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Research article

A modified projection and contraction method for solving a variational inequality problem in Hilbert spaces

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Abstract: In this paper, we proposed a modified projection and contraction method for solving a variational inequality problem in Hilbert spaces. In the existing projection and contraction methods for solving the variational inequality problem, the sequence $\{\beta_n\}$ has a similar computation manner, which is computed in a self-adaptive manner, but the sequence $\{\beta_n\}$ in our method is a sequence of numbers in (0,1) given in advance, which is the main difference of our method with the existing projection and contraction methods. A line search is used to deal with the unknown Lipschitz constant of the mapping. The strong convergence of the proposed method is proved under certain conditions. Finally, some numerical examples are presented to illustrate the effectiveness of our method and compare the computation results with some related methods in the literature. The numerical results show that our method has an obvious competitive advantage compared with the related methods.

Keywords: variational inequality; projection method; golden ratio; Hilbert space
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1. Introduction

Let H be a real Hilbert space, C be a nonempty, closed, and convex subset of H, and $F : H \to H$ be a nonlinear mapping. The variational inequality problem (VIP in short) is to find a point $x^* \in C$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \ \forall y \in C. \tag{1.1}$$

Denote the set of the solutions of VIP (1.1) by S. VIP has many applications in economics, optimization, mechanics, and so on; see [1, 2].

In recent decades, many methods have been proposed for solving the various classes of VIPs. Korpelevich [3] proposed the famous extragradient method and proved the weak convergence of the method under certain conditions. It is well known that the extragradient method needs compute two projections on the feasible set C at each iteration, which adds the computation load especially when C has a complex construction. To overcome the drawback, some authors proposed improved methods where only one projection is computed at each iteration, such as Tseng's method [4], the subgradient extragradient method [5], and the projection and contraction method [6]. To the content of this paper, we focus on the projection and contraction method. In [6], He proposed a projection and contraction method for solving the monotone VIP (1.1) as follows: $x_0 \in H$, and

$$\begin{cases} y_n = P_C(x_n - \tau F(x_n)), \\ d(x_n, y_n) = x_n - y_n - \tau(F(x_n) - F(y_n)), \\ x_{n+1} = x_n - \gamma\beta_n d(x_n, y_n), \ n \geq 1, \end{cases} \tag{1.2}$$

where F is a monotone and L-Lipschitz continuous mapping, $\tau \in \left(0, \frac{1}{L}\right)$, $\gamma \in (0, 2)$ and

$$\beta_n = \begin{cases} \dfrac{\langle x_n - y_n, d(x_n, y_n)\rangle}{\|d(x_n, y_n)\|^2}, & \text{if } d(x_n, y_n) \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

The author proved that under certain conditions the sequence $\{x_n\}$ generated by (1.2) converges weakly to a solution of the VIP (1.1).

Since He's result, many modified versions of the projection and contraction method (1.2) have been introduced. For example, Dong et al. [7] proposed the following inertial projection and contraction method for solving the monotone VIP (1.1): $x_0 \in H$, and

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \tau F(w_n)), \\ d(w_n, y_n) = w_n - y_n - \tau(F(w_n) - F(y_n)), \\ x_{n+1} = w_n - \gamma\beta_n d(w_n, y_n), \ n \geq 1, \end{cases} \tag{1.4}$$

where F is a monotone and L-Lipschitz continuous mapping, $\tau \in \left(0, \frac{1}{L}\right)$, $\gamma \in (0, 2)$, $\{\alpha_n\} \subset [0, \alpha]$ with $\alpha < 1$, and

$$\beta_n = \begin{cases} \dfrac{\langle w_n - y_n, d(w_n, y_n)\rangle}{\|d(w_n, y_n)\|^2}, & \text{if } d(w_n, y_n) \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1.5}$$

The authors proved that under certain conditions the sequence $\{x_n\}$ generated by (1.4) converges weakly to a solution of the VIP (1.1).

In 2023, Zhang and Chu [8] proposed a new extrapolation projection and contraction method by combining the golden ratio technique [9] for solving the VIP (1.1) as follows: $x_0, y_1 \in H$, and

$$\begin{cases} y_n = \dfrac{\phi - 1}{\phi} x_n + \dfrac{1}{\phi} y_{n-1}, \\ \bar{y}_n = P_C(y_n - \lambda_n F(y_n)), \\ x_{n+1} = y_n - \gamma\beta_n d(y_n, \bar{y}_n), \ n \geq 1, \end{cases} \tag{1.6}$$

where F is a pseudomonotone and L-Lipschitz continuous mapping, $\mu \in (0,1)$, $\phi \in (1,\infty)$, $\gamma \in (0,2)$,

$$\beta_n = \begin{cases} \dfrac{\langle y_n - \bar{y}_n, d(y_n, \bar{y}_n)\rangle}{\|d(y_n, \bar{y}_n)\|^2}, & d(y_n, \bar{y}_n) \neq 0, \\ 0, & \|d(y_n, \bar{y}_n)\| = 0 \end{cases} \tag{1.7}$$

with

$$d(y_n, \bar{y}_n) = y_n - \bar{y}_n - \lambda_n(F(y_n) - F(\bar{y}_n))$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{ \dfrac{\mu\|y_n - \bar{y}_n\|}{\|F(y_n) - F(\bar{y}_n)\|}, \xi_n\lambda_n + \tau_n \right\}, & F(y_n) \neq F(\bar{y}_n), \\ \xi_n\lambda_n + \tau_n, & \text{otherwise}, \end{cases} \tag{1.8}$$

where $\{\xi_n\} \subset [1,\infty)$ with $\sum_{n=1}^{\infty}(\xi_n - 1) < \infty$ and $\{\tau_n\} \subset (0,\infty)$ with $\sum_{n=1}^{\infty} \tau_n < \infty$. The authors proved that under certain conditions, the sequence $\{x_n\}$ generated by (1.6) converges weakly to a solution of the VIP (1.1). Note the sequence $\{\lambda_n\}$ in (1.8) is permitted to be increasing. The similar definition with λ_n in (1.8) can be found in [10, 11].

On the more projection and contraction methods and golden ratio methods for solving VIPs, the interested readers may refer to [12–14].

