
Research article

A new flexible Weibull distribution for modeling real-life data: Improved estimators, properties, and applications

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Abstract: In this paper, we proposed a novel and flexible lifetime model, the generalized Kavya–Manoharan Weibull distribution, which can be interpreted as a proportional reversed hazard model. The most remarkable feature of the proposed model is its ability to effectively capture a wide range of hazard rate patterns using only three parameters. These include decreasing, J-shaped, reverse J-shaped, and increasing patterns, as well as key nonmonotonic shapes such as the bathtub, modified bathtub, and upside-down bathtub shapes. Additionally, its density can exhibit right-skewness, left-skewness, symmetry, and reversed-J shapes. We explored several distributional properties of the proposed model and estimated its parameters using eight methods. The effectiveness of these estimators was validated through extensive simulation studies. Furthermore, we assessed the versatility of the proposed distribution using three real-world datasets, demonstrating its exceptional capacity to fit the data accurately. Our results indicated that the proposed distribution outperforms several existing generalizations of the Weibull distribution in terms of fit quality.

Keywords: GKM-G family; Weibull distribution; quantile function; order statistics; parameter estimation; failure time data

Mathematics Subject Classification: 60E05, 62F10, 62N05

1. Introduction

The Weibull distribution is extensively utilized for analyzing lifetime data and is particularly effective for modeling monotonic hazard rates (HRs). Its density functions are typically right or left-skewed, making it ideal for reliability and survival analysis. However, it falls short when dealing with non-monotonic HRs, such as those exhibiting bathtub-shaped or upside-down bathtub-shaped patterns. While the Weibull distribution is highly effective in modeling monotonic HRs, it lacks the flexibility necessary to capture more complex failure rate behaviors, which are commonly observed in real-world data across domains, such as medicine, engineering, and industrial reliability. Although numerous extensions of the Weibull distribution have been proposed to address this limitation, many of these alternatives require more than four parameters to accurately represent intricate HR patterns. Additionally, existing models often struggle to effectively capture non-monotonic HR behaviors, including J-shaped or modified bathtub curves.

Some recent extensions of the Weibull distribution, introduced to expand its modeling capabilities across a wider range of lifetime data, include the beta Weibull [1], Kumaraswamy–Weibull [2], truncated Weibull [3], transmuted Weibull [4], exponentiated generalized Weibull [5], new extended Weibull [6], modified beta Weibull [7], Kumaraswamy complementary Weibull geometric [8], Weibull–Weibull [9], alpha power Weibull [10], odd log-logistic exponentiated Weibull [11], Lindley Weibull [12], exponentiated Weibull [13], alpha logarithmic transformed Weibull [14], alpha power exponentiated Weibull [15], odd Lomax–Weibull [16], Maxwell–Weibull [17], exponentiated additive Weibull [18], new generalized modified Weibull [19] new flexible Weibull [20], odd Burr exponentiated Weibull [21], odd log-logistic Lindley–Weibull [22], alpha power Kumaraswamy–Weibull [23], new exponentiated inverse Weibull [24], entropy-transformed Weibull [25], extended Weibull [26], and odd beta prime Weibull [27] distributions. These enhanced distributions offer increased flexibility, enