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Research article

A fixed point approach to predator-prey dynamics via nonlinear mixed Volterra–Fredholm integral equations in complex-valued suprametric spaces

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Abstract: This study explores the concept of complex-valued suprametric spaces, a recent and generalized framework in fixed point theory. This concept extends both complex-valued spaces and classical metric spaces. The article aims to establish common fixed point results for rational contractions governed by control functions dependent on two variables within the setting of complex-valued suprametric spaces. These findings generalize several well-known results in the field. An illustrative example is provided to validate the novelty of the key result. Furthermore, the study derives common fixed point theorems for rational contractions with control functions of a single variable under the same framework. As a practical application, the findings are applied to study the solution of a nonlinear mixed Volterra–Fredholm integral equation within the context of predator-prey dynamics.

Keywords: complex-valued suprametric spaces; common fixed point; control functions; Volterra–Fredholm integral equation; rational expressions; predator-prey dynamics
Mathematics Subject Classification: 46S40, 47H10, 54H25

1. Introduction

Fixed point (FP) theory is an extensive and multifaceted area of mathematics, primarily divided into three major branches: metric, topological, and discrete FP theory. Among these, metric FP theory plays a crucial role as a foundational framework, focusing on establishing the existence and uniqueness of fixed points for self-mappings defined on metric spaces (MSs). This theory is inherently linked to the concepts of distance and convergence, which are the core features of MSs.

The notion of an MS was first introduced by Maurice Fréchet [1] in 1906, describing a set equipped with a metric or distance function that measures the distance between any two points in the space. Over the years, this concept has been generalized and extended by various mathematicians, resulting in the development of advanced structures. Key generalizations include the partial MSs by Matthews [2], the b-MSs introduced by Czerwik [3], the rectangular MSs by Branciari [4], the cone MSs developed by Huang [5], and the supra-metric spaces (SMS) introduced by Berzig [6].

These generalizations have expanded the scope of metric FP theory, facilitating the analysis of more intricate mathematical systems and their applications. The SMS framework, proposed by Berzig [6], employs a relaxed version of the triangle inequality to prove FP results, such as the Banach contraction theorem, within this setting. Additional research in ordered vector spaces and nonlinear integral and matrix equations has also been conducted within this framework. Building upon SMSs, Berzig [7–9] later extended the concept by generalizing the triangle inequality, introducing two new spaces: generalized SMSs and b-SMSs.

On the other hand, complex numbers, introduced by the Italian mathematician Gerolamo Cardano in the 16th century while solving cubic equations, have become an integral part of mathematics and its applications. Represented in the form $z = a + bi$, where a and b are real numbers and i is the imaginary unit satisfying $i^2 = -1$, complex numbers extend the concept of real numbers and enable the solution of equations that lack real solutions, such as $x^2 + 1 = 0$. The importance of complex numbers spans various fields: in mathematics, they underpin the fundamental theorem of algebra and play a central role in analysis, geometry, and number theory; in physics, they are essential in quantum mechanics, electromagnetism, and wave theory; in engineering, they simplify computations in signal processing, electrical circuit analysis, and control systems; and in computer science, they are utilized in graphics, fractals, and algorithms for modeling and simulations. Building upon the foundation of complex numbers, the idea of complex-valued metric spaces (CVMS) was given by Azam et al. [10] to extend classical MSs by defining the distance between points as a complex number rather than a real number. A complex-valued MS consists of a set Ψ and a mapping $\mathfrak{c} : \Psi \times \Psi \to \mathbb{C}$ satisfying properties analogous to those of real-valued MSs, including non-negativity, symmetry, and a generalized triangle inequality. These spaces have proven valuable in FP theory, enabling the exploration of more general contraction mappings and providing a framework for solving advanced mathematical problems, including nonlinear equations and integral equations. Rouzkard et al. [11] extended Azam et al.'s results [10] by including a rational expression to the contractive condition. Following this, Sintunavarat et al. [12] further extended these results using control functions with one variable. Sitthikul et al. [13] advanced the theory by employing control functions with two variables in the context of CVMSs. Very recently, Panda et al. [14] merged the concepts of SMSs and CVMSs and pioneered the notion of complex-valued suprametric spaces (CVSMSs). In this way, they subsequently proved several common FP theorems for rational contractions within this framework. They applied their results to solve complex nonlinear integral equations via contractive mappings. For a more in-depth discussion, the readers may consult References [15–18].

FP theorems serve as a cornerstone in functional analysis, offering a systematic approach to validate the existence and uniqueness of solutions across a wide spectrum of mathematical equations, including integral equations. These theorems are particularly valuable in resolving complex nonlinear problems, such as mixed Volterra–Fredholm integral equations, where they furnish a rigorous methodology to demonstrate solvability and ensure a unique solution under certain conditions. The nonlinear mixed Volterra–Fredholm integral equation, in turn, acts as a flexible mathematical tool for simulating dynamic ecological systems, enabling researchers to analyze species' interactions, including predator-prey dynamics and competition. For further information, the readers are encouraged to consult

References [19, 20].

In this research article, we prove common FP (CFP) theorems for rational contractions involving control functions dependent on multiple variables within the framework of CVSMSs. The results are further extended to the case of single-variable control functions and constants. Through this approach, we derive some important FP results in SMSs, CVMSs, and CVSMSs, which emerge as special cases of our main theorem. To underscore the novelty of the main result, an illustrative example is provided. The results are practically demonstrated by solving a nonlinear mixed Volterra–Fredholm integral equation that models predator-prey dynamics.

2. Preliminaries

The concept of MS was introduced by Fréchet [1] in 1906, defined as follows:

Definition 1. *([6]) Let* $\Psi \neq \emptyset$ *and* $\mathfrak{c} : \Psi \times \Psi \to \mathbb{R}^+$ *be a function such that*

(i) $0 \leq \mathfrak{c}(\mathfrak{v}, \mathfrak{o})$ and $\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) = 0 \iff \mathfrak{v} = \mathfrak{o}$,
(ii) $\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) = \mathfrak{c}(\mathfrak{o}, \mathfrak{v})$,
(iii) $\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) \leq \mathfrak{c}(\mathfrak{v}, \varpi) + \mathfrak{c}(\varpi, \mathfrak{o})$,

for all $\mathfrak{v}, \mathfrak{o}, \varpi \in \Psi$; then $(\Psi, \mathfrak{c})$ is said to be an MS.

Berzig [6] introduced the concept of SMS in this way.

Definition 2. *([6]) Let* $\Psi \neq \emptyset$ *and* $\curlywedge \geq 0$*. Consider a function* $\mathfrak{c} : \Psi \times \Psi \to \mathbb{R}^+$ *such that*

(i) $0 \leq \mathfrak{c}(\mathfrak{v}, \mathfrak{o})$ and $\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) = 0 \iff \mathfrak{v} = \mathfrak{o}$,
(ii) $\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) = \mathfrak{c}(\mathfrak{o}, \mathfrak{v})$,
(iii) $\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) \leq \mathfrak{c}(\mathfrak{v}, \varpi) + \mathfrak{c}(\varpi, \mathfrak{o}) + \curlywedge\mathfrak{c}(\mathfrak{v}, \varpi)\mathfrak{c}(\varpi, \mathfrak{o})$,

for all $\mathfrak{v}, \mathfrak{o}, \varpi \in \Psi$; then $(\Psi, \mathfrak{c})$ is called an SMS.

Example 1. *Let* $\Psi = \{0, 1, 2\}$ *and define the distance function* $\mathfrak{c} : \Psi \times \Psi \to \mathbb{R}^+$ *as follows:*

$$\mathfrak{c}(0, 1) = \mathfrak{c}(1, 0) = 0.5,$$

$$\mathfrak{c}(0, 2) = \mathfrak{c}(2, 0) = 1,$$

$$\mathfrak{c}(1, 2) = \mathfrak{c}(2, 1) = 2,$$

and

$$\mathfrak{c}(0, 0) = \mathfrak{c}(1, 1) = \mathfrak{c}(2, 2) = 0.$$

Let's choose $\curlywedge = 1.5$. Then, $(\Psi, \mathfrak{c})$ is an SMS but not an MS because the triangle of an MS is not satisfied, that is,

$$2 = \mathfrak{c}(1, 2) > \mathfrak{c}(1, 0) + \mathfrak{c}(0, 2) = 0.5 + 1.$$

In the context of the newly introduced SMS, Berzig [6] proved the Banach contraction principle.

Theorem 1. *([6]) Let* $(\Psi, \mathfrak{c})$ *be a complete SMS and* $\mathfrak{A} : \Psi \to \Psi$*. If* $\Bbbk \in [0, 1)$ *exists such that*

$$\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{A}\mathfrak{o}) \leq \Bbbk\mathfrak{c}(\mathfrak{v}, \mathfrak{o}),$$

for all $\mathfrak{v}, \mathfrak{o} \in \Psi$, then $\mathfrak{A}$ has a unique FP.

On the other hand, Azam et al. [10] introduced the concept of a CVMS by replacing the real numbers with complex numbers in the codomain, thereby extending the classical definition of an MS to the complex number system.

Let z_1 and z_2 be complex numbers. A partial order $\precsim$ on $\mathbb{C}$ is defined by:

$$z_1 \precsim z_2 \;\Leftrightarrow\; \operatorname{Re}(z_1) \leqslant \operatorname{Re}(z_2),\; \operatorname{Im}(z_1) \leqslant \operatorname{Im}(z_2).$$

It follows that

$$z_1 \precsim z_2,$$

if one of these conditions is met:

$$\begin{aligned}
&\text{(a) } \operatorname{Re}(z_1) = \operatorname{Re}(z_2),\; \operatorname{Im}(z_1) < \operatorname{Im}(z_2),\\
&\text{(b) } \operatorname{Re}(z_1) < \operatorname{Re}(z_2),\; \operatorname{Im}(z_1) = \operatorname{Im}(z_2),\\
&\text{(c) } \operatorname{Re}(z_1) < \operatorname{Re}(z_2),\; \operatorname{Im}(z_1) < \operatorname{Im}(z_2),\\
&\text{(d) } \operatorname{Re}(z_1) = \operatorname{Re}(z_2),\; \operatorname{Im}(z_1) = \operatorname{Im}(z_2).
\end{aligned}$$

Azam et al. [10] defined the concept of CVMS as follows:

Definition 3. *([10]) Let $\Psi \neq \emptyset$. Let $\mathfrak{c} : \Psi \times \Psi \to \mathbb{C}$ be a function such that*

(i) $0 \precsim \mathfrak{c}(\mathfrak{v}, \mathfrak{o})$ and $\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) = 0 \Longleftrightarrow \mathfrak{v} = \mathfrak{o}$,
(ii) $\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) = \mathfrak{c}(\mathfrak{o}, \mathfrak{v})$,
(iii) $\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) \precsim \mathfrak{c}(\mathfrak{v}, \varpi) + \mathfrak{c}(\varpi, \mathfrak{o})$,

for all $\mathfrak{v}, \mathfrak{o}, \varpi \in \Psi$; then $(\Psi, \mathfrak{c})$ is said to be a CVMS.

Example 2. *([10]) Let $\Psi = [0, 1]$ and $\mathfrak{v}, \mathfrak{o} \in \Psi$. Define $\mathfrak{c} : \Psi \times \Psi \to \mathbb{C}$ by*

$$\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) = \begin{cases} 0, & \text{if } \mathfrak{v} = \mathfrak{o}, \\ \frac{i}{2}, & \text{if } \mathfrak{v} \neq \mathfrak{o}. \end{cases}$$

Then, $(\Psi, \mathfrak{c})$ is a CVMS.

Azam et al. [10] established the following FP theorem in the context of CVMSs.

Theorem 2. *([10]) Let $(\Psi, \mathfrak{c})$ be a complete CVMS and $\mathfrak{A}, \mathfrak{B} : \Psi \to \Psi$. If the non-negative real numbers $\mathbb{k}_1$ and $\mathbb{k}_2$ with $\mathbb{k}_1 + \mathbb{k}_2 < 1$ exist such that*

$$\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{B}\mathfrak{o}) \preceq \mathbb{k}_1 \mathfrak{c}(\mathfrak{v}, \mathfrak{o}) + \mathbb{k}_2 \frac{\mathfrak{c}(\mathfrak{v}, \mathfrak{A}\mathfrak{v})\, \mathfrak{c}(\mathfrak{o}, \mathfrak{B}\mathfrak{o})}{1 + \mathfrak{c}(\mathfrak{v}, \mathfrak{o})},$$

for all $\mathfrak{v}, \mathfrak{o} \in \Psi$, then $\mathfrak{A}$ and $\mathfrak{B}$ possess a unique CFP.

Rouzkard et al. [11] refined the contractive condition of Azam et al. [10] through the inclusion of a rational expression, subsequently demonstrating the following result.

Theorem 3. *([11]) Let $(\Psi, \mathfrak{c})$ be a complete CVMS and $\mathfrak{A}, \mathfrak{B} : \Psi \to \Psi$. If $\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_3 \in [0, 1)$ with $\mathbb{k}_1 + \mathbb{k}_2 + \mathbb{k}_3 < 1$ exist such that*

$$\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{B}\mathfrak{o}) \preceq \mathbb{k}_1 \mathfrak{c}(\mathfrak{v}, \mathfrak{o}) + \mathbb{k}_2 \frac{\mathfrak{c}(\mathfrak{v}, \mathfrak{A}\mathfrak{v})\, \mathfrak{c}(\mathfrak{o}, \mathfrak{B}\mathfrak{o})}{1 + \mathfrak{c}(\mathfrak{v}, \mathfrak{o})} + \mathbb{k}_3 \frac{\mathfrak{c}(\mathfrak{o}, \mathfrak{A}\mathfrak{v})\, \mathfrak{c}(\mathfrak{v}, \mathfrak{B}\mathfrak{o})}{1 + \mathfrak{c}(\mathfrak{v}, \mathfrak{o})},$$

for all $\mathfrak{v}, \mathfrak{o} \in \Psi$, then $\mathfrak{A}$ and $\mathfrak{B}$ possess a unique CFP.

Sintunavarat et al. [12] extended the primary finding of Azam et al. [10] by substituting the constants with single-variable control functions.

Theorem 4. *([12]) Let* $(\Psi, \mathfrak{c})$ *be a complete CVMS and* $\mathfrak{A}, \mathfrak{B} : \Psi \to \Psi$. *Assuming the existence of the functions* $\Bbbk_1, \Bbbk_2 : \Psi \to [0, 1)$ *such that*

(a) $\Bbbk_1 (\mathfrak{B}\mathfrak{A}\mathfrak{v}) \leq \Bbbk_1 (\mathfrak{v}) \; \Bbbk_2 (\mathfrak{B}\mathfrak{A}\mathfrak{v}) \leq \Bbbk_2 (\mathfrak{v})$,
(b) $\Bbbk_1 (\mathfrak{v}) + \Bbbk_2 (\mathfrak{v}) < 1$,
(c)

$$\mathfrak{c} (\mathfrak{A}\mathfrak{v}, \mathfrak{B}\mathfrak{o}) \leq \Bbbk_1 (\mathfrak{v}) \, \mathfrak{c} (\mathfrak{v}, \mathfrak{o}) + \Bbbk_2 (\mathfrak{v}) \frac{\mathfrak{c} (\mathfrak{v}, \mathfrak{A}\mathfrak{v}) \, \mathfrak{c} (\mathfrak{o}, \mathfrak{B}\mathfrak{o})}{1 + \mathfrak{c} (\mathfrak{v}, \mathfrak{o})},$$

for all $\mathfrak{v}, \mathfrak{o} \in \Psi$, then $\mathfrak{A}$ and $\mathfrak{B}$ possess a unique CFP.

