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Research article

Advances on fractal measures of Cartesian product sets in Euclidean space

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Abstract: Let μ and ν be two compactly supported Borel probability measures on $\mathbb{R}^d$ and $\mathbb{R}^l$, respectively, and let $q \in \mathbb{R}$ and h, g be two Hausdorff functions. In this paper, we are concerned with evaluation of the lower and upper Hewitt-Stromberg measure of Cartesian product sets, denoted, respectively, by $\mathsf{H}_\mu^{q,g}$ and $\mathsf{P}_\nu^{q,h}$, by means of the measure of their components. This is done by the construction of new multifractal measures in a similar manner to Hewitt-Stomberg measures but using the class of all (semi-) half-open binary cubes of covering sets in the definition rather than the class of all balls. Our derived product formula excludes the 0–∞ case, and our approach is uniquely applied within an Euclidean space, distinguishing it from those previously utilized in metric spaces. Furthermore, by examining the measures of symmetric generalized Cantor sets, we establish that the exclusion of the 0–∞ condition is essential and cannot be omitted.

Keywords: iterated function system; generalized affine fractal interpolation function; linear transformation
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1. Introduction and main results

Let $\mathcal{X}$ and $\mathcal{Y}$ be two separable metric spaces and let $\mathcal{M}(\mathcal{X})$ be the family of compactly supported Borel probability measures on $\mathcal{X}$. We say that $\mu \in \mathcal{M}(\mathcal{X})$ satisfies the doubling condition if

$$\limsup_{r \searrow 0} \left(\sup_{x \in \mathsf{supp}(\mu)} \frac{\mu(B(x, ar))}{\mu(\mathsf{B}(x, r))} \right) < \infty$$

for some $a > 1$ (or, equivalently, any $a > 1$), where $\mathsf{B}(x, r)$ is closed ball with a center x and a radius r. We use $\mathcal{M}_0(\mathcal{X})$ to denote the family of compactly supported Borel probability measures on $\mathcal{X}$ that fulfill the doubling condition [1]. To study the multifractal analysis of measures introduced by Mandelbrot in [2, 3], we must turn back to the study of sets related to the local behavior of such measures, called level sets, defined, for $\beta \in \mathbb{R}$, as :

$$E_\mu(\beta) = \left\{x \in \mathsf{supp}(\mu);\ \ \lim_{r\to 0} \frac{\log \mu(\mathsf{B}(x,r))}{\log r} = \beta\right\},$$

where $\mathsf{supp}(\mu)$ is the topologic support of μ, and $\mathsf{B}(x,r)$ stands for the closed ball with a center x and a radius $r > 0$. Thus, this study is essentially linked to its punctual nature and falls under set theory. However, some geometric sets are essentially known by means of the measures that are supported by them, i.e.,

$$\nu(A) = \sup\{\nu(B),\ \ B \subset A\},$$

for a given measure ν and a given set A. Hence, when we consider a set A, we focus on the properties of the measure ν rather than the geometric structure of A. The set A is thus partitioned into α-level sets $E_\mu(\beta)$. This allows the inclusion of μ into the computation of the fractal measures and dimensions. Olsen, in [1], introduced the multifractal generalizations of the fractal dimensions. This is achieved by constructing the generalization of Hausdorff and the packing measures, denoted $\mathcal{H}_\mu^{q,t}$ and $\mathcal{P}_\mu^{q,t}$ in $\mathbb{R}^d$, where $d \geq 1$, respectively. Later, in [4], the authors introduced a new multifractal formalism that deviates from the classical approach. To achieve this, they constructed two distinct measures known as the lower and upper Hewitt-Stromberg (H-S) measures, denoted, respectively, by $\mathsf{H}_\mu^{q,t}$ and $\mathsf{P}_\mu^{q,t}$. These measures serve as fundamental tools in the analysis of multifractal structures. Given the importance of these measures in this study, it is crucial to examine their properties, including their behavior on product sets and their density characteristics both of which play a critical role in understanding the broader implications of this new formalism. In particular, in [5], the author proved the existence of a constant $c > 0$ such that, for any measurable sets $A \subseteq \mathbb{R}^d$ and $B \subseteq \mathbb{R}^l$, the following inequality holds:

$$\mathcal{H}_{\mu\times\nu}^{q,s+t}(A \times B) \leq c_1\ \ \mathcal{H}_\mu^{q,s}(A)\, \mathcal{H}_\nu^{q,t}(B) \leq c_2\ \ \mathcal{P}_{\mu\times\nu}^{q,s+t}(A \times B) \leq c_3\ \ \mathcal{P}_\mu^{q,s}(A)\, \mathcal{P}_\nu^{q,t}(B), \tag{1.1}$$

provided that we have the measures $\mu \in \mathcal{M}_0(\mathbb{R}^d)$ and $\nu \in \mathcal{M}_0(\mathbb{R}^l)$ and with the convention that

$$0 \times \infty = 0.$$

The constant c_i $(i = 1, 2, 3)$ depends only on certain structural parameters, such as the dimensions d and l, but is independent of the specific choice of E and F. Moreover, in the specific case $q = 0$, the associated dimensional inequalities for the products of these measures have been derived in [6–8]. For additional related discussions, the readers may consult [9, 10]. Furthermore, the inequalities above are explicitly stated in this case in [8, 11–13]. In particular, if

$$d = l = 1 \quad \text{and} \quad \mu = \nu$$

are basically the Lebesgue measure on $\mathbb{R}$, one has, for

$$q + s = q + t = \log 2/ \log 3$$

and $E = F$ as the middle third Cantor [14, 15]

$$\mathcal{H}_\mu^{q,s}(A)\, \mathcal{P}_\mu^{q,t}(B) = 1 \times 4^t < \mathcal{P}_{\mu\times\nu}^{q,s+t}(A \times B) = 4^{s+t} = \mathcal{P}_\mu^{q,s}(A)\, \mathcal{P}_\nu^{q,t}(B).$$

Remark 1. *The equation of* (1.1) *has important physical interpretations depending on the context. Note, for* $q = 0$*, that*

$$\mathcal{H}^1 = \mathcal{L}^1,$$

is the one-dimensional Lebesgue measure. In particular, if $A, B \subseteq \mathbb{R}$*, then the product set* $A \times B$ *forms a subset of* $\mathbb{R}^2$*, and then* (1.1) *gives an approximation of the Hausdorff measure of* $A \times B$ *using the area of the region covered by the Cartesian product. These prove, in particular, that*

$$\mathcal{H}^2(A \times B) \neq \mathcal{H}^1(A)\mathcal{H}^1(B).$$

A Hausdorff function

$$h : \mathbb{R}_+ \to \mathbb{R}_+$$

is a function that is increasing, continuous, and satisfies

$$h(0) = 0.$$

These functions are often used in the context of geometric measure theory, particularly in defining Hausdorff measures. Let $\mathcal{F}$ denote the set of all such dimension functions, i.e., the set of all Hausdorff functions. Additionally, a Hausdorff function h is considered to fulfill the doubling condition if a positive constant γ exists such that the following inequality holds:

$$h(2r) \leq \gamma h(r), \quad \text{for all } r > 0.$$

This condition essentially ensures that h does not grow too quickly and is often used to ensure specific regularity properties of the corresponding measures. The subset of $\mathcal{F}$ consisting of all Hausdorff functions that satisfy the doubling condition is denoted by $\mathcal{F}_0$. Recently, in [16], the authors introduced the generalized pseudo-packing measure $\mathsf{R}_\mu^{q,h}$ and they proved that

$$\mathcal{H}_{\mu\times\nu}^{q,hg}(A \times B) \leq \mathcal{H}_\mu^{q,h}(A)\ \mathsf{R}_\nu^{q,g}(B) \leq \mathsf{R}_{\mu\times\nu}^{q,hg}(A \times B), \tag{1.2}$$

for all $A \subseteq \mathcal{X}$ and $B \subseteq \mathbb{Y}$, provided that we do not have 0–∞ case; that is, the product on the medium side does not take the form $0 \times \infty$ or $\infty \times 0$. Note that we do not any restriction on the measures μ, ν, h, and g; that is, they do not satisfy necessary the doubling condition. In addition, one has (see again [16])

$$\mathcal{P}_{\mu\times\nu}^{q,hg}(A \times B) \leq \mathsf{Q}_\mu^{q,h}(A)\ \mathcal{P}_\nu^{q,g}(B), \tag{1.3}$$

except in the 0–∞ case, where $\mathsf{Q}_\mu^{q,h}$ is the weighted generalized packing measure. In particular, one can obtain (1.1) under appropriate geometric conditions on $\mathcal{X}$ and $\mathcal{Y}$ (amenable to packing) [8, 16].

Traditional packing and Hausdorff measures are defined using packings and coverings made up of collections of balls with diameters less than a given positive value δ. An alternative approach to constructing fractal measures utilizes packings and coverings by using families of balls with a fixed diameter δ. These measures, known as H-S measures, were first introduced in [17, Exercise (10.51)]. They were later explicitly described in Pesin's monograph [18] and are also referenced, albeit in an implicit manner, in foundational works such as Mattila's [19]. The importance of H-S measures goes beyond their theoretical definition; they offer a flexible framework for analyzing fractals and their complex characteristics. Numerous studies, including [20–23] for H-S measures and [24–26] for

Standard measures, have demonstrated their utility in exploring the local properties of fractals and the behavior of fractal products. These works underscore the adaptability of H-S measures across various contexts, thus enriching the field of fractal geometry and its applications. Furthermore, Edgar's comprehensive exposition of these measures [27, pp. 32–36] provides a clear and accessible introduction, thoroughly detailing their construction, properties, and potential applications.

In Section 3, we are interested in studying the counterpart of the formula (1.1) related to the lower and upper H-S measures in Euclidean space. This result was shown for $q = 0$ in [28] in Euclidean space. We will prove the following theorem.

Theorem 1. *Let $A \subseteq \mathbb{R}^d$, $B \subseteq \mathbb{R}^l$, $\mu \in \mathcal{M}_0(\mathbb{R}^d)$, $\nu \in \mathcal{M}_0(\mathbb{R}^l)$, $h, g \in \mathcal{F}_0$ and $q \in \mathbb{R}$. Positive constants c_1–c_4 exist such that*

$$\mathsf{H}_\mu^{q,h}(A)\mathsf{H}_\nu^{q,g}(B) \leq c_1 \mathsf{H}_{\mu\times\nu}^{q,hg}(A \times B) \leq c_2\, \mathsf{H}_\mu^{q,h}(A)\mathsf{P}_\nu^{q,g}(B) \leq c_3\, \mathsf{P}_{\mu\times\nu}^{q,hg}(A \times B) \leq c_4 \mathsf{P}_\mu^{q,h}(A)\mathsf{P}_\nu^{q,g}(B), \qquad (1.4)$$

except in the 0–∞ *case.*

To prove the first inequality, we introduce a new multifractal measure that parallels the lower H-S measure and is notably simpler to analyze. This is achieved by utilizing a class of half-open dyadic cubes as covering sets in the definition, instead of using closed balls. The use of half-open dyadic cubes provides a new framework for the analysis, simplifying the structure of the measure. For the second inequality, we extend the technique by replacing the traditional dyadic cubes with half-open semi-dyadic cubes. This adjustment leads to the definition of two distinct measures that correspond to the upper and lower H-S measures. This choice arises from the fact that semi-dyadic cubes $v_n(x)$ are less sensitive to the position of x compared with the corresponding dyadic cubes $u_n(x)$. Semi-dyadic cubes have been utilized in works such as [5, 13, 29]. It is important to note that this construction is specific to Euclidean space, making our proof distinct from those in [30].

Remark 2. *It is important to emphasize that our analysis was not conducted for an arbitrary subset $\Gamma \subset \mathbb{R}^2$, but specifically for cases where Γ takes the form of a Cartesian product*

$$\Gamma = A \times B.$$

Addressing such a problem is far from straightforward, as it necessitates the application of integral versions of product set. For a deeper exploration of these techniques and their implications, we refer the reader to [5, 31, 32].

