{P_2} + \ell_{Q_2}\right)\right\} \\ +\ell_h\left\{(\nu_1+\nu_3) + (\nu_2+\nu_4)\left(\ell_{P_1} + \ell_{Q_1}\right)\right\} \end{array}\right] \|\omega - \omega^{*}\|_{G_0} \\
=\ & \ell^{*}\|\omega - \omega^{*}\|_{G_0}.
\end{aligned}
$$

It follows that

$$
\left\|\mathfrak{R}_2\omega - \mathfrak{R}_2\omega^{*}\right\|_{G_0} \leq \ell^{*}\|\omega - \omega^{*}\|_{G_0}.
$$

Since $\ell^{*} < 1$, then, $\mathfrak{R}_2$ is contraction. Since g, P_1, P_2, Q_1 and Q_2 are continuous, then $\mathfrak{R}_1$ is continuous. Furthermore,

$$
\begin{aligned}
& \left\|\left(\mathfrak{R}_1\omega\right)(\varsigma)\right\|_{\Xi} \\
\leq\ & \int_0^{\varsigma} \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} \left\|g\left(\vartheta, \varrho_{\vartheta} + \widetilde{\omega}_{\vartheta}, P_2\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right], Q_2\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right)\right\|_{\Xi} d\vartheta \\
& + \int_0^{\varsigma} \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} \left\|h\left(\vartheta, \varrho_{\vartheta} + \widetilde{\omega}_{\vartheta}, P_1\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right], Q_1\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right)\right\|_{\Xi} d\vartheta \\
\leq\ & \int_0^{\varsigma} \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} \left[V_1(\vartheta)\|\varrho_{\vartheta} + \widetilde{\omega}_{\vartheta}\|_{\Theta} + V_2(\vartheta)\left\|P_2\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right\|_{\Xi} + V_3(\vartheta)\left\|Q_2\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right\|_{\Xi}\right] d\vartheta \\
& + \int_0^{\varsigma} \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)} \left[V_4(\vartheta)\|\varrho_{\vartheta} + \widetilde{\omega}_{\vartheta}\|_{\Theta} + V_5(\vartheta)\left\|P_1\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right\|_{\Xi} + V_6(\vartheta)\left\|Q_1\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right\|_{\Xi}\right] d\vartheta \\
\leq\ & 2\|q^{*}\|_{L^1}\widehat{s}^{*}(\nu_1 + \nu_2).
\end{aligned}
$$

Hence, $\mathfrak{R}_1$ is uniformly bounded on H_s. Ultimately, we claim that $\mathfrak{R}_1$ is compact. Indeed, we prove that $\mathfrak{R}_1$ is equicontinuous. For $\varsigma_1, \varsigma_2 \in U$, with $\varsigma_1 < \varsigma_2$ and $\omega \in H_{\widehat{s}}$, one has

$$
\begin{aligned}
& \left\|\left(\mathfrak{R}_1\omega\right)(\varsigma_2) - \left(\mathfrak{R}_1\omega\right)(\varsigma_1)\right\|_{\Xi} \\
\leq\ & \int_0^{\varsigma_1} \frac{(\varsigma_2-\vartheta)^{p-1} - (\varsigma_1-\vartheta)^{p-1}}{\Gamma(p)} \Big[\Big\|g\left(\vartheta, \varrho_{\vartheta} + \widetilde{\omega}_{\vartheta}, P_2\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right], Q_2\left[\varrho(\vartheta) + \widetilde{\omega}(\vartheta)\right]\right)\Big\|_{\Xi}
\end{aligned}
$$

$$
\begin{aligned}
&+\left\|h\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right\|_\Xi\Big]d\vartheta\\
&+\int_{\varsigma_1}^{\varsigma_2}\frac{(\varsigma_2-\vartheta)^{p-1}}{\Gamma(p)}\Big[\left\|g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right\|_\Xi\\
&+\left\|h\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right\|_\Xi\Big]d\vartheta\\
\le\;&\int_0^{\varsigma_1}\frac{(\varsigma_2-\vartheta)^{p-1}-(\varsigma_1-\vartheta)^{p-1}}{\Gamma(p)}\left[V_1(\vartheta)\widehat{s}^*+V_2(\vartheta)V_8(\vartheta)\widehat{s}^*\vartheta+V_3(\vartheta)V_{10}(\vartheta)\widehat{s}^*\vartheta\right.\\
&\left.+V_4(\vartheta)\widehat{s}^*+V_5(\vartheta)V_7(\vartheta)\widehat{s}^*\vartheta+V_6(\vartheta)V_9(\vartheta)\widehat{s}^*\vartheta\right]d\vartheta\\
&+\int_{\varsigma_1}^{\varsigma_2}\frac{(\varsigma_2-\vartheta)^{p-1}}{\Gamma(p)}\left[V_1(\vartheta)\widehat{s}^*+V_2(\vartheta)V_8(\vartheta)\widehat{s}^*\vartheta+V_3(\vartheta)V_{10}(\vartheta)\widehat{s}^*\vartheta\right.\\
&\left.+V_4(\vartheta)\widehat{s}^*+V_5(\vartheta)q_7(\vartheta)\widehat{s}^*\vartheta+V_6(\vartheta)V_9(\vartheta)\widehat{s}^*\vartheta\right]d\vartheta\\
\le\;&2\left\|V^*\right\|_{L^1}\widehat{s}^*\left(\int_0^{\varsigma_1}\frac{(\varsigma_2-\vartheta)^{p-1}-(\varsigma_1-\vartheta)^{p-1}}{\Gamma(p)}(1+\vartheta)d\vartheta+\int_{\varsigma_1}^{\varsigma_2}\frac{(\varsigma_2-\vartheta)^{p-1}}{\Gamma(p)}(1+\vartheta)d\vartheta\right).
\end{aligned}
$$