In (1.3), (1.5), and (1.7), the sequence $\{\beta_n\}$ is computed in a self-adaptive manner involving $d(\cdot,\cdot)$ and the constant γ is chosen from (0,2). In fact, besides the projection and contraction methods mentioned above, in almost all versions of projection and contraction methods, the sequence $\{\beta_n\}$ is computed in a similar self-adaptive manner. A natural question is if the range of γ can be relaxed or omitted and $\{\beta_n\}$ can be computed by other mains or is replaced by a sequence of numbers that is given in advance at the initial part of the proposed method. It seems that no authors consider the question. In addition, find that y_n in (1.6) can be written as $y_n = \left(1 - \frac{1}{\phi}\right) x_n + \frac{1}{\phi} y_{n-1}$, which is a convex combination of x_n and y_{n-1}. Replacing ϕ with a sequence $\{\phi_n\}$ in $(1,\infty)$, y_n will be $\left(1 - \frac{1}{\phi_n}\right) x_n + \frac{1}{\phi_n} y_{n-1}$. In this case, if $\phi_n = \phi$, then y_n is reduced to y_n in (1.6) and so such that y_n in (1.6) is the special case. Based on the above idea, in this paper, we propose a modified version of the method (1.6) where the constant γ is omitted and the sequence $\{\beta_n\}$ replaced by a sequence of numbers in (0,1). The main difference of our method with (1.6) is as follows:

(1) the constant ϕ in (1.6) is replaced by a sequence $\{\phi_n\}$;

(2) the constant γ in (0,2) is not needed, and the sequence $\{\beta_n\}$ in (1.7) is replaced by a sequence of numbers in (0,1), which is the main novelty of our method.

We prove that under certain conditions the proposed method converges strongly to a solution of the VIP (1.1). The numerical examples are presented to illustrate the effectiveness of our method. Although our method is a slight modification of the method (1.6), the numerical results show that our method has a faster convergence rate than the method (1.6) and other related methods.

2. Preliminaries

In this section, let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. Denote the weak convergence by the symbol $\rightharpoonup$. For any $x, y \in H$, it holds that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle. \tag{2.1}$$

Definition 2.1. *[15]* A mapping $F : H \to H$ is said to be
(i) Pseudomonotone on C if

$$\forall x, y \in C, \langle F(x), y - x\rangle \geq 0 \Rightarrow \langle F(y), y - x\rangle \geq 0;$$

(ii) Pseudomonotone with respect to a subset $B \subset C$ on C if

$$\forall x \in B, y \in C, \langle F(x), y - x\rangle \geq 0 \Rightarrow \langle F(y), y - x\rangle \geq 0;$$

(iii) L-Lipschitz continuous on H if there exists a constant $L > 0$ such that

$$\forall x, y \in H, \|F(x) - F(y)\| \leq L\|x - y\|;$$

(iv) Sequentially weakly continuous if for each sequence $\{x_n\}$, one has

$$x_n \rightharpoonup x \Rightarrow F(x_n) \rightharpoonup F(x).$$

Let C be a nonempty closed convex subset of H. For any $x \in H$, there exists a unique $z \in C$ such that

$$z = \min_{y \in C} \|y - x\|.$$

Denote the element z by $P_C(x)$. The mapping $P_C : H \to C$ is called the metric projection from H onto C.
Lemma 2.1. *[15] P_C has the following properties:*

$$z = P_C(x) \Leftrightarrow \langle x - z, z - y\rangle \geq 0, \ \forall y \in C.$$

Lemma 2.2. *[16] Assume that* $\{a_n\} \subset [0, \infty)$, $\{b_n\} \subset (0, 1)$, *and* $\{c_n\} \subset (0, \infty)$ *satisfy the following*

$$a_{n+1} \leq (1 - b_n)a_n + c_n, \quad \forall n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} b_n = \infty$ *and* $\limsup_{n\to\infty} \frac{c_n}{a_n} \leq 0$ *hold, then* $\lim_{n\to+\infty} a_n = 0$.
Lemma 2.3. *[17] Let $\{a_n\}$ be a sequence of non-negative real numbers such that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exists a non-decreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number, $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

3. Main result

In this section, let C be a nonempty, closed and convex subset of a real Hilbert space H and $F : H \to H$ be a mapping. Considering the following conditions:
(A1) $S \neq \emptyset$.
(A2) $\langle F(y), y - z\rangle \geq 0$ for all $z \in S$ and $y \in C$.
(A3) F is L-Lipschitz continuous.

(A4) For any sequence $\{u_n\} \subset C$ with $u_n \rightharpoonup u$, it has $\langle F(u), u - y\rangle \le \limsup_{n\to\infty}\langle F(u_n), u_n - y\rangle$ for all $y \in C$.

Remark 3.1. Condition (A2), used for example in [18], is weaker than the pseudomonotonicity assumption. In fact, from Conditions (A1) and (A2), it follows that F is pseudomonotone with respect to S on C. In addition, it needs to be noted that Condition (A4) is not related to the condition that F is sequentially weakly continuous on C.

For solving the VIP (1.1), we propose the projection and contraction method as follows.

Algorithm 3.1

Initialization. Choose the initial points $y_0, x_1 \in H$, the parameters $\tau \in (0,1), \mu \in (0,\infty)$, $\gamma \in (0,\infty)$, the sequences $\{\phi_n\} \subset [\phi,\infty))$ with $\phi > 1$, $\{\beta_n\} \subset [\beta,\infty)$ with $\beta > 0$ and $\{\psi_n\}_{n=1}^{\infty} \subset (0,1)$ satisfy

$$\lim_{n\to\infty} \psi_n = 0 \text{ and } \sum_{n=1}^{\infty} \psi_n = \infty.$$

Set $n = 1$.

Step 1. Compute $w_n = (1-\psi_n)y_n$ with

$$y_n = \frac{\phi_n - 1}{\phi_n} x_n + \frac{1}{\phi_n} y_{n-1}.$$

Step 2. Compute

$$\bar{y}_n = P_C(w_n - \lambda_n F(w_n)),$$

where λ_n is chosen to be the largest $\lambda \in \{\gamma\tau^l : l = 0,1,2,\cdots\}$ such that

$$\lambda\|F(w_n) - F(\bar{y}_n)\| \le \mu\|w_n - \bar{y}_n\|. \tag{3.1}$$

If $w_n = \bar{y}_n$ or $F(\bar{y}_n) = 0$, then stop and $\bar{y}_n \in S$. Otherwise, go to Step 3.