When

$$h(r) = r^t,$$

the measures $\mathsf{H}_\mu^{q,h}$ and $\mathsf{P}_\mu^{q,h}$ are simply denoted as $\mathsf{H}_\mu^{q,t}$ and $\mathsf{P}_\mu^{q,t}$, respectively. In this case, these measures assign, in the standard manner, a multifractal dimension to each subset A of $\mathbb{R}^d$, defined as follows:

$$\mathsf{b}_\mu^q(A) = \inf\{t \in \mathbb{R},\ \mathsf{H}_\mu^{q,t}(A) = \infty\} \quad \text{and} \quad \mathsf{B}_\mu^q(A) = \inf\{t \in \mathbb{R},\ \mathsf{P}_\mu^{q,t}(A) = \infty\}.$$

If $q = 0$, $\mathsf{b}_\mu(A)$ and $\mathsf{B}_\mu(A)$ do not depend on μ and are simply denoted b and B, respectively. Theorem A implies, when all the hypothesis are satisfied, that

$$\mathsf{b}_{\mu_1}^q(A) + \mathsf{b}_{\mu_2}^q(B) \leq \mathsf{b}_{\mu_1\times\mu_2}^q(A \times B) \leq \mathsf{b}_{\mu_1}^q(A) + \mathsf{B}_{\mu_2}^q(B) \leq \mathsf{B}_{\mu_1\times\mu_2}^q(A \times B). \qquad (1.5)$$

Moreover, all these inequalities may be strict. Indeed, one can construct two sets A and B such that

$$\mathsf{b}^q_{\mu_1}(A) + \mathsf{b}^q_{\mu_2}(B) < \mathsf{b}^q_{\mu_1\times\mu_2}(A \times B),$$

(see [33] for $q = 0$). However, in Example 2, we give a sufficient condition to get the first equality in Eq (1.5):

$$\mathsf{b}^q_{\mu_1}(A) + \mathsf{b}^q_{\mu_2}(B) = \mathsf{b}^q_{\mu_1\times\mu_2}(A \times B).$$

One can define also the multifractal separator functions

$$\mathsf{b}_\mu(q) = \mathsf{b}^q_\mu(\mathsf{supp}(\mu))$$

and

$$\mathsf{B}_\mu(q) = \mathsf{B}^q_\mu(\mathsf{supp}(\mu)).$$

Where b_μ is known to be a decreasing function, while B_μ is both a decreasing and convex function [4]. In addition, it holds that

$$\mathsf{b}_\mu \leq \mathsf{B}_\mu.$$

As a consequence, since

$$\mathsf{supp}(\mu_1 \times \mu_2) = \mathsf{supp}(\mu_1) \times \mathsf{supp}(\mu_2),$$

we get the following result:

$$\mathsf{b}_{\mu_1}(q) + \mathsf{b}_{\mu_2}(q) \leq \mathsf{b}_{\mu_1\times\mu_2}(q) \leq \mathsf{b}_{\mu_1}(q) + \mathsf{B}_{\mu_2}(q) \leq \mathsf{B}_{\mu_1\times\mu_2}(q), \tag{1.6}$$

by taking

$$E = \mathsf{supp}(\mu_1)$$

and

$$F = \mathsf{supp}(\mu_2)$$

in Theorem 1. Similar results were also proven for the s-dimensional Hausdorff measure $\mathcal{H}^s$ and the s-dimensional packing measure $\mathcal{P}^s$ [6, 13, 34, 35]. In addition, a variety of related results and further developments on this problem can be found in the works of [36, 37].

Now, given $\mu, \theta \in \mathcal{P}(\mathbb{R}^d)$, $q \in \mathbb{R}$, $h, g \in \mathcal{F}_0$, and $x \in \mathsf{supp}(\mu)$, we define the upper and lower (q, s)-densities of θ at x with respect to μ as

$$\overline{d}^{q,h}_\mu(x, \theta) = \limsup_{r\to 0} \frac{\theta(\mathsf{B}(x, r))}{\mu(\mathsf{B}(x, r))^q h(2r)} \quad \text{and} \quad \underline{d}^{q,h}_\mu(x, \theta) = \liminf_{r\to 0} \frac{\theta(\mathsf{B}(x, r))}{\mu(\mathsf{B}(x, r))^q h(2r)}. \tag{1.7}$$

If

$$\underline{d}^{q,h}_\mu(x, \theta) = \overline{d}^{q,h}_\mu(x, \theta),$$

we use $d^{q,h}_\mu(x, \theta)$ to denote the common value. In [30], the authors used some density inequalities as "local versions" of the product inequalities. In particular, they proved that the inequality

$$\mathsf{P}^{q,hg}_{\mu\times\nu}(A \times B) \leq c\mathsf{P}^{q,h}_\mu(A)\mathsf{P}^{q,g}_\nu(B)$$

may be deduced from the following density inequality:

$$c\,\underline{d}^{q,hg}_{\mu\times\nu}((x,y),\theta_1\times\theta_2)\geq \underline{d}^{q,h}_{\mu}(x,\theta_1)\underline{d}^{q,g}_{\nu}(y,\theta_2),$$

where θ_1 is the restriction of $\mathsf{P}^{q,h}_{\mu}$ to E and θ_2 is the restriction of $\mathsf{P}^{q,g}_{\nu}$ to B.

The set A satisfies the condition

$$\mathsf{b}^q_\mu(A)=\mathsf{B}^q_\mu(A)$$

for measures μ under consideration, which will be called regular set. Regularity is defined with respect to various measures, such as the packing measure [29, 38], the Hausdorff measure [39–42], and the H-S measure [43–45]. Notably, Tricot et al. [38, 46] demonstrated that a subset A of $\mathbb{R}^d$ has integer Hausdorff and packing dimensions if it is strongly regular, meaning that

$$\mathcal{H}^t(A)=\mathcal{P}^t(A)$$

for $t\geq 0$. Furthermore, as a consequence of (1.5), it follows that if either E or F is regular, then

$$\mathsf{b}^q_{\mu_1}(A)+\mathsf{b}^q_{\mu_2}(B)=\mathsf{b}^q_{\mu_1\times\mu_2}(A\times B)=\mathsf{B}^q_{\mu_1\times\mu_2}(A\times B). \tag{1.8}$$

In Theorem 1, we assume that the products do not take the form $0\times\infty$ or $\infty\times 0$. In Section 4, by estimating the measure of d-dimensional symmetric generalized Cantor sets, we demonstrate that this assumption is essential and can not be omitted. Specifically, let $0<\alpha,\beta<1$, to establish the second inequality in (1.4), we then need to prove that

$$\overline{\mathsf{H}}^{q,\alpha+\beta}_{\mu\times\nu,0}(H)\leq c\,\overline{\mathsf{H}}^{q,\alpha}_{\mu,0}(A)\,\overline{\mathsf{P}}^{q,\beta}_{\nu}(B),$$

for all

$$H\subseteq A\times B$$

and some positive constant c, where $\overline{\mathsf{H}}^{q,\alpha}_{\mu,0}$ and $\overline{\mathsf{P}}^{q,\beta}_{\nu}$ are the pre-lower and pre-upper H-S measures, respectively (see Section 3.2 and Eq (3.2)). We establish the following result.

Theorem 2. *One-dimensional generalized Cantor sets $\mathcal{K}_1$, $\mathcal{K}_2^1$, $\mathcal{K}_2^2$, and $\mathcal{K}_2^3$ such that*

$$\overline{\mathsf{H}}^{q,\alpha}_{\mu,0}(\mathcal{K}_1)=0,\quad \overline{\mathsf{P}}^{q,\beta}_{\mu}(\mathcal{K}_2^j)=\infty,$$

and $\overline{\mathsf{H}}^{q,\alpha+\beta}_{\mu\times\nu,0}(\mathcal{K}_1\times\mathcal{K}_2)$ and $\overline{\mathsf{P}}^{q,\alpha+\beta}_{\mu\times\nu}(\mathcal{K}_1\times\mathcal{K}_2)$ are infinite, positive finite, and zero according as $j=1,2,3$, respectively.

2. Construction of fractal measures and preliminary results

2.1. Construction of fractal measures

In this paper, we use formulas containing too many different variables, which is unpleasant, and omitting these extra parameters will create no confusion. To this end, for $\mu\in\mathcal{P}(\mathbb{R}^d)$, $\nu\in\mathcal{P}(\mathbb{R}^l)$, and $h,g\in\mathcal{F}$, we define the gauge functions ξ and ζ as

$$\xi(x,r)=\mu(\mathsf{B}(x,r)^q h(2r)\quad\text{and}\quad \zeta(x,r)=\nu(\mathsf{B}(x,r)^q g(2r), \tag{2.1}$$

where $q \in \mathbb{R}$, $r > 0$, with the conventions

$$0^q = \infty$$

for $q \leq 0$ and

$$0^q = 0$$

for $q > 0$. The reader should note that we have simply used ξ (respectively, ζ) to denote the gauge function depending on μ (respectively, ν), q, and h (respectively, g). If

$$h(r) = r^s \quad \text{and} \quad g(r) = r^t$$

for $s, t \in \mathbb{R}$, then ξ and ζ will be denoted as ξ_s and ζ_t respectively. In this section, we construct the different fractal measures used in this paper. Let $\delta > 0$,

$$A \subseteq \mathsf{supp}(\mu),$$

and $\{\mathsf{B}(x_i,\ r_i)\}_i$ is a δ-packing of the A, that is, a countable family of disjoint closed balls such that $x_i \in A$ and

$$0 < 2r_i < \delta$$

for all i. Write

$$\mathcal{P}^{\xi}_{\delta}(A) = \sup \sum \xi(x_i, r_i) \quad \text{and} \quad \mathcal{P}^{\xi}_{0}(A) = \inf_{\delta>0} \mathcal{P}^{\xi}_{\delta}(A),$$

where the supremum is taken over all δ-packings of the set E. The generalized packing measure $\mathcal{P}^{\xi}$ of A with respect to ξ is defined by

$$\mathcal{P}^{\xi}(A) = \inf_{A \subseteq \bigcup_i A_i} \sum \mathcal{P}^{\xi}_{0}(A_i)$$

and

$$\mathcal{P}^{\xi}(\emptyset) = 0.$$

In a similar way, we define

$$\mathcal{H}^{\xi}_{\delta}(A) = \inf \sum \xi(x_i, r_i) \quad \text{and} \quad \mathcal{H}^{\xi}_{0}(A) = \sup_{\delta>0} \mathcal{H}^{\xi}_{\delta}(A),$$

where the infimum is taken over all δ-coverings $\left\{\mathsf{B}(x_i, r_i)\right\}_i$ of E; that is, $x_i \in E$, $0 < 2r_i < \delta$, and

$$A \subseteq \bigcup_i \mathsf{B}(x_i, r_i).$$

We define the generalized Hausdorff measure as

$$\mathcal{H}^{\xi}(A) = \sup_{E \subseteq A} \mathcal{H}^{\xi}_{0}(E)$$

and

$$\mathcal{H}^{\xi}(\emptyset) = 0.$$

We refer to [1, 5] for more details (see also [46, 47] for $q = 0$). Moreover, an integer $\kappa \in \mathbb{N}$ exists such that

$$\mathcal{H}^{\xi} \leq \kappa \mathcal{P}^{\xi}.$$

Similarly, we define

$$\overline{\mathsf{P}}^{\xi}(A) = \limsup_{r\to 0} M^q_{\mu,r}(A)h(2r),$$

where

$$M^q_{\mu,r}(A) = \sup\left\{\sum_i \mu(\mathsf{B}(x_i,r))^q \;\middle|\; \left\{\mathsf{B}(x_i,r)\right\}_i \text{ is a centered packing of } A\right\}.$$

It is clear that $\overline{\mathsf{P}}^{\xi}$ is increasing and

$$\overline{\mathsf{P}}^{\xi}(\emptyset) = 0.$$

However it is not σ-additive. For this, we define the P^{ξ}-measure defined as

$$\mathsf{P}^{\xi}(A) = \inf\left\{\sum_i \overline{\mathsf{P}}^{\xi}(A_i) \;\middle|\; A \subseteq \bigcup_i A_i \text{ and the } A_i'\text{s are bounded}\right\}.$$

In a similar way, we define

$$\overline{\mathsf{H}}^{\xi}_r(A) = N^q_{\mu,r}(A)h(2r) \quad \text{and} \quad \overline{\mathsf{H}}^{\xi}_0(A) = \liminf_{r\to 0} \overline{\mathsf{H}}^{\xi}_r(A),$$

where

$$N^q_{\mu,r}(A) = \inf\left\{\sum_i \mu(\mathsf{B}(x_i,r))^q \;\middle|\; \left\{\mathsf{B}(x_i,r)\right\}_i \text{ is a centered covering of } A\right\}.$$

Clearly, $\overline{\mathsf{H}}^{\xi}_0$ is not countably subadditive and not increasing; one needs some modification to obtain an outer measure. More precisely, let

$$\overline{\mathsf{H}}^{\xi}(A) = \inf\left\{\sum_i \overline{\mathsf{H}}^{\xi}_0(A_i) \;\middle|\; A \subseteq \bigcup_i A_i \text{ and the } A_i'\text{s are bounded}\right\}$$

and

$$\mathsf{H}^{\xi}(A) = \sup_{E\subseteq A} \overline{\mathsf{H}}^{\xi}(E).$$

It is well known (see, for instance, [48]) that H^{ξ} and P^{ξ} are metric outer measures, which implies that they are measures on the Borel algebra. Moreover, for some integer $\kappa \in \mathbb{N}$, the following inequality holds:

$$\mathcal{H}^{\xi}(A) \le \mathsf{H}^{\xi}(A) \le \kappa\mathsf{P}^{\xi}(A) \le \kappa\mathcal{P}^{\xi}(A).$$