Therefore, $\left\|(\mathfrak{R}_1\omega)(\varsigma_2)-(\mathfrak{R}_1\omega)(\varsigma_2)\right\|_\Xi\to 0$ as $\varsigma_1\to\varsigma_2$. Hence, Φ_1 is equicontinuous. By the Arzelà-Ascoli theorem, we establish that $\mathfrak{R}_1$ is compact on $H_{\widehat{s}}$. Consequently, invoking Krasnoselskii's FP theorem, $\omega\in G_0$ exists such that $\mathfrak{R}\omega=\omega$, which is a solution the neutral BVP (1.2). □

For the uniqueness, we have the following theorem:

Theorem 4.3. *Via Assertions ($\mathbf{A}_4$) and ($\mathbf{A}_6$), the neutral BVP (1.2) has a unique solution on* $(-\infty,\sigma]$.

Proof. Define the set $H_{\widehat{s}}=\left\{\omega\in G_0:\|\omega\|_{G_0}\le s\right\}$ and assume that $\omega,\omega^*\in G_0$. For $\varsigma\in U$, we get

$$
\begin{aligned}
&\left\|(\mathfrak{R}\omega)(\varsigma)-(\mathfrak{R}\omega^*)(\varsigma)\right\|_\Xi\\
\le\;&\int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)}\left\|g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right.\\
&\left.-g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}^*_\vartheta,P_2\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right],Q_2\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right)\right\|_\Xi d\vartheta\\
&+\int_0^{\varsigma}\frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)}\left\|h\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right.\\
&\left.-h\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}^*_\vartheta,P_1\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right],Q_1\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right)\right\|_\Xi d\vartheta\\
&+\frac{\varsigma}{|B|}\left(\sum_{j=1}^{u}\left|b_j\right|\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)}\left\|g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right.\right.\\
&\left.-g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}^*_\vartheta,P_2\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right],Q_2\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right)\right\|_\Xi d\vartheta\\
&+\sum_{j=1}^{u}\left|b_j\right|\int_0^{\lambda_j}\frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)}\left\|h\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right.\\
&\left.-h\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}^*_\vartheta,P_1\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right],Q_1\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right)\right\|_\Xi d\vartheta\\
&+\int_0^{\sigma}\frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)}\left\|g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_2\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right.\\
&\left.-g\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}^*_\vartheta,P_2\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right],Q_2\left[\varrho(\vartheta)+\widetilde{\omega}^*(\vartheta)\right]\right)\right\|_\Xi d\vartheta\\
&+\int_0^{\sigma}\frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)}\left\|h\left(\vartheta,\varrho_\vartheta+\widetilde{\omega}_\vartheta,P_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right],Q_1\left[\varrho(\vartheta)+\widetilde{\omega}(\vartheta)\right]\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
& - h\left(\vartheta, \varrho_\vartheta + \widetilde{\omega}^*_\vartheta, P_1\left[\varrho(\vartheta) + \widetilde{\omega}^*(\vartheta)\right], Q_1\left[\varrho(\vartheta) + \widetilde{\omega}^*(\vartheta)\right]\right)\Big\|_\Xi d\vartheta\Big) \\
\le\ & \int_0^\varsigma \frac{(\varsigma-\vartheta)^{p-1}}{\Gamma(p)}\left[\ell_g\left(\left\|\widetilde{\omega}_\vartheta - \widetilde{\omega}^*_\vartheta\right\|_\Theta + (\ell_{P_2}+\ell_{Q_2})\left\|\widetilde{\omega}_\mu - \widetilde{\omega}^*_\mu\right\|_\Theta \vartheta\right)\right. \\
& \left. + \ell_h\left(\left\|\widetilde{\omega}_\vartheta - \widetilde{\omega}^*_\vartheta\right\|_\Theta + (\ell_{P_1}+\ell_{Q_1})\left\|\widetilde{\omega}_\mu - \widetilde{\omega}^*_\mu\right\|_\Theta \vartheta\right)\right] d\vartheta \\
& + \frac{\varsigma}{|B|}\left(\sum_{j=1}^u \left|b_j\right| \int_0^{\lambda_j} \frac{(\lambda_j-\vartheta)^{q_j+p-1}}{\Gamma(q_j+p)}\left[\ell_g\left(\left\|\widetilde{\omega}_\vartheta - \widetilde{\omega}^*_\vartheta\right\|_\Theta + (\ell_{P_2}+\ell_{Q_2})\left\|\widetilde{\omega}_\mu - \widetilde{\omega}^*_\mu\right\|_\Theta \vartheta\right)\right.\right. \\
& \left. + \ell_h\left(\left\|\widetilde{\omega}_\vartheta - \widetilde{\omega}^*_\vartheta\right\|_\Theta + (\ell_{P_1}+\ell_{Q_1})\left\|\widetilde{\omega}_\mu - \widetilde{\omega}^*_\mu\right\|_\Theta \vartheta\right)\right] d\vartheta \\
& + \int_0^\sigma \frac{(\sigma-\vartheta)^{p-1}}{\Gamma(p)}\left[\ell_g\left(\left\|\widetilde{\omega}_\vartheta - \widetilde{\omega}^*_\vartheta\right\|_\Theta + (\ell_{P_2}+\ell_{Q_2})\left\|\widetilde{\omega}_\mu - \widetilde{\omega}^*_\mu\right\|_\Theta \vartheta\right)\right. \\
& \left.\left. + \ell_h\left(\left\|\widetilde{\omega}_\vartheta - \widetilde{\omega}^*_\vartheta\right\|_\Theta + (\ell_{P_1}+\ell_{Q_1})\left\|\widetilde{\omega}_\mu - \widetilde{\omega}^*_\mu\right\|_\Theta \vartheta\right)\right] d\vartheta\right) \\
\le\ & \left\{\ell_g N_1^*\left[\frac{\sigma^p}{\Gamma(p+1)} + \frac{\sigma}{|B|}\sum_{j=1}^u \left|b_j\right| \frac{(\lambda_j)^{q_j+p}}{\Gamma(q_j+p+1)} + \frac{\sigma^{p+1}}{|B|\,\Gamma(p+1)}\right.\right. \\
& \left. + (\ell_{P_2}+\ell_{Q_2})\left(\frac{\sigma^{p+1}}{\Gamma(p+2)} + \frac{\sigma}{|B|}\sum_{j=1}^u \left|b_j\right| \frac{(\lambda_j)^{q_j+p+1}}{\Gamma(q_j+p+2)} + \frac{\sigma^{p+2}}{|B|\,\Gamma(p+2)}\right)\right] \\
& + \ell_h N_1^*\left[\frac{\sigma^p}{\Gamma(p+1)} + \frac{\sigma}{|B|}\sum_{j=1}^u \left|b_j\right| \frac{(\lambda_j)^{q_j+p}}{\Gamma(q_j+p+1)} + \frac{\sigma^{p+1}}{|B|\,\Gamma(p+1)}\right. \\
& \left.\left. + (\ell_{P_1}+\ell_{Q_1})\left(\frac{\sigma^{p+1}}{\Gamma(p+2)} + \frac{\sigma}{|B|}\sum_{j=1}^u \left|b_j\right| \frac{(\lambda_j)^{q_j+p+1}}{\Gamma(q_j+p+2)} + \frac{\sigma^{p+2}}{|B|\,\Gamma(p+2)}\right)\right]\right\} \|\omega - \omega^*\|_{G_0} \\
\le\ & \ell_g N_1^*\left\{\xi_1 + \xi_2(\ell_{P_2}+\ell_{Q_2})\right\} + \ell_h N_1^*\left\{\xi_1 + \xi_2(\ell_{P_1}+\ell_{Q_1})\right\} \|\omega - \omega^*\|_{G_0} \\
=\ & S^* \|\omega - \omega^*\|_{G_0}.
\end{aligned}
$$