Step 3. Compute

$$x_{n+1} = w_n - \beta_n d(w_n, \bar{y}_n),$$

where

$$d(w_n, \bar{y}_n) = w_n - \bar{y}_n - \lambda_n(F(w_n) - F(\bar{y}_n)).$$

Step 4. Set $n = n + 1$ and go to Step 1.

Remark 3.2. In Algorithm 3.1, the definition of w_n has the same manner as the ones in [19–22]. Clearly, if $\phi_n \equiv \phi$ with $\phi > 1$, then y_n is reduced to the one in (1.4). In particular, if $\phi_n = \frac{1+\sqrt{5}}{2}$, then ϕ_n is the golden ratio. In the existing projection and contraction methods, $\{\beta_n\}$ is computed in a self-adaptive manner like (1.3), (1.5), and (1.7) and its value strictly depends on the result, which is computed by the self-adaptive formula. However, in our method, $\{\beta_n\}$ is a sequence of numbers in (0,1) given in advance. Since there is no other restriction imposed on $\{\beta_n\}$, we have the greater freedom to choose or adjust $\{\beta_n\}$ in advance. It is the main difference of our method with the existing projection and contraction methods.

The following remark shows that the stopping criterion in Step 2 can work well.

Remark 3.3. By the definition of $\bar{y}_n$ and Lemma 2.1, we have

$$\langle \bar{y}_n - (w_n - \lambda_n F(w_n)), y - \bar{y}_n \rangle \geq 0, \ \forall y \in C.$$

If $\bar{y}_n = w_n$ for some $n \in \mathbb{N}$, then it has

$$\lambda_n \langle F(\bar{y}_n), y - \bar{y}_n \rangle \geq 0, \ \forall y \in C.$$

Since $\lambda_n > 0$, we have

$$\langle F(\bar{y}_n), y - \bar{y}_n \rangle \geq 0, \ \forall y \in C.$$

It follows that $\bar{y}_n \in S$. In addition, it is clear that $F(\bar{y}_n) = 0$ leads to $\bar{y}_n \in S$.

In the rest of this section, for showing the convergence of Algorithm 3.1, we assume that $\{x_n\}$, $\{w_n\}$, $\{y_n\}$, and $\{\bar{y}_n\}$ are infinite sequences.

Lemma 3.1. *[23] Assume that* (A3) *holds. Then the Armijo-line search rule* (3.1) *is well defined, and* $\min\left\{\gamma, \frac{\mu\tau}{L}\right\} \leq \lambda_n \leq \gamma$ *for all* $n \in \mathbb{N}$.

Lemma 3.2. *Assume that* (A1)–(A3) *hold. Then the sequences* $\{x_n\}$, $\{y_n\}$, $\{\bar{y}_n\}$ *and* $\{w_n\}$ *are bounded.*

Proof. For each $z \in S$ and $n \in \mathbb{N}$, by the definition of $\bar{y}_n$ and Lemma 2.1, it follows that

$$\langle \bar{y}_n - z, w_n - \bar{y}_n - \lambda_n F(w_n) \rangle \geq 0. \tag{3.2}$$

By (A2) it holds that

$$\langle \bar{y}_n - z, F(\bar{y}_n) \rangle \geq 0,$$

which, together with $\lambda_n > 0$, leads to that

$$\lambda_n \langle \bar{y}_n - z, F(\bar{y}_n) \rangle \geq 0. \tag{3.3}$$

Combining (3.2) with (3.3), we have

$$\langle \bar{y}_n - z, d(w_n, \bar{y}_n) \rangle = \langle \bar{y}_n - z, w_n - \bar{y}_n - \lambda_n (F(w_n) - F(\bar{y}_n)) \rangle \geq 0.$$

So

$$\langle w_n - z, d(w_n, \bar{y}_n) \rangle \geq \langle w_n - \bar{y}_n, d(w_n, \bar{y}_n) \rangle.$$

Multiplying β_n on both sides of the above inequality, we get

$$\beta_n \langle w_n - z, d(w_n, \bar{y}_n) \rangle \geq \beta_n \langle w_n - \bar{y}_n, d(w_n, \bar{y}_n) \rangle. \tag{3.4}$$

Substituting (3.4) into

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|w_n - z - \beta_n d(w_n, \bar{y}_n)\|^2 \\ &= \|w_n - z\|^2 - 2\beta_n \langle w_n - z, d(w_n, \bar{y}_n) \rangle + \beta_n^2 \|d(w_n, \bar{y}_n)\|^2, \end{aligned}$$

we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \|w_n - z\|^2 - 2\beta_n \langle w_n - \bar{y}_n, d(w_n, \bar{y}_n) \rangle + \beta_n^2 \|d(w_n, \bar{y}_n)\|^2 \\ &\leq \|w_n - z\|^2 - 2\beta_n \langle w_n - \bar{y}_n, d(w_n, \bar{y}_n) \rangle + \beta_n \|d(w_n, \bar{y}_n)\|^2 \\ &= \|w_n - z\|^2 - \beta_n \|w_n - \bar{y}_n\|^2 + \beta_n \lambda_n^2 \|F(w_n) - F(\bar{y}_n)\|^2. \end{aligned}$$

Furthermore, by (3.1) we get

$$\begin{aligned} \|x_{n+1}-z\|^2 &\le \|w_n-z\|^2-\beta_n\|w_n-\bar{y}_n\|^2+\beta_n\mu^2\|w_n-\bar{y}_n\|^2 \\ &= \|w_n-z\|^2-\beta_n(1-\mu^2)\|w_n-\bar{y}_n\|^2. \end{aligned} \tag{3.5}$$