2.2. *Construction of the generalized Cantor set*

In the following, we recall the construction of the one-dimensional generalized Cantor set $\mathcal{K}$. Let L be a positive number, let $\{n_k\}_{k\ge 1}$ be a sequence of integers, and let $\{\lambda_k\}_{k\ge 1}$ be a sequence of positive numbers such that

$$n_k > 1, \quad n_1\lambda_1 < \mathsf{L} \quad \text{and} \quad \lambda_{k+1}\, n_{k+1} < \lambda_k \tag{2.2}$$

for all $k \geq 1$. The construction of the generalized Cantor set $\left\{\mathsf{L}, \{n_k\}_{k\geq 1}, \{\lambda_k\}_{k\geq 1}\right\}$ is as follows. In the first step, from a given closed interval with the length L, remove $(n_1 - 1)$ open intervals and then leaves n_1 closed intervals with the length λ_1, denoted by $I_1, \ldots, I_{n_1}$. Let

$$J_1 = \bigcup_{j_1=1}^{n_1} I_{j_1}.$$

In the second step, from each remaining closed interval with the length λ_1, remove $(n_2 - 1)$ open intervals and leaves n_2 closed intervals with the length λ_2. These are denoted as I_{j_1,j_2}, and we can write

$$J_2 = \bigcup_{j_1=1}^{n_1} \bigcup_{j_2=1}^{n_2} I_{j_1,j_2}.$$

We continue this process and, in the k-th step, obtain $n_1 n_2 \cdots n_k$ closed intervals with the length λ_k, denoted $I_{j_1,j_2,\ldots,j_k}$ and denote their union as J_k. Then let

$$\mathcal{K} = \bigcap_{k=0}^{\infty} J_k.$$

Let

$$\mu = \nu$$

be the uniform measure on $\mathcal{K}$, that is

$$\mu(Q^k) = \lambda_k$$

and define

$$S_k = \frac{\mu(Q^{k+1})}{\mu(Q^k)} = \frac{\lambda_{k+1}}{\lambda_k}. \tag{2.3}$$

This construction can be generalized in $\mathbb{R}^d$ and $\mathcal{K}^d$, denoting the generalized Cantor set. Let F_k be the product set of d copies J_k. Thus, F_k is the union of $(n_1 n_2 \ldots n_k)^d$ closed cubes with the side λ_k, each of which may be denoted as $Q^{(k)}$, and

$$\mathcal{K}^d = \bigcap_{k=0}^{\infty} F_k.$$

The next lemma will be used in Section 4 to estimate the measure of $\mathcal{K}^d$.

Lemma 1. *Let $\mathcal{K}^d$ be the d-dimensional symmetric generalized Cantor set ($d \geq 1$). A set function Ψ, defined on every non-empty closed subset in $\mathbb{R}^d$ and r_0, exists such that, for every open cube I with the side $r \leq r_0$, we have*

$$\Psi(I) \leq 2^{3d} h(r) \lambda_k^q, \tag{2.4}$$

where k is the unique integer such that

$$\lambda_{k+1} \leq r < \lambda_k.$$

Proof. We start the proof by constructing the set function Ψ. Assume that

$$\liminf_{k\to\infty} (n_1 n_2 \ldots n_k)^d h(\lambda_k) \lambda_k^q > 0.$$

Let

$$0 < b < \liminf_{k\to\infty}(n_1 n_2 \dots n_k)^d h(\lambda_k)\lambda_k^q,$$

then there is a k_0 such that

$$\lambda_{k_0} < t_0$$

and

$$h(\lambda_k) > b/(n_1 n_2 \dots n_k)^d \lambda_k^q$$

for all $k > k_0$. We define the sequence (λ'_k) such that

$$h(\lambda'_k) = \frac{b}{(n_1 n_2 \dots n_k)^d \lambda_k^q}. \tag{2.5}$$

Clearly, we have $\lambda_k > \lambda'_k$ (since h is increasing) and

$$h(\lambda'_{k+1}) = \frac{b}{(n_1 n_2 \dots n_{k+1})^d \lambda_{k+1}^q} =^{(2.3)} \frac{b}{(n_1 n_2 \dots n_{k+1})^d S_k^q \lambda_k^q} = \frac{h(\lambda'_k)}{n_{k+1}^d S_k^q}.$$

Let A be any open set and define

$$N_{\mu,k}^q(A) = \inf\Big\{ \sum_i \mu(Q_i)^q, \quad Q_i \in F_k, \quad \text{and meeting} \quad A \Big\}.$$

Then, we have

$$\begin{aligned} N_{\mu,k+1}^q(A) &= \inf\Big\{ \sum_i \mu(Q_i)^q, \quad Q_i \in F_{k+1}, \quad \text{and meeting} \quad A \Big\} \\ &\le \inf\Big\{ \sum_i \mu(Q_i)^q \frac{\mu(Q^k)^q}{\mu(Q^k)^q} \quad Q_i \in F_{k+1}, \quad \text{and meeting} \quad A \Big\} \\ &\le k_{k+1}^d S_k^q \inf\Big\{ \sum_i \mu(Q_i)^q, \quad Q_i \in F_k, \quad \text{and meeting} \quad A \Big\} \\ &= k_{k+1}^d S_k^q N_{\mu,k}^q(A). \end{aligned}$$

It follows that the sequence $\{N_{\mu,k}^q(A) h(\lambda'_k)\}$ is decreasing, and we may define the function

$$\Psi(A) = \lim_{k\to 0} N_{\mu,k}^q(A) h(\lambda'_k).$$

Now, we will prove (2.4). Let I be an open cube, k exists such that

$$1 \le j \le n_{k+1} \qquad \text{and} \qquad \lambda_{k+1} \le r < \lambda_k.$$

Moreover, take a positive sequence $(\delta_k)_{k\ge 1}$ such that

$$n_k \lambda_k + (n_k - 1)\delta_k = \lambda_{k-1}. \tag{2.6}$$

Then the following exists:

$$1 \le j \le n_{k+1},$$

such that

$$j\lambda_{k+1} + (j-1)\delta_{k+1} \le r < (j+1)\lambda_{k+1} + j\delta_{k+1}. \tag{2.7}$$

Observe that

$$\begin{aligned} N^q_{\mu,k+1}(I) &= \inf\Big\{ \sum_i \mu(Q_i)^q, \quad Q_i \in F_{k+1}, \quad \text{and meeting } \ A\Big\} \\ &\le 2^d (j+1)^d \mu(Q^{k+1})^q \le 2^{2d} j^d \mu(Q^{k+1})^q. \end{aligned}$$

It follows that

$$\Psi(I) \le 2^{2d} j^d \mu(Q^{k+1})^q h(\lambda'_{k+1}),$$

- If $j = 1$, then

$$\Psi(I) \le 2^{2d} \mu(Q^{k+1})^q h(\lambda_{k+1}) \le 2^{2d} \mu(Q^{k+1})^q h(r);$$

- If $1 < j < n_{k+1}$, then

$$\begin{aligned} j^d \mu(Q^{k+1})^q h(\lambda'_{k+1}) &=^{(2.5)} j^d \mu(Q^{k+1})^q \frac{b}{(n_1 n_2 \dots n_{k+1})} \\ &= (j/n_1 n_2 \dots n_{k+1})^d b \\ &= (j/n_{k+1})^d h(\lambda'_k) \mu(Q^k)^q. \end{aligned}$$

Since

$$\lambda'_k \frac{j}{k_{r+1}} \le \lambda'_k,$$

and

$$t \mapsto h(t)/t^d$$

is decreasing, we get

$$(j/n_{k+1})^d h(\lambda'_k) \le h(j\lambda'_k/n_{k+1}),$$

and then

$$j^d \mu(Q^{k+1})^q h(\lambda'_{k+1}) \le \mu(Q^k)^q h(j\lambda'_k/n_{k+1}).$$

Now, observe that

$$\begin{aligned} j\lambda'_k/n_{k+1} \le j\lambda_k/n_{k+1} &\le^{(2.6)} \frac{j}{n_{k+1}} n_{k+1}(\lambda_{k+1} + \delta_{k+1}) \\ &\le 2(j\lambda_{k+1} + (j-1)\delta_{k+1}) \le^{(2.7)} 2r. \end{aligned}$$

As a consequence, we obtain

$$\Psi(I) \le 2^{2d} j^d \mu(Q^{k+1})^q h(\lambda'_{k+1}) \le 2^{2d} \mu(Q^k)^q h(2r) \le 2^{3d} \mu(Q^k)^q h(r).$$

This completes the proof. □

3. Product formula: proof of Theorem 1

We set, for $n \in \mathbb{N}$,

$$\mathcal{U}_n = \Big\{ \prod_{i=1}^{d} \Big[\frac{l_i}{2^n}, \frac{l_i+1}{2^n} \Big[, \quad l_1, \dots, l_d \in \mathbb{Z} \Big\}$$

and

$$\mathcal{V}_n = \Big\{ \prod_{i=1}^{d} \Big[\frac{\frac{1}{2} l_i}{2^n}, \frac{\frac{1}{2} l_i + 1}{2^n} \Big[, \quad l_1, \dots, l_d \in \mathbb{Z} \Big\}.$$

The family $\mathcal{U}_n$ denotes the set of half-open dyadic cubes of order n. For $x \in \mathbb{R}^d$, let $\mathrm{u}_n(x)$ denote the unique cube $\mathrm{u} \in \mathcal{U}_n$ that contains x. Similarly, the family $\mathcal{V}_n$ consists of half-open dyadic semi-cubes of order n. For $x \in \mathbb{R}^d$, let $\mathrm{v}_n(x)$ represent the unique semi-cube $\mathrm{v} \in \mathcal{V}_n$ that contains x and has its complement at a distance of 2^{-n-2} from $\mathrm{u}_{n+2}(x)$. Define

$$\mathcal{K} = \{(k_1, \dots, k_d) \mid k_i = 0 \ \text{ or } \ \frac{1}{2}\}.$$

For each

$$\mathbf{k} = (k_1, \dots, k_d) \in \mathcal{K},$$

let

$$\mathcal{V}_{\mathbf{k},n} = \Big\{ \prod_{i=1}^{d} \Big[\frac{k_i + l_i}{2^n}, \frac{k_i + l_i + 1}{2^n} \Big[, \quad l_1, \dots, l_d \in \mathbb{Z} \Big\}.$$

Note that for

$$\mathrm{v} \neq \mathrm{v}' \in \mathcal{V}_{\mathbf{k},n},$$

we have

$$\mathrm{v} \cap \mathrm{v}' = \emptyset.$$

Additionally, the collection $(\mathcal{V}_{\mathbf{k},n})_{\mathbf{k} \in \mathcal{K}}$ forms a partition of the family $\mathcal{V}_n$. Moreover, if

$$\mathrm{v}, \mathrm{v}' \in \mathcal{V}_{\mathbf{k}} := \bigcup_{n \geq 0} \mathcal{V}_{\mathbf{k},n},$$

then either

$$\mathrm{v} \cap \mathrm{v}' = \emptyset$$

or one is contained within the other. Finally, for $A \subset \mathbb{R}^d$, define

$$\mathcal{V}_n(A) = \{\mathrm{v}_n(x) : x \in A\} \quad \text{and} \quad \mathcal{V}_{\mathbf{k},n}(A) = \mathcal{V}_n(A) \cap \mathcal{V}_{\mathbf{k},n}.$$

In what follows, we construct measures on $\mathbb{R}^d$ analogous to the generalized lower and upper H-S measures. However, instead of using the collection of all closed balls in the definition, we employ the class of all half-open dyadic semi-cubes. For $A \subseteq \mathbb{R}^d$, we define

$$\overline{\mathsf{H}}^{*\xi}(A) = \liminf_{n \to +\infty} N_{\mu,n}^{*q}(A)\, h(2^{-n}) \quad \text{and} \quad \overline{\mathsf{P}}^{*\xi}(A) = \limsup_{n \to +\infty} M_{\mu,n}^{*q}(A)\, h(2^{-n}),$$

where the numbers $N_n^*(A)$ and $M_n^*(A)$ are defined as

$$N_{\mu,n}^{*q}(A) = \inf\left\{\sum_i \mu(v_i)^q \;\Big|\; \left(\mathrm{v}_i\right)_{i\in I} \text{ is a family of coverings of } A \text{ such that } \mathrm{v}_i \in \mathcal{V}_n(A)\right\}$$

and

$$M_{\mu,n}^{*q}(A) = \sup\left\{\sum_i \mu(v_i)^q \;\; \mathrm{v}_i \in \mathcal{V}_n(A),\; i \in I, \quad \text{and} \quad \overline{v}_i \cap \overline{\mathrm{v}}_j = \emptyset \quad \text{for } i \neq j\right\}.$$

The functions $\overline{\mathsf{H}}^{*\xi}$ and $\overline{\mathsf{P}}^{*\xi}$ are increasing and satisfy

$$\overline{\mathsf{H}}^{*\xi}(\emptyset) = \overline{\mathsf{P}}^{*\xi}(\emptyset) = 0.$$

However these functions are not σ-additive. For this, we consider

$$\mathsf{H}^{*\xi}(A) = \inf\left\{\sum_i \overline{\mathsf{H}}^{*\xi}(A_i) \;\Big|\; A \subseteq \bigcup_i A_i \;\; A_i \text{ is bounded}\right\},$$

$$\mathsf{P}^{*\xi}(A) = \inf\left\{\sum_i \overline{\mathsf{P}}^{*\xi}(A_i) \;\Big|\; A \subseteq \bigcup_i A_i \;\; A_i \text{ is bounded}\right\}.$$

Lemma 2. *For every set $A \subset \mathbb{R}^d$, a constant $c > 0$ exists such that*

$$c^{-1}\mathsf{P}^{\xi}(A) \leq \mathsf{P}^{*\xi}(A) \leq c\mathsf{P}^{\xi}(A) \quad \textit{and} \quad c^{-1}\mathsf{H}^{\xi}(A) \leq \mathsf{H}^{*\xi}(A) \leq c\mathsf{H}^{\xi}(A). \tag{3.1}$$

Proof. This arises from the fact that

$$\mathsf{B}(x, 2^{-n-2}) \subseteq \mathrm{v}_n(x) \subseteq \mathsf{B}(x, \sqrt{d}2^{-n}).$$

This completes the proof. □

Similarly, we may define $\mathsf{H}^{**\xi}$ and $\mathsf{P}^{**\xi}$, by using the class of all half-open dyadic cubes in the definition instead of the class of all half-open dyadic semi-cubes. However, it is important to note the resulting pre-measure, denoted $\overline{\mathsf{P}}^{**\xi}$, is not equivalent to the pre-measure $\overline{\mathsf{P}}^{\xi}$. For more discussion, consult [29, Example 3.5], where the interplay between the two pre-measures is explored. This highlights how seemingly minor changes in the class of sets used can lead to significant differences in the resulting pre-measures and their properties.