Hence,

$$\left\|\mathfrak{R}(\omega) - \mathfrak{R}(\omega^*)\right\|_{G_0} \le S^* \|\omega - \omega^*\|_{G_0}.$$

By (A_6), $S^* < 1$. Thus, $\mathfrak{R}$ is a contraction. By Banach's FP theorem, $\mathfrak{R}$ has a unique FP, which is a unique solution to the problem (1.2) on $(-\infty, \sigma]$. □

5. Supportive examples

This section is devoted to testing the conditions of the proposed systems and their effectiveness, which leads to supporting and enhancing the theoretical results we obtained.

Example 5.1. Assume the following FFIDE:

$$
\begin{cases}
{}^{LC}D^{\frac{5}{2}}\varpi(\varsigma) = \frac{1}{(\varsigma+7)^2}e^{-\nu\varsigma}\frac{|\varpi_\varsigma|}{1+|\varpi_\varsigma|} + \frac{1}{32}\int_0^\varsigma \frac{\varsigma\vartheta e^{-\nu\varsigma}}{16(1+\vartheta)}\frac{\cos(\varpi_\vartheta)}{1+\cos(\varpi_\vartheta)}d\vartheta + \frac{1}{64}\int_0^\sigma \frac{\varsigma\vartheta e^{-\nu\varsigma}}{20(1+\vartheta)}\frac{\sin(\varpi_\vartheta)}{1+\sin(\varpi_\vartheta)}d\vartheta,\ \varsigma\in[0,1], \\
\varpi(\varsigma) = \psi(\varsigma),\ \varsigma\in(-\infty,0], \\
\varpi(1) = \sum_{j=1}^3 b_j\left(I_{0^+}^{q_j}\varpi\right)(\lambda_j),\ 0<\lambda_1<\lambda_2<\lambda_3<1,
\end{cases}
\tag{5.1}
$$

where $\nu > 0$ is a real constant and define the set H_ν as

$$H_\nu = \left\{ \omega \in C\left((-\infty, 0], \mathbb{R}\right) : \lim_{\phi \to -\infty} e^{\nu\phi}\omega(\phi) \text{ exists in } \mathbb{R} \right\},$$

under the norm

$$\|\omega\|_\nu = \sup_{\phi \in (-\infty, 0]} e^{\nu\phi} |\omega(\phi)| .$$

Assume that $\varpi : (-\infty, \sigma] \to \Xi$ in order that $\varpi_0 = \psi \in H_\nu$. Then

$$\lim_{\phi \to -\infty} e^{\nu\phi}\omega_\varsigma(\phi) = \lim_{\phi \to -\infty} e^{\nu\phi}\omega(\varsigma + \phi) = \lim_{\phi \to -\infty} e^{\nu(\phi - \varsigma)}\omega(\phi) = e^{-\nu\varsigma} \lim_{\phi \to -\infty} e^{\nu\phi}\omega_0(\phi) < \infty.$$