On the other hand, by the definition of y_{n+1} we have

$$\begin{aligned} \|x_{n+1}-z\|^2 &= \left\|\frac{\phi_{n+1}}{\phi_{n+1}-1}y_{n+1}-\frac{1}{\phi_{n+1}-1}y_n-z\right\|^2 \\ &= \frac{\phi_{n+1}}{\phi_{n+1}-1}\|y_{n+1}-z\|^2-\frac{1}{\phi_{n+1}-1}\|y_n-z\|^2+\frac{\phi_{n+1}}{(\phi_{n+1}-1)^2}\|y_{n+1}-y_n\|^2 \\ &= \frac{\phi_{n+1}}{\phi_{n+1}-1}\|y_{n+1}-z\|^2-\frac{1}{\phi_{n+1}-1}\|y_n-z\|^2+\frac{1}{\phi_{n+1}}\|x_{n+1}-y_n\|^2. \end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.5), we obtain

$$\begin{aligned} &\frac{\phi_{n+1}}{\phi_{n+1}-1}\|y_{n+1}-z\|^2-\frac{1}{\phi_{n+1}-1}\|y_n-z\|^2+\frac{1}{\phi_{n+1}}\|x_{n+1}-y_n\|^2 \\ &\quad\le \|w_n-z\|^2-\beta_n(1-\mu^2)\|w_n-\bar{y}_n\|^2 \\ &\quad= \|(1-\psi_n)(y_n-z)+\psi_n(-z)\|^2-\beta_n(1-\mu^2)\|w_n-\bar{y}_n\|^2 \\ &\quad\le (1-\psi_n)\|y_n-z\|^2+\psi_n\|z\|^2-\beta_n(1-\mu^2)\|w_n-\bar{y}_n\|^2. \end{aligned} \tag{3.7}$$

It follows that

$$\begin{aligned} \|y_{n+1}-z\|^2 &\le \left[1-\psi_n\left(1-\frac{1}{\phi_{n+1}}\right)\right]\|y_n-z\|^2+\psi_n\left(1-\frac{1}{\phi_{n+1}}\right)\|z\|^2 \\ &\quad -\beta_n\left(1-\mu^2\right)\left(1-\frac{1}{\phi_{n+1}}\right)\|w_n-\bar{y}_n\|^2 \\ &\le \left[1-\psi_n\left(1-\frac{1}{\phi_{n+1}}\right)\right]\|y_n-z\|^2+\psi_n\left(1-\frac{1}{\phi_{n+1}}\right)\|z\|^2 \\ &\le \max\{\|y_n-z\|^2,\|z\|^2\} \\ &\le \ldots \le \max\{\|y_1-z\|^2,\|z\|^2\},\ \forall n\in\mathbb{N}. \end{aligned} \tag{3.8}$$

Hence $\{y_n\}$ is bounded. The boundedness of $\{w_n\}$ is from $\|w_n\|=(1-\psi_n)\|y_n\|\le\|y_n\|$, which implies that $\{P_C w_n\}$ is also bounded. In addition, $\{F(w_n)\}$ is bounded because of (A3). By the definition of $\bar{y}_n$ we have

$$\begin{aligned} \|\bar{y}_n\| &\le \|\bar{y}_n-P_C w_n\|+\|P_C w_n\| = \|P_C(w_n-\lambda_n F(w_n))-P_C w_n\|+\|P_C w_n\| \\ &\le \lambda_n\|F(w_n)\|+\|P_C w_n\|,\ \forall n\in\mathbb{N}. \end{aligned}$$

From Lemma 3.1 it follows that $\{\bar{y}_n\}$ is bounded. Finally, by (3.5) and the boundedness of $\{w_n\}$, we obtain the boundedness of $\{x_n\}$. The proof is complete.

In the following lemma, we denote the set of weak cluster points of $\{w_n\}$ by $\omega_w(w_n)$.

Lemma 3.3. *Assume that* (A1)–(A3) *hold. If* $\lim_{n\to\infty}\|w_n-\bar{y}_n\|=0$, *then* $\omega_w(w_n)\subset S$.

Proof. Since $\{w_n\}$ is bounded, $\omega_w(w_n)\ne\emptyset$. Choose $\bar{w}\in\omega_w(w_n)$ arbitrarily and assume that w_{n_k} is a subsequence of $\{w_n\}$ such that $w_{n_k}\rightharpoonup\bar{w}$ as $k\to\infty$. From $\lim_{k\to\infty}\|w_{n_k}-\bar{y}_{n_k}\|=0$ it follows that $\bar{y}_{n_k}\rightharpoonup\bar{w}$ and so $\bar{w}\in C$. For each $y\in C$, from Lemma 2.1 and the definition of $\bar{y}_n$ we have

$$\langle\bar{y}_{n_k}-(w_{n_k}-\lambda_{n_k}F(w_{n_k})),\bar{y}_{n_k}-y\rangle\le 0,$$

or equivalently,

$$\langle F(w_{n_k}), w_{n_k} - y\rangle \le \langle F(w_{n_k}), w_{n_k} - \bar{y}_{n_k}\rangle + \frac{1}{\lambda_{n_k}}\langle w_{n_k} - \bar{y}_{n_k}, \bar{y}_{n_k} - y\rangle. \tag{3.9}$$

Taking $\limsup_{k\to\infty}$ in (3.9), by $\lim_{k\to\infty}\|w_{n_k} - \bar{y}_{n_k}\| = 0$ and Lemma 3.1, we get

$$\limsup_{k\to\infty}\langle F(w_{n_k}), w_{n_k} - y\rangle \le 0. \tag{3.10}$$

Since F is Lipschitz continuous and $\lim_{k\to\infty}\|w_{n_k} - \bar{y}_{n_k}\| = 0$, we have

$$\lim_{k\to\infty}\|F(w_{n_k}) - F(\bar{y}_{n_k})\| = 0. \tag{3.11}$$

Note that

$$\langle F(\bar{y}_{n_k}), \bar{y}_{n_k} - y\rangle = \langle F(\bar{y}_{n_k}) - F(w_{n_k}), w_{n_k} - y\rangle + \langle F(w_{n_k}), w_{n_k} - y\rangle + \langle F(\bar{y}_{n_k}), \bar{y}_{n_k} - w_{n_k}\rangle. \tag{3.12}$$

Taking $\limsup_{k\to\infty}$ in (3.12) and using (3.11) and (A4) we get

$$\langle F(\bar{w}), \bar{w} - y\rangle \le 0,$$

which together with $\bar{w} \in C$ and the arbitrariness of $y \in C$ leads to $\bar{w} \in S$. The proof is complete.

In the position we give the main result on Algorithm 3.1 as follows:

Theorem 3.1. *Assume that* (A1)–(A4) *hold. The sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *converges strongly to* $x^* = P_S(0)$.