3.1. Proof of the first inequality

In this section, for the sake of simplicity and clarity, we focus on results that pertain specifically to subsets of the plane. However, it is worth noting that these results can be extended to higher-dimensional spaces without significant complications. Let $\Pi \subset \mathbb{R}^2$ represent a subset of the plane. For a given x-coordinate, we use Π_x to denote the set of all points in Π whose abscissa (x-coordinate) equals x. Given an arbitrary subset A of the x-axis, we will only prove that, if $x \in A$, we have

$$\mathsf{H}^{\zeta}(\Pi_x) > a$$

or some constant a, and then

$$\overline{\mathsf{H}}^{\xi\zeta}(\Pi) \geq \Pi a\mathsf{H}^{\xi}(A).$$

Let n be a non-negative integer and let $\{I_i \times I_j\}_{i,j}$ be a collection of half-open dyadic cubes of order n covering Π. Set

$$A_n = \left\{x \in E, \quad N^{**q}_{\nu,n}(\Pi_x)g(2^{-n}) > b_1^{-1}a\right\}.$$

Note that

$$\begin{aligned} N^{**q}_{\mu\times\nu,n}(\Pi)h(2^{-n})g(2^{-n}) &\geq N^{**q}_{\mu,n}(A_n)\inf\left\{N^{**q}_{\nu,n}(\Pi_x),\ x \in A_n\right\}h(2^{-n})g(2^{-n}) \\ &\geq b_1^{-1}aN^{**q}_{\mu,n}(A_n)h(2^{-n}). \end{aligned}$$

This holds for any covering of Π by the binary squares $\{I_i \times I_j\}_{i,j}$ with 2^{-n} sides . Hence,

$$b_1^{-1}a\overline{\mathsf{H}}^{*\xi}_n(A_n) \leq \overline{\mathsf{H}}^{*\xi\zeta}_n(\Pi) \leq \overline{\mathsf{H}}^{*\xi\zeta}(\Pi).$$

Since A_n increases to A as $n \to +\infty$, then for any $p \leq n$, we have

$$b_1^{-1}a\overline{\mathsf{H}}^{*\xi}_n(A_p) \leq b_1^{-1}a\overline{\mathsf{H}}^{*\xi}_n(A_k) \leq \overline{\mathsf{H}}^{*\xi\zeta}(\Pi).$$

Thus, we obtain

$$b_1^{-1}a\mathsf{H}^{*\xi}_n(A_p) \leq b_1^{-1}a\overline{\mathsf{H}}^{*\xi}_n(E_p) \leq \overline{\mathsf{H}}^{*q,hg}_{\mu\times\nu}(\Pi) \leq \alpha_1\overline{\mathsf{H}}^{q,hg}_{\mu\times\nu}(\Pi)$$

for $p \geq 1$. Thereby, the continuity of the measure H^* implies that

$$b_1^{-1}a\mathsf{H}^{*\xi}(A) \leq \alpha_1\overline{\mathsf{H}}^{q,hg}_{\mu\times\nu}(\Pi).$$

Thus, using Lemma 2, we get

$$b_1^{-2}a\mathsf{H}^{\xi}(A) \leq b_1^{-1}a\mathsf{H}^{*q,h}_{\mu}(A) \leq \alpha_1\overline{\mathsf{H}}^{\xi\zeta}(\Pi).$$

Finally, by taking

$$\Pi = b_1^{-2}\alpha_1^{-1},$$

we get the result.

3.2. Proof of the second inequality

Let $A \subseteq \mathbb{R}^d$ and $B \subseteq \mathbb{R}^l$. We prove that a constant $c > 0$ exists such that

$$\mathsf{H}^{\xi\zeta}(A \times B) \leq c\mathsf{H}^{\xi}(A)\mathsf{P}^{\zeta}(B).$$

Let

$$H \subseteq A \times B,$$

$r > 0$, and let $\{\mathsf{B}(x_i, r)\}_i$ be a centered r-covering of A. We denote n as the integer such that

$$\sqrt{l}2^{-n} < r \leq \sqrt{l}2^{-n+1}.$$

For $\mathrm{v} \in \mathcal{V}_n(B)$ with

$$(\mathsf{B}(x_i, r) \times \mathrm{v}) \cap H) \neq \emptyset$$

and each i, choose a point

$$y_{i,\mathrm{v}} \in B(x_i, r)$$

and a point $y'_{i,\mathrm{v}} \in \mathrm{v}$ such that

$$(y_{i,\mathrm{v}}, y'_{i,\mathrm{v}}) \in (\mathsf{B}(x_i, r) \times \mathrm{v}) \cap H).$$

Note that

$$\begin{aligned} H &\subseteq \bigcup_i \Big(\bigcup_{\substack{\mathrm{v} \in \mathcal{V}_n(B) \\ (\mathsf{B}(x_i,r)\times \mathrm{v}) \cap H \neq \emptyset}} \mathsf{B}(x_i, r) \times \mathrm{v} \Big) \\ &\subseteq \bigcup_i \Big(\bigcup_{\substack{\mathrm{v} \in \mathcal{V}_n(B) \\ (\mathsf{B}(x_i,r)\times \mathrm{v}) \cap H \neq \emptyset}} \mathsf{B}(y_{i,\mathrm{v}}, 2r) \times \mathsf{B}(y'_{i,\mathrm{v}}, 2r) \Big) \\ &\subseteq \bigcup_i \Big(\bigcup_{\substack{\mathrm{v} \in \mathcal{V}_n(B) \\ (\mathsf{B}(x_i,r)\times \mathrm{v}) \cap H \neq \emptyset}} \mathsf{B}((y_{i,\mathrm{v}}, y'_{i,\mathrm{v}}), 2r) \Big). \end{aligned}$$

As a consequence, we have the family $(\mathsf{B}((y_{i,\mathrm{v}}, y'_{i,\mathrm{v}}), 2r))_{i \in \mathbb{N}, \mathrm{v} \in \mathcal{V}_n(B), \mathsf{B}(x_i,r)\times \mathrm{v}) \cap H \neq \emptyset}$, which forms a centered $(2r)$-covering of H. Furthermore, we get

$$\mathsf{B}(y'_{i,\mathrm{v}}, \eta_r) \subseteq \mathsf{B}(y'_{i,\mathrm{v}}, 2^{-n-2})$$

for

$$\eta_r = 2^{-3} \sqrt{l} r.$$

It follows, for each $\mathbf{k} \in \mathcal{K}$, that the family

$$(B(y'_{i,\mathrm{v}}, \eta_r), \quad i \in \mathbb{N}, \mathrm{v} \in \mathcal{V}_{\mathbf{k},n}(B), \mathsf{B}(x_i, r) \times \mathrm{v}) \cap H \neq \emptyset$$

is a centered η_r-packing of B. It follows that

$$\begin{aligned} \overline{\mathsf{H}}^{\xi\zeta}_{2r}(H) &\leq \sum_i \Big(\sum_{\substack{\mathrm{v} \in \mathcal{V}_n(B) \\ (\mathsf{B}(x_i,r)\times v) \cap H \neq \emptyset}} \mu(\mathsf{B}(y_{i,\mathrm{v}}, 2r))^q \nu(y'_{i,\mathrm{v}}, 2r)^q h(4r) g(4r) \Big) \\ &\leq m_h m_g m_\nu^q \sum_i \mu(\mathsf{B}(y_{i,\mathrm{v}}, 2r))^q h(2r) \Big(\sum_{\mathbf{k} \in \mathcal{K}} \sum_{\substack{\mathrm{v} \in \mathcal{V}_{\mathbf{k},n}(B) \\ (\mathsf{B}(x_i,r)\times \mathrm{v}) \cap H \neq \emptyset}} \nu(y'_{i,\mathrm{v}}, \eta_r)^q g(2\eta_r) \Big) \\ &\leq m_h m_g m_\nu^q \sum_i \mu(\mathsf{B}(y_{i,\mathrm{v}}, 2r))^q h(2r) \Big(\sum_{\mathbf{k} \in \mathcal{K}} \overline{\mathsf{P}}^{\zeta}_{\eta_r}(B) \Big) \\ &\leq 2^l m_h m_g m_\nu^q \overline{\mathsf{P}}^{\zeta}_{\eta_r}(B) \sum_i \mu(\mathsf{B}(y_{i,\mathrm{v}}, 2r))^q h(2r). \end{aligned}$$

Thus, by considering the infimum over all possible centered r-coverings of the set A, we get

$$\overline{\mathsf{H}}^{\xi\zeta}_{2r}(H) \leq 2^l m_h m_g m_\nu^q \overline{\mathsf{H}}^{\xi}_{r}(A) \overline{\mathsf{P}}^{\zeta}_{\eta_r}(B).$$

Therefore,

$$\overline{\mathsf{H}}_0^{\xi\zeta}(H) \leq c \liminf_{r\to 0} \overline{\mathsf{H}}_r^{\xi}(A) \limsup_{r\to 0} \overline{\mathsf{P}}_{\eta_r}^{\zeta}(B) = c\overline{\mathsf{H}}_0^{\xi}(A)\overline{\mathsf{P}}^{\zeta}(B), \tag{3.2}$$

where

$$c = 2^l m_h m_g m_\nu^q.$$

Now, assume that

$$A \subseteq \bigcup_i A_i$$

and

$$B \subseteq \bigcup_j B_j.$$

Then

$$H \subseteq A \times B \subseteq \bigcup_{i,j} A_i \times B_j.$$

It follows that

$$\begin{aligned}\overline{\mathsf{H}}^{\xi\zeta}(H) &\leq \sum_{i,j} \overline{\mathsf{H}}_{\mu\times\nu,0}^{q,hg}(A_i \times B_j)\\ &\leq c\sum_{i,j} \overline{\mathsf{H}}_{\mu,0}^{q,h}(A_i)\overline{\mathsf{P}}_{\nu}^{q,g}(B_j).\\ &\leq c\Big(\sum_i \overline{\mathsf{H}}_0^{\xi}(A_i)\Big)\Big(\sum_j \overline{\mathsf{P}}^{\zeta}(B_j)\Big).\end{aligned}$$

Since the cover (A_i) of A and the cover (B_j) of B were arbitrarily chosen, we obtain

$$\overline{\mathsf{H}}^{\xi\zeta}(H) \leq c\overline{\mathsf{H}}^{\xi}(A)\mathsf{P}^{\zeta}(B) \leq c\mathsf{H}^{\xi}(A)\mathsf{P}^{\zeta}(B).$$

This holds for all for all

$$H \subseteq A \times B$$

which implies that

$$\mathsf{H}^{\xi\zeta}(A \times B) \leq c\mathsf{H}^{\xi}(A)\mathsf{P}^{\zeta}(B).$$

3.3. Proof of the third inequality

Let

$$A \subseteq \mathbb{R}^d \quad \text{and} \quad B \subseteq \mathbb{R}^l.$$

We aim to show that a constant $c > 0$ exists such that the following inequality holds:

$$\mathsf{P}^{\xi\zeta}(A \times B) \geq c\mathsf{H}^{\xi}(A)\mathsf{P}^{\zeta}(B).$$

For simplicity, we limit our discussion to subsets of the plane, although the result can be extended to higher dimensions without without significant complications. Let Q be any packing of B consisting of semi-dyadic intervals, and let C be any covering of A composed of semi-dyadic intervals. We define the following

$$C_1 = \left\{\mathrm{u}_i \in C, \mathrm{u}_i \text{ is dyadic and } \bar{\mathrm{u}}_i \cap \bar{\mathrm{u}}_j = \emptyset \text{ for } i \neq j\right\},$$
$$C_2 = \left\{\mathrm{u}_i \in C, \mathrm{u}_i \text{ is not dyadic and } \bar{\mathrm{u}}_i \cap \bar{\mathrm{u}}_j = \emptyset \text{ for } i \neq j\right\},$$
$$C_3 = \left\{\mathrm{u}_i \in C, \mathrm{u}_i \text{ is dyadic}\right\} \bigcap C \backslash C_1,$$
$$C_4 = \left\{\mathrm{u}_i \in C, \mathrm{u}_i \text{ is not dyadic}\right\} \bigcap C \backslash C_2.$$