Therefore, $\omega_\varsigma \in H_\nu$. Select $N_1 = N_2 = \kappa = 1$. Hence, we show the condition

$$\left\|\omega_\varsigma\right\|_\nu \le N_1(\varsigma) \sup\left\{|\varpi(\vartheta)| : 0 \le \vartheta \le \varsigma\right\} + N_2(\varsigma) \|\varpi_0\|_\nu .$$

Clearly, $\left|\omega_\varsigma(\phi)\right| = |\omega(\varsigma + \phi)|$. If $\varsigma + \phi \le 0$, we get

$$\left|\omega_\varsigma(\phi)\right| \le \sup\left\{|\varpi(\vartheta)| : -\infty < \vartheta \le 0\right\} .$$

In the case of $\varsigma + \phi \ge 0$, we have

$$\left|\omega_\varsigma(\phi)\right| \le \sup\left\{|\varpi(\vartheta)| : 0 < \vartheta \le \varsigma\right\} .$$

Hence, if $\varsigma + \phi \in [0, 1]$, we can write

$$\left|\omega_\varsigma(\phi)\right| \le \sup\left\{|\varpi(\vartheta)| : -\infty < \vartheta \le 0\right\} + \sup\left\{|\varpi(\vartheta)| : 0 \le \vartheta \le \varsigma\right\} ,$$

which implies that

$$\left\|\omega_\varsigma\right\|_\nu \le \sup\left\{|\varpi(\vartheta)| : 0 \le \vartheta \le \varsigma\right\} + \|\varpi_0\|_\nu .$$

Furthermore, the pair $(H_\nu, \|\omega\|)$ is a BS and H_ν is a phase space. Here, $p = \frac{5}{2}$, $u = 3$, and we choose

$$\begin{aligned} b_1 &= \tfrac{1}{6}, & b_2 &= \tfrac{1}{8}, & b_3 &= 4, \\ \lambda_1 &= \tfrac{1}{9}, & \lambda_2 &= \tfrac{1}{4}, & \lambda_3 &= \tfrac{7}{11}, \\ q_1 &= \tfrac{1}{3}, & q_2 &= \tfrac{1}{2}, & q_3 &= \tfrac{6}{5}. \end{aligned}$$

By simple calculation, we have

$$\begin{cases} B = \sigma - \sum\limits_{j=1}^{u} \frac{b_j \lambda_j^{q_i+1}}{\Gamma(q_i+2)} = 1 - \sum\limits_{j=1}^{3} \frac{b_j \lambda_j^{q_i+1}}{\Gamma(q_i+2)} \approx 0.3729 \neq 0, \\ \nu_1 = \frac{\sigma^p}{\Gamma(1+p)} \approx 0.3009, \ \nu_2 = \frac{\sigma^{p+1}}{\Gamma(2+p)} \approx 0.0859, \\ \nu_3 = \sum\limits_{j=1}^{u} \left|b_j\right| \frac{\lambda_j^{q_i+p}}{\Gamma(q_i+p+1)} \approx 0.0088, \ \nu_4 = \sum\limits_{j=1}^{u} \left|b_j\right| \frac{\lambda_j^{q_i+p+1}}{\Gamma(q_i+p+2)} \approx 0.0008, \\ \xi_1 = \left(1 + \frac{\sigma}{|B|}\right)\nu_1 + \frac{\sigma}{|B|}\nu_3 \approx 1.1314, \\ \xi_2 = \left(1 + \frac{\sigma}{|B|}\right)\nu_2 + \frac{\sigma}{|B|}\nu_4 \approx 0.3184. \end{cases}$$

From (5.1), we have

$$g\left(\varsigma, \varpi_{\varsigma}, P(\varsigma), Q(\varsigma)\right) = \frac{1}{(\varsigma+7)^2} e^{-\nu\varsigma} \frac{\left|\varpi_{\varsigma}\right|}{1+\left|\varpi_{\varsigma}\right|} + \frac{1}{32} P\varpi(\varsigma) + \frac{1}{64} Q\varpi(\varsigma),$$

where

$$\begin{aligned} P\varpi(\varsigma) &= \int_0^{\varsigma} \frac{\varsigma\vartheta e^{-\nu\varsigma}}{16(1+\vartheta)} \frac{\cos(\varpi_{\vartheta})}{1+\cos(\varpi_{\vartheta})} d\vartheta, \\ Q\varpi(\varsigma) &= \int_0^{\sigma} \frac{\varsigma\vartheta e^{-\nu\varsigma}}{20(1+\vartheta)} \frac{\sin(\varpi_{\vartheta})}{1+\sin(\varpi_{\vartheta})} d\vartheta. \end{aligned}$$