Sitthikul et al. [13] presented the following result for the functions of two variables in the framework of CVMSs.

Theorem 5. *([13]) Let* $(\Psi, \mathfrak{c})$ *be a complete CVMS and* $\mathfrak{A}, \mathfrak{B} : \Psi \to \Psi$. *Assuming the existence of the functions* $\Bbbk_1, \Bbbk_2, \Bbbk_3 : \Psi \times \Psi \to [0, 1)$ *such that for all* $\mathfrak{v}, \mathfrak{o} \in \Psi$,

(a) $\Bbbk_1 (\mathfrak{B}\mathfrak{A}\mathfrak{v}, \mathfrak{o}) \leq \Bbbk_1 (\mathfrak{v}, \mathfrak{o})$ and $\Bbbk_1 (\mathfrak{v}, \mathfrak{A}\mathfrak{B}\mathfrak{o}) \leq \Bbbk_1 (\mathfrak{v}, \mathfrak{o})$,
$\Bbbk_2 (\mathfrak{B}\mathfrak{A}\mathfrak{v}, \mathfrak{o}) \leq \Bbbk_2 (\mathfrak{v}, \mathfrak{o})$ and $\Bbbk_2 (\mathfrak{v}, \mathfrak{A}\mathfrak{B}\mathfrak{o}) \leq \Bbbk_2 (\mathfrak{v}, \mathfrak{o})$,
$\Bbbk_3 (\mathfrak{B}\mathfrak{A}\mathfrak{v}, \mathfrak{o}) \leq \Bbbk_3 (\mathfrak{v}, \mathfrak{o})$ and $\Bbbk_3 (\mathfrak{v}, \mathfrak{A}\mathfrak{B}\mathfrak{o}) \leq \Bbbk_3 (\mathfrak{v}, \mathfrak{o})$,
(b) $\Bbbk_1 (\mathfrak{v}, \mathfrak{o}) + \Bbbk_2 (\mathfrak{v}, \mathfrak{o}) + \Bbbk_3 (\mathfrak{v}, \mathfrak{o}) < 1$,
(c)

$$\mathfrak{c} (\mathfrak{A}\mathfrak{v}, \mathfrak{B}\mathfrak{o}) \leq \Bbbk_1 (\mathfrak{v}, \mathfrak{o}) \, \mathfrak{c} (\mathfrak{v}, \mathfrak{o}) + \Bbbk_2 (\mathfrak{v}, \mathfrak{o}) \frac{\mathfrak{c} (\mathfrak{v}, \mathfrak{A}\mathfrak{v}) \, \mathfrak{c} (\mathfrak{o}, \mathfrak{B}\mathfrak{o})}{1 + \mathfrak{c} (\mathfrak{v}, \mathfrak{o})} + \Bbbk_3 (\mathfrak{v}, \mathfrak{o}) \frac{\mathfrak{c} (\mathfrak{o}, \mathfrak{A}\mathfrak{v}) \, \mathfrak{c} (\mathfrak{v}, \mathfrak{B}\mathfrak{o})}{1 + \mathfrak{c} (\mathfrak{v}, \mathfrak{o})},$$

for all $\mathfrak{v}, \mathfrak{o} \in \Psi$, then $\mathfrak{A}$ and $\mathfrak{B}$ possess a unique CFP.

Recently, Panda et al. [14] merged the concepts of SMSs and CVMSs and introduced the notion of CVSMS in this manner.

Definition 4. *([14]) Let* $\Psi \neq \emptyset$ *and* $\curlywedge \geq 0$. *Consider a function* $\mathfrak{c} : \Psi \times \Psi \to \mathbb{C}$ *such that*

(i) $0 \precsim \mathfrak{c}(\mathfrak{v}, \mathfrak{o})$ and $\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) = 0 \Longleftrightarrow \mathfrak{v} = \mathfrak{o}$,
(ii) $\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) = \overline{\mathfrak{c}(\mathfrak{o}, \mathfrak{v})}$,
(iii) $\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) \precsim \mathfrak{c}(\mathfrak{v}, \varpi) + \mathfrak{c}(\varpi, \mathfrak{o}) + \curlywedge \mathfrak{c}(\mathfrak{v}, \varpi)\mathfrak{c}(\varpi, \mathfrak{o})$,

for all $\mathfrak{v}, \mathfrak{o}, \varpi \in \Psi$; then $(\Psi, \mathfrak{c})$ is called a CVSMS.

Example 3. *Let* $\Psi = \mathbb{C}$ *(the set of complex numbers) and* $\curlywedge = 1$. *Define the function* $\mathfrak{c} : \Psi \times \Psi \to \mathbb{C}$ *by*

$$\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) = |\mathfrak{v} - \mathfrak{o}| \, (1 + i),$$

where $|\mathfrak{v} - \mathfrak{o}|$ is the modulus of the difference between $\mathfrak{v}$ and $\mathfrak{o}$, and i is the imaginary unit. Then, $(\Psi, \mathfrak{c})$ is a CVSMS.

Panda et al. [14] proved the following result in the setting of CVSMSs.

Theorem 6. *([14]) Let* $(\Psi, \mathfrak{c})$ *be a complete CVSMS and* $\mathfrak{A}, \mathfrak{B} : \Psi \to \Psi$. *If the non-negative real numbers* $\Bbbk_1, \Bbbk_2$ *and* $\Bbbk_3$ *with* $\Bbbk_1 + \Bbbk_2 + \Bbbk_3 < 1$ *exist such that*

$$\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{B}\mathfrak{o}) \le \Bbbk_1 \mathfrak{c}(\mathfrak{v}, \mathfrak{o}) + \Bbbk_2 \frac{\mathfrak{c}(\mathfrak{v}, \mathfrak{A}\mathfrak{v})\, \mathfrak{c}(\mathfrak{o}, \mathfrak{B}\mathfrak{o})}{1 + \mathfrak{c}(\mathfrak{v}, \mathfrak{o})} + \Bbbk_3 \frac{\mathfrak{c}(\mathfrak{o}, \mathfrak{A}\mathfrak{v})\, \mathfrak{c}(\mathfrak{v}, \mathfrak{B}\mathfrak{o})}{1 + \mathfrak{c}(\mathfrak{v}, \mathfrak{o})},$$

for all $\mathfrak{v}, \mathfrak{o} \in \Psi$, then $\mathfrak{A}$ and $\mathfrak{B}$ possess a unique CFP.

The following lemma plays a key role in the proof of our main result.

Lemma 1. *([14]) Let* $(\Psi, \mathfrak{c})$ *be a CVSMS, and let* $\{\mathfrak{v}_r\} \subseteq \Psi$ *be a sequence. Then, the following hold:*

(i). Convergence: The sequence $\{\mathfrak{v}_r\}$ converges to $\mathfrak{v}$ if and only if $|\mathfrak{c}(\mathfrak{v}_r, \mathfrak{v})| \to 0$ as $r \to \infty$.

(ii). Cauchy Condition: The sequence $\{\mathfrak{v}_r\}$ is a Cauchy sequence if and only if $|\mathfrak{c}(\mathfrak{v}_r, \mathfrak{v}_{r+m})| \to 0$ as $r \to \infty$, for any $m \in \mathbb{N}$.

In this research work, we prove common FP theorems for rational contractions involving control functions of multiple variables in the framework of CVSMSs. In this way, we derive some important FP results, including the fundamental theorem of Panda et al. [14] in the context of CVSMSs, along with the significant findings of Azam et al. [10], Rouzkard et al. [11], Sintunavarat et al. [12], and Sitthikul et al. [13] within the scope of CVMSs, and the core theorem of Berzig [6] in the setting of SMSs. To emphasize the novelty of the main result, we also provide an illustrative example. Furthermore, the results are practically demonstrated by solving a nonlinear mixed Volterra–Fredholm integral equation, which models predator-prey dynamics.

3. Main results

We establish the following proposition, which is necessary for the proof of our prime result:

Proposition 1. *Let* $\mathfrak{A}, \mathfrak{B} : \Psi \to \Psi$ *be self-mappings and* $\mathfrak{v}_0 \in \Psi$. *Define the sequence* $\{\mathfrak{v}_r\}$ *by*

$$\mathfrak{v}_{2r+1} = \mathfrak{A}\mathfrak{v}_{2r} \ \text{ and } \ \mathfrak{v}_{2r+2} = \mathfrak{B}\mathfrak{v}_{2r+1},$$

for all $r = 0, 1, 2, ...$

Suppose a function $\Bbbk_1 : \Psi \times \Psi \to [0, 1)$ exist such that

$$\Bbbk_1(\mathfrak{B}\mathfrak{A}\mathfrak{v}, \mathfrak{o}) \le \Bbbk_1(\mathfrak{v}, \mathfrak{o}) \text{ and } \Bbbk_1(\mathfrak{v}, \mathfrak{A}\mathfrak{B}\mathfrak{o}) \le \Bbbk_1(\mathfrak{v}, \mathfrak{o}),$$

for all $\mathfrak{v}, \mathfrak{o} \in \Psi$. Then

$$\Bbbk_1(\mathfrak{v}_{2r}, \mathfrak{o}) \le \Bbbk_1(\mathfrak{v}_0, \mathfrak{o}) \text{ and } \Bbbk_1(\mathfrak{v}, \mathfrak{v}_{2r+1}) \le \Bbbk_1(\mathfrak{v}, \mathfrak{v}_1),$$

for all $\mathfrak{v}, \mathfrak{o} \in \Psi$ and $r = 0, 1, 2, ...$

We now state and prove the following lemma, which plays a crucial role in establishing our main result.

Lemma 2. *Let* $\Bbbk_1, \Bbbk_2 : \Psi \times \Psi \to [0, 1)$ *and* $\mathfrak{v}, \mathfrak{o} \in \Psi$. *If* $\mathfrak{A}, \mathfrak{B} : \Psi \to \Psi$ *satisfy*

$$\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{B}\mathfrak{A}\mathfrak{v}) \le \Bbbk_1(\mathfrak{v}, \mathfrak{A}\mathfrak{v})\, \mathfrak{c}(\mathfrak{v}, \mathfrak{A}\mathfrak{v}) + \Bbbk_2(\mathfrak{v}, \mathfrak{A}\mathfrak{v}) \frac{\mathfrak{c}(\mathfrak{v}, \mathfrak{A}\mathfrak{v})\, \mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{B}\mathfrak{A}\mathfrak{v})}{1 + \mathfrak{c}(\mathfrak{v}, \mathfrak{A}\mathfrak{v})},$$

$$\mathfrak{c}(\mathfrak{A}\mathfrak{B}\mathfrak{o},\mathfrak{B}\mathfrak{o})\leq \Bbbk_1(\mathfrak{B}\mathfrak{o},\mathfrak{o})\,\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{o})+\Bbbk_2(\mathfrak{B}\mathfrak{o},\mathfrak{o})\,\frac{\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{A}\mathfrak{B}\mathfrak{o})\,\mathfrak{c}(\mathfrak{o},\mathfrak{B}\mathfrak{o})}{1+\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{o})},$$

in which case

$$|\mathfrak{c}(\mathfrak{A}\mathfrak{v},\mathfrak{B}\mathfrak{A}\mathfrak{v})|\leq \Bbbk_1(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,|\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})|+\Bbbk_2(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,|\mathfrak{c}(\mathfrak{A}\mathfrak{v},\mathfrak{B}\mathfrak{A}\mathfrak{v})|,$$

$$|\mathfrak{c}(\mathfrak{A}\mathfrak{B}\mathfrak{o},\mathfrak{B}\mathfrak{o})|\leq \Bbbk_1(\mathfrak{B}\mathfrak{o},\mathfrak{o})\,|\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{o})|+\Bbbk_2(\mathfrak{B}\mathfrak{o},\mathfrak{o})\,|\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{A}\mathfrak{B}\mathfrak{o})|.$$

Proof. We can write

$$\begin{aligned}
|\mathfrak{c}(\mathfrak{A}\mathfrak{v},\mathfrak{B}\mathfrak{A}\mathfrak{v})| &\leq \left|\Bbbk_1(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})+\Bbbk_2(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,\frac{\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{A}\mathfrak{v},\mathfrak{B}\mathfrak{A}\mathfrak{v})}{1+\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})}\right| \\
&\leq \Bbbk_1(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,|\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})|+\Bbbk_2(\mathfrak{v},\mathfrak{A}\mathfrak{v})\left|\frac{\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})}{1+\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})}\right||\mathfrak{c}(\mathfrak{A}\mathfrak{v},\mathfrak{B}\mathfrak{A}\mathfrak{v})| \\
&\leq \Bbbk_1(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,|\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})|+\Bbbk_2(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,|\mathfrak{c}(\mathfrak{A}\mathfrak{v},\mathfrak{B}\mathfrak{A}\mathfrak{v})|.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|\mathfrak{c}(\mathfrak{A}\mathfrak{B}\mathfrak{o},\mathfrak{B}\mathfrak{o})| &\leq \left|\Bbbk_1(\mathfrak{B}\mathfrak{o},\mathfrak{o})\,\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{o})+\Bbbk_2(\mathfrak{B}\mathfrak{o},\mathfrak{o})\,\frac{\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{A}\mathfrak{B}\mathfrak{o})\,\mathfrak{c}(\mathfrak{o},\mathfrak{B}\mathfrak{o})}{1+\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{o})}\right| \\
&\leq \Bbbk_1(\mathfrak{B}\mathfrak{o},\mathfrak{o})\,|\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{o})|+\Bbbk_2(\mathfrak{B}\mathfrak{o},\mathfrak{o})\left|\frac{\mathfrak{c}(\mathfrak{o},\mathfrak{B}\mathfrak{o})}{1+\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{o})}\right||\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{A}\mathfrak{B}\mathfrak{o})| \\
&\leq \Bbbk_1(\mathfrak{B}\mathfrak{o},\mathfrak{o})\,|\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{o})|+\Bbbk_2(\mathfrak{B}\mathfrak{o},\mathfrak{o})\,|\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{A}\mathfrak{B}\mathfrak{o})|.
\end{aligned}$$

We now present the leading theorem of this section. □

Theorem 7. *Let* $(\Psi,\mathfrak{c})$ *be a complete CVSMS and* $\mathfrak{A},\mathfrak{B}:\Psi\to\Psi$. *Assuming the existence of the functions* $\Bbbk_1,\Bbbk_2,\Bbbk_3:\Psi\times\Psi\to[0,1)$ *such that for all* $\mathfrak{v},\mathfrak{o}\in\Psi$,

(a) $\Bbbk_1(\mathfrak{B}\mathfrak{A}\mathfrak{v},\mathfrak{o})\leq\Bbbk_1(\mathfrak{v},\mathfrak{o})$ and $\Bbbk_1(\mathfrak{v},\mathfrak{A}\mathfrak{B}\mathfrak{o})\leq\Bbbk_1(\mathfrak{v},\mathfrak{o})$,
$\Bbbk_2(\mathfrak{B}\mathfrak{A}\mathfrak{v},\mathfrak{o})\leq\Bbbk_2(\mathfrak{v},\mathfrak{o})$ and $\Bbbk_2(\mathfrak{v},\mathfrak{A}\mathfrak{B}\mathfrak{o})\leq\Bbbk_2(\mathfrak{v},\mathfrak{o})$,
$\Bbbk_3(\mathfrak{B}\mathfrak{A}\mathfrak{v},\mathfrak{o})\leq\Bbbk_3(\mathfrak{v},\mathfrak{o})$ and $\Bbbk_3(\mathfrak{v},\mathfrak{A}\mathfrak{B}\mathfrak{o})\leq\Bbbk_3(\mathfrak{v},\mathfrak{o})$,
(b) $\Bbbk_1(\mathfrak{v},\mathfrak{o})+\Bbbk_2(\mathfrak{v},\mathfrak{o})+\Bbbk_3(\mathfrak{v},\mathfrak{o})<1$,
(c)

$$\mathfrak{c}(\mathfrak{A}\mathfrak{v},\mathfrak{B}\mathfrak{o})\leq\Bbbk_1(\mathfrak{v},\mathfrak{o})\,\mathfrak{c}(\mathfrak{v},\mathfrak{o})+\Bbbk_2(\mathfrak{v},\mathfrak{o})\,\frac{\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{o},\mathfrak{B}\mathfrak{o})}{1+\mathfrak{c}(\mathfrak{v},\mathfrak{o})}+\Bbbk_3(\mathfrak{v},\mathfrak{o})\,\frac{\mathfrak{c}(\mathfrak{o},\mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{v},\mathfrak{B}\mathfrak{o})}{1+\mathfrak{c}(\mathfrak{v},\mathfrak{o})}, \tag{3.1}$$

for all $\mathfrak{v},\mathfrak{o}\in\Psi$, then $\mathfrak{A}$ and $\mathfrak{B}$ admit a unique CFP.