Clearly, we have each of C_i is a packing of E and $C_i \times Q$ is a packing of $A \times B$. Therefore,

$$4M^{*q,hg}_{\mu\times\nu,n}(A \times B)h(2^{-n})g(2^{-n}) \geq \sum_{\mathrm{u}\in Q} \nu(\mathrm{u})^q h(2^{-n})g(2^{-n})\Big(\sum_{\mathrm{v}\in C_1}\mu(\mathrm{v})^q + \sum_{\mathrm{v}\in C_2}\mu(\mathrm{v})^q + \sum_{\mathrm{v}\in C_3}\mu(\mathrm{v})^q + \sum_{\mathrm{v}\in C_4}\mu(\mathrm{v})^q\Big).$$

This holds for any packing Q of B and

$$C = \bigcup_i C_i,$$

so we have

$$4M^{*q,hg}_{\mu\times\nu,n}(A \times B)h(2^{-n})g(2^{-n}) \geq M^{*q,g}_{\nu,n}(B)g(2^{-n}) \sum_{\mathrm{v}\in C} \mu(v)^q h(2^{-n}) \geq M^{*q,g}_{\nu,n}(B)g(2^{-n})N^{*q,h}_{\mu,n}(A)h(2^{-n}).$$

Thus,

$$\overline{\mathsf{P}}^{*\xi\zeta}(A \times B) \geq \frac{1}{4}\overline{\mathsf{P}}^{*\zeta}(B)\overline{\mathsf{H}}^{*\xi}(A) \geq \frac{1}{4}\mathsf{P}^{*\zeta}(B)\mathsf{H}^{*\xi}(A).$$

Finally, we get the desired result using (3.1).

3.4. Proof of the fourth inequality

Let $A \subseteq \mathbb{R}^d$ and $B \subseteq \mathbb{R}^l$. We will prove that a constant $c > 0$ exists such that

$$\mathsf{P}^{*\xi\zeta}(A \times B) \leq c\mathsf{P}^{*\xi}(A)\mathsf{P}^{*\zeta}(B).$$

Here again, we limit our study to subsets of the plane, since the extension to higher dimensions does not involve significant complications. Let $\mathcal{B}$ represent any packing of the set $A \times B$ containing semi-dyadic squares, where each square is formed as the Cartesian product of two semi-dyadic intervals. We define the sets as follows:

$$C = \left\{\mathrm{u}_n(x) : \exists \mathrm{v}_n(y) \text{ such that } w_n(x,y) = \mathrm{u}_n(x) \times \mathrm{v}_n(y) \in \mathcal{B}, \quad x \in A, y \in B\right\}$$

and

$$Q = \left\{\mathrm{v}_n(x) : \exists \mathrm{u}_n(y) \text{ such that } w_n(x,y) = \mathrm{u}_n(x) \times \mathrm{v}_n(y) \in \mathcal{B}, \quad x \in A, y \in B\right\}.$$

Next, we examine the subclasses

$$C_1 = \left\{\mathrm{u}_n(x) \in C, \quad \mathrm{u}_n(x) \text{ is dyadic}\right\},$$
$$Q_1 = \left\{\mathrm{v}_n(x) \in Q, \quad \mathrm{v}_n(x) \text{ is dyadic}\right\},$$

$$C_2 = \Big\{ \mathrm{u}_n(x) \in C, \quad \mathrm{u}_n(x) \text{ is not dyadic} \Big\},$$
$$Q_2 = \Big\{ \mathrm{v}_n(x) \in Q, \quad \mathrm{v}_n(x) \text{ is not dyadic} \Big\}.$$

It is not difficult to note that each of C_1, C_2 is a packing of A and, similarly, each of Q_1, Q_2 is a packing of B. Moreover, each square of the packing $\mathcal{B}$ is in the collection $C_i \times Q_j$, $i, j \in \{1, 2\}$. Therefore,

$$\begin{aligned}\sum_{(u,\mathrm{v})\in\mathcal{B}} \mu(\mathrm{u})^q \nu(\mathrm{v})^q h(2^{-n}) g(2^{-n}) \leq & \Big[\sum_{\mathrm{u}\in C_1} \mu(\mathrm{u})^q h(2^{-n}) + \sum_{\mathrm{u}\in C_2} \mu(\mathrm{u})^q h(2^{-n}) \Big] \\ & \cdot \Big[\sum_{\mathrm{v}\in Q_1} \nu(\mathrm{u})^q g(2^{-n}) + \sum_{\mathrm{v}\in Q_2} \nu(\mathrm{u})^q g(2^{-n}) \Big] \\ \leq & 4 M^{*q,h}_{\mu,n}(A) h(2^{-n}) M^{*q,g}_{\nu,n}(B) g(2^{-n}).\end{aligned}$$

This holds, for any packing of $A \times B$, so we have

$$M^{*q,hg}_{\mu\times\nu,n}(A \times B) h(2^{-n)} g(2^{-n)} \leq 4 M^{*q,h}_{\mu,n}(A) h(2^{-n}) M^{*q,g}_{\nu,n}(B) g(2^{-n})$$

and then

$$\overline{\mathsf{P}}^{*\xi\zeta}(A \times B) \leq 4 \overline{\mathsf{P}}^{*\xi}_n(A)\ \overline{\mathsf{P}}^{*\zeta}_n(B).$$

Let

$$A \subseteq \bigcup_i A_i$$

for

$$B \subseteq \bigcup_j B_j,$$

we have:

$$\begin{aligned}\mathsf{P}^{*\xi\zeta}(A \times B) &\leq \sum_{i,j} \overline{\mathsf{P}}^{*\xi\zeta}(A_i \times B_j) \leq 4 \sum_{i,j} \overline{\mathsf{P}}^{*\xi}(A_i)\ \overline{\mathsf{P}}^{*\zeta}(B_j). \\ &\leq 4 \Big(\sum_i \overline{\mathsf{P}}^{*\xi}(A_i) \Big) \Big(\sum_j \overline{\mathsf{P}}^{*\zeta}(B_j) \Big).\end{aligned}$$

Since (A_i) represents an arbitrary covering of E and (B_j) represents an arbitrary covering of B, we can deduce that

$$\mathsf{P}^{*\xi\zeta}(A \times B) \leq 4 \mathsf{P}^{*\xi}(A) \mathsf{P}^{*\zeta}(B).$$

Finally, by applying (3.1), we obtain the desired conclusion.

3.5. Applications of Theorem 1

Let $\mu, \theta \in \mathcal{M}(\mathbb{R}^d)$, $q, s, t \in \mathbb{R}$, and $x \in \mathsf{supp}(\mu)$, and recall the upper and lower (q, h)-densities of θ at x with respect to μ as defined in (1.7). In this section, we assume that

$$\mathcal{P}^{\xi_s}(A) < \infty$$

and

$$\mathcal{P}^{\zeta_t}(B) < \infty.$$

When studying fractal measures, a common question that naturally arises is whether we can guarantee the existence of subsets that possess finite or positive Hausdorff measures. This question becomes crucial in understanding the intricate structure of fractals, as it involves determining whether certain subsets exhibit measurable properties in terms of the Hausdorff measure, either finite or positive. Assume that

$$\inf_{0<r\le\delta} \frac{q\ln\mu(\mathsf{B}(x,r)+s\ln(2r)}{\ln\delta} \le -\alpha \quad \text{and} \quad \inf_{0<r\le\delta} \frac{q\ln\nu(\mathsf{B}(x,r)+t\ln(2r)}{\ln\delta} \le -\alpha \tag{3.3}$$

for some positive real number α. The assumption (3.3) implies, for every $\delta > 0$ that is small enough, that

$$\mu(\mathsf{B}(x,r))^q \nu(\mathsf{B}(x,r))^q (2r)^{t+s} \ge \delta^{-2\alpha}.$$

It follows that for

$$G = \{x\} \times \{y\}, \quad \delta > 0,$$

we then have

$$\overline{\mathcal{H}}^{q,s+t}_{\mu\times\nu,2\delta}(G) \ge (2\delta)^{-2\alpha}.$$

Letting δ tend to zero, we get

$$\mathsf{H}^{\xi_s\zeta_t}(\{G\}) \ge \mathcal{H}^{\xi_s\zeta_t}(\{G\}) = \overline{\mathcal{H}}^{\xi_s\zeta_t}(\{G\}) = +\infty.$$

Note that the assumption (3.3) is satisfied; for instance, if we take

$$\mu = \nu$$

to be the Lebesgue measure with

$$q + t < 0.$$

In this case, we see that the Hausdorff measure constructed above is the standard Hausdorff measure $\mathcal{H}^\varphi$ with

$$\varphi(r) = (2r)^{q+t}.$$

Thus, for any closed nonempty set

$$G \subseteq A \times B,$$

every subset of G, including the empty set, is a subset of infinite measures. Thus, we may construct the measures $\mathsf{H}^{\xi_s\zeta_t}$ for which the subset of finite measure properties can fail to hold for every closed set of infinite measures. One can assume also that for every $\delta > 0$, the following exists:

$$0 < r \le \delta/2,$$

such that

$$\mu(\mathsf{B}(x,r))^q (2r)^t \le \delta.$$

Using Theorem 1, we formulate a sufficient condition to obtain

$$0 \le \mathsf{H}^{\xi_s\zeta_t}(G) \le \mathsf{P}^{\xi_s\zeta_t}(G) < \infty.$$

First, we will state the following result, which is a direct consequence of Theorem 1.

Corollary 1. *Let $A \subset \mathbb{R}^d$ and $B \subset \mathbb{R}^l$, $\mu, \theta \in \mathcal{P}(\mathbb{R}^d)$, and $\nu, \theta' \in \mathcal{P}(\mathbb{R}^l)$ such that μ and ν satisfy the doubling condition. Let*

$$G' \subset G \subseteq A \times B,$$

such that

$$\mathcal{H}^{q,s+t}_{\mu\times\nu}(G) = \infty.$$

(1) Assume that if $\inf_{(x,y)\in G'} \underline{d}^{q,h_s}_{\mu}(x,\theta) < \infty$ and $\inf_{(x,y)\in G'} \underline{d}^{q,h_t}_{\nu}(x,\theta') > 0$, then $\mathsf{P}^{\xi_s\zeta_t}(G') < \infty$.

(2) Assume that if $\sup_{(x,y)\in G'} \overline{d}^{q,h_s}_{\mu}(x,\theta) < \infty$ and $\sup_{(x,y)\in G'} \overline{d}^{q,h_t}_{\nu}(x,\theta') > 0$, then $\mathsf{H}^{\xi_s\zeta_t}(G') > 0$.

Proof. Using [30, Lemma 3], we have

$$\mathsf{H}^{\xi_s}(A) \geq \gamma\theta(A)$$

if

$$\sup_{x\in A} \overline{d}^{q,h_s}_{\mu}(x,\theta) < \infty$$

and

$$\mathsf{P}^{\xi_s}(A) \leq \tilde{\gamma}\theta(A),$$

whenever

$$\inf_{x\in A} \underline{d}^{q,h_s}_{\mu}(x,\theta) > 0,$$

where $\gamma, \tilde{\gamma}$ are positive constants. for all $\theta \in \mathcal{P}(\mathbb{R}^d)$. Thus, the result follows from Theorem 1.