Now, for $\varpi_{\varsigma}, \varrho_{\varsigma} \in H_{\nu}$, we have

$$\begin{aligned} \left|P\left(\varsigma, \vartheta, \varpi_{\vartheta}\right) - P\left(\varsigma, \vartheta, \varrho_{\vartheta}\right)\right| &= \left|\frac{\varsigma\vartheta e^{-\nu\varsigma}}{16(1+\vartheta)} \frac{\cos(\varpi_{\vartheta})}{1+\cos(\varpi_{\vartheta})} - \frac{\varsigma\vartheta e^{-\nu\varsigma}}{16(1+\vartheta)} \frac{\cos(\varrho_{\vartheta})}{1+\cos(\varrho_{\vartheta})}\right| \\ &\leq \frac{1}{16} \left\|\varpi - \varrho\right\|_{\nu}, \end{aligned} \tag{5.2}$$

$$\begin{aligned} \left|Q\left(\varsigma, \vartheta, \varpi_{\vartheta}\right) - Q\left(\varsigma, \vartheta, \varrho_{\vartheta}\right)\right| &= \left|\frac{\varsigma\vartheta e^{-\nu\varsigma}}{20(1+\vartheta)} \frac{\sin(\varpi_{\vartheta})}{1+\sin(\varpi_{\vartheta})} - \frac{\varsigma\vartheta e^{-\nu\varsigma}}{20(1+\vartheta)} \frac{\sin(\varrho_{\vartheta})}{1+\sin(\varrho_{\vartheta})}\right| \\ &\leq \frac{1}{20} \left\|\varpi - \varrho\right\|_{\nu}, \end{aligned} \tag{5.3}$$

$$\begin{aligned} &\left|g\left(\varsigma, \varpi_{\varsigma}, P\varpi(\varsigma), Q\varpi(\varsigma)\right) - g\left(\varsigma, \varrho_{\varsigma}, P\varrho(\varsigma), Q\varrho(\varsigma)\right)\right| \\ \leq\ & \frac{1}{(\varsigma+7)^2} e^{-\nu\varsigma} \frac{\left|\varpi_{\varsigma} - \varrho_{\varsigma}\right|}{\left(1+\left|\varpi_{\varsigma}\right|\right)\left(1+\left|\varrho_{\varsigma}\right|\right)} + \frac{1}{32} \left|P\varpi(\varsigma) - P\varrho(\varsigma)\right| + \frac{1}{64} \left|Q\varpi(\varsigma) - P\varrho(\varsigma)\right| \\ \leq\ & \frac{1}{64} \left(\left\|\varpi - \varrho\right\|_{\nu} + \frac{1}{8} \left\|\varpi - \varrho\right\|_{\nu} + \frac{1}{20} \left\|\varpi - \varrho\right\|_{\nu}\right), \end{aligned} \tag{5.4}$$

$$\begin{aligned} &\left|g\left(\varsigma, \psi, \varpi, \varrho\right)\right| \\ =\ & \left|\frac{1}{(\varsigma+7)^2} e^{-\nu\varsigma} \frac{\left|\psi_{\varsigma}\right|}{1+\left|\psi_{\varsigma}\right|} + \frac{1}{32} \int_0^{\varsigma} \frac{\varsigma\vartheta e^{-\nu\varsigma}}{16(1+\vartheta)} \frac{\cos(\varpi_{\vartheta})}{1+\cos(\varpi_{\vartheta})} d\vartheta + \frac{1}{64} \int_0^{\sigma} \frac{\varsigma\vartheta e^{-\nu\varsigma}}{20(1+\vartheta)} \frac{\sin(\varrho_{\vartheta})}{1+\sin(\varrho_{\vartheta})} d\vartheta\right| \\ \leq\ & \frac{1}{64} \left|\psi\right| + \frac{1}{32} \left|\varpi\right| + \frac{1}{64} \left|\varrho\right|, \end{aligned} \tag{5.5}$$

$$\left|P\left(\varsigma, \vartheta, \varpi\right)\right| = \left|\frac{\varsigma\vartheta e^{-\nu\varsigma}}{16(1+\vartheta)} \frac{\cos(\varpi_{\vartheta})}{1+\cos(\varpi_{\vartheta})}\right| \leq \frac{1}{32} \left|\varpi\right|, \tag{5.6}$$

and

$$\left|Q\left(\varsigma, \vartheta, \varpi\right)\right| = \left|\frac{\varsigma\vartheta e^{-\nu\varsigma}}{20(1+\vartheta)} \frac{\sin(\varpi_{\vartheta})}{1+\sin(\varpi_{\vartheta})}\right| \leq \frac{1}{40} \left|\varpi\right|. \tag{5.7}$$

It follows from (5.2)–(5.7) that $\ell_g = \frac{1}{64}$, $\ell_P = \frac{1}{16}$, $\ell_Q = \frac{1}{20}$, $V_1(\varsigma) = \frac{1}{64}$, $V_2(\varsigma) = \frac{1}{32}$, $V_3(\varsigma) = \frac{1}{64}$, $V_4(\varsigma) = \frac{1}{32}$, $V_5(\varsigma) = \frac{1}{40}$, and $N_1^* = 1$. Hence

$$\ell = \frac{\sigma}{|B|} \ell_g N_1^* \left\{(\nu_1 + \nu_3) + (\nu_2 + \nu_4)\left(\ell_P + \ell_Q\right)\right\} \approx 0.0138 < 1,$$

and

$$S = \ell_g N_1^* \{\xi_1 + \xi_2 (\ell_P + \ell_Q)\} \approx 0.0182 < 1.$$

Therefore, all the requirements of Theorems 3.3 and 3.4 are satisfied. Then, the considered problem (5.1) has a unique solution on $(-\infty, \sigma]$.