Proof. Let $\mathfrak{v},\mathfrak{o}\in\Psi$. From Eq (3.1), we have

$$\begin{aligned}
\mathfrak{c}(\mathfrak{A}\mathfrak{v},\mathfrak{B}\mathfrak{A}\mathfrak{v}) &\leq \Bbbk_1(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})+\Bbbk_2(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,\frac{\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{A}\mathfrak{v},\mathfrak{B}\mathfrak{A}\mathfrak{v})}{1+\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})} \\
&\quad+\Bbbk_3(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,\frac{\mathfrak{c}(\mathfrak{A}\mathfrak{v},\mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{v},_2\mathfrak{A}\mathfrak{v})}{1+\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})} \\
&= \Bbbk_1(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})+\Bbbk_2(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,\frac{\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{A}\mathfrak{v},\mathfrak{B}\mathfrak{A}\mathfrak{v})}{1+\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})}.
\end{aligned}$$

By Lemma 2, we get

$$|\mathfrak{c}(\mathfrak{A}\mathfrak{v},\mathfrak{B}\mathfrak{A}\mathfrak{v})| \leq \mathbb{k}_1(\mathfrak{v},\mathfrak{A}\mathfrak{v})|\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})| + \mathbb{k}_2(\mathfrak{v},\mathfrak{A}\mathfrak{v})|\mathfrak{c}(\mathfrak{A}\mathfrak{v},\mathfrak{B}\mathfrak{A}\mathfrak{v})|. \tag{3.2}$$

Similarly, we have

$$\begin{aligned}
\mathfrak{c}(\mathfrak{A}\mathfrak{B}\mathfrak{o},\mathfrak{B}\mathfrak{o}) &\leq \mathbb{k}_1(\mathfrak{B}\mathfrak{o},\mathfrak{o})\,\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{o}) + \mathbb{k}_2(\mathfrak{B}\mathfrak{o},\mathfrak{o})\frac{\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{A}\mathfrak{B}\mathfrak{o})\,\mathfrak{c}(\mathfrak{o},\mathfrak{B}\mathfrak{o})}{1+\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{o})} \\
&\quad + \mathbb{k}_3(\mathfrak{B}\mathfrak{o},\mathfrak{o})\frac{\mathfrak{c}(\mathfrak{o},\mathfrak{A}\mathfrak{B}\mathfrak{o})\,\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{B}\mathfrak{o})}{1+\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{o})} \\
&= \mathbb{k}_1(\mathfrak{B}\mathfrak{o},\mathfrak{o})\,\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{o}) + \mathbb{k}_2(\mathfrak{B}\mathfrak{o},\mathfrak{o})\frac{\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{A}\mathfrak{B}\mathfrak{o})\,\mathfrak{c}(\mathfrak{o},\mathfrak{B}\mathfrak{o})}{1+\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{o})}.
\end{aligned}$$

By Lemma 2, we get

$$|\mathfrak{c}(\mathfrak{A}\mathfrak{B}\mathfrak{o},\mathfrak{B}\mathfrak{o})| \leq \mathbb{k}_1(\mathfrak{B}\mathfrak{o},\mathfrak{o})|\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{o})| + \mathbb{k}_2(\mathfrak{B}\mathfrak{o},\mathfrak{o})|\mathfrak{c}(\mathfrak{B}\mathfrak{o},\mathfrak{A}\mathfrak{B}\mathfrak{o})|. \tag{3.3}$$

Let $\mathfrak{v}_0 \in \Psi$ be given. Define the sequence $\{\mathfrak{v}_r\}$ inductively by

$$\mathfrak{v}_{2r+1} = \mathfrak{A}\mathfrak{v}_{2r} \text{ and } \mathfrak{v}_{2r+2} = \mathfrak{B}\mathfrak{v}_{2r+1}.$$

From Proposition 1, Eqs (3.2), and (3.3), for all $r = 0, 1, 2, ...$

$$\begin{aligned}
|\mathfrak{c}(\mathfrak{v}_{2r+1},\mathfrak{v}_{2r})| &= |\mathfrak{c}(\mathfrak{A}\mathfrak{B}\mathfrak{v}_{2r-1},\mathfrak{B}\mathfrak{v}_{2r-1})| \leq \mathbb{k}_1(\mathfrak{B}\mathfrak{v}_{2r-1},\mathfrak{v}_{2r-1})|\mathfrak{c}(\mathfrak{B}\mathfrak{v}_{2r-1},\mathfrak{v}_{2r-1})| \\
&\quad + \mathbb{k}_2(\mathfrak{B}\mathfrak{v}_{2r-1},\mathfrak{v}_{2r-1})|\mathfrak{c}(\mathfrak{B}\mathfrak{v}_{2r-1},\mathfrak{A}\mathfrak{B}\mathfrak{v}_{2r-1})| \\
&= \mathbb{k}_1(\mathfrak{v}_{2r},\mathfrak{v}_{2r-1})|\mathfrak{c}(\mathfrak{v}_{2r},\mathfrak{v}_{2r-1})| + \mathbb{k}_2(\mathfrak{v}_{2r},\mathfrak{v}_{2r-1})|\mathfrak{c}(\mathfrak{v}_{2r},\mathfrak{v}_{2r+1})| \\
&\leq \mathbb{k}_1(\mathfrak{v}_0,\mathfrak{v}_{2r-1})|\mathfrak{c}(\mathfrak{v}_{2r},\mathfrak{v}_{2r-1})| + \mathbb{k}_2(\mathfrak{v}_0,\mathfrak{v}_{2r-1})|\mathfrak{c}(\mathfrak{v}_{2r},\mathfrak{v}_{2r+1})| \\
&\leq \mathbb{k}_1(\mathfrak{v}_0,\mathfrak{v}_1)|\mathfrak{c}(\mathfrak{v}_{2r},\mathfrak{v}_{2r-1})| + \mathbb{k}_2(\mathfrak{v}_0,\mathfrak{v}_1)|\mathfrak{c}(\mathfrak{v}_{2r},\mathfrak{v}_{2r+1})|,
\end{aligned}$$

which implies that

$$|\mathfrak{c}(\mathfrak{v}_{2r+1},\mathfrak{v}_{2r})| \leq \frac{\mathbb{k}_1(\mathfrak{v}_0,\mathfrak{v}_1)}{1-\mathbb{k}_2(\mathfrak{v}_0,\mathfrak{v}_1)}|\mathfrak{c}(\mathfrak{v}_{2r},\mathfrak{v}_{2r-1})|. \tag{3.4}$$

Similarly, we have

$$\begin{aligned}
|\mathfrak{c}(\mathfrak{v}_{2r+2},\mathfrak{v}_{2r+1})| &= |\mathfrak{c}(\mathfrak{B}\mathfrak{A}\mathfrak{v}_{2r},\mathfrak{A}\mathfrak{v}_{2r})| \leq \mathbb{k}_1(\mathfrak{v}_{2r},\mathfrak{A}\mathfrak{v}_{2r})|\mathfrak{c}(\mathfrak{v}_{2r},\mathfrak{A}\mathfrak{v}_{2r})| \\
&\quad + \mathbb{k}_2(\mathfrak{v}_{2r},\mathfrak{A}\mathfrak{v}_{2r})|\mathfrak{c}(\mathfrak{A}\mathfrak{v}_{2r},\mathfrak{B}\mathfrak{A}\mathfrak{v}_{2r})| \\
&= \mathbb{k}_1(\mathfrak{v}_{2r},\mathfrak{v}_{2r+1})|\mathfrak{c}(\mathfrak{v}_{2r},\mathfrak{v}_{2r+1})| + \mathbb{k}_2(\mathfrak{v}_{2r},\mathfrak{v}_{2r+1})|\mathfrak{c}(\mathfrak{v}_{2r+1},\mathfrak{v}_{2r+2})| \\
&\leq \mathbb{k}_1(\mathfrak{v}_0,\mathfrak{v}_{2r+1})|\mathfrak{c}(\mathfrak{v}_{2r},\mathfrak{v}_{2r+1})| + \mathbb{k}_2(\mathfrak{v}_0,\mathfrak{v}_{2r+1})|\mathfrak{c}(\mathfrak{v}_{2r+1},\mathfrak{v}_{2r+2})| \\
&\leq \mathbb{k}_1(\mathfrak{v}_0,\mathfrak{v}_1)|\mathfrak{c}(\mathfrak{v}_{2r},\mathfrak{v}_{2r+1})| + \mathbb{k}_2(\mathfrak{v}_0,\mathfrak{v}_1)|\mathfrak{c}(\mathfrak{v}_{2r+1},\mathfrak{v}_{2r+2})|,
\end{aligned}$$

which implies that

$$\begin{aligned}
|\mathfrak{c}(\mathfrak{v}_{2r+2},\mathfrak{v}_{2r+1})| &\leq \frac{\mathbb{k}_1(\mathfrak{v}_0,\mathfrak{v}_1)}{1-\mathbb{k}_2(\mathfrak{v}_0,\mathfrak{v}_1)}|\mathfrak{c}(\mathfrak{v}_{2r},\mathfrak{v}_{2r+1})| \\
&= \frac{\mathbb{k}_1(\mathfrak{v}_0,\mathfrak{v}_1)}{1-\mathbb{k}_2(\mathfrak{v}_0,\mathfrak{v}_1)}|\mathfrak{c}(\mathfrak{v}_{2r+1},\mathfrak{v}_{2r})|. \qquad (3.5)
\end{aligned}$$

Let $\mu = \frac{\mathbb{k}_1(\mathfrak{v}_0,\mathfrak{v}_1)}{1-\mathbb{k}_2(\mathfrak{v}_0,\mathfrak{v}_1)} < 1$. Then from Eqs (3.4) and (3.5), we have

$$|\mathfrak{c}(\mathfrak{v}_{r+1},\mathfrak{v}_r)| \leq \mu |\mathfrak{c}(\mathfrak{v}_r,\mathfrak{v}_{r-1})|,$$

for all $r \in \mathbb{N}$. Recursively, we can generate a sequence $\{\mathfrak{v}_r\}$ in Ψ such that

$$\begin{aligned} |\mathfrak{c}(\mathfrak{v}_{r+1},\mathfrak{v}_r)| &\leq \mu |\mathfrak{c}(\mathfrak{v}_r,\mathfrak{v}_{r-1})| \\ &\leq \mu^2 |\mathfrak{c}(\mathfrak{v}_{r-1},\mathfrak{v}_{r-2})| \\ &\quad . \\ &\quad . \\ &\quad . \\ &\leq \mu^r |\mathfrak{c}(\mathfrak{v}_1,\mathfrak{v}_0)| = \mu^r |\mathfrak{c}(\mathfrak{v}_0,\mathfrak{v}_1)|. \end{aligned}$$

That is,

$$|\mathfrak{c}(\mathfrak{v}_r,\mathfrak{v}_{r+1})| = |\mathfrak{c}(\mathfrak{v}_{r+1},\mathfrak{v}_r)| \leq \mu^r |\mathfrak{c}(\mathfrak{v}_0,\mathfrak{v}_1)|, \tag{3.6}$$

for all $r \in \mathbb{N}$. Now for $m > r$, we get

$$\begin{aligned} |\mathfrak{c}(\mathfrak{v}_r,\mathfrak{v}_m)| &\leq |\mathfrak{c}(\mathfrak{v}_r,\mathfrak{v}_{r+1})| + |\mathfrak{c}(\mathfrak{v}_{r+1},\mathfrak{v}_m)| + \wedge |\mathfrak{c}(\mathfrak{v}_r,\mathfrak{v}_{r+1})| |\mathfrak{c}(\mathfrak{v}_{r+1},\mathfrak{v}_m)| \\ &= |\mathfrak{c}(\mathfrak{v}_r,\mathfrak{v}_{r+1})| + (1 + \wedge |\mathfrak{c}(\mathfrak{v}_r,\mathfrak{v}_{r+1})|) |\mathfrak{c}(\mathfrak{v}_{r+1},\mathfrak{v}_m)|. \end{aligned} \tag{3.7}$$

We now apply the triangle inequality again to the second term $|\mathfrak{c}(\mathfrak{v}_{r+1},\mathfrak{v}_m)|$, and we have

$$\begin{aligned} |\mathfrak{c}(\mathfrak{v}_{r+1},\mathfrak{v}_m)| &\leq |\mathfrak{c}(\mathfrak{v}_{r+1},\mathfrak{v}_{r+2})| + |\mathfrak{c}(\mathfrak{v}_{r+2},\mathfrak{v}_m)| + \wedge |\mathfrak{c}(\mathfrak{v}_{r+1},\mathfrak{v}_{r+2})| |\mathfrak{c}(\mathfrak{v}_{r+2},\mathfrak{v}_m)| \\ &= |\mathfrak{c}(\mathfrak{v}_{r+1},\mathfrak{v}_{r+2})| + (1 + \wedge |\mathfrak{c}(\mathfrak{v}_{r+1},\mathfrak{v}_{r+2})|) |\mathfrak{c}(\mathfrak{v}_{r+2},\mathfrak{v}_m)|. \end{aligned}$$