Proof. For each $n \in \mathbb{N}$, from the definition of $\{w_n\}$ and (2.1) it follows that

$$\begin{aligned} \|w_n - x^*\|^2 &= \|(1-\psi_n)y_n - x^*\|^2 \\ &= \|(1-\psi_n)(y_n - x^*) + \psi_n(-x^*)\|^2 \\ &\le (1-\psi_n)\|y_n - x^*\|^2 + 2\psi_n\langle -x^*, w_n - x^*\rangle. \end{aligned} \tag{3.13}$$

Replacing z in (3.5) and (3.7), and using (3.13), we get

$$\begin{aligned} &\frac{\phi_{n+1}}{\phi_{n+1}-1}\|y_{n+1} - x^*\|^2 - \frac{1}{\phi_{n+1}-1}\|y_n - x^*\|^2 + \frac{1}{\phi_{n+1}}\|x_{n+1} - y_n\|^2 \\ &= \|x_{n+1} - x^*\|^2 \le \|w_n - x^*\|^2 \\ &\le (1-\psi_n)\|y_n - x^*\|^2 + 2\psi_n\langle -x^*, w_n - x^*\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \|y_{n+1} - x^*\|^2 &\le \left[1 - \psi_n\left(1 - \frac{1}{\phi_{n+1}}\right)\right]\|y_n - x^*\|^2 - \frac{\phi_{n+1}-1}{\phi_{n+1}^2}\|x_{n+1} - y_n\|^2 \\ &\quad + 2\psi_n\left(1 - \frac{1}{\phi_{n+1}}\right)\langle -x^*, w_n - x^*\rangle \\ &\le \left[1 - \psi_n\left(1 - \frac{1}{\phi_{n+1}}\right)\right]\|y_n - x^*\|^2 + 2\psi_n\left(1 - \frac{1}{\phi_{n+1}}\right)\langle -x^*, w_n - x^*\rangle. \end{aligned} \tag{3.14}$$

Next we divide the following two cases to prove $\lim_{n\to\infty} \|y_n - x^*\| = 0$.
Case 1. Suppose there exists $N \in \mathbb{N}$ such that $\{\|y_n - x^*\|^2\}$ is monotonically nonincreasing for all $n > N$. Since $\{w_n\}$ is bounded, there exists a subsequence $\{w_{n_k}\}$ such that $w_{n_k} \rightharpoonup \bar{w}$. Without loss generality we may assume that

$$\limsup_{n\to\infty} \langle -x^*, w_n - x^* \rangle = \lim_{k\to\infty} \langle -x^*, w_{n_k} - x^* \rangle = \langle -x^*, \bar{w} - x^* \rangle. \tag{3.15}$$

Hence $\{\|y_n - x^*\|^2\}$ is convergent. From (3.8) with $z = x^*$ we have

$$\begin{aligned} \beta(1-\mu^2)\left(1 - \frac{1}{\phi_{n+1}}\right)\|w_n - \bar{y}_n\|^2 &\leq \beta_n(1-\mu^2)\left(1 - \frac{1}{\phi_{n+1}}\right)\|w_n - \bar{y}_n\|^2 \\ &\leq \left[1 - \psi_n\left(1 - \frac{1}{\phi_{n+1}}\right)\right]\|y_n - x^*\|^2 - \|y_{n+1} - x^*\|^2 \\ &\quad + \psi_n\left(1 - \frac{1}{\phi_{n+1}}\right)\|x^*\|^2. \end{aligned} \tag{3.16}$$

Since $\phi_n \geq \phi > 1$ and $\lim_{n\to\infty} \psi_n = 0$, letting $n \to \infty$ in (3.16), we get

$$\lim_{n\to\infty} \|w_n - \bar{y}_n\| = 0. \tag{3.17}$$

By Lemma 3.3 and (3.17) we have $\bar{w} \in S$. From Lemma 2.1 it follows that

$$\langle -x^*, \bar{w} - x^* \rangle \leq 0,$$

which together with (3.15) leads that

$$\limsup_{n\to\infty} \langle -x^*, w_n - x^* \rangle \leq 0. \tag{3.18}$$

By the hypothesis on $\{\phi_n\}$ and $\{\psi_n\}$, we have

$$\sum_{n=1}^{\infty} \psi_n \left(1 - \frac{1}{\phi_n}\right) = \infty. \tag{3.19}$$

Applying Lemma 2.2 with $a_n = \|y_n - x^*\|^2$, $b_n = \psi_n\left(1 - \frac{1}{\phi_{n+1}}\right)$ and $c_n = 2\psi_n\left(1 - \frac{1}{\phi_{n+1}}\right)\langle -x^*, w_n - x^* \rangle$ to (3.17) and using (3.18) and (3.19), we obtain $\lim_{n\to\infty} \|y_n - x^*\| = 0$.
Case 2. Suppose that there exists a subsequence $\{m_k\} \subset \mathbb{N}$ with $m_k \to \infty$ such that

$$\|y_{m_k} - x^*\| \leq \|y_{m_k+1} - x^*\|, \ \forall k \in \mathbb{N}. \tag{3.20}$$

From Lemma 2.3 it follows that

$$\|y_k - x^*\| \leq \|y_{m_k+1} - x^*\|, \ \forall k \in \mathbb{N}. \tag{3.21}$$

By replacing n in (3.14) with m_k and using (3.20) we get

$$\begin{aligned} \|y_{m_k+1} - x^*\|^2 &\leq \left[1 - \psi_{m_k}\left(1 - \frac{1}{\phi_{m_k+1}}\right)\right]\|y_{m_k} - x^*\|^2 + 2\psi_{m_k}\left(1 - \frac{1}{\phi_{m_k+1}}\right)\langle -x^*, w_{m_k} - x^* \rangle \\ &\leq \left[1 - \psi_{m_k}\left(1 - \frac{1}{\phi_{m_k+1}}\right)\right]\|y_{m_k+1} - x^*\|^2 + 2\psi_{m_k}\left(1 - \frac{1}{\phi_{m_k+1}}\right)\langle -x^*, w_{m_k} - x^* \rangle, \end{aligned}$$

which implies that

$$\|y_{m_k+1} - x^*\|^2 \le 2\langle -x^*, w_{m_k} - x^*\rangle.$$

By a similar process of showing (3.18), we can get

$$\limsup_{k\to\infty}\langle -x^*, w_{m_k} - x^*\rangle \le 0.$$

Hence

$$\limsup_{k\to\infty}\|y_{m_k+1} - x^*\|^2 \le 2\limsup_{k\to\infty}\langle -x^*, w_{m_k} - x^*\rangle \le 0. \tag{3.22}$$