Example 5.2. Assume the following neutral FFIDE:

$$\begin{cases} {}^{LC}D_{\varsigma}^{\frac{5}{2}}\left[\varpi(\varsigma) - \int_0^{\varsigma} \sqrt{\frac{(\varsigma-\vartheta)^{p-1}}{\pi}} \left(\frac{e^{-\nu\varsigma}}{20}\frac{\varpi_{\varsigma}^2}{1+\varpi_{\varsigma}^2} + \frac{1}{20}\int_0^{\varsigma} \frac{e^{-\nu\varsigma}}{6} \ln(1+\varpi_{\varsigma})\, d\vartheta + \frac{1}{25}\int_0^{\sigma} \frac{e^{-\nu\varsigma}}{4}\frac{\tan^{-1}(\varpi_{\varsigma})}{1+\tan^{-1}(\varpi_{\varsigma})} d\vartheta\right)\right] \\ = \frac{(1+e^{-\varsigma})e^{-\nu\varsigma}}{(34+e^{\varsigma})}\frac{|\varpi_{\varsigma}|}{1+|\varpi_{\varsigma}|} + \frac{1}{15}\int_0^{\varsigma} e^{-\nu\varsigma}\cos(\frac{\varpi_{\vartheta}}{5})d\vartheta + \frac{1}{35}\int_0^{\sigma} e^{-\nu\varsigma}\sin(\frac{\varpi_{\vartheta}}{6})d\vartheta,\ \varsigma \in [0,1], \\ \varpi(\varsigma) = \psi(\varsigma),\ \varsigma \in (-\infty, 0], \\ \varpi(1) = \sum_{j=1}^{3} b_j \left(I_{0^+}^{q_j}\varpi\right)\left(\lambda_j\right),\ 0 < \lambda_1 < \lambda_2 < \lambda_3 < 1. \end{cases} \tag{5.8}$$

Assume that H_ν is the phase space, which is defined in Example 5.1, where $p = \frac{5}{2}$, $u = 3$, and

$$\begin{array}{lll} b_1 = \frac{1}{5}, & b_2 = \frac{1}{4}, & b_3 = 5, \\ \lambda_1 = \frac{1}{7}, & \lambda_2 = \frac{1}{2}, & \lambda_3 = \frac{7}{12}, \\ q_1 = \frac{1}{2}, & q_2 = \frac{1}{3}, & q_3 = \frac{7}{2}. \end{array}$$

By simple calculation, we have

$$\begin{cases} B \approx 0.5231 \neq 0, \\ \nu_1 = \frac{\sigma^p}{\Gamma(1+p)} \approx 0.4187,\ \nu_2 = \frac{\sigma^{p+1}}{\Gamma(2+p)} \approx 0.3979, \\ \nu_3 = \sum_{j=1}^{u} \left|b_j\right| \frac{\lambda_j^{q_i+p}}{\Gamma(q_i+p+1)} \approx 0.0132,\ \nu_4 = \sum_{j=1}^{u} \left|b_j\right| \frac{\lambda_j^{q_i+p+1}}{\Gamma(q_i+p+2)} \approx 0.0037, \\ \xi_1 = \left(1 + \frac{\sigma}{|B|}\right)\nu_1 + \frac{\sigma}{|B|}\nu_3 \approx 1.5841, \\ \xi_2 = \left(1 + \frac{\sigma}{|B|}\right)\nu_2 + \frac{\sigma}{|B|}\nu_4 \approx 0.7239. \end{cases}$$

From (5.8), one can write

$$\begin{aligned} g(\varsigma, \varpi_{\varsigma}, P_2(\varsigma), Q_2(\varsigma)) &= \frac{(1+e^{-\varsigma})\, e^{-\nu\varsigma}}{(34+e^{\varsigma})}\frac{\left|\varpi_{\varsigma}\right|}{1+\left|\varpi_{\varsigma}\right|} + \frac{1}{15}P_2\varpi(\varsigma) + \frac{1}{35}Q_2\varpi(\varsigma), \\ h(\varsigma, \varpi_{\varsigma}, P_1(\varsigma), Q_1(\varsigma)) &= \frac{e^{-\nu\varsigma}}{20}\frac{\varpi_{\varsigma}^2}{1+\varpi_{\varsigma}^2} + \frac{1}{20}P_1\varpi(\varsigma) + \frac{1}{25}Q_1\varpi(\varsigma), \end{aligned}$$

where

$$\begin{aligned} P_2\varpi(\varsigma) &= \int_0^{\varsigma} \frac{e^{-\nu\varsigma}}{6} \ln(1+\varpi_{\varsigma})\, d\vartheta, \\ Q_2\varpi(\varsigma) &= \int_0^{\sigma} \frac{e^{-\nu\varsigma}}{4}\frac{\tan^{-1}(\varpi_{\varsigma})}{1+\tan^{-1}(\varpi_{\varsigma})} d\vartheta, \\ P_1\varpi(\varsigma) &= \int_0^{\varsigma} e^{-\nu\varsigma}\cos(\frac{\varpi_{\vartheta}}{5})d\vartheta, \end{aligned}$$

$$Q_1\varpi(\varsigma) = \int_0^{\sigma} e^{-\nu\varsigma}\sin(\frac{\varpi_\vartheta}{6})d\vartheta.$$