□

Example 1. *Recall the construction of the Moran set given in Section 2.2.*

Lemma 3. *[49] Let $A \subset I$ be a Moran set that satisfies the strong separation condition, and let θ be a finite Borel measure with*

$$\mathsf{supp}(\theta) \subset A.$$

Then there are some positive constants c_i $(1 \leq i \leq 4)$ depending on δ and t, such that the following inequalities hold for any $\varphi(i) \in A$:

$$c_1 \underline{\lim}_{n\to\infty} \frac{\theta(I_n(i))}{\mu(I_n(i))^q |I_n(i)|^t} \leq \underline{\lim}_{r\to 0} \frac{\theta(B(\varphi(i),r))}{\mu(B(\varphi(i),r))^q (2r)^t} \leq c_2 \underline{\lim}_{n\to\infty} \frac{\theta(I_n(i))}{\mu(I_n(i))^q |I_n(i)|^t},$$
$$c_3 \overline{\lim}_{n\to\infty} \frac{\theta(I_n(i))}{\mu(I_n(i))^q |I_n(i)|^t} \leq \overline{\lim}_{r\to 0} \frac{\theta(B(\varphi(i),r))}{\mu(B(\varphi(i),r))^q (2r)^t} \leq c_4 \overline{\lim}_{n\to\infty} \frac{\theta(I_n(i))}{\mu(I_n(i))^q |I_n(i)|^t}.$$

Now consider the special case $I = [0,1]$, $n_k = 2$, and $c_{kj} = \frac{1}{3}$ for all $k \geq 1$ and $1 \leq j \leq n_k$. In this case, the Moran set $A = B$ is the classical ternary Cantor set. Let

$$\alpha = \frac{\log 2}{\log 3}$$

and θ and θ' be probability measures on I defined by

$$\theta(I_n(i)) = \begin{cases} |I_n(i)|^\alpha, & \text{if } i \in D, \\ 0, & \text{otherwise}, \end{cases}$$

$$\theta'(I_n(i)) = \begin{cases} |I_n(i)|^{\beta}, & \text{if } i \in D, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\alpha = q + s \text{ and } \beta = q + t.$$

It is clear that

$$\mathsf{supp}(\theta) \subset E \text{ and } \mathsf{supp}(\theta)' \subset E.$$

Moreover, we have

$$\lim_{n\to\infty} \frac{\theta(I_n(i))}{\mu(I_n(i))^q |I_n(i)|^s} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\theta'(I_n(i))}{\mu(I_n(i))^q |I_n(i)|^t} = 1.$$

It follows, using Lemma 3, that

$$0 < \underline{d}_{\mu}^{q,h_s}(x,\theta) \le \overline{d}_{\mu}^{q,h_s}(x,\theta) < \infty$$

and

$$0 < \underline{d}_{\nu}^{q,h_t}(x,\theta) \le \overline{d}_{\nu}^{q,h_t}(x,\theta) < \infty.$$

Corollary 1 implies that

$$0 < \mathsf{H}_{\mu\times\nu}^{q,s+t}(A\times A) \le \mathsf{P}_{\mu\times\nu}^{q,s+t}(A\times A) < \infty.$$

Example 2. *Let $\mu, \nu \in \mathcal{M}(\mathbb{R})$, $q \in \mathbb{R}$, and let A and B be two sets of points in the x-axis and y-axis, respectively. In this example, we give a sufficient condition to obtain*

$$\mathsf{b}_{\mu\times\nu}^{q}(A\times B) = \mathsf{b}_{\mu}^{q}(A) + \mathsf{b}_{\nu}^{q}(B).$$

From Theorem 1, we have

$$\mathsf{b}_{\mu\times\nu}^{q}(A\times B) \ge \mathsf{b}_{\mu}^{q}(A) + \mathsf{b}_{\nu}^{q}(B),$$

so we only have to prove the inverse inequality. For this, for $t, s \in \mathbb{R}$, we define the lower ζ_t-dimensional density of A at the point y as

$$D^{\zeta_t}(y) = \liminf_{r\to 0} \inf_{x\in B} \frac{\mathsf{H}_{\nu}^{q,t}\big(A \cap \mathsf{B}(y,r)\big)}{\nu\big(\mathsf{B}(x,r)\big)^q (2r)^t}.$$

Fix $r > 0$ and define the set $I_y(r)$ as the centered interval on y with the length r. For $n \ge 1$, consider the set

$$B_n = \left\{ y \in B,\ \ \mathsf{H}^{\zeta_t}\Big(B \cap I_y(r)\Big) > \sup_{x\in B} \nu(I_x(r))^q r^t/n, \quad \forall r \le n^{-1} \right\}.$$

Assume that $D^{\zeta_t}(y) > 0$ for all $y \in F$, which implies clearly that $B_n \nearrow B$. In addition, if we prove that

$$\mathsf{H}_{\mu\times\nu}^{q,s+t}(A\times B_n) < +\infty \tag{3.4}$$

for some $n \in \mathbb{N}$, then we deduce that

$$\mathsf{b}_{\mu\times\nu}(A\times B) = s + t.$$

This gives the result if we choose

$$t = \mathsf{b}_{\nu}^{q}(B) \ \ and \ \ s = \mathsf{b}_{\mu}^{q}(A).$$

Now, we will prove (3.4). *Let*

$$\widetilde{A} \subseteq A \ \ and \ \ \widetilde{B}_n \subseteq B_n.$$

Let n be an integer and $0 < r \leq 1/n$*; we then define*

$$I(r) = \{I_y(r), \quad y \in \widetilde{B}_n\}.$$

We can extract f a finite subset $J(r)$ *rom* $I(r)$ *such that* $\widetilde{B}_n \subset J(r)$ *and no three intervals of* $J(r)$ *have points in it.*

Lemma 4. *For* $0 < r \leq 1/n$*, we have*

$$J(r) \leq 2nr^{-t}\big(\sup_{x\in B} \nu(I_x(r))\big)^{-q}\mathsf{H}^{\zeta_t}(B). \tag{3.5}$$

Proof. Divide the set $J(r)$ into $J_1(r)$ and $J_2(r)$ such that in each of them the intervals do not overlap. Using the definition of the set F_n, we get

$$\big(\sup_{x\in F} \nu(I_x(r))\big)^{-q}r^{-t}n\mathsf{H}^{\zeta_t}(B) \geq \sum_{I\in J_1(r)} \big(\sup_{x\in B} \nu(I_x(r))\big)^{-q}r^{-t}n\mathsf{H}^{\zeta_t}(B \cap I) > \#J_1(r).$$

Similarly, we obtain

$$\#J_2(r) \leq \big(\sup_{x\in F} \nu(I_x(r))\big)^{-q}r^{-t}n\mathsf{H}^{\zeta_t}(B)$$

as required.

In the other hand, for $\epsilon > 0$, a sequence of sets $\{A_i\}$ exists such that

$$\widetilde{A} \subseteq \bigcup_i A_i$$

and that

$$\sum_i \overline{\mathsf{H}}_0^{\xi_s}(A_i) \leq \mathsf{H}^{\xi_s}(A) + \epsilon.$$

Thus, we have a sequence $\{B_{i,j}\}$ of intervals of length r covering $\widetilde{A}$ such that the family $\{B_{i,j}\}_j$, for each i, is a covering of A_i and

$$\sum_i N_{\mu,r/2}^{q}(A_i)r^s \leq \mathsf{H}^{\xi_s}(A) + 2\epsilon. \tag{3.6}$$

Let $[a, b]$ represent any interval within the set $\{B_{i,j}\}$. Enclose all points in this set that fall between the lines $x = a$ and $x = b$ with squares whose sides are parallel to these lines. The projections of these squares onto the y-axis correspond to intervals in $J(r)$. In a similar manner, construct sets of squares for each interval in $\{B_{i,j}\}$, and denote the set of squares associated with the interval $[a, b]$ as $C(a, b)$. Since the number of squares in $C(a, b)$ does not exceed the number of intervals in $J(r)$, and each square intersecting $\widetilde{A} \times \widetilde{B}_n$ can be inscribed within a centered ball of diameter $r' = 3r$, it follows that:

$$N_{\mu\times\nu,r'/2}^{q}(\widetilde{A} \times \widetilde{B}_n) \leq \#J(r) \ \sup_{x\in F} \nu(I_x(r))^q \sum_{i,j} \mu(B_{i,j})^q.$$

Thus, using (3.5) and (3.6), we get

$$\begin{aligned}\overline{\mathsf{H}}^{q,s+t}_{\mu\times\nu,r'/2}(\widetilde{A}\times\widetilde{B}_n) &\le 2nr^{-t}\mathsf{H}^{\xi_t}(B)(3r)^{s+t}\sum_{i,j}\mu(B_{i,j})^q\\ &\le 2\times 3^{s+t}n\mathsf{H}^{\zeta_t}(B)\sum_i N^q_{\mu,r/2}(A_i)r^s\\ &\le 2\times 3^{s+t}n\mathsf{H}^{\zeta_t}(B)(\mathsf{H}^{\xi_s}(A)+2\epsilon).\end{aligned}$$

Since ϵ is arbitrarily, we get

$$\mathsf{H}^{s+t}_0(\widetilde{A}\times\widetilde{B}_n)\le 2\times 3^{s+t}n\mathsf{H}^{\zeta_t}(B)\mathsf{H}^{\xi_s}(A).$$

Finally, we have

$$\overline{\mathsf{H}}^{s+t}(A\times B_n)\le 2\times 3^{s+t}n\mathsf{H}^{\zeta_t}(B)\mathsf{H}^{\xi_s}(A),$$

from which the Eq (3.4) follows. □

The result given in this example can be summarized in the next theorem.

Theorem 3. *Let E and F be sets of points in x-axis and y-axis, respectively. Set*

$$s=\mathsf{b}^q_\mu(A)\ \ \textit{and}\ \ t=\mathsf{b}^q_\nu(B)$$

and assume that $\mathsf{H}^{\xi_s}(A),\mathsf{H}^{\zeta_t}(B)\in(0,\infty)$*, and, for all* $y\in F$*,* $D^{\zeta_t}(y)>0$*. In this case,*

$$\mathsf{b}^q_{\mu\times\nu}(A\times B)=\mathsf{b}^q_\mu(A)+\mathsf{b}^q_\nu(B).$$

4. Estimation of the measure of symmetric generalized Cantor sets

We define the set $\mathcal{G}$ of all continuous and increasing functions h on $[0,t_0)$ for some $t_0>0$ satisfying $h(0)=0$, and the function

$$t\longmapsto h(t)/t^d$$

is decreasing. We assume in this section that $h\in\mathcal{G}$ and that it t satisfies the doubling condition

$$h(2t)\le 2^d h(t),\quad\text{for}\ \ 0<t<t_0/2.$$

4.1. Estimation of the generalized Hausdorff measure

A cube $I(x,r)$ in $\mathbb{R}^d$ is a subset of the form

$$I(x,r)=\prod_{i=1}^{n}\left[x_i-r,\ x_i+r\right].$$

For a cube I, we use $l(I)$ to denote its side length. In this section, using cubes with sides of a length less than δ rather than closed balls, we define a generalized Hausdorff measure $\widetilde{\mathcal{H}}^{q,h}_\mu$ equivalent to the generalized Hausdorff measure $\mathcal{H}^{q,h}_\mu$. We prove that this measure is appropriate for estimating the measure of the generalized Cantor set. Let $\mu\in\mathcal{P}_D(\mathbb{R}^d)$, $h\in\mathcal{F}_0$, and $q\in\mathbb{R}$. Define

$$\widetilde{\mathcal{H}}^{q,h}_{\mu,0}(A)=\lim_{\delta\to 0}\widetilde{\mathcal{H}}^{q,h}_{\mu,\delta}(A),$$

where

$$\widetilde{\mathcal{H}}^{q,h}_{\mu,\delta}(A) = \inf \sum_i \mu(\mathsf{B}(x_i,\ r_i))^q h(|I_i|)$$

with the infimum being taken over all coverings of A by cubes with sides of a length $\leq \delta$. Then a constant C exists such that

$$C^{-1}\ \widetilde{\mathcal{H}}^{q,h}_{\mu,0}(A) \leq \mathcal{H}^{q,h}_{\mu,0}(A) \leq C\ \widetilde{\mathcal{H}}^{q,h}_{\mu,0}(A).$$

We will compute the estimation of the generalized Hausdorff measure of the $\mathcal{K}^d$. More precisely, we have the following result.

Theorem 4. *Let $\mathcal{K}^d$ be the d-dimensional symmetric generalized Cantor set ($d \geq 1$) constructed by the system $\left\{\mathsf{L}, \{n_k\}_{k\geq 1}, \{\lambda_k\}_{k\geq 1}\right\}$. We then have*

$$2^{-3d} \liminf_{k\to\infty}(n_1 n_2 \dots n_k)^d \lambda_k^q h(\lambda_k) \leq \mathsf{H}^{q,h}_{\mu,0}(\mathcal{K}^d) \leq \mathsf{P}^{q,h}_{\mu,0}(\mathcal{K}^d) \leq M \limsup_{k\to\infty}(n_1 n_2 \dots n_k)^d \lambda_k^q h(\lambda_k).$$

Proof. We focus on proving only the left-hand inequality; the validity of the right-hand inequality can be established using similar argument. Let Ψ be the set function in Lemma 1. Let ε be a positive number with $\varepsilon \leq r_0$ and $\{I_i\}$ be a ε-covering of $\mathcal{K}^d$ by open cubes with the sides $r_i \leq \varepsilon$. We have

$$\begin{aligned}\sum_i \mu(I_i)^q h(r_i) &\geq 2^{-3d} \sum_i \Psi(I_i)\\ &\geq 2^{-3d}\Psi(\bigcup_i I_i) \geq 2^{-3d} b.\end{aligned}$$

Since b is an arbitrary number such that

$$b < \liminf_{k\to\infty}(n_1 n_2 \dots n_k)^d \lambda_k^q h(\lambda_k),$$

then we get the desired result. □

4.2. Example: study of the case where 0–∞

In this example, we take $d = 1$ and we consider the one-dimensional generalized Cantor set $\mathcal{K}_1$ (resp. $\mathcal{K}_2$) constructed by the system $\left\{\mathsf{L}, \{n_k\}_{k\geq 1}, \{\lambda_k\}_{k\geq 1}\right\}$ (resp. $\left\{\mathsf{L}, \{n_k\}_{k\geq 1}, \{\Lambda_k\}_{k\geq 1}\right\}$) In the following, we consider $l = 1, d = 1, n_k = 2$, and

$$\lambda_k = k^{\xi_1} 2^{-k/\alpha} \qquad \Lambda_k = k^{-\xi_2} 2^{-k/\beta} \qquad h(t) = t^\alpha \qquad g(t) = t^\beta$$