We now apply the triangle inequality again to the second term $|\mathfrak{c}(\mathfrak{v}_{r+2},\mathfrak{v}_m)|$, and we have

$$\begin{aligned} |\mathfrak{c}(\mathfrak{v}_{r+2},\mathfrak{v}_m)| &\leq |\mathfrak{c}(\mathfrak{v}_{r+2},\mathfrak{v}_{r+3})| + |\mathfrak{c}(\mathfrak{v}_{r+3},\mathfrak{v}_m)| + \wedge |\mathfrak{c}(\mathfrak{v}_{r+2},\mathfrak{v}_{r+3})| |\mathfrak{c}(\mathfrak{v}_{r+3},\mathfrak{v}_m)| \\ &= |\mathfrak{c}(\mathfrak{v}_{r+2},\mathfrak{v}_{r+3})| + (1 + \wedge |\mathfrak{c}(\mathfrak{v}_{r+2},\mathfrak{v}_{r+3})|) |\mathfrak{c}(\mathfrak{v}_{r+3},\mathfrak{v}_m)|, \end{aligned}$$

and so on

$$\begin{aligned} |\mathfrak{c}(\mathfrak{v}_{m-2},\mathfrak{v}_m)| &\leq |\mathfrak{c}(\mathfrak{v}_{m-2},\mathfrak{v}_{m-1})| + |\mathfrak{c}(\mathfrak{v}_{m-1},\mathfrak{v}_m)| + \wedge |\mathfrak{c}(\mathfrak{v}_{m-2},\mathfrak{v}_{m-1})| |\mathfrak{c}(\mathfrak{v}_{m-1},\mathfrak{v}_m)| \\ &= |\mathfrak{c}(\mathfrak{v}_{m-2},\mathfrak{v}_{m-1})| + (1 + \wedge |\mathfrak{c}(\mathfrak{v}_{m-2},\mathfrak{v}_{m-1})|) |\mathfrak{c}(\mathfrak{v}_{m-1},\mathfrak{v}_m)|. \end{aligned}$$

Recursively substituting each inequality into the preceding one (3.7) and simplifying, we have

$$|\mathfrak{c}(\mathfrak{v}_r,\mathfrak{v}_m)| \leq \sum_{k=r}^{m-1} |\mathfrak{c}(\mathfrak{v}_k,\mathfrak{v}_{k+1})| \prod_{i=r}^{k-1} (1 + \wedge |\mathfrak{c}(\mathfrak{v}_i,\mathfrak{v}_{i+1})|). \tag{3.8}$$

By the inequality (3.6), we have

$$|\mathfrak{c}(\mathfrak{v}_k,\mathfrak{v}_{k+1})| \leq \mu^k |\mathfrak{c}(\mathfrak{v}_0,\mathfrak{v}_1)| \quad \text{and} \quad |\mathfrak{c}(\mathfrak{v}_i,\mathfrak{v}_{i+1})| \leq \mu^i |\mathfrak{c}(\mathfrak{v}_0,\mathfrak{v}_1)|.$$

From the inequality (3.8), we get

$$|\mathfrak{c}(\mathfrak{v}_r,\mathfrak{v}_m)| \leq \sum_{k=r}^{m-1} \mu^k |\mathfrak{c}(\mathfrak{v}_0,\mathfrak{v}_1)| \prod_{i=r}^{k-1} \left(1 + \curlywedge \mu^i |\mathfrak{c}(\mathfrak{v}_0,\mathfrak{v}_1)|\right),$$

which implies

$$|\mathfrak{c}(\mathfrak{v}_r,\mathfrak{v}_m)| \leq |\mathfrak{c}(\mathfrak{v}_0,\mathfrak{v}_1)| \sum_{k=r}^{m-1} \mu^k \prod_{i=r}^{k-1} \left(1 + \curlywedge \mu^i |\mathfrak{c}(\mathfrak{v}_0,\mathfrak{v}_1)|\right). \tag{3.9}$$

Now, observe that $\left(1 + \curlywedge \mu^i |\mathfrak{c}(\mathfrak{v}_0,\mathfrak{v}_1)|\right) \geq 1$, for all i. Therefore,

$$\prod_{i=r}^{k-1} \left(1 + \curlywedge \mu^i |\mathfrak{c}(\mathfrak{v}_0,\mathfrak{v}_1)|\right) \geq 1.$$

With the product term bounded below by 1, we can simplify the summation

$$|\mathfrak{c}(\mathfrak{v}_0,\mathfrak{v}_1)| \sum_{k=r}^{m-1} \mu^k \prod_{i=r}^{k-1} \left(1 + \curlywedge \mu^i |\mathfrak{c}(\mathfrak{v}_0,\mathfrak{v}_1)|\right) \geq |\mathfrak{c}(\mathfrak{v}_0,\mathfrak{v}_1)| \sum_{k=r}^{m-1} \mu^k. \tag{3.10}$$

Since $\sum_{k=r}^{m-1} \mu^k$ is a finite geometric series with the first term μ^r and the common ratio μ, the sum of a finite geometric series is given by

$$\sum_{k=r}^{m-1} \mu^k = \mu^r \frac{(1-\mu^{m-r})}{1-\mu}.$$

As $m \to \infty$, $\mu^{m-r} \to 0$. Therefore, the sum converges to $\frac{\mu^r}{1-\mu}$; that is,

$$\lim_{m\to\infty} \mu^r \frac{(1-\mu^{m-r})}{1-\mu} = \frac{\mu^r}{1-\mu}.$$

Since $\mu < 1$, the expression $\frac{\mu^r}{1-\mu}$ approaches 0 as $r \to \infty$. Letting $r \to \infty$ in inequality (3.9) and employing the established facts, we arrive at

$$\lim_{r\to\infty} |\mathfrak{c}(\mathfrak{v}_r,\mathfrak{v}_m)| = 0.$$

Therefore, by Lemma 1 (ii), $\{\mathfrak{v}_r\}$ is Cauchy. As Ψ is complete, so $\mathfrak{v}^* \in \Psi$ exist such that $\mathfrak{v}_r \to \mathfrak{v}^*$ as $r \to \infty$. Thus

$$\lim_{r\to\infty} \mathfrak{v}_r = \mathfrak{v}^*.$$

Now, we show that $\mathfrak{v}^*$ is a FP of $\mathfrak{A}$. From (3.1), we have

$$\begin{aligned}
\mathfrak{c}(\mathfrak{v}^*,\mathfrak{A}\mathfrak{v}^*) &\leq \mathfrak{c}(\mathfrak{v}^*,\mathfrak{v}_{2r+2}) + \mathfrak{c}(\mathfrak{v}_{2r+2},\mathfrak{A}\mathfrak{v}^*) + \curlywedge \mathfrak{c}(\mathfrak{v}^*,\mathfrak{v}_{2r+2})\, \mathfrak{c}(\mathfrak{v}_{2r+2},\mathfrak{A}\mathfrak{v}^*) \\
&= \mathfrak{c}(\mathfrak{v}^*,\mathfrak{v}_{2r+2}) + \mathfrak{c}(\mathfrak{B}\mathfrak{v}_{2r+1},\mathfrak{A}\mathfrak{v}^*) + \curlywedge \mathfrak{c}(\mathfrak{v}^*,\mathfrak{B}\mathfrak{v}_{2r+1})\, \mathfrak{c}(\mathfrak{B}\mathfrak{v}_{2r+1},\mathfrak{A}\mathfrak{v}^*) \\
&= \mathfrak{c}(\mathfrak{v}^*,\mathfrak{B}\mathfrak{v}_{2r+1}) + \mathfrak{c}(\mathfrak{A}\mathfrak{v}^*,\mathfrak{B}\mathfrak{v}_{2r+1}) \\
&\leq \mathfrak{c}(\mathfrak{v}^*,\mathfrak{v}_{2r+2}) + \begin{pmatrix} \mathbb{k}_1(\mathfrak{v}^*,\mathfrak{v}_{2r+1})\, \mathfrak{c}(\mathfrak{v}^*,\mathfrak{v}_{2r+1}) \\ +\mathbb{k}_2(\mathfrak{v}^*,\mathfrak{v}_{2r+1}) \frac{\mathfrak{c}(\mathfrak{v}^*,\mathfrak{A}\mathfrak{v}^*)\mathfrak{c}(\mathfrak{v}_{2r+1},\mathfrak{B}\mathfrak{v}_{2r+1})}{1+\mathfrak{c}(\mathfrak{v}^*,\mathfrak{v}_{2r+1})} \\ +\mathbb{k}_3(\mathfrak{v}^*,\mathfrak{v}_{2r+1}) \frac{\mathfrak{c}(\mathfrak{v}_{2r+1},\mathfrak{A}\mathfrak{v}^*)\mathfrak{c}(\mathfrak{v}^*,\mathfrak{B}\mathfrak{v}_{2r+1})}{1+\mathfrak{c}(\mathfrak{v}^*,\mathfrak{v}_{2r+1})} \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&+\curlywedge\, \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{B}\mathfrak{v}_{2r+1}\right) \mathfrak{c}\left(\mathfrak{B}\mathfrak{v}_{2r+1}, \mathfrak{A}\mathfrak{v}^*\right)\\
\leq\ & \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+2}\right)+\left(\begin{array}{c}
\Bbbk_1\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right) \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right) \\
+\Bbbk_2\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right) \frac{\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{A}\mathfrak{v}^*\right) \mathfrak{c}\left(\mathfrak{v}_{2r+1}, \mathfrak{v}_{2r+2}\right)}{1+\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right)} \\
+\Bbbk_3\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right) \frac{\mathfrak{c}\left(\mathfrak{v}_{2r+1}, \mathfrak{A}\mathfrak{v}^*\right) \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+2}\right)}{1+\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right)}
\end{array}\right)\\
&+\curlywedge\, \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+2}\right) \mathfrak{c}\left(\mathfrak{v}_{2r+2}, \mathfrak{A}\mathfrak{v}^*\right).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\leq\ & \left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+2}\right)\right|+\left(\begin{array}{c}
\Bbbk_1\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right)\left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right)\right| \\
+\Bbbk_2\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right) \frac{\left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{A}\mathfrak{v}^*\right)\right|\left|\mathfrak{c}\left(\mathfrak{v}_{2r+1}, \mathfrak{v}_{2r+2}\right)\right|}{\left|1+\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right)\right|} \\
+\Bbbk_3\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right) \frac{\left|\mathfrak{c}\left(\mathfrak{v}_{2r+1}, \mathfrak{A}\mathfrak{v}^*\right)\right|\left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+2}\right)\right|}{\left|1+\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right)\right|}
\end{array}\right)\\
&+\curlywedge\left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+2}\right)\right|\left|\mathfrak{c}\left(\mathfrak{v}_{2r+2}, \mathfrak{A}\mathfrak{v}^*\right)\right|.
\end{aligned}
$$

Letting $r \to \infty$, we have $\left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{A}\mathfrak{v}^*\right)\right| = 0$. Thus $\mathfrak{v}^* = \mathfrak{A}\mathfrak{v}^*$. To demonstrate that $\mathfrak{v}^*$ is a FP of $\mathfrak{B}$, we observe from (3.1) that

$$
\begin{aligned}
\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{B}\mathfrak{v}^*\right) \leq\ & \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right)+\mathfrak{c}\left(\mathfrak{v}_{2r+1}, \mathfrak{B}\mathfrak{v}^*\right)+\curlywedge \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right) \mathfrak{c}\left(\mathfrak{v}_{2r+1}, \mathfrak{B}\mathfrak{v}^*\right)\\
\leq\ & \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right)+\mathfrak{c}\left(\mathfrak{A}\mathfrak{v}_{2r}, \mathfrak{B}\mathfrak{v}^*\right)+\curlywedge \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right) \mathfrak{c}\left(\mathfrak{v}_{2r+1}, \mathfrak{B}\mathfrak{v}^*\right)\\
\leq\ & \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right)+\left(\begin{array}{c}
\Bbbk_1\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right) \mathfrak{c}\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right) \\
+\Bbbk_2\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right) \frac{\mathfrak{c}\left(\mathfrak{v}_{2r}, \mathfrak{A}\mathfrak{v}_{2r}\right) \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{B}\mathfrak{v}^*\right)}{1+\mathfrak{c}\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right)} \\
+\Bbbk_3\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right) \frac{\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{A}\mathfrak{v}_{2r}\right) \mathfrak{c}\left(\mathfrak{v}_{2r}, \mathfrak{B}\mathfrak{v}^*\right)}{1+\mathfrak{c}\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right)}
\end{array}\right)\\
&+\curlywedge\, \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right) \mathfrak{c}\left(\mathfrak{v}_{2r+1}, \mathfrak{B}\mathfrak{v}^*\right)\\
\leq\ & \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right)+\left(\begin{array}{c}
\Bbbk_1\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right) \mathfrak{c}\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right) \\
+\Bbbk_2\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right) \frac{\mathfrak{c}\left(\mathfrak{v}_{2r}, \mathfrak{v}_{2r+1}\right) \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{B}\mathfrak{v}^*\right)}{1+\mathfrak{c}\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right)} \\
+\Bbbk_3\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right) \frac{\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right) \mathfrak{c}\left(\mathfrak{v}_{2r}, \mathfrak{B}\mathfrak{v}^*\right)}{1+\mathfrak{c}\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right)}
\end{array}\right)\\
&+\curlywedge\, \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right) \mathfrak{c}\left(\mathfrak{v}_{2r+1}, \mathfrak{B}\mathfrak{v}^*\right).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{B}\mathfrak{v}^*\right)\right| \leq\ & \left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right)\right|+\left(\begin{array}{c}
\Bbbk_1\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right)\left|\mathfrak{c}\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right)\right| \\
+\Bbbk_2\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right) \frac{\left|\mathfrak{c}\left(\mathfrak{v}_{2r}, \mathfrak{v}_{2r+1}\right)\right|\left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{B}\mathfrak{v}^*\right)\right|}{\left|1+\mathfrak{c}\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right)\right|} \\
+\Bbbk_3\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right) \frac{\left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right)\right|\left|\mathfrak{c}\left(\mathfrak{v}_{2r}, \mathfrak{B}\mathfrak{v}^*\right)\right|}{\left|1+\mathfrak{c}\left(\mathfrak{v}_{2r}, \mathfrak{v}^*\right)\right|}
\end{array}\right)\\
&+\curlywedge\left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}_{2r+1}\right)\right|\left|\mathfrak{c}\left(\mathfrak{v}_{2r+1}, \mathfrak{B}\mathfrak{v}^*\right)\right|.
\end{aligned}
$$

□

Letting $r \to \infty$, we have $\left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{B}\mathfrak{v}^*\right)\right| = 0$, a therefore, $\mathfrak{v}^* = \mathfrak{B}\mathfrak{v}^*$. Thus, $\mathfrak{v}^*$ is a CFP of $\mathfrak{A}$ and $\mathfrak{B}$. To establish the uniqueness of $\mathfrak{v}^*$, suppose, to the contrary, that another CFP $\mathfrak{v}^/$ of $\mathfrak{A}$ and $\mathfrak{B}$ exist. This implies that

$$
\mathfrak{v}^/ = \mathfrak{A}\mathfrak{v}^/ = \mathfrak{B}\mathfrak{v}^/,
$$

but $\mathfrak{v}^* \neq \mathfrak{v}^/$. From (3.1), we have

$$
\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^/\right) = \mathfrak{c}\left(\mathfrak{A}\mathfrak{v}^*, \mathfrak{B}\mathfrak{v}^/\right)
$$

$$
\begin{aligned}
&\leq \mathbb{k}_1\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right) \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)+\mathbb{k}_2\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right) \frac{\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{A} \mathfrak{v}^*\right) \mathfrak{c}\left(\mathfrak{v}^{/}, \mathfrak{B} \mathfrak{v}^{/}\right)}{1+\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)} \\
&\quad+\mathbb{k}_3\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right) \frac{\mathfrak{c}\left(\mathfrak{v}^{/}, \mathfrak{A} \mathfrak{v}^*\right) \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{B} \mathfrak{v}^{/}\right)}{1+\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)} \\
&= \mathbb{k}_1\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right) \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)+\mathbb{k}_2\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right) \frac{\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^*\right) \mathfrak{c}\left(\mathfrak{v}^{/}, \mathfrak{v}^{/}\right)}{1+\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)} \\
&\quad+\mathbb{k}_3\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right) \frac{\mathfrak{c}\left(\mathfrak{v}^{/}, \mathfrak{v}^*\right) \mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)}{1+\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)}.
\end{aligned}
$$

This implies that we have

$$
\begin{aligned}
\left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)\right| &\leq \mathbb{k}_1\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)\left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)\right| \\
&\quad+\mathbb{k}_3\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)\left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)\right|\left|\frac{\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)}{1+\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)}\right| \\
&\leq \mathbb{k}_1\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)\left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)\right|+\mathbb{k}_3\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)\left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)\right| \\
&= \left(\mathbb{k}_1\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)+\mathbb{k}_3\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)\right)\left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)\right|.
\end{aligned}
$$

As $\mathbb{k}_1\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)+\mathbb{k}_3\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)<1$, we have

$$
\left|\mathfrak{c}\left(\mathfrak{v}^*, \mathfrak{v}^{/}\right)\right|=0.
$$

Thus $\mathfrak{v}^*=\mathfrak{v}^{/}$.