Now, for $\varpi_\varsigma, \varrho_\varsigma \in H_\nu$, we have

$$\begin{aligned} |P_2(\varsigma,\vartheta,\varpi_\vartheta) - P_2(\varsigma,\vartheta,\varrho_\vartheta)| &= \left|\frac{e^{-\nu\varsigma}}{6}\ln(1+\varpi_\varsigma) - \frac{e^{-\nu\varsigma}}{6}\ln(1+\varrho_\varsigma)\right| \\ &\leq \frac{1}{6}\|\varpi-\varrho\|_\nu, \end{aligned} \tag{5.9}$$

$$\begin{aligned} |Q_2(\varsigma,\vartheta,\varpi_\vartheta) - Q_2(\varsigma,\vartheta,\varrho_\vartheta)| &= \left|\frac{e^{-\nu\varsigma}}{4}\frac{\tan^{-1}(\varpi_\varsigma)}{1+\tan^{-1}(\varpi_\varsigma)}d\vartheta - \frac{e^{-\nu\varsigma}}{4}\frac{\tan^{-1}(\varpi_\varsigma)}{1+\tan^{-1}(\varpi_\varsigma)}d\vartheta\right| \\ &\leq \frac{1}{4}\|\varpi-\varrho\|_\nu, \end{aligned} \tag{5.10}$$

$$\begin{aligned} |P_1(\varsigma,\vartheta,\varpi_\vartheta) - P_1(\varsigma,\vartheta,\varrho_\vartheta)| &= \left|e^{-\nu\varsigma}\cos(\frac{\varpi_\vartheta}{5}) - e^{-\nu\varsigma}\cos(\frac{\varrho_\vartheta}{5})\right| \\ &\leq \frac{1}{5}\|\varpi-\varrho\|_\nu, \end{aligned} \tag{5.11}$$

$$\begin{aligned} |Q_1(\varsigma,\vartheta,\varpi_\vartheta) - Q_1(\varsigma,\vartheta,\varrho_\vartheta)| &= \left|e^{-\nu\varsigma}\sin(\frac{\varpi_\vartheta}{6}) - e^{-\nu\varsigma}\sin(\frac{\varrho_\vartheta}{6})\right| \\ &\leq \frac{1}{6}\|\varpi-\varrho\|_\nu, \end{aligned} \tag{5.12}$$

$$\begin{aligned} &\left|g(\varsigma,\varpi_\varsigma,P_2\varpi(\varsigma),Q_2\varpi(\varsigma)) - g(\varsigma,\varrho_\varsigma,P_2\varrho(\varsigma),Q_2\varrho(\varsigma))\right| \\ \leq\ & \frac{(1+e^{-\varsigma})e^{-\nu\varsigma}}{(34+e^{\varsigma})}\frac{|\varpi_\varsigma-\varrho_\varsigma|}{(1+|\varpi_\varsigma|)(1+|\varrho_\varsigma|)} + \frac{1}{15}|P_2\varpi(\varsigma)-P_2\varrho(\varsigma)| + \frac{1}{35}|Q_2\varpi(\varsigma)-P_2\varrho(\varsigma)| \\ \leq\ & \frac{1}{5}\left(\|\varpi-\varrho\|_\nu + \frac{1}{3}\|\varpi-\varrho\|_\nu + \frac{1}{7}\|\varpi-\varrho\|_\nu\right), \end{aligned} \tag{5.13}$$

$$\begin{aligned} &\left|h(\varsigma,\varpi_\varsigma,P_1\varpi(\varsigma),Q_1\varpi(\varsigma)) - h(\varsigma,\varrho_\varsigma,P_1\varrho(\varsigma),Q_1\varrho(\varsigma))\right| \\ \leq\ & \frac{e^{-\nu\varsigma}}{20}\frac{\varpi_\varsigma^2}{1+\varpi_\varsigma^2} + \frac{1}{20}|P_2\varpi(\varsigma)-P_2\varrho(\varsigma)| + \frac{1}{25}|Q_2\varpi(\varsigma)-P_2\varrho(\varsigma)| \\ \leq\ & \frac{1}{4}\left(\|\varpi-\varrho\|_\nu + \frac{1}{5}\|\varpi-\varrho\|_\nu + \frac{1}{5}\|\varpi-\varrho\|_\nu\right), \end{aligned} \tag{5.14}$$

$$\begin{aligned} &|g(\varsigma,\psi,\varpi,\varrho)| \\ =\ & \left|\frac{(1+e^{-\varsigma})e^{-\nu\varsigma}}{(34+e^{\varsigma})}\frac{|\psi_\varsigma|}{1+|\psi_\varsigma|} + \frac{1}{15}\int_0^{\varsigma} e^{-\nu\varsigma}\cos(\frac{\varpi_\vartheta}{5})d\vartheta + \frac{1}{35}\int_0^{\sigma} e^{-\nu\varsigma}\sin(\frac{\varrho_\vartheta}{6})d\vartheta\right| \end{aligned}$$

$$\leq \frac{1}{35}|\psi| + \frac{1}{15}|\varpi| + \frac{1}{35}|\varrho|, \tag{5.15}$$