Theorem 5. *The constants M and M' exist such that*

$$M \liminf_{k\to\infty} 2^k \lambda_k^{q+\alpha} 2^k \Lambda_k^{q+\beta} \leq \overline{\mathsf{H}}^{q,hg}_{\mu\times\nu,0}(\mathcal{K}_1 \times \mathcal{K}_2) \leq \overline{\mathsf{P}}^{q,hg}_{\mu\times\nu}(\mathcal{K}_1 \times \mathcal{K}_2) \leq M' \limsup_{k\to\infty} 2^k \lambda_k^{q+\alpha} 2^k \Lambda_k^{q+\beta}.$$

Proof. We focus on proving only the left-hand inequality; the validity of the right-hand inequality can be deduced using the same idea. Assume that

$$A := \liminf_{k\to\infty} 2^k \lambda_k^{q+\alpha} 2^k \Lambda_k^{q+\beta} > 0;$$

otherwise, the result remains trivial. Let

$$0 < B < A$$

and choose a positive integer k_1 satisfying the following inequality:

$$B < 2^k \lambda_k^{q+\alpha} 2^k \Lambda_k^{q+\beta}$$

for all $k \geq k_1$. Now we define the sequence $(\tilde{\Lambda}_k)_{k \geq k_1}$ as

$$B = 2^k \lambda_k^{q+\alpha} 2^k \tilde{\Lambda}_k^{q+\beta}.$$

It follows that

$$\tilde{\Lambda}_k < \Lambda_k \quad \text{and} \quad 2^2 \lambda_{k+1}^{q+\alpha}\ \tilde{\Lambda}_{k+1}^{q+\beta} = \lambda_k^{q+\alpha}\ \tilde{\Lambda}_k^{q+\beta} \tag{4.1}$$

for all $k \geq k_1$.

Let

$$K \subset \mathcal{K}_1 \times \mathcal{K}_2$$

and use I_1^k (resp. I_2^k) to denote any of the closed intervals of the generation r of $\mathcal{K}_1$ (resp. $\mathcal{K}_2$). Then

$$\begin{aligned} N^q_{\mu\times\nu,k+1}(K) &= \inf\Big\{ \sum_i \mu \times \nu(I_1^{r+1} \times I_2^{k+1})^q, \ \ I_1^{k+1} \times I_2^{k+1} \quad \text{meeting} \ \ K \Big\} \\ &\leq \inf\Big\{ \sum_i \mu(I_1^{r+1})^q \nu(I_2^{k+1})^q, \ \ I_1^{k+1} \times I_2^{k+1} \quad \text{meeting} \ \ K \Big\}. \end{aligned}$$

Note that

$$\lambda_{k+1} = \lambda_k (\frac{k+1}{k})^{\xi_1} 2^{-1/\alpha} \quad \text{and} \quad \Lambda_{r+1} = \Lambda_k (\frac{k}{k+1})^{\xi_2} 2^{-1/\beta},$$

and then

$$\begin{aligned} N^q_{\mu\times\nu,k+1}(K) &\leq \inf\Big\{ \sum_i \mu(I_1^{k+1})^q \nu(I_2^{k+1})^q, \ \ I_1^{k+1} \times I_2^{k+1} \quad \text{meeting} \ \ K \Big\} \\ &\leq k_{k+1} 2^{-q/\alpha} 2^{-q/\beta} (\frac{k+1}{k})^{q\xi_1 - q\xi_2} \inf\Big\{ \sum_i \mu(I_1^k)^q \nu(I_2^k)^q, \ \ I_1^k \times I_2^k \quad \text{meeting} \ \ K \Big\} \\ &= 2^2 2^{-q/\alpha} 2^{-q/\beta} (\frac{k+1}{k})^{q\xi_1 - q\xi_2} N^q_{\mu\times\nu,k}(K), \\ N^q_{\mu\times\nu,k+1}(K) \lambda_{k+1}^\alpha \tilde{\Lambda}_{k+1}^\beta &\leq 2^2 2^{-q/\alpha} 2^{-q/\beta} (\frac{k+1}{k})^{q\xi_1 - q\xi_2} \lambda_{k+1}^\alpha \tilde{\Lambda}_{k+1}^\beta N^q_{\mu\times\nu,k}(K) \\ &\leq \frac{2^2 2^{-q/\alpha} 2^{-q/\beta}}{\lambda_{k+1}^q \tilde{\Lambda}_{r+1}^q} (\frac{k+1}{k})^{q\xi_1 - q\xi_2} \lambda_{k+1}^{q+\alpha} \tilde{\Lambda}_{k+1}^{q+\beta} N^q_{\mu\times\nu,k}(K) \\ &\leq^{(4.1)} \frac{2^{-q/\alpha} 2^{-q/\beta}}{\lambda_{k+1}^q \tilde{\Lambda}_{k+1}^q} (\frac{k+1}{k})^{q\xi_1 - q\xi_2} \lambda_k^{q+\alpha} \tilde{\Lambda}_k^{q+\beta} N^q_{\mu\times\nu,k}(K) \\ &\leq^{(4.1)} \Big[\frac{2^{-1/\alpha} 2^{-1/\beta} \lambda_k \tilde{\Lambda}_k}{\lambda_{k+1} \tilde{\Lambda}_{k+1}} (\frac{k+1}{k})^{\xi_1 - \xi_2} \Big]^q \lambda_k^\alpha \tilde{\Lambda}_k^\beta N^q_{\mu\times\nu,k}(K). \end{aligned}$$

It follows that the sequence $\{N^q_{\mu\times\nu,k}(K) \lambda_k^\alpha \tilde{\Lambda}_r^\beta\}$ is decreasing, and we may define the function

$$\Phi(A) = \lim_{k\to 0} N^q_{\mu\times\nu,k}(A) \lambda_k^\alpha \tilde{\Lambda}_k^\beta.$$

Case $\beta \le \alpha$**.** We can choose $k_2 > k_1$ such that

$$\Lambda_k < \lambda_{k+1} \qquad \text{for all} \quad k \ge k_2.$$

Let

$$s_o = \Lambda_{k_2}$$

and consider any two-dimensional open cube I with the side $s \le s_0$. Let p and k be the unique positive integers such that

$$\lambda_{p+1} < s \le \lambda_p \qquad \text{and} \qquad \Lambda_{k+1} < s \le \Lambda_k.$$

Since

$$\lambda_{p+1} < \Lambda_k < \lambda_{k+1}$$

for $k \ge k_2$, we deduce that $k < p$. Moreover, the open cube I meets at most 2^2 rectangles of the form $I_1^p \times I_2^k$ and so meets at most 2^4 rectangles of the form $I_1^{p+1} \times I_2^{k+1}$. Therefore, since $p > k$, it follows that

$$\begin{aligned} N^q_{\mu\times\nu,p+1}(I) &\le \inf\Big\{ \sum_i \mu(I_1^{p+1})^q \nu(I_2^{p+1})^q, \quad I_1^{p+1} \times I_2^{p+1} \quad \text{meeting } I \Big\} \\ &\le 2^4 2^{p-r} \lambda^q_{p+1} \Lambda^q_{p+1}. \end{aligned}$$

Since $2^k \Lambda_k^\beta$ decrease as r increases, note that

$$2^{p-k} \Lambda^\beta_{p+1} < \Lambda^\beta_{k+1}.$$

Then,

$$\begin{aligned} \Phi(I) &\le N^q_{\mu\times\nu,p+1}(I) \lambda^\alpha_{p+1} \tilde{\Lambda}^\beta_{p+1} \le 2^{4+p-k} \lambda^q_{p+1} \Lambda^q_{p+1} \lambda^\alpha_{p+1} \tilde{\Lambda}^\beta_{p+1} \\ &\le 2^{4+p-k} \mu(Q^{p+1})^q \nu(Q^{(p+1)})^q \lambda^\alpha_{(p+1)} \, \Lambda^\beta_{p+1} \\ &\le 2^4 \mu(Q^{(p+1)})^q \nu(Q^{(p+1)})^q s^{\alpha+\beta}. \end{aligned}$$

□

Example 3. *As a consequence, we construct an estimate of the generalized packing measures of product sets of one-dimensional generalized Cantor sets. Let* $0 < \alpha$ *and* $\beta < 1$. *In this example, we consider the one-dimensional generalized Cantor set* $\mathcal{K}_1$ *(resp.* $\mathcal{K}_2$*) constructed by the system* $\big\{l, \{k_r\}_{r\ge1}, \{\lambda_r\}_{r\ge1}\big\}$ *(resp.* $\big\{l, \{k_r\}_{r\ge1}, \{\Pi_r\}_{r\ge1}\big\}$*). Set* $l = 1$ *and* $n_k = 2$ *and consider in the following:*

$$\lambda_k = \left(k^2 2^{-k}\right)^{\frac{1}{\alpha+q}}, \quad \Lambda_k = \left(k^{-j} 2^{-k}\right)^{\frac{1}{\beta+q}}, \quad h(t) = t^\alpha, \ \textit{and} \ \ g(t) = t^\beta.$$

(1) We have

$$\begin{aligned} \lim_{k\to\infty} 2^k \lambda_k^{q+\alpha} 2^k \Lambda_k^{q+\beta} &= \lim_{k\to\infty} 2^k \lambda_k^{q+\alpha} 2^k \Lambda_k^{q+\beta} \\ &= \lim_{k\to\infty} (k^2 2^{-k})(k^{-j} 2^{-k}) 2^k \\ &= \lim_{k\to\infty} k^{2-j}. \end{aligned}$$

Therefore, $\overline{\mathsf{H}}^{q,hg}_{\mu\times\nu,0}(\mathcal{K}_1\times\mathcal{K}_2)$ *(resp.* $\overline{\mathsf{P}}^{q,hg}_{\mu\times\nu}(\mathcal{K}_1\times\mathcal{K}_2)$*) is infinite, positive finite, and zero for* $j = 1, 2, 3,$ *respectively.*

(2) We have

$$\lim_{k\to\infty} 2^k \lambda_k^{q+\alpha} = \lim_{k\to\infty} 2^k (k^2 2^{-k}) = \infty.$$

Therefore,

$$\overline{\mathsf{H}}_{\mu,0}^{q,h}(\mathcal{K}_1) = \overline{\mathsf{P}}_{\mu}^{q,h}(\mathcal{K}_1) = \infty.$$

(3) We have

$$\lim_{k\to\infty} 2^k \Lambda_k^{q+\alpha} = \lim_{k\to\infty} 2^k (k^{-j} 2^{-k}) = 0.$$

Therefore,

$$\overline{\mathsf{H}}_{\nu,0}^{q,g}(\mathcal{K}_2) = \overline{\mathsf{P}}_{\nu}^{q,g}(\mathcal{K}_2) = 0.$$

5. Conclusions and perspectives

Let $A \subseteq \mathbb{R}^d$ and $B \subseteq \mathbb{R}^l$. In this work, we present a novel approach that is distinct from that in [30], as it is specifically tailored for Euclidean spaces, to establish the following inequality:

$$\mathsf{H}_{\mu}^{q,h}(A)\mathsf{H}_{\nu}^{q,g}(B) \le c_1 \mathsf{H}_{\mu\times\nu}^{q,hg}(A\times B) \le c_2\ \mathsf{H}_{\mu}^{q,h}(A)\mathsf{P}_{\nu}^{q,g}(B) \le c_3\ \mathsf{P}_{\mu\times\nu}^{q,hg}(A\times B) \le c_4 \mathsf{P}_{\mu}^{q,h}(A)\mathsf{P}_{\nu}^{q,g}(B).$$

This result holds under the assumption that μ, ν, h, g satisfy the doubling condition and that none of the products is of the form $0\times\infty$ or $\infty\times 0$. Furthermore, by analyzing the measures of symmetric generalized Cantor sets, we demonstrate that the exclusion of the 0–∞ condition is indispensable and thus cannot be omitted. Let $(\mathcal{X}, \rho)$ and $(\mathcal{X}', \rho')$ be two separable metric spaces. The result presented in this paper holds true for both $\mathcal{X}$ and $\mathcal{X}'$, though the approach used in our proof does not extend to metric spaces.