Corollary 1. *Let* $(\Psi, \mathfrak{c})$ *be a complete CVSMS and* $\mathfrak{A}, \mathfrak{B}: \Psi \rightarrow \Psi$. *Assuming the existence of the functions* $\mathbb{k}_1, \mathbb{k}_2: \Psi \times \Psi \rightarrow [0,1)$ *such that*

(a) $\mathbb{k}_1(\mathfrak{B}\mathfrak{A}\mathfrak{v}, \mathfrak{o}) \leq \mathbb{k}_1(\mathfrak{v}, \mathfrak{o})$ and $\mathbb{k}_1(\mathfrak{v}, \mathfrak{A}\mathfrak{B}\mathfrak{o}) \leq \mathbb{k}_1(\mathfrak{v}, \mathfrak{o})$,
$\mathbb{k}_2(\mathfrak{B}\mathfrak{A}\mathfrak{v}, \mathfrak{o}) \leq \mathbb{k}_2(\mathfrak{v}, \mathfrak{o})$ and $\mathbb{k}_2(\mathfrak{v}, \mathfrak{A}\mathfrak{B}\mathfrak{o}) \leq \mathbb{k}_2(\mathfrak{v}, \mathfrak{o})$,
(b) $\mathbb{k}_1(\mathfrak{v}, \mathfrak{o})+\mathbb{k}_2(\mathfrak{v}, \mathfrak{o})<1$,
(c)

$$
\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{B}\mathfrak{o}) \leq \mathbb{k}_1(\mathfrak{v}, \mathfrak{o}) \mathfrak{c}(\mathfrak{v}, \mathfrak{o})+\mathbb{k}_2(\mathfrak{v}, \mathfrak{o}) \frac{\mathfrak{c}(\mathfrak{v}, \mathfrak{A}\mathfrak{v}) \mathfrak{c}(\mathfrak{o}, \mathfrak{B}\mathfrak{o})}{1+\mathfrak{c}(\mathfrak{v}, \mathfrak{o})},
$$

for all $\mathfrak{v}, \mathfrak{o} \in \Psi$, then $\mathfrak{A}$ and $\mathfrak{B}$ admit a unique CFP.

Proof. Define $\mathbb{k}_3: \Psi \times \Psi \rightarrow [0,1)$ by $\mathbb{k}_3(\mathfrak{v}, \mathfrak{o})=0$ in Theorem 7. □

Corollary 2. *Let* $(\Psi, \mathfrak{c})$ *be a complete CVSMS and* $\mathfrak{A}, \mathfrak{B}: \Psi \rightarrow \Psi$. *Assuming the existence of the functions* $\mathbb{k}_1, \mathbb{k}_3: \Psi \times \Psi \rightarrow [0,1)$ *such that*

(a) $\mathbb{k}_1(\mathfrak{B}\mathfrak{A}\mathfrak{v}, \mathfrak{o}) \leq \mathbb{k}_1(\mathfrak{v}, \mathfrak{o})$ and $\mathbb{k}_1(\mathfrak{v}, \mathfrak{A}\mathfrak{B}\mathfrak{o}) \leq \mathbb{k}_1(\mathfrak{v}, \mathfrak{o})$,
$\mathbb{k}_3(\mathfrak{B}\mathfrak{A}\mathfrak{v}, \mathfrak{o}) \leq \mathbb{k}_3(\mathfrak{v}, \mathfrak{o})$ and $\mathbb{k}_3(\mathfrak{v}, \mathfrak{A}\mathfrak{B}\mathfrak{o}) \leq \mathbb{k}_3(\mathfrak{v}, \mathfrak{o})$,
(b) $\mathbb{k}_1(\mathfrak{v}, \mathfrak{o})+\mathbb{k}_3(\mathfrak{v}, \mathfrak{o})<1$,

(c)

$$\mathfrak{c}(\mathfrak{A}\mathfrak{v},\mathfrak{B}\mathfrak{o}) \leq \Bbbk_1(\mathfrak{v},\mathfrak{o})\,\mathfrak{c}(\mathfrak{v},\mathfrak{o}) + \Bbbk_3(\mathfrak{v},\mathfrak{o})\frac{\mathfrak{c}(\mathfrak{o},\mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{v},\mathfrak{B}\mathfrak{o})}{1+\mathfrak{c}(\mathfrak{v},\mathfrak{o})},$$

for all $\mathfrak{v},\mathfrak{o} \in \Psi$, then $\mathfrak{A}$ and $\mathfrak{B}$ admit a unique CFP.

Proof. Define $\Bbbk_2 : \Psi\times\Psi \to [0,1)$ by $\Bbbk_2(\mathfrak{v},\mathfrak{o}) = 0$ in Theorem 7. □

Corollary 3. *Let* $(\Psi,\mathfrak{c})$ *be a complete CVSMS and* $\mathfrak{A},\mathfrak{B} : \Psi \to \Psi$. *Assuming the existence of the function* $\Bbbk_1 : \Psi\times\Psi \to [0,1)$ *such that*

(a) $\Bbbk_1(\mathfrak{B}\mathfrak{A}\mathfrak{v},\mathfrak{o}) \leq \Bbbk_1(\mathfrak{v},\mathfrak{o})$ and $\Bbbk_1(\mathfrak{v},\mathfrak{A}\mathfrak{B}\mathfrak{o}) \leq \Bbbk_1(\mathfrak{v},\mathfrak{o})$,
(b)

$$\mathfrak{c}(\mathfrak{A}\mathfrak{v},\mathfrak{B}\mathfrak{o}) \leq \Bbbk_1(\mathfrak{v},\mathfrak{o})\,\mathfrak{c}(\mathfrak{v},\mathfrak{o}),$$

for all $\mathfrak{v},\mathfrak{o} \in \Psi$, then $\mathfrak{A}$ and $\mathfrak{B}$ admit a unique CFP.

Proof. Define $\Bbbk_2,\Bbbk_3 : \Psi\times\Psi \to [0,1)$ by $\Bbbk_2(\mathfrak{v},\mathfrak{o}) = \Bbbk_3(\mathfrak{v},\mathfrak{o}) = 0$ in Theorem 7. □

Corollary 4. *Let* $(\Psi,\mathfrak{c})$ *be a complete CVSMS and* $\mathfrak{A} : \Psi \to \Psi$. *Assuming the existence of the functions* $\Bbbk_1,\Bbbk_2,\Bbbk_3 : \Psi\times\Psi \to [0,1)$ *such that*

(a) $\Bbbk_1(\mathfrak{A}\mathfrak{v},\mathfrak{o}) \leq \Bbbk_1(\mathfrak{v},\mathfrak{o})$ and $\Bbbk_1(\mathfrak{v},\mathfrak{A}\mathfrak{o}) \leq \Bbbk_1(\mathfrak{v},\mathfrak{o})$,
$\Bbbk_2(\mathfrak{A}\mathfrak{v},\mathfrak{o}) \leq \Bbbk_2(\mathfrak{v},\mathfrak{o})$ and $\Bbbk_2(\mathfrak{v},\mathfrak{A}\mathfrak{o}) \leq \Bbbk_2(\mathfrak{v},\mathfrak{o})$,
$\Bbbk_3(\mathfrak{A}\mathfrak{v},\mathfrak{o}) \leq \Bbbk_3(\mathfrak{v},\mathfrak{o})$ and $\Bbbk_3(\mathfrak{v},\mathfrak{A}\mathfrak{o}) \leq \Bbbk_3(\mathfrak{v},\mathfrak{o})$,
(b) $\Bbbk_1(\mathfrak{v},\mathfrak{o}) + \Bbbk_2(\mathfrak{v},\mathfrak{o}) + \Bbbk_3(\mathfrak{v},\mathfrak{o}) < 1$,
(c)

$$\mathfrak{c}(\mathfrak{A}\mathfrak{v},\mathfrak{A}\mathfrak{o}) \leq \Bbbk_1(\mathfrak{v},\mathfrak{o})\,\mathfrak{c}(\mathfrak{v},\mathfrak{o}) + \Bbbk_2(\mathfrak{v},\mathfrak{o})\frac{\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{o},\mathfrak{A}\mathfrak{o})}{1+\mathfrak{c}(\mathfrak{v},\mathfrak{o})} + \Bbbk_3(\mathfrak{v},\mathfrak{o})\frac{\mathfrak{c}(\mathfrak{o},\mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{o})}{1+\mathfrak{c}(\mathfrak{v},\mathfrak{o})},$$

for all $\mathfrak{v},\mathfrak{o} \in \Psi$, then $\mathfrak{A}$ has a unique FP.

Proof. Take $\mathfrak{A} = \mathfrak{B}$ in Theorem 7. □

Corollary 5. *Let* $(\Psi,\mathfrak{c})$ *be a complete CVSMS and* $\mathfrak{A} : \Psi \to \Psi$. *Assuming the existence of the functions* $\Bbbk_1,\Bbbk_2 : \Psi\times\Psi \to [0,1)$ *such that*

(a) $\Bbbk_1(\mathfrak{A}\mathfrak{v},\mathfrak{o}) \leq \Bbbk_1(\mathfrak{v},\mathfrak{o})$ and $\Bbbk_1(\mathfrak{v},\mathfrak{A}\mathfrak{o}) \leq \Bbbk_1(\mathfrak{v},\mathfrak{o})$,
$\Bbbk_2(\mathfrak{A}\mathfrak{v},\mathfrak{o}) \leq \Bbbk_2(\mathfrak{v},\mathfrak{o})$ and $\Bbbk_2(\mathfrak{v},\mathfrak{A}\mathfrak{o}) \leq \Bbbk_2(\mathfrak{v},\mathfrak{o})$,
(b) $\Bbbk_1(\mathfrak{v},\mathfrak{o}) + \Bbbk_2(\mathfrak{v},\mathfrak{o}) < 1$,
(c)

$$\mathfrak{c}(\mathfrak{A}\mathfrak{v},\mathfrak{A}\mathfrak{o}) \leq \Bbbk_1(\mathfrak{v},\mathfrak{o})\,\mathfrak{c}(\mathfrak{v},\mathfrak{o}) + \Bbbk_2(\mathfrak{v},\mathfrak{o})\frac{\mathfrak{c}(\mathfrak{v},\mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{o},\mathfrak{A}\mathfrak{o})}{1+\mathfrak{c}(\mathfrak{v},\mathfrak{o})},$$

for all $\mathfrak{v},\mathfrak{o} \in \Psi$, then $\mathfrak{A}$ has a unique FP.

Proof. Define $\Bbbk_3 : \Psi\times\Psi \to [0,1)$ by $\Bbbk_3(\mathfrak{v},\mathfrak{o}) = 0$ in Corollary 4.

□

Corollary 6. *Let* $(\Psi,\mathfrak{c})$ *be a complete CVSMS and* $\mathfrak{A} : \Psi \to \Psi$. *Assuming the existence of the function* $\Bbbk_1 : \Psi\times\Psi \to [0,1)$ *such that*

(a) $\Bbbk_1(\mathfrak{A}\mathfrak{v}, \mathfrak{o}) \leq \Bbbk_1(\mathfrak{v}, \mathfrak{o})$ and $\Bbbk_1(\mathfrak{v}, \mathfrak{A}\mathfrak{o}) \leq \Bbbk_1(\mathfrak{v}, \mathfrak{o})$,

(b)

$$\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{A}\mathfrak{o}) \preceq \Bbbk_1(\mathfrak{v}, \mathfrak{o})\, \mathfrak{c}(\mathfrak{v}, \mathfrak{o}),$$

for all $\mathfrak{v}, \mathfrak{o} \in \Psi$, then $\mathfrak{A}$ has a unique FP.

Proof. Define $\Bbbk_2, \Bbbk_3 : \Psi \times \Psi \to [0, 1)$ by $\Bbbk_2(\mathfrak{v}, \mathfrak{o}) = \Bbbk_3(\mathfrak{v}, \mathfrak{o}) = 0$ in Corollary 4. □

Example 4. *Let* $\Psi = [0, 1]$, *define* $\mathfrak{c} : \Psi \times \Psi \to \mathbb{C}$ *by*

$$\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) = |\mathfrak{v} - \mathfrak{o}| + i\,|\mathfrak{v} - \mathfrak{o}|,$$

for all $\mathfrak{v}, \mathfrak{o} \in \Psi$. Now, we prove that $(\Psi, \mathfrak{c})$ is complete CVSMS.

Condition (i).

- **Non-negativity:** For $\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) = |\mathfrak{v} - \mathfrak{o}| + i\,|\mathfrak{v} - \mathfrak{o}|$:

$$Re\,(\mathfrak{c}(\mathfrak{v}, \mathfrak{o})) = |\mathfrak{v} - \mathfrak{o}| \geq 0 \text{ and } Im\,(\mathfrak{c}(\mathfrak{v}, \mathfrak{o})) = |\mathfrak{v} - \mathfrak{o}| \geq 0.$$

  Thus, $0 + 0i \preceq \mathfrak{c}(\mathfrak{v}, \mathfrak{o})$ is satisfied.
- **Zero iff equality**:

$\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) = 0 + 0i$ implies $|\mathfrak{v} - \mathfrak{o}| = 0$, which means $\mathfrak{v} = \mathfrak{o}$. Conversely, if $\mathfrak{v} = \mathfrak{o}$, then $|\mathfrak{v} - \mathfrak{o}| = 0$ implies $\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) = 0 + 0i$.

Condition (ii).

Since

$$\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) = |\mathfrak{v} - \mathfrak{o}| + i\,|\mathfrak{v} - \mathfrak{o}| = |\mathfrak{o} - \mathfrak{v}| + i\,|\mathfrak{o} - \mathfrak{v}| = \mathfrak{c}(\mathfrak{o}, \mathfrak{v}),$$

symmetry is satisfied.

Condition (iii).

We now satisfy the triangle inequality, that is,

$$\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) \precsim \mathfrak{c}(\mathfrak{v}, \varpi) + \mathfrak{c}(\varpi, \mathfrak{o}) + \curlywedge \mathfrak{c}(\mathfrak{v}, \varpi)\mathfrak{c}(\varpi, \mathfrak{o}),$$

for all $\mathfrak{v}, \varpi, \mathfrak{o} \in \Psi$ and for some $\curlywedge \geq 0$.

(1). $\mathfrak{c}(\mathfrak{v}, \varpi) + \mathfrak{c}(\varpi, \mathfrak{o}) = (|\mathfrak{v} - \varpi| + i\,|\mathfrak{v} - \varpi|) + (|\varpi - \mathfrak{o}| + i\,|\varpi - \mathfrak{o}|)$. Simplifying, we get

$$Re\,(\mathfrak{c}(\mathfrak{v}, \varpi) + \mathfrak{c}(\varpi, \mathfrak{o})) = |\mathfrak{v} - \varpi| + |\varpi - \mathfrak{o}|,$$

and

$$Im\,(\mathfrak{c}(\mathfrak{v}, \varpi) + \mathfrak{c}(\varpi, \mathfrak{o})) = |\mathfrak{v} - \varpi| + |\varpi - \mathfrak{o}|.$$

(2). $\mathfrak{c}(\mathfrak{v}, \varpi)\mathfrak{c}(\varpi, \mathfrak{o}) = (|\mathfrak{v} - \varpi| + i\,|\mathfrak{v} - \varpi|)(|\varpi - \mathfrak{o}| + i\,|\varpi - \mathfrak{o}|)$. Simplifying, we get

$$\mathfrak{c}(\mathfrak{v}, \varpi)\mathfrak{c}(\varpi, \mathfrak{o}) = 2i\,|\mathfrak{v} - \varpi|\,|\varpi - \mathfrak{o}|.$$

Now,

$$\mathfrak{c}(\mathfrak{v}, \varpi) + \mathfrak{c}(\varpi, \mathfrak{o}) + \curlywedge \mathfrak{c}(\mathfrak{v}, \varpi)\mathfrak{c}(\varpi, \mathfrak{o}) = (|\mathfrak{v} - \varpi| + |\varpi - \mathfrak{o}|) + i\,(|\mathfrak{v} - \varpi| + |\varpi - \mathfrak{o}| + 2 \curlywedge |\mathfrak{v} - \varpi|\,|\varpi - \mathfrak{o}|).$$

Compare $\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) = |\mathfrak{v} - \mathfrak{o}| + i\,|\mathfrak{v} - \mathfrak{o}|$.

Real part:

$$|\mathfrak{v} - \mathfrak{o}| \leq |\mathfrak{v} - \varpi| + |\varpi - \mathfrak{o}|,$$

by the triangle inequality in $\mathbb{R}$.

Imaginary part:

$$|\mathfrak{v} - \mathfrak{o}| \leq |\mathfrak{v} - \varpi| + |\varpi - \mathfrak{o}| + 2 \curlywedge |\mathfrak{v} - \varpi|\,|\varpi - \mathfrak{o}|,$$

which follows from non-negativity of $\curlywedge$ and the terms. Thus, Condition (iii) is satisfied. Hence, $(\Psi, \mathfrak{c})$ is a complete CVSMS.

Define a self mapping $\mathfrak{A} : \Psi \to \Psi$ by

$$\mathfrak{A}\mathfrak{v} = \frac{1}{3}(2 - \mathfrak{v}).$$

Consider $\mathbb{k}_1 : \Psi \times \Psi \to [0, 1)$, where

$$\mathbb{k}_1(\mathfrak{v}, \mathfrak{o}) = \frac{1}{1 + \mathfrak{v} + \mathfrak{o}}.$$

In this case

$$\mathfrak{A}\mathfrak{v} - \mathfrak{A}\mathfrak{o} = \frac{1}{3}(2 - \mathfrak{v}) - \frac{1}{3}(2 - \mathfrak{o}) = \frac{1}{3}(\mathfrak{o} - \mathfrak{v}).$$

Thus,

$$|\mathfrak{A}\mathfrak{v} - \mathfrak{A}\mathfrak{o}| = \frac{1}{3}\,|\mathfrak{o} - \mathfrak{v}| = \frac{1}{3}\,|\mathfrak{v} - \mathfrak{o}|,$$

and

$$\mathfrak{c}\,(\mathfrak{A}\mathfrak{v}, \mathfrak{A}\mathfrak{o}) = \frac{1}{3}\,|\mathfrak{v} - \mathfrak{o}| + i\frac{1}{3}\,|\mathfrak{v} - \mathfrak{o}|.$$

Now assume that

$$\begin{aligned} \mathbb{k}_1(\mathfrak{v}, \mathfrak{o})\mathfrak{c}\,(\mathfrak{v}, \mathfrak{o}) &= \frac{1}{1 + \mathfrak{v} + \mathfrak{o}}\left(|\mathfrak{v} - \mathfrak{o}| + i\,|\mathfrak{v} - \mathfrak{o}|\right) \\ &= \frac{|\mathfrak{v} - \mathfrak{o}|}{1 + \mathfrak{v} + \mathfrak{o}} + i\frac{|\mathfrak{v} - \mathfrak{o}|}{1 + \mathfrak{v} + \mathfrak{o}}. \end{aligned}$$

Since $1 + \mathfrak{v} + \mathfrak{o} > 1$ for all $\mathfrak{v}, \mathfrak{o} \in [0, 1]$, we have

$$\frac{1}{3} < \frac{1}{1 + \mathfrak{v} + \mathfrak{o}}.$$

Therefore,

$$\frac{1}{3}\,|\mathfrak{v} - \mathfrak{o}| \leq \frac{|\mathfrak{v} - \mathfrak{o}|}{1 + \mathfrak{v} + \mathfrak{o}}.$$

Thus,

$$\begin{aligned} \mathfrak{c}\,(\mathfrak{A}\mathfrak{v}, \mathfrak{A}\mathfrak{o}) &= \frac{1}{3}\,|\mathfrak{v} - \mathfrak{o}| + i\frac{1}{3}\,|\mathfrak{v} - \mathfrak{o}| \\ &\leq \frac{|\mathfrak{v} - \mathfrak{o}|}{1 + \mathfrak{v} + \mathfrak{o}} + i\frac{|\mathfrak{v} - \mathfrak{o}|}{1 + \mathfrak{v} + \mathfrak{o}} \\ &= \mathbb{k}_1(\mathfrak{v}, \mathfrak{o})\mathfrak{c}\,(\mathfrak{v}, \mathfrak{o}). \end{aligned}$$

Hence, all the conditions of Corollary 6 are fulfilled, establishing that $\frac{1}{2}$ is the unique FP of the mapping $\mathfrak{A}$.

Corollary 7. *Let* $(\Psi, \mathfrak{c})$ *be a complete CVSMS and* $\mathfrak{A} : \Psi \to \Psi$. *Assuming the existence of the functions* $\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_3 : \Psi \times \Psi \to [0, 1)$ *such that*

(a) $\mathbb{k}_1(\mathfrak{A}\mathfrak{v}, \mathfrak{o}) \leq \mathbb{k}_1(\mathfrak{v}, \mathfrak{o})$ and $\mathbb{k}_1(\mathfrak{v}, \mathfrak{A}\mathfrak{o}) \leq \mathbb{k}_1(\mathfrak{v}, \mathfrak{o})$,
$\mathbb{k}_2(\mathfrak{A}\mathfrak{v}, \mathfrak{o}) \leq \mathbb{k}_2(\mathfrak{v}, \mathfrak{o})$ and $\mathbb{k}_2(\mathfrak{v}, \mathfrak{A}\mathfrak{o}) \leq \mathbb{k}_2(\mathfrak{v}, \mathfrak{o})$,
$\mathbb{k}_3(\mathfrak{A}\mathfrak{v}, \mathfrak{o}) \leq \mathbb{k}_3(\mathfrak{v}, \mathfrak{o})$ and $\mathbb{k}_3(\mathfrak{v}, \mathfrak{A}\mathfrak{o}) \leq \mathbb{k}_3(\mathfrak{v}, \mathfrak{o})$,
(b) $\mathbb{k}_1(\mathfrak{v}, \mathfrak{o}) + \mathbb{k}_2(\mathfrak{v}, \mathfrak{o}) + \mathbb{k}_3(\mathfrak{v}, \mathfrak{o}) < 1$,
(c)

$$\mathfrak{c}(\mathfrak{A}^r\mathfrak{v}, \mathfrak{A}^r\mathfrak{o}) \leq \mathbb{k}_1(\mathfrak{v}, \mathfrak{o})\,\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) + \mathbb{k}_2(\mathfrak{v}, \mathfrak{o})\frac{\mathfrak{c}(\mathfrak{v}, \mathfrak{A}^r\mathfrak{v})\,\mathfrak{c}(\mathfrak{o}, \mathfrak{A}^r\mathfrak{o})}{1 + \mathfrak{c}(\mathfrak{v}, \mathfrak{o})} + \mathbb{k}_3(\mathfrak{v}, \mathfrak{o})\frac{\mathfrak{c}(\mathfrak{o}, \mathfrak{A}^r\mathfrak{v})\,\mathfrak{c}(\mathfrak{v}, \mathfrak{A}^r\mathfrak{o})}{1 + \mathfrak{c}(\mathfrak{v}, \mathfrak{o})}, \tag{3.11}$$

for all $\mathfrak{v}, \mathfrak{o} \in \Psi$, then $\mathfrak{A}$ has a unique FP.

Proof. By Corollary (4), $\mathfrak{v} \in \Psi$ exist such that $\mathfrak{A}^r\mathfrak{v} = \mathfrak{v}$. In this case,

$$\begin{aligned}
\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{v}) &= \mathfrak{c}(\mathfrak{A}\mathfrak{A}^r\mathfrak{v}, \mathfrak{A}^r\mathfrak{v}) \\
&= \mathfrak{c}(\mathfrak{A}^r\mathfrak{A}\mathfrak{v}, \mathfrak{A}^r\mathfrak{v}) \leq \mathbb{k}_1(\mathfrak{A}\mathfrak{v}, \mathfrak{v})\,\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{v}) + \mathbb{k}_2(\mathfrak{A}\mathfrak{v}, \mathfrak{v})\frac{\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{A}^r\mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{v}, \mathfrak{A}^r\mathfrak{v})}{1 + \mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{v})} \\
&\quad + \mathbb{k}_3(\mathfrak{A}\mathfrak{v}, \mathfrak{v})\frac{\mathfrak{c}(\mathfrak{v}, \mathfrak{A}^r\mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{A}^r\mathfrak{v})}{1 + \mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{v})} \\
&\leq \mathbb{k}_1(\mathfrak{A}\mathfrak{v}, \mathfrak{v})\,\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{v}) + \mathbb{k}_2(\mathfrak{A}\mathfrak{v}, \mathfrak{v})\frac{\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{v}, \mathfrak{v})}{1 + \mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{v})} + \mathbb{k}_3(\mathfrak{A}\mathfrak{v}, \mathfrak{v})\frac{\mathfrak{c}(\mathfrak{v}, \mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{v})}{1 + \mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{v})} \\
&= \mathbb{k}_1(\mathfrak{A}\mathfrak{v}, \mathfrak{v})\,\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{v}) + \mathbb{k}_3(\mathfrak{A}\mathfrak{v}, \mathfrak{v})\frac{\mathfrak{c}(\mathfrak{v}, \mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{v})}{1 + \mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{v})},
\end{aligned}$$

which implies that

$$\begin{aligned}
|\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{v})| &\leq \mathbb{k}_1(\mathfrak{A}\mathfrak{v}, \mathfrak{v})\,|\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{v})| + \mathbb{k}_3(\mathfrak{A}\mathfrak{v}, \mathfrak{v})\,|\mathfrak{c}(\mathfrak{v}, \mathfrak{A}\mathfrak{v})|\left|\frac{\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{v})}{1 + \mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{v})}\right| \\
&\leq \mathbb{k}_1(\mathfrak{A}\mathfrak{v}, \mathfrak{v})\,|\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{v})| + \mathbb{k}_3(\mathfrak{A}\mathfrak{v}, \mathfrak{v})\,|\mathfrak{c}(\mathfrak{v}, \mathfrak{A}\mathfrak{v})| \\
&= (\mathbb{k}_1(\mathfrak{A}\mathfrak{v}, \mathfrak{v}) + \mathbb{k}_3(\mathfrak{A}\mathfrak{v}, \mathfrak{v}))\,|\mathfrak{c}(\mathfrak{v}, \mathfrak{A}\mathfrak{v})|,
\end{aligned}$$

which is possible only whenever $|\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{v})| = 0$. Thus $\mathfrak{A}\mathfrak{v} = \mathfrak{v}$. □

We now turn our attention to deriving FP results for contractive conditions encompassing control functions of a single variable.

Corollary 8. *Let* $(\Psi, \mathfrak{c})$ *be a complete CVSMS and* $\mathfrak{A}, \mathfrak{B} : \Psi \to \Psi$. *Assuming the existence of the functions* $\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_3 : \Psi \to [0, 1)$ *such that*

(a) $\mathbb{k}_1(\mathfrak{B}\mathfrak{A}\mathfrak{v}) \leq \mathbb{k}_1(\mathfrak{v})$,
$\mathbb{k}_2(\mathfrak{B}\mathfrak{A}\mathfrak{v}) \leq \mathbb{k}_2(\mathfrak{v})$,
$\mathbb{k}_3(\mathfrak{B}\mathfrak{A}\mathfrak{v}) \leq \mathbb{k}_3(\mathfrak{v})$,
(b) $\mathbb{k}_1(\mathfrak{v}) + \mathbb{k}_2(\mathfrak{v}) + \mathbb{k}_3(\mathfrak{v}) < 1$,
(c)

$$\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{B}\mathfrak{o}) \leq \mathbb{k}_1(\mathfrak{v})\,\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) + \mathbb{k}_2(\mathfrak{v})\frac{\mathfrak{c}(\mathfrak{v}, \mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{o}, \mathfrak{B}\mathfrak{o})}{1 + \mathfrak{c}(\mathfrak{v}, \mathfrak{o})} + \mathbb{k}_3(\mathfrak{v})\frac{\mathfrak{c}(\mathfrak{o}, \mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{v}, \mathfrak{B}\mathfrak{o})}{1 + \mathfrak{c}(\mathfrak{v}, \mathfrak{o})},$$

for all $\mathfrak{v}, \mathfrak{o} \in \Psi$, then $\mathfrak{A}$ and $\mathfrak{B}$ possess a unique CFP.

Proof. Define $\Bbbk_1, \Bbbk_2, \Bbbk_3 : \Psi\times\Psi\to [0,1)$ by

$$\Bbbk_1(\mathfrak{v}, \mathfrak{o}) = \Bbbk_1(\mathfrak{v}), \quad \Bbbk_2(\mathfrak{v}, \mathfrak{o}) = \Bbbk_2(\mathfrak{v}), \quad \text{and} \quad \Bbbk_3(\mathfrak{v}, \mathfrak{o}) = \Bbbk_3(\mathfrak{v})$$

for all $\mathfrak{v}, \mathfrak{o} \in \Psi$. For all $\mathfrak{v}, \mathfrak{o} \in \Psi$, we then have □

(a) $\Bbbk_1(\mathfrak{B}\mathfrak{A}\mathfrak{v}, \mathfrak{o}) = \Bbbk_1(\mathfrak{B}\mathfrak{A}\mathfrak{v}) \leq \Bbbk_1(\mathfrak{v}) = \Bbbk_1(\mathfrak{v}, \mathfrak{o})$ and $\Bbbk_1(\mathfrak{v}, \mathfrak{A}\mathfrak{B}\mathfrak{o}) = \Bbbk_1(\mathfrak{v}) = \Bbbk_1(\mathfrak{v}, \mathfrak{o})$,
$\Bbbk_2(\mathfrak{B}\mathfrak{A}\mathfrak{v}, \mathfrak{o}) = \Bbbk_2(\mathfrak{B}\mathfrak{A}\mathfrak{v}) \leq \Bbbk_2(\mathfrak{v}) = \Bbbk_2(\mathfrak{v}, \mathfrak{o})$ and $\Bbbk_2(\mathfrak{v}, \mathfrak{A}\mathfrak{B}\mathfrak{o}) = \Bbbk_2(\mathfrak{v}) = \Bbbk_2(\mathfrak{v}, \mathfrak{o})$,
$\Bbbk_3(\mathfrak{B}\mathfrak{A}\mathfrak{v}, \mathfrak{o}) = \Bbbk_3(\mathfrak{B}\mathfrak{A}\mathfrak{v}) \leq \Bbbk_3(\mathfrak{v}) = \Bbbk_3(\mathfrak{v}, \mathfrak{o})$ and $\Bbbk_3(\mathfrak{v}, \mathfrak{A}\mathfrak{B}\mathfrak{o}) = \Bbbk_3(\mathfrak{v}) = \Bbbk_3(\mathfrak{v}, \mathfrak{o})$,
(b) $\Bbbk_1(\mathfrak{v}, \mathfrak{o}) + \Bbbk_2(\mathfrak{v}, \mathfrak{o}) + \Bbbk_3(\mathfrak{v}, \mathfrak{o}) = \Bbbk_1(\mathfrak{v}) + \Bbbk_2(\mathfrak{v}) + \Bbbk_3(\mathfrak{v}) < 1$,
(c)

$$\begin{aligned} \mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{B}\mathfrak{o}) &\leq \Bbbk_1(\mathfrak{v})\,\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) + \Bbbk_2(\mathfrak{v})\,\frac{\mathfrak{c}(\mathfrak{v}, \mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{o}, \mathfrak{B}\mathfrak{o})}{1 + \mathfrak{c}(\mathfrak{v}, \mathfrak{o})} + \Bbbk_3(\mathfrak{v})\,\frac{\mathfrak{c}(\mathfrak{o}, \mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{v}, \mathfrak{B}\mathfrak{o})}{1 + \mathfrak{c}(\mathfrak{v}, \mathfrak{o})} \\ &= \Bbbk_1(\mathfrak{v}, \mathfrak{o})\,\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) + \Bbbk_2(\mathfrak{v}, \mathfrak{o})\,\frac{\mathfrak{c}(\mathfrak{v}, \mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{o}, \mathfrak{B}\mathfrak{o})}{1 + \mathfrak{c}(\mathfrak{v}, \mathfrak{o})} + \Bbbk_3(\mathfrak{v}, \mathfrak{o})\,\frac{\mathfrak{c}(\mathfrak{o}, \mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{v}, \mathfrak{B}\mathfrak{o})}{1 + \mathfrak{c}(\mathfrak{v}, \mathfrak{o})}. \end{aligned}$$

By Theorem 7, $\mathfrak{A}$ and $\mathfrak{B}$ possess a unique CFP.

Corollary 9. *Let* $(\Psi, \mathfrak{c})$ *be a complete CVSMS and* $\mathfrak{A}, \mathfrak{B} : \Psi \to \Psi$. *Assuming the existence of the functions* $\Bbbk_1, \Bbbk_2 : \Psi\to [0,1)$ *such that*

(a) $\Bbbk_1(\mathfrak{B}\mathfrak{A}\mathfrak{v}) \leq \Bbbk_1(\mathfrak{v})$,
$\Bbbk_2(\mathfrak{B}\mathfrak{A}\mathfrak{v}) \leq \Bbbk_2(\mathfrak{v})$,
(b) $\Bbbk_1(\mathfrak{v}) + \Bbbk_2(\mathfrak{v}) < 1$,
(c)

$$\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{B}\mathfrak{o}) \leq \Bbbk_1(\mathfrak{v})\,\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) + \Bbbk_2(\mathfrak{v})\,\frac{\mathfrak{c}(\mathfrak{v}, \mathfrak{A}\mathfrak{v})\,\mathfrak{c}(\mathfrak{o}, \mathfrak{B}\mathfrak{o})}{1 + \mathfrak{c}(\mathfrak{v}, \mathfrak{o})},$$

for all $\mathfrak{v}, \mathfrak{o} \in \Psi$, then $\mathfrak{A}$ and $\mathfrak{B}$ possess a unique CFP.

Proof. Define $\Bbbk_3 : \Psi\to [0,1)$ by $\Bbbk_3(\mathfrak{v}) = 0$ in Corollary 8. □

Corollary 10. *Let* $(\Psi, \mathfrak{c})$ *be a complete CVSMS and* $\mathfrak{A}, \mathfrak{B} : \Psi \to \Psi$. *Assuming the existence of the function* $\Bbbk_1 : \Psi\to [0,1)$ *such that*

(a) $\Bbbk_1(\mathfrak{B}\mathfrak{A}\mathfrak{v}) \leq \Bbbk_1(\mathfrak{v})$,
(b)

$$\mathfrak{c}(\mathfrak{A}\mathfrak{v}, \mathfrak{B}\mathfrak{o}) \leq \Bbbk_1(\mathfrak{v})\,\mathfrak{c}(\mathfrak{v}, \mathfrak{o}),$$

for all $\mathfrak{v}, \mathfrak{o} \in \Psi$, then $\mathfrak{A}$ and $\mathfrak{B}$ possess a unique CFP.

Proof. Define $\Bbbk_2, \Bbbk_3 : \Psi\to [0,1)$ by $\Bbbk_2(\mathfrak{v}) = \Bbbk_3(\mathfrak{v}) = 0$ in Corollary 8. □

Remark 1. *The main result of Panda et al. [14] follows directly from Corollary 8 by defining the functions* $\Bbbk_1, \Bbbk_2$ *and* $\Bbbk_3$ *as* $\Bbbk_1, \Bbbk_2, \Bbbk_3 : \Psi\to [0,1)$ *with* $\Bbbk_1(\cdot) = \Bbbk_1$, $\Bbbk_2(\cdot) = \Bbbk_2$ *and* $\Bbbk_3(\cdot) = \Bbbk_3$.

Corollary 11. *Let* $(\Psi, \mathfrak{c})$ *be a complete CVSMS and let* $\mathfrak{A} : \Psi \to \Psi$. *If the non-negative real numbers* $\Bbbk_1, \Bbbk_2$ *and* $\Bbbk_3$ *with* $\Bbbk_1 + \Bbbk_2 + \Bbbk_3 < 1$ *exist such that*

$$\mathfrak{c}\left(\mathfrak{A}\mathfrak{v},\mathfrak{A}\mathfrak{o}\right)\preceq\mathbb{k}_1\mathfrak{c}\left(\mathfrak{v},\mathfrak{o}\right)+\mathbb{k}_2\frac{\mathfrak{c}\left(\mathfrak{v},\mathfrak{A}\mathfrak{v}\right)\mathfrak{c}\left(\mathfrak{o},\mathfrak{A}\mathfrak{o}\right)}{1+\mathfrak{c}\left(\mathfrak{v},\mathfrak{o}\right)}+\mathbb{k}_3\frac{\mathfrak{c}\left(\mathfrak{o},\mathfrak{A}\mathfrak{v}\right)\mathfrak{c}\left(\mathfrak{v},\mathfrak{A}\mathfrak{o}\right)}{1+\mathfrak{c}\left(\mathfrak{v},\mathfrak{o}\right)},$$

for all $\mathfrak{v},\mathfrak{o}\in\Psi$, then $\mathfrak{A}$ has a unique FP.

Proof. The proof follows directly from Remark 1 by setting $\mathfrak{A}=\mathfrak{B}$, which ensures the existence of a unique FP for $\mathfrak{A}$. □

Corollary 12. *Let* $(\Psi,\mathfrak{c})$ *be a complete CVSMS and* $\mathfrak{A},\mathfrak{B}:\Psi\to\Psi$. *If* $\mathbb{k}_1,\mathbb{k}_2\in[0,1)$ *with* $\mathbb{k}_1+\mathbb{k}_2<1$ *exist such that*

$$\mathfrak{c}\left(\mathfrak{A}\mathfrak{v},\mathfrak{B}\mathfrak{o}\right)\preceq\mathbb{k}_1\mathfrak{c}\left(\mathfrak{v},\mathfrak{o}\right)+\mathbb{k}_2\frac{\mathfrak{c}\left(\mathfrak{v},\mathfrak{A}\mathfrak{v}\right)\mathfrak{c}\left(\mathfrak{o},\mathfrak{B}\mathfrak{o}\right)}{1+\mathfrak{c}\left(\mathfrak{v},\mathfrak{o}\right)},$$

for all $\mathfrak{v},\mathfrak{o}\in\Psi$, then $\mathfrak{A}$ and $\mathfrak{B}$ possess a unique CFP.

Proof. Setting $\mathbb{k}_3=0$ in Remark 1 yields the desired result. □

Corollary 13. *Let* $(\Psi,\mathfrak{c})$ *be a complete CVSMS and* $\mathfrak{A},\mathfrak{B}:\Psi\to\Psi$. *If* $\mathbb{k}_1\in[0,1)$ *exists such that*

$$\mathfrak{c}\left(\mathfrak{A}\mathfrak{v},\mathfrak{B}\mathfrak{o}\right)\preceq\mathbb{k}_1\mathfrak{c}\left(\mathfrak{v},\mathfrak{o}\right),$$

for all $\mathfrak{v},\mathfrak{o}\in\Psi$, then $\mathfrak{A}$ and $\mathfrak{B}$ possess a unique CFP.

Proof. Take $\mathbb{k}_2=0$ in Corollary 12. □

Remark 2. *By choosing* $\mathfrak{A}=\mathfrak{B}$ *and restricting the complex numbers to real numbers by setting the imaginary part to* 0 *in Corollary 13, the principal result of Berzig [6] follows.*

Remark 3. *The primary result presented by Sitthikul et al. [13] is a direct consequence of Theorem 7 when the parameter* $\curlywedge=0$.

Remark 4. *The principal finding of Sintunavarat et al. [12] emerges as a special case of Corollary 9 by taking* $\curlywedge=0$.

Remark 5. *By applying Remark 1 and choosing* $\curlywedge=0$, *the main result of Rouzkard et al. [11] follows naturally.*

Remark 6. *Taking* $\curlywedge=0$ *in Corollary 12, we obtain the leading result of Azam et al. [10].*

4. Applications

FP theorems are a powerful tool in functional analysis, providing a framework to prove the existence and uniqueness of solutions to various mathematical equations, including integral equations. These theorems are particularly valuable in addressing complex problems such as nonlinear mixed Volterra–Fredholm integral equations, where their application provides a systematic approach to establish solvability and ensure uniqueness under specific conditions.

The ensuing section is concerned with the investigation of the following nonlinear mixed Volterra–Fredholm integral equation:

$$\mathfrak{v}(t) = f(t) + \int_a^b K_1(t, \mathfrak{o})\Phi_1(\mathfrak{v}(\mathfrak{o}))\mathfrak{c}\mathfrak{o} + \int_a^t K_2(t, \mathfrak{o})\Phi_2(\mathfrak{v}(\mathfrak{o})\mathfrak{c}\mathfrak{o}, \tag{4.1}$$

where $\mathfrak{v}(t)$ is the unknown function to be determined; f is a given function from $[a, b]$ to $\mathbb{C}$; $K_1(t, \mathfrak{o})$ is a specific form of the kernel representing the Fredholm-type integral term, with fixed limits of integration over $[a, b]$; $K_2(t, \mathfrak{o})$ is a specific form of the kernel representing the Volterra-type integral term, with the upper limit of integration dependent on t. The function $\mathfrak{o}$ is a transformation function that maps the integration variable t to a new variable. $\Phi_1, \Phi_2 : \mathbb{R} \to \mathbb{C}$ are nonlinear functions of the unknown function $\mathfrak{v}$.

Nonlinear mixed Volterra–Fredholm integral equations, such as (4.1), play a crucial role in modeling biological and ecological systems, particularly in the study of species interactions. Predator-prey dynamics, competition, and other ecological processes often exhibit both temporal and spatial dependencies, which can be effectively captured using this integral framework. In a predator-prey system, Eq (4.1) accounts for both local (time-dependent) and global (space-dependent) effects as follows:

- $\mathfrak{v}(t) -$: Predator population at time t;
- $f(t)-$: External influences (e.g., seasonal effects on predator reproduction);
- $K_1(t, \mathfrak{o})-$: Kernel representing spatial interactions (e.g., prey's availability across regions);
- $\Phi_1(\mathfrak{v}(\mathfrak{o}))-$: Nonlinear effects of resource competition or habitat constraints;
- $K_2(t, \mathfrak{o})-$: Kernel for temporal interactions, representing predator-prey encounter rates;
- $\Phi_2(\mathfrak{v}(\mathfrak{o})-$: Nonlinear predator response to prey density (e.g., functional response).

This integral formulation provides a robust mathematical foundation for analyzing predator-prey interactions and can be extended to other ecological and biological systems exhibiting similar spatial-temporal dependencies. By employing FP theory, we establish the conditions under which (4.1) admits a unique solution, ensuring its theoretical and practical relevance in ecological modeling.

We now present and prove a theorem that establishes the solution of a mixed Volterra–Fredholm integral equation using the FP result obtained in this study.

Theorem 8. *Consider* $\Psi = C([a, b])$. *Define* $\mathfrak{c} : C([a, b]) \times C([a, b]) \to \mathbb{C}$ *by*

$$\mathfrak{c}(\mathfrak{v}, \mathfrak{o}) = \max_{t \in [a,b]} \|\mathfrak{v} - \mathfrak{o}\| \, e^{i \tan^{-1}(a)},$$

in which case, $(\Psi, \mathfrak{c})$ is a CVSMS. Assume the following:

(i) $M > 0$ exists such that

$$\|K_1(t, \mathfrak{o}(t))\| \leq M \text{ and } \|K_2(t, \mathfrak{o}(t))\| \leq M,$$

for all $t \in [a, b]$.

(ii) The nonlinear functions Φ_1 and Φ_2 satisfy the contractive condition

$$\|\Phi_1(\mathfrak{v}_1) - \Phi_1(\mathfrak{v}_2)\| \leq \lambda \|\mathfrak{v}_1 - \mathfrak{v}_2\| \text{ and } \|\Phi_2(\mathfrak{v}_1) - \Phi_2(\mathfrak{v}_2)\| \leq \beta \|\mathfrak{v}_1 - \mathfrak{v}_2\|,$$

for some constants $0 < \lambda, \beta < 1$ and for all $\mathfrak{v}_1, \mathfrak{v}_2 \in C([a, b])$.