Combining (3.21) and (3.22), we have

$$\limsup_{k\to\infty}\|y_k - x^*\|^2 \le \limsup_{k\to\infty}\|y_{m_k+1} - x^*\|^2 = 0.$$

It follows that $\lim_{k\to\infty}\|y_k - x^*\|^2 = 0$. By the above two cases, we obtain $\lim_{n\to\infty}\|y_n - x^*\| = 0$. Since $\lim_{n\to\infty}\psi_n = 0$, one has

$$\|w_n - x^*\| = \|(1-\psi_n)y_n - x^*\| \le \|y_n - x^*\| + \psi_n\|y_n\| \to 0 \quad \text{as} \quad n\to\infty. \tag{3.23}$$

Finally, by (3.5) and (3.23) we have

$$\|x_{n+1} - x^*\| \le \|w_n - x^*\| \to x^*, \quad \text{as } n\to\infty.$$

The proof is complete.

4. Numerical examples

In this section, we provide the numerical examples to test the convergence of Algorithm 3.1 and compare the numerical experimental results with some related methods. The codes are executed in MATLAB 2016a using a PC (Surface Pro 5) with an Intel(R) Core(TM) i5-7300U CPU running at 2.60GHz and 8.00 GB of RAM.

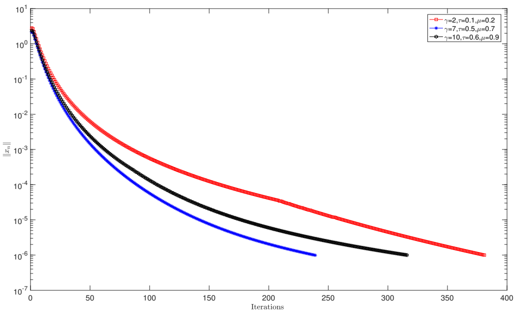
We first present the following example to test the effectiveness of Algorithm 3.1 for solving the non-monotone VIP (1.1).

Example 4.1. *[24]* Let $H = \mathbb{R}^m$, $C = [0,\pi]\times[0,\pi]\times[0,1]\times\cdots\times[0,1]$, $F : H\to H$ be a mapping defined by

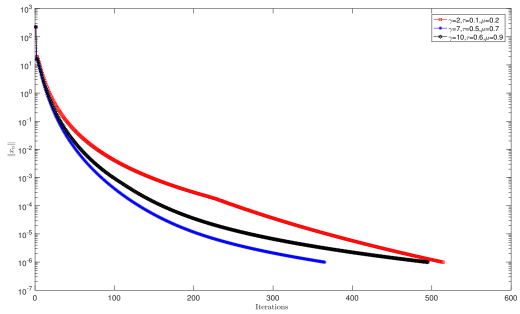
$$F(x) = (x_2 + \cos(x_2), x_1 + \sin(x_1), x_3, \cdots, x_m),\ \forall x = (x_1,\cdots,x_m)\in H.$$

It follows that $S = \{\mathbf{0}\}$, Conditions (A1)-(A4) hold and F is not pseudomonotone; see [24].

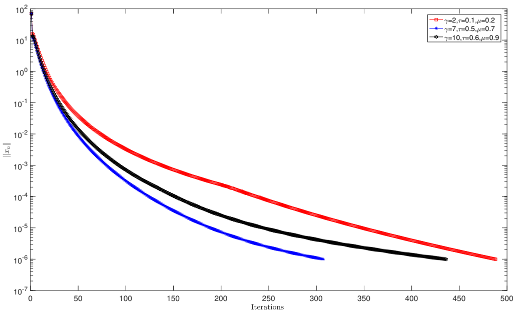
We choose $\phi_n = \frac{1+\sqrt{5}}{2} + \frac{1}{n}$, $\psi_n = \frac{10}{10+n}$, $\beta_n = \frac{110+n}{100+10n)}$ and use $\|x_n\| \le 10^{-6}$ as the stopping criterion of Algorithm 3.1. The computed results for $\{x_n\}$ by Algorithm 3.1 with the different initial points y_0, x_1, different dimension m, and different parameters γ, τ and μ are shown in Figure 1. Table 1 gives CPU time (in seconds) of Algorithm 3.1 for this example. From the curves in Figure 1 we see that the sequence $\{x_n\}$ converges to $\mathbf{0}$.



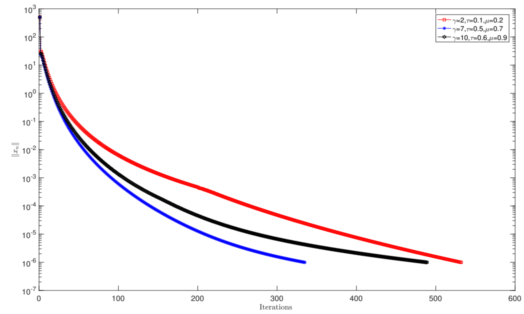
(a) $m = 1000, y_0 = x_1 = (1, \cdots, 1)^T$



(b) $m = 2000, y_0 = x_1 = (5, \cdots, 5)^T$



(c) $m = 5000, y_0 = (1, \cdots, 1)^T, x_1 = (5, \cdots, 5)^T$



(d) $m = 10000, y_0 = (5, \cdots, 5)^T, x_1 = (1, \cdots, 1)^T$

Figure 1. Numerical results of Algorithm 3.1 for Example 4.1.

Table 1. CPU time of Algorithm 3.1 for Example 4.1.

(γ, μ, τ)	$m = 1000$	$m = 2000$	$m = 5000$	$m = 10000$
$(2, 0.1, 0.2)$	0.0546	0.0718	0.2739	0.3488
$(7, 0.5, 0.7)$	0.0853	0.0743	0.2751	0.2165
$(10, 0.6, 0.9)$	0.0989	0.0758	0.2687	0.3210

Next we use the following example to compare the numerical results by our Algorithm 3.1 and the algorithm (1.2) (denoted by Algorithm H), the algorithm (1.6) (denoted by Algorithm Z) and Algorithm 3.1 in [13] (denoted by Algorithm T).