(1) Let $\mathcal{B}(\mathcal{X})$ denote the family of closed balls in $\mathcal{X}$, and let $\Phi(\mathcal{X})$ represent the class of pre-measures. A pre-measure is any increasing function

$$\xi : \mathcal{B}(\mathcal{X}) \to [0, +\infty]$$

satisfying

$$\xi(\emptyset) = 0.$$

It is natural to consider a general construction of $\mathcal{H}_{\mu}^{q,\xi}$, defined using a measure μ and a pre-measure ξ. Specifically, our result applies when

$$\xi(\mathsf{B}(x,r)) = h(2r)$$

and allows for the choice

$$\xi(\mathsf{B}) = h(|\mathsf{B}|)$$

for all $\mathsf{B} \in \mathcal{B}(\mathcal{X})$. Let

$$\xi \in \Phi(\mathcal{X}) \ \text{ and } \ \xi' \in \Phi(\mathcal{X}').$$

We define ξ_0, the Cartesian product measure generated from the functions ξ and ξ', on $\mathcal{B}(\mathcal{X} \times \mathcal{X}')$ as

$$\xi_0(\mathsf{B} \times \mathsf{B}') = \xi(\mathsf{B})\xi'(\mathsf{B}'), \quad \text{for all} \quad \mathsf{B} \in \mathcal{B}(\mathcal{X}),\ \mathsf{B}' \in \mathcal{B}(\mathcal{X}').$$

We strongly believe that the resulting measure is particularly well-suited for studying Cartesian product sets. Under a suitable doubling condition, we obtain the following result:

$$\mathsf{H}^{q,\xi_0}_{\mu\times\nu}(A \times B) = \mathsf{H}^{q,\xi}_{\mu}(A)\mathsf{H}^{q,\xi'}_{\nu}(B), \tag{5.1}$$

for all $A \subset \mathcal{X}$ and $B \subset \mathcal{X}'$. This construction was first introduced by Kelly in [50]; see also [51].

(2) To establish the equality presented in Eq (5.1), we draw inspiration from the work of Kelly [50]. Specifically, we propose constructing a weighted lower H-W measure, denoted $\mathcal{W}^{q,h}_{\mu}$, for any given Hausdorff measure h. This approach involves assigning non-negative weights to the covering sets, adhering to what is commonly referred to as the third method for constructing an outer measure. On the basis of this framework, we conjecture that the equality in (5.1) holds if the constructed weighted measure satisfies

$$\mathcal{W}^{q,\xi}_{\mu} = \mathsf{H}^{q,\xi}_{\mu}.$$

Similarly, one can construct a weighted upper H-W measure, denoted $\mathcal{Q}^{q,h}_{\mu}$, by following the same approach used for the weighted lower H-W measure but replacing covering with packing [8]. We conjecture that the equality

$$\mathsf{P}^{q,\xi_0}_{\mu\times\nu}(A \times B) = \mathsf{P}^{q,\xi}_{\mu}(A)\mathsf{P}^{q,\xi'}_{\nu}(B),$$

for all $A \subset \mathcal{X}$ and $B \subset \mathcal{X}'$, holds if the constructed weighted measure satisfies

$$\mathcal{Q}^{q,\xi}_{\mu} = \mathsf{P}^{q,\xi}_{\mu}.$$

(3) A similar result to (1.2) and (1.3) can be achieved by examining fractal pseudo-packings and weighted measures of the H-S type. The purpose of employing these generalizations is to eliminate the need for assuming the doubling condition.

(4) Our results in this paper can be readily extended to the setting of generalized lower and upper H-S measures, denoted $\mathsf{H}^{q,h}_{\mu_1\times\mu_2}$ and $\mathsf{P}^{q,h}_{\mu_1\times\mu_2}$. These fractal measures play a crucial role in the multifractal analysis of a measure relative to another measure [52].

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

References

1. L. Olsen, A multifractal formalism, *Adv. Math.*, **116** (1995), 82–196. https://doi.org/10.1006/aima.1995.1066
2. B. Mandelbrot, *Les objects fractales: forme, hasard et dimension*, Flammarion, 1975.
3. B. Mandelbrot, The fractal geometry of nature, *Amer. J. Phys.*, **51** (1983), 286–287. https://doi.org/10.1119/1.13295
4. N. Attia, B. Selmi, A multifractal formalism for Hewitt-Stromberg measures, *J. Geom. Anal.*, **31** (2021), 825–862. https://doi.org/10.1007/s12220-019-00302-3
5. L. Olsen, Multifractal dimensions of product measures, *Math. Proc. Camb. Philos. Soc.*, **120** (1996), 709–734. https://doi.org/10.1017/S0305004100001675
6. J. M. Marstrand, The dimension of Cartesian product sets, *Proc. Cambridge Philos. Soc.*, **50** (1954), 198–202. https://doi.org/10.1017/S0305004100029236
7. C. Tricot, Two definitions of fractional dimension, *Math. Proc. Camb. Philos. Soc.*, **91** (1982), 57–74. https://doi.org/10.1017/S0305004100059119
8. J. D. Howroyd, On Hausdorff and packing dimension of product spaces, *Math. Proc. Camb. Philos. Soc.*, **119** (1996), 715–727. https://doi.org/10.1017/S0305004100074545
9. X. Jiang, Q. Liu, Z. Wen, An intermediate value property of fractal dimensions of Cartesian product, *Fractals*, **25** (2017), 1750052. https://doi.org/10.1142/S0218348X17500529
10. C. Wei, S. Wen, Z. Wen, Remark on dimension of cartesian product sets, *Fractals*, **24** (2016), 1650031. https://doi.org/10.1142/S0218348X16500316
11. H. Haase, Dimension of measures, *Ada Univ. Carolinae Math. Phys.*, **31** (1990), 29–34.
12. H. Haase, On the dimension of product measures, *Mathematika*, **37** (1990), 316–323. https://doi.org/10.1112/S0025579300013024
13. X. Hu, S. J. Taylor, Fractal properties of products and projections of measures in R^d, *Math. Proc. Camb. Philos. Soc.*, **115** (1994), 527–544. https://doi.org/10.1017/S0305004100072285
14. D. J. Feng, S. Hua, Z. Y. Wen, The pointwise densities of the Cantor measure, *J. Math. Anal. Appl.*, **250** (2000), 692–705. https://doi.org/10.1006/jmaa.2000.7137
15. B. Jia, Z. Zhou, Z. Zhu, J. Luo, The packing measure of the Cartesian product of the middle third Cantor set with itself, *J. Math. Anal. Appl.*, **288** (2003), 424–441. https://doi.org/10.1016/j.jmaa.2003.09.001
16. R. Guedri, N. Attia, A note on the generalized Hausdorff and packing measures of product sets in metric space, *Math. Inequal. Appl.*, **25** (2022), 335–358. https://doi.org/10.7153/mia-2022-25-20
17. E. Hewitt, K. Stromberg, *Real and abstract analysis: a modern treatment of the theory of functions of a real variable*, Springer-Verlag, 1965. https://doi.org/10.1007/978-3-642-88044-5

18. Y. Pesin, *Dimension theory in dynamical systems, contemporary views and applications*, University of Chicago Press, 1997.

19. P. Mattila, *Geometry of sets and measures in Euclidian spaces: fractals and rectifiability*, Cambridge University Press, 1995. https://doi.org/10.1017/CBO9780511623813

20. O. Guizani, M. Amal, N. Attia, Some relations between Hewitt-Stromberg premeasure and Hewitt-Stromberg measure, *Filomat*, **37** (2023), 13–20. https://doi.org/10.2298/FIL2301013A

21. O. Guizani, N. Attia, A note on scaling properties of Hewitt Stromberg measure, *Filomat*, **36** (2022), 3551–3559. https://doi.org/10.2298/FIL2210551A

22. L. Olsen, On the average Hewitt-Stromberg measures of typical compact spaces, *Math. Z.*, **293** (2019), 1201–1225. https://doi.org/10.1007/s00209-019-02239-3

23. H. Haase, A contribution to measure and dimension of metric spaces, *Math. Nachr.*, **124** (1985), 45–55. https://doi.org/10.1002/mana.19851240104

24. H. Haase, Open-invariant measures and the covering number of sets, *Math. Nachr.*, **134** (1987), 295–307. https://doi.org/10.1002/mana.19871340121

25. S. Jurina, N. MacGregor, A. Mitchell, L. Olsen, A. Stylianou, On the Hausdorff and packing measures of typical compact metric spaces, *Aequationes Math.*, **92** (2018), 709–735. https://doi.org/10.1007/s00010-018-0548-5

26. O. Zindulka, Packing measures and dimensions on Cartesian products, *ArXiv*, 2012. https://doi.org/10.48550/arXiv.1208.5522

27. G. A. Edgar, *Integral, probability, and fractal measures*, Springer-Verlag, 1998. https://doi.org/10.1007/978-1-4757-2958-0

28. N. Attia, H. Jebali, M. H. Khalifa, A note on fractal measures of cartesian product sets, *Bull. Malays. Math. Sci. Soc.*, **44** (2021), 4383–4404. https://doi.org/10.1007/s40840-021-01172-1

29. S. J. Taylor, C. Tricot, Packing measure and its evaluation for a brownian path, *Trans. Amer. Math. Soc.*, **288** (1985), 679–699. https://doi.org/10.1090/S0002-9947-1985-0776398-8

30. N. Attia, R. Guedri, O. Guizani, Note on the multifractal measures of cartesian product sets, *Commun. Korean Math. Soc.*, **37** (2022), 1073–1097. https://doi.org/10.4134/CKMS.c210350

31. K. J. Falconer, *Fractal geometry: mathematical foundations and applications*, Wiley, 1990.

32. J. Mullins, N. Mullins, *Derivation bases, interval functions, and fractal measures*, The Ohio State University, 1996.

33. O. Guizani, M. Amal, N. Attia, On the Hewitt-Stromberg measure of product sets, *Ann. Mat. Pura Appl.*, **200** (2020), 867–879. https://doi.org/10.1007/s10231-022-01248-0

34. P. A. P. Mohan, A. S. Besicovitch, The measure of product and cylinder sets, *J. London Math. Soc.*, **20** (1945), 110–120. https://doi.org/10.1112/jlms/s1-20.2.110

35. C. Tricot, Two definitions of fractional dimension, *Math. Proc. Camb. Philos. Soc.*, **91** (1982), 57–74. https://doi.org/10.1017/S0305004100059119

36. M. Ohtsuka, Capacite des ensembles produits, *Nagoya Math. J.*, **12** (1957), 95–130. https://doi.org/10.1017/S0027763000021966

37. Y. M. Xiao, Packing dimension, Hausdorff dimension and Cartesian product sets, *Math. Proc. Camb. Philos. Soc.*, **120** (1996), 535–546. https://doi.org/10.1017/S030500410007506X

38. S. J. Taylor, C. Tricot, The packing measure of rectifiable subsets of the plane, *Math. Proc. Camb. Philos. Soc.*, **99** (1986), 285–296. https://doi.org/10.1017/S0305004100064203

39. C. Cutler, The density theorem and Hausdorff inequality for packing measure in general metric spaces, *Illinois J. Math.*, **39** (1995), 676–694. https://doi.org/10.1215/ijm/1255986272

40. G. A. Edgar, Centered densities and fractal measures, *New York J. Math.*, **13** (2007), 33–87.

41. P. Mattila, R. D. Mauldin, Measure and dimension functions: measurablility and densities, *Math. Proc. Camb. Philos. Soc.*, **121** (1997), 81–100. https://doi.org/10.1090/S0002-9947-1944-0009975-6

42. A. P. Morse, J. F. Randolph, The ϕ-rectifiable subsets of the plane, *Amer. Math. Soc. Trans.*, **55** (1944), 236–305. https://doi.org/10.2307/1990191

43. H. H. Lee, I. S. Beak, The relations of Hausdorff, *-Hausdorff, and packing measures, *Real Anal. Exch.*, **16** (1990/1991), 497–507. https://doi.org/10.2307/44153728

44. H. H. Lee, I. S. Baek, On d-measure and d-dimension, *Real Anal. Exch.*, **17** (1992), 590–596. https://doi.org/10.2307/44153752

45. H. H. Lee, I. S. Baek, The comparison of d-meuasure with packing and Hausdorff measures, *Kyungpook Math. J.*, **32** (1992), 523–531.

46. X. S. Raymond, C. Tricot, Packing regularity of sets in n-space, *Math. Proc. Camb. Philos. Soc.*, **103** (1988), 133–145. https://doi.org/10.1017/S0305004100064690

47. H. Joyce, D. Preiss, On the existence of subsets of positive finite packing measure, *Mathematika*, **42** (1995), 14–24. https://doi.org/10.1112/S002557930001130X

48. N. Attia, B. Selmi, Regularities of multifractal Hewitt-Stromberg measures, *Commun. Korean Math. Soc.*, **34** (2019), 213–230. https://doi.org/10.4134/CKMS.c180030

49. M. Dai, The equivalence of measures on Moran set in general metric space, *Chaos Solitons Fract.*, **29** (2006), 55–64. https://doi.org/10.1016/j.chaos.2005.10.016

50. J. D. Kelly, A method for constructing measures appropriate for the study of Cartesian products, *Proc. London Math. Soc.*, **s3-26** (1973), 521–546. https://doi.org/10.1112/plms/s3-26.3.521

51. N. Attia, H. Jebali, R. Guedri, On a class of Hausdorff measure of cartesian product sets in metric spaces, *Topol. Methods Nonlinear Anal.*, **62** (2023), 601–623. https://doi.org/10.12775/TMNA.2023.016

52. N. Attia, Relative multifractal spectrum, *Commun. Korean Math. Soc.*, **33** (2018), 459–471. https://doi.org/10.4134/CKMS.c170143

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