$$\begin{aligned} & |h(\varsigma, \psi, \varpi, \varrho)| \\ = & \left| \frac{e^{-\nu\varsigma}}{20} \frac{\psi_\varsigma^2}{1+\psi_\varsigma^2} + \frac{1}{20} \int_0^\varsigma \frac{e^{-\nu\varsigma}}{6} \ln(1+\varpi_\varsigma)\, d\vartheta + \frac{1}{25} \int_0^\sigma \frac{e^{-\nu\varsigma}}{4} \frac{\tan^{-1}(\varrho_\varsigma)}{1+\tan^{-1}(\varrho_\varsigma)} d\vartheta \right| \\ \leq & \frac{1}{20}|\psi| + \frac{1}{160}|\varpi| + \frac{1}{100}|\varrho|, \end{aligned} \tag{5.16}$$

$$|P_2(\varsigma, \vartheta, \varpi)| = \left| \frac{e^{-\nu\varsigma}}{6} \ln(1+\varpi_\varsigma) \right| \leq \frac{1}{6}|\varpi|, \tag{5.17}$$

$$|Q_2(\varsigma, \vartheta, \varpi)| = \left| \frac{e^{-\nu\varsigma}}{4} \frac{\tan^{-1}(\varpi_\varsigma)}{1+\tan^{-1}(\varpi_\varsigma)} \right| \leq \frac{1}{4}|\varpi|, \tag{5.18}$$

$$|P_1(\varsigma, \vartheta, \varpi)| = \left| e^{-\nu\varsigma} \cos(\frac{\varpi_\vartheta}{5}) \right| \leq \frac{1}{5}|\varpi|, \tag{5.19}$$

and

$$|Q_1(\varsigma, \vartheta, \varpi)| = \left| e^{-\nu\varsigma} \sin(\frac{\varpi_\vartheta}{6}) \right| \leq \frac{1}{60}|\varpi|. \tag{5.20}$$

From (5.9)–(5.20), we have $\ell_g = \ell_{P_1} = V_7(\varsigma) = \frac{1}{5}$, $\ell_h = \ell_{Q_2} = V_{10}(\varsigma) = \frac{1}{4}$, $\ell_{P_2} = \ell_{Q_1} = V_8(\varsigma) = \frac{1}{6}$, $V_1(\varsigma) = V_3(\varsigma) = \frac{1}{35}$, $V_2(\varsigma) = \frac{1}{15}$, $V_4(\varsigma) = \frac{1}{20}$, $V_5(\varsigma) = \frac{1}{160}$, $V_6(\varsigma) = \frac{1}{100}$,, $V_9(\varsigma) = \frac{1}{60}$, $N_1^* = 1$. Thus, we can write

$$S^* = \ell_g N_1^* \{\xi_1 + \xi_2(\ell_{P_2} + \ell_{Q_2})\} + \ell_h N_1^* \{\xi_1 + \xi_2(\ell_{P_1} + \ell_{Q_1})\} \approx 0.8395 < 1,$$

and

$$\ell^* = \frac{\sigma N_1^*}{|B|} \Big(\ell_g [(\nu_1 + \nu_3) + (\ell_{P_2} + \ell_{Q_2})(\nu_2 + \nu_4)] + \ell_h [(\nu_1 + \nu_3) + (\ell_{P_1} + \ell_{Q_1})(\nu_2 + \nu_4)] \Big) \approx 0.5059 < 1.$$

Hence, all the assertions of Theorems 3.4 and 4.2 are fulfilled. Therefore, the supposed problem (5.8) has a unique solution on $(-\infty, \sigma]$.

6. Conclusions and future work

The study of FFIDEs presents a formidable challenge due to the inherent complexities arising from the interplay of fractional-order derivatives, functional arguments, and integral operators. Traditional methods often fall short in addressing these equations due to the non-local nature of fractional derivatives and the intricate dependence on past states introduced by functional arguments. Overcoming these difficulties requires the development and application of sophisticated mathematical tools, including specialized FP theorems tailored for fractional settings, careful treatment of infinite delay, and the construction of appropriate function spaces that accommodate the combined effects of these operators. Furthermore, the presence of multi-term fractional integral boundary conditions adds another layer of complexity, demanding innovative techniques for handling the non-local and distributed nature of the boundary constraints. Successfully navigating these hurdles necessitates a deep understanding of fractional calculus, functional analysis, and operator theory, ultimately paving the way for a more comprehensive understanding of the dynamics governed by FFIDEs. This paper investigates the existence and uniqueness of solutions for a class of hybrid fractional-order functional and neutral functional integrodifferential equations, featuring infinite delay and multi-term fractional integral boundary conditions. A rigorous mathematical framework is developed, leveraging FP theorems, to analyze these complex equations. The LC definition of fractional derivatives is employed, facilitating a comprehensive study of nonlocal dynamics. Illustrative examples are provided to demonstrate the applicability and practical relevance of the theoretical results. Future work includes exploring more complex equations (e.g., variable-order, generalized functional arguments), investigating stability, controllability, and numerical methods, and applying these equations to real-world problems. Developing new fixed point theorems tailored for fractional functional integrodifferential equations and studying associated inverse problems are also promising research avenues. Finally, we also look forward to extending the study period outside the proposed period [2,3].

Author contributions

Manal Elzain Mohamed Abdalla: Writing–review-editing, formal analysis, funding acquisition; Hasanen A. Hammad: Writing–original draft, conceptualization, investigation, methodology. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Data availability statement

Data sharing is not applicable to the article as no data sets were generated or analyzed during the current study.

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Conflict of interest

All authors confirm that they have no conflict of interest.

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