Example 4.2. *[19, 25]* Let $F : \mathbb{R}^m \to \mathbb{R}^m$ be a mapping defined by $F(x) = Mx + q$ with

$$M = NN^T + B + D,$$

where N is an $m \times m$ matrix, B is an $m \times m$ skew-symmetric matrix, and D is an $m \times m$ diagonal matrix with its diagonal entries being nonnegative (hence M is positive semidefinite). The feasible set is

$$C = \{x \in \mathbb{R}^m : Ex \leq f\},$$

where E is a $k \times m$ matrix and $f \in \mathbb{R}^k$ is a vector. It follows that F is monotone and Lipschitz continuous with the constant $L = \|M\|$. All entries of N, B are generated randomly and uniformly in [-2,2], those of E and f are generated randomly and uniformly in [0,1], those of D are generated randomly and uniformly in (0,2). Conditions (A1)–(A4) are satisfied.

We choose the initial points $y_0 = x_1 = (1, \cdots, 1)^T$ for Algorithm 3.1 and Algorithm Z, and $x_0 = (1, \cdots, 1)^T$ for Algorithm H, and $x_0 = x_1 = (1, \cdots, 1)^T$ for Algorithm T. The parameters and control sequences for Algorithm 3.1, Algorithm H and Algorithm Z are as follows:

Algorithm 3.1: $\gamma = 2, \tau = 0.3, \mu = 0.2, \psi_n = \frac{100}{100+n}, \phi_n = \frac{\sqrt{5}+1}{2} + \frac{1}{n}, \beta_n = \frac{1}{10} + \frac{10}{10+n}$;

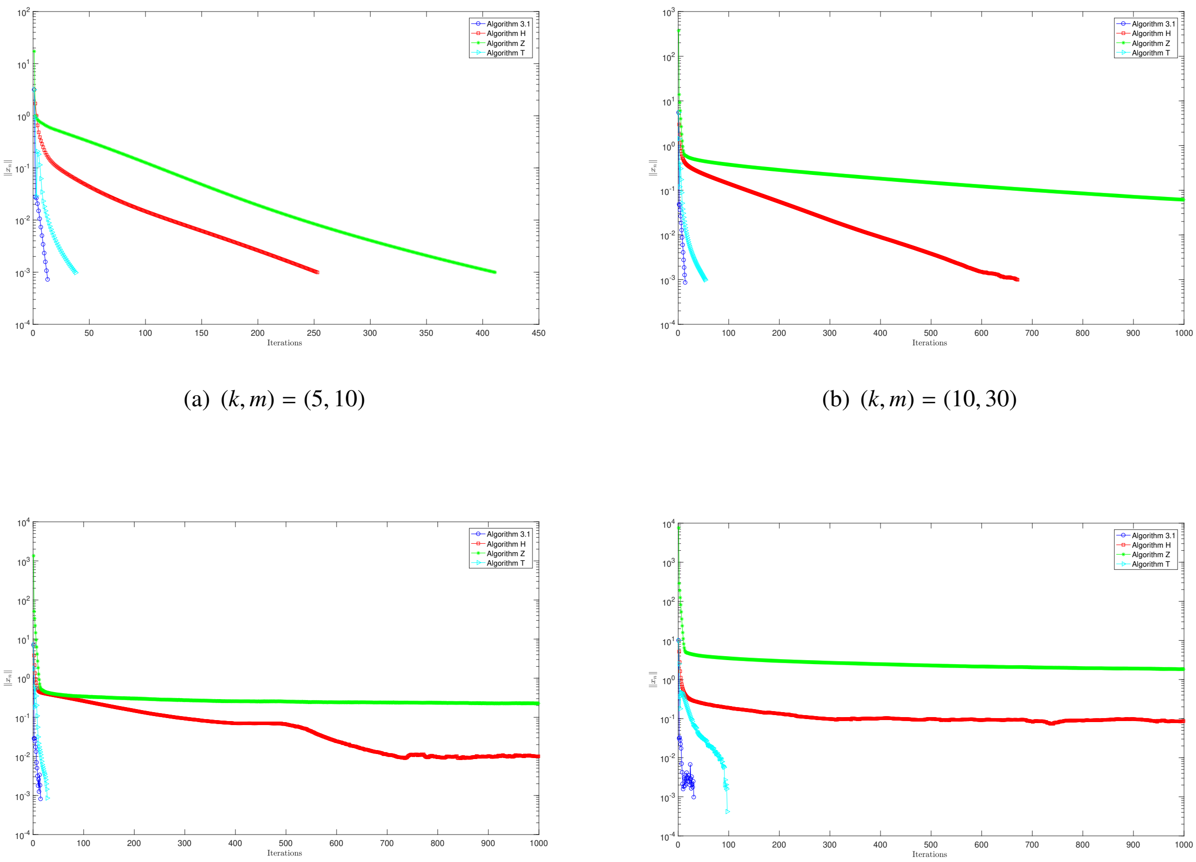
Algorithm H: $\tau = \frac{7L}{10}, \gamma = \frac{3}{2}$;

Algorithm Z: $\phi = \frac{\sqrt{5}+1}{2}, \lambda_1 = \frac{1}{20}, \gamma = \frac{3}{2}, \mu = \frac{3}{5}, \xi_n = 1 + \frac{1}{n^2}, \tau_n = \frac{1}{n^2}$;

Algorithm T: $\phi = 0.6, \beta = 0.8, \gamma = 2, \delta = 0.4, l = 0.5, \tau = 1.5, \epsilon_n = \frac{100}{(1+n)^2}, \xi_n = \frac{1}{1+n}, \sigma_n = 0.5 * (1 - \xi_n)$.

We consider the following two cases for this example.

Case 1 $q = \mathbf{0}$. In this case, $S = \{\mathbf{0}\}$. We use $\|x_n\| < 10^{-3}$ or the maximum number of iterations $n = 1000$ as the common stopping criterion of these algorithms. The numerical results computed are shown in Figure 2 and CPU time (in seconds) is given in Table 2.