(iii) There is a function $\Bbbk_1 : C([a,b]) \times C([a,b]) \to [0,1)$ defined by

$$\Bbbk_1(\mathfrak{v}_1, \mathfrak{v}_2) = \min\left\{1, \frac{M(\lambda+\beta)(b-a)}{e^{i\tan^{-1}(a)}} \frac{\|\mathfrak{v}_1 - \mathfrak{v}_2\|}{\mathfrak{c}(\mathfrak{v}_1, \mathfrak{v}_2)}\right\},$$

where M, λ, β, a, b are such that $M(\lambda+\beta)(b-a) < 1$.

Then the integral equation

$$\mathfrak{v}(t) = f(t) + \int_a^b K_1(t, \mathfrak{o}(t))\Phi_1(\mathfrak{v}(\mathfrak{o}(t)))\mathfrak{c}t + \int_a^t K_2(t, \mathfrak{o}(t))\Phi_2(\mathfrak{v}(\mathfrak{o}(t)))\mathfrak{c}t,$$

possesses a unique solution in $C([a,b])$.

Proof. Define the mapping $\mathfrak{A} : C([a,b]) \to C([a,b])$ by

$$\mathfrak{A}\mathfrak{v}(t) = f(t) + \int_a^b K_1(t, \mathfrak{o}(t))\Phi_1(\mathfrak{v}(\mathfrak{o}(t)))\mathfrak{c}t + \int_a^t K_2(t, \mathfrak{o}(t))\Phi_2(\mathfrak{v}(\mathfrak{o}(t)))\mathfrak{c}t.$$

Consider $\mathfrak{v}_1, \mathfrak{v}_2 \in C([a,b])$. In this case,

$$\mathfrak{c}(\mathfrak{A}\mathfrak{v}_1, \mathfrak{A}\mathfrak{v}_2) = \max_{t\in[a,b]} \|\mathfrak{A}\mathfrak{v}_1(t) - \mathfrak{A}\mathfrak{v}_2(t)\| \, e^{i\tan^{-1}(a)}.$$

Now, for each $t \in [a,b]$, we have

$$\|\mathfrak{A}\mathfrak{v}_1(t) - \mathfrak{A}\mathfrak{v}_2(t)\| = \left\| \begin{array}{c} f(t) + \int_a^b K_1(t, \mathfrak{o}(s))\Phi_1(\mathfrak{v}_1(\mathfrak{o}(s)))\mathfrak{c}s + \int_a^t K_2(t, \mathfrak{o}(s))\Phi_2(\mathfrak{v}_1(\mathfrak{o}(s)))\mathfrak{c}s \\ -\left(f(t) + \int_a^b K_1(t, \mathfrak{o}(s))\Phi_1(\mathfrak{v}_2(\mathfrak{o}(s)))\mathfrak{c}s + \int_a^t K_2(t, \mathfrak{o}(s))\Phi_2(\mathfrak{v}_2(\mathfrak{o}(s)))\mathfrak{c}s\right) \end{array} \right\|.$$

This simplifies to

$$\begin{aligned} \|\mathfrak{A}\mathfrak{v}_1(t) - \mathfrak{A}\mathfrak{v}_2(t)\| \;\leq\; & \int_a^b \|K_1(t, \mathfrak{o}(s))\| \, \|\Phi_1(\mathfrak{v}_1(\mathfrak{o}(s))) - \Phi_1(\mathfrak{v}_2(\mathfrak{o}(s)))\| \, \mathfrak{c}s \\ & + \int_a^t \|K_2(t, \mathfrak{o}(s))\| \, \|\Phi_2(\mathfrak{v}_1(\mathfrak{o}(s))) - \Phi_2(\mathfrak{v}_2(\mathfrak{o}(s)))\| \, \mathfrak{c}s. \end{aligned}$$

Using the contractive property of Φ_1 and Φ_2, we obtain

$$\|\mathfrak{A}\mathfrak{v}_1(t) - \mathfrak{A}\mathfrak{v}_2(t)\| \leq M\left(\begin{array}{c} \lambda \int_a^b \|\mathfrak{v}_1(\mathfrak{o}(s)) - \mathfrak{v}_2(\mathfrak{o}(s))\| \, \mathfrak{c}s \\ +\beta \int_a^t \|\mathfrak{v}_1(\mathfrak{o}(s)) - \mathfrak{v}_2(\mathfrak{o}(s))\| \, \mathfrak{c}s \end{array} \right).$$

Since $\|\mathfrak{v}_1(\mathfrak{o}(s)) - \mathfrak{v}_2(\mathfrak{o}(s))\| \leq \max_{t\in[a,b]} \|\mathfrak{v}_1(\mathfrak{o}(t)) - \mathfrak{v}_2(\mathfrak{o}(t))\| = \frac{\mathfrak{c}(\mathfrak{v}_1, \mathfrak{v}_2)}{e^{i\tan^{-1}(a)}}$,

$$\|\mathfrak{A}\mathfrak{v}_1(t) - \mathfrak{A}\mathfrak{v}_2(t)\| \leq M\left(\begin{array}{c} \lambda \int_a^b \frac{\mathfrak{c}(\mathfrak{v}_1, \mathfrak{v}_2)}{e^{i\tan^{-1}(a)}} \mathfrak{c}s \\ +\beta \int_a^t \frac{\mathfrak{c}(\mathfrak{v}_1, \mathfrak{v}_2)}{e^{i\tan^{-1}(a)}} \mathfrak{c}s \end{array} \right). \tag{4.2}$$

Therefore,

$$\int_a^b \frac{\mathfrak{c}(\mathfrak{v}_1, \mathfrak{v}_2)}{e^{i\tan^{-1}(a)}} \mathfrak{c}s = (b-a)\frac{\mathfrak{c}(\mathfrak{v}_1, \mathfrak{v}_2)}{e^{i\tan^{-1}(a)}},$$

and

$$\int_a^t \frac{\mathfrak{c}(\mathfrak{v}_1, \mathfrak{v}_2)}{e^{i\tan^{-1}(a)}} \mathfrak{c}s = (t-a) \frac{\mathfrak{c}(\mathfrak{v}_1, \mathfrak{v}_2)}{e^{i\tan^{-1}(a)}}.$$

Substituting these into the inequality (4.2), we have

$$\begin{aligned} \|\mathfrak{A}\mathfrak{v}_1(t) - \mathfrak{A}\mathfrak{v}_2(t)\| &\le M \begin{pmatrix} \lambda (b-a) \frac{\mathfrak{c}(\mathfrak{v}_1,\mathfrak{v}_2)}{e^{i\tan^{-1}(a)}} \\ +\beta (t-a) \frac{\mathfrak{c}(\mathfrak{v}_1,\mathfrak{v}_2)}{e^{i\tan^{-1}(a)}} \end{pmatrix} \\ &= M \frac{\mathfrak{c}(\mathfrak{v}_1, \mathfrak{v}_2)}{e^{i\tan^{-1}(a)}} \begin{pmatrix} \lambda (b-a) \\ +\beta (t-a) \end{pmatrix}. \end{aligned}$$

Taking the maximum for $t \in [a, b]$, the term $\lambda (b-a) + \beta (t-a)$ achieves its maximum at $t = b$, giving

$$\begin{aligned} \|\mathfrak{A}\mathfrak{v}_1(t) - \mathfrak{A}\mathfrak{v}_2(t)\| &\le M \frac{\mathfrak{c}(\mathfrak{v}_1, \mathfrak{v}_2)}{e^{i\tan^{-1}(a)}} (\lambda (b-a) + \beta (b-a)) \\ &= M (\lambda + \beta) (b-a) \frac{\mathfrak{c}(\mathfrak{v}_1, \mathfrak{v}_2)}{e^{i\tan^{-1}(a)}}. \end{aligned} \tag{4.3}$$

Since $\mathfrak{c}$ is defined as the maximum over $t \in [a, b]$,

$$\mathfrak{c} (\mathfrak{A}\mathfrak{v}_1, \mathfrak{A}\mathfrak{v}_2) = \max_{t\in[a,b]} \|\mathfrak{A}\mathfrak{v}_1(t) - \mathfrak{A}\mathfrak{v}_2(t)\| \, e^{i\tan^{-1}(a)}.$$

Including the exponential factor $e^{i\tan^{-1}(a)}$ in the inequality (4.3), we have

$$\mathfrak{c} (\mathfrak{A}\mathfrak{v}_1, \mathfrak{A}\mathfrak{v}_2) \le \Bbbk_1(\mathfrak{v}_1, \mathfrak{v}_2)\mathfrak{c} (\mathfrak{v}_1, \mathfrak{v}_2).$$

Moreover, since $M (\lambda + \beta) (b-a) < 1$ and $\frac{\|\mathfrak{v}_1-\mathfrak{v}_2\|}{\mathfrak{c}(\mathfrak{v}_1,\mathfrak{v}_2)} \le 1$, it follows by the definition of $\Bbbk_1{:}C([a, b]) \times C([a, b]) \to [0, 1)$ that $\Bbbk_1(\mathfrak{v}_1, \mathfrak{v}_2) <1$. Hence, all the conditions of Corollary 6 are satisfied. Therefore, the mixed Volterra–Fredholm integral equation admits a unique solution, identical to its unique FP. □

Example 5. *Consider the nonlinear mixed Volterra–Fredholm integral equation*

$$\mathfrak{v}(t) = f(t) + \int_0^1 e^{t-\mathfrak{o}}\mathfrak{v}^2(\mathfrak{o})\mathfrak{c}\mathfrak{o} + \int_0^t (t-\mathfrak{o})\, \mathfrak{v}(\mathfrak{o})\mathfrak{c}\mathfrak{o},$$

where $f(t) = 1 + t + e^t(1-e)$, $K_1(t, \mathfrak{o}) = e^{t-\mathfrak{o}}$, $K_2(t, \mathfrak{o}) = (t-\mathfrak{o})$, $\Phi_1(\mathfrak{v}) = \mathfrak{v}^2$, and $\Phi_2(\mathfrak{v}) = \mathfrak{v}$. This integral equation has a unique solution $\mathfrak{v}(t) = e^t$.

5. Conclusion and future work

In this research work, we have explored the concept of CVSMSs and have established CFP theorems for rational contractions incorporating two variables control functions. Additionally, the study has derived CFP theorems for rational contractions with control functions of a single variable in the same setting. Our results directly imply the main findings of Berzig [6], Azam et al. [10], Rouzkard et al. [11], Sintunavarat et al. [12], Sitthikul et al. [13], and Panda et al. [14]. A demonstrative example has been provided to showcase the novelty of the primary results. The results are practically demonstrated by solving a nonlinear mixed Volterra–Fredholm integral equation that models predator-prey dynamics.

Future research directions include extending CFP theorems to multi-valued, fuzzy, and L-fuzzy mappings within the framework of CVSMSs. Additionally, exploring differential and integral inclusions in this setting presents a promising avenue for further study. It is expected that the findings of this research will inspire continued investigations and refinements, potentially broadening the scope of applications for these results.

Use of AI tools declaration

The author confirms that no Artificial Intelligence (AI) tools were utilized in the creation of this article.

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Conflicts of interest

The author declares no conflicts of interest.

References

1. M. M. Frechet, Sur quelques points du calcul fonctionnel, *Rend. Circ. Matem. Palermo*, **22** (1906), 1–72. https://doi.org/10.1007/BF03018603
2. S. G. Matthews, Partial metric topology, *Ann. NY. Acad. Sci.*, **728** (1994), 183–197. https://doi.org/10.1111/j.1749-6632.1994.tb44144.x
3. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostraviensis*, **1** (1993), 5–11
4. A. Branciari, A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces, *Publ. Math. Debrecen*, **57** (2000), 31–37. https://doi.org/10.5486/PMD.2000.2133
5. L. G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **332** (2007), 1468–1476. https://doi.org/10.1016/j.jmaa.2005.03.087
6. M. Berzig, First results in suprametric spaces with applications, *Mediterr. J. Math.*, **19** (2022), 226. https://doi.org/10.1007/s00009-022-02148-6
7. M. Berzig, Fixed point results in generalized suprametric spaces, *Topological Algebra and its Applications*, **11** (2023), 20230105. https://doi.org/10.1515/taa-2023-0105
8. M. Berzig, Nonlinear contraction in b-suprametric spaces, *J. Anal.*, **32** (2024), 2401–2414. https://doi.org/10.1007/s41478-024-00732-5
9. M. Berzig, Strong b-suprametric spaces and fixed point principles, *Complex Anal. Oper. Theory*, **18** (2024), 148. https://doi.org/10.1007/s11785-024-01594-2

10. A. Azam, B. Fisher, M. Khan, Common fixed point theorems in complex valued metric spaces, *Numer. Func. Anal. Opt.*, **32** (2011), 243–253. https://doi.org/10.1080/01630563.2011.533046

11. F. Rouzkard, M. Imdad, Some common fixed point theorems on complex valued metric spaces, *Comp. Math. Appl.*, **64** (2012), 1866–1874. https://doi.org/10.1016/j.camwa.2012.02.063

12. W. Sintunavarat, P. Kumam, Generalized common fixed point theorems in complex valued metric spaces and applications, *J. Inequal. Appl.*, **2012** (2012), 84. https://doi.org/10.1186/1029-242X-2012-84

13. K. Sitthikul, S. Saejung, Some fixed point theorems in complex valued metric spaces, *Fixed Point Theory Appl.*, **2012** (2012), 189. https://doi.org/10.1186/1687-1812-2012-189

14. S. K. Panda, V. Vijayakumar, R. P. Agarwal, Complex-valued suprametric spaces, related fixed point results, and their applications to Barnsley Fern fractal generation and mixed Volterra–Fredholm integral equations, *Fractal Fract.*, **8** (2024), 410. https://doi.org/10.3390/fractalfract8070410

15. S. T. Zubair, M. Aphane, A. Mukheimer, T. Abdeljawad, A fixed point technique for solving boundary value problems in branciari suprametric spaces, *Results in Nonlinear Analysis*, **7** (2024), 80–93. https://doi.org/10.31838/rna/2024.07.03.008

16. Y. Hao, J. Gou, H. Guan, The uniqueness of fixed points for two new classes of contractive mappings of integral type in b-metric spaces, *J. Nonlinear Funct. Anal.*, **2024** (2024), 30. https://doi.org/10.23952/jnfa.2024.30

17. J. Appell, N. Merentes, S. Reinwand, G. Vinti, How to reduce some fixed point theorems, *Appl. Set-Valued Anal. Optim.*, **6** (2024), 1–11.

18. A. Latif, A. H. Alotaibi, M. Noorwali, Fixed point results via multivalued contractive type mappings involving a generalized distance on metric type spaces, *J. Nonlinear Var. Anal.*, **8** (2024), 787–798. https://doi.org/10.23952/jnva.8.2024.5.06

19. M. Bohner, M. Fan, J. M. Zhang, Existence of periodic solutions in predator-prey and competition dynamic systems, *Nonlinear Anal.-Real*, **7** (2006) 1193–1204. https://doi.org/10.1016/j.nonrwa.2005.11.002

20. M. S. Boyce, Modeling predator-prey dynamics, In: *Research techniques in animal ecology: controversies and consequences*, New York: Columbia University Press, 2000, 253–287.