(a) $(k, m) = (5, 10)$

(b) $(k, m) = (10, 30)$

(c) $(k, m) = (30, 50)$

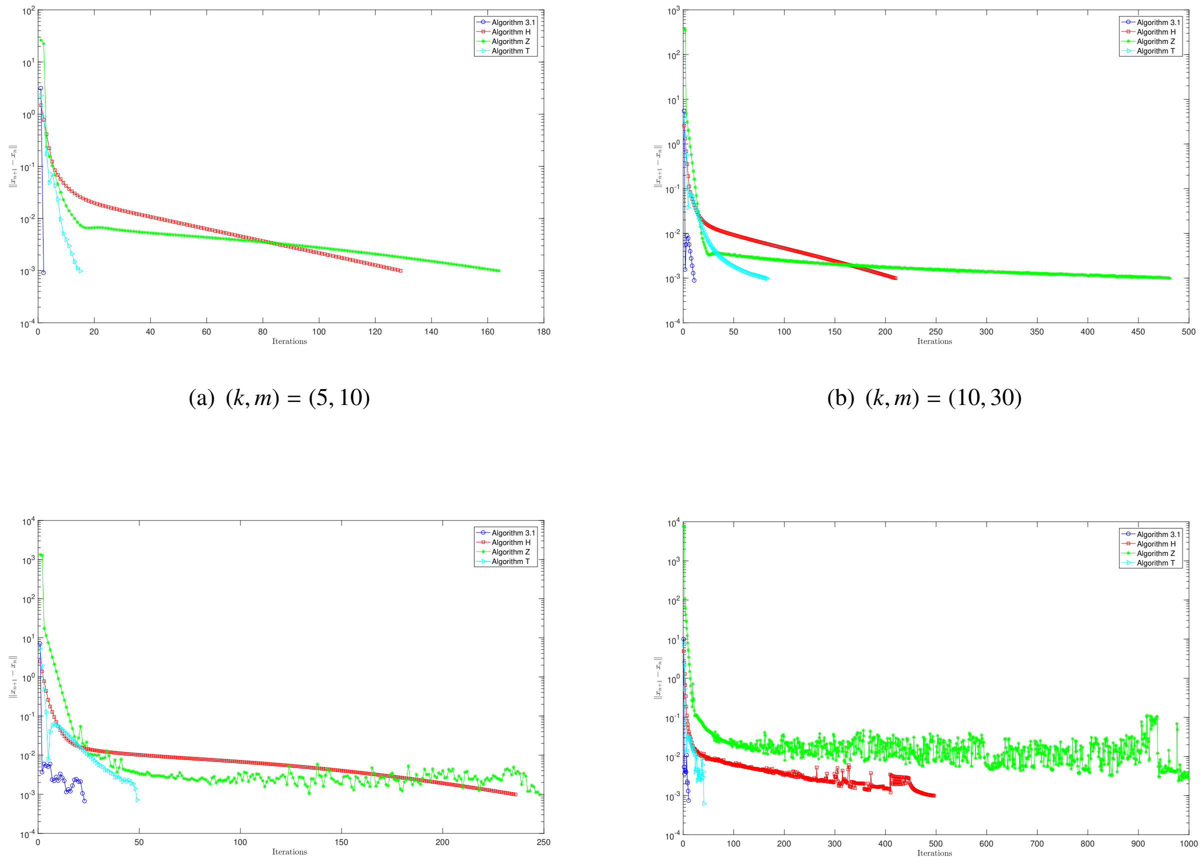
(d) $(k, m) = (50, 100)$

Figure 2. Numerical results of used algorithms for Example 4.2 with Case 1.

Table 2. CPU time of used algorithms for Example 4.2 with Case 1.

	Algorithm 3.1	Algorithm H	Algorithm Z	Algorithm T
	CPU time	CPU time	CPU time	CPU time
$(k, m) = (5, 10)$	16.1047	55.6034	88.2832	68.8139
$(k, m) = (10, 30)$	30.9358	197.4414	297.8164	189.2787
$(k, m) = (30, 50)$	33.6395	250.7686	236.5039	78.8249
$(k, m) = (50, 100)$	66.9299	210.3711	182.2526	274.8092

Case 2. All entries of q are generated randomly and uniformly in (0,2). In this case the solution of VIP (1.1) is not known. We use $\|x_n - x_{n-1}\| < 10^{-3}$ as the common stopping criterion of these algorithms. The numerical results computed are shown in Figure 3 and CPU time (in seconds) is given in Table 3.



(a) $(k, m) = (5, 10)$

(b) $(k, m) = (10, 30)$

(c) $(k, m) = (30, 50)$

(d) $(k, m) = (50, 100)$

Figure 3. Numerical results of used algorithms for Example 4.2 with Case 2.

Table 3. CPU time of used algorithms for Example 4.2 with Case 2.

	Algorithm 3.1	Algorithm H	Algorithm Z	Algorithm T
	CPU time	CPU time	CPU time	CPU time
$(k, m) = (5, 10)$	2.2174	21.4457	34.7764	23.5107
$(k, m) = (10, 30)$	21.7007	50.0565	141.1718	136.3548
$(k, m) = (30, 50)$	49.0367	59.7578	58.7833	58.1909
$(k, m) = (50, 100)$	36.1365	108.7671	188.5018	123.2509

From Figures 2 and 3, Tables 2 and 3, we see that our Algorithm 3.1 needs the lesser iteration numbers and CPU time than the other three algorithms, which shows that our method has the obvious competitive advantage compared with the algorithms for this numerical example.

5. Conclusions

In this paper, we proposed a modified projection and contraction method for solving a nonmonotone variational inequality problem in a Hilbert space. Without prior knowledge of the Lipschitz constant of the mapping, we use a line search rule to update the step-size in our algorithm. The main novelty of our method lies in that we use a sequence of numbers in (0,1) to replace the sequence $\{\beta_n\}$, which is computed in a self-adaptive manner involving $d(\cdot, \cdot)$ in other projection and contraction methods. Under certain conditions, we prove the strong convergence of the proposed method. The numerical examples are given to illustrate the effectiveness of our method. The numerical results show that our method has the obvious competitive advantage compared with the other related algorithms.

Author contributions

Limei Xue : Methodology, computing numerical examples, writing-original draft; Jianmin Song: Formal analysis, writing-original draft; Shenghua Wang: Design of algorithm, proof of conclusions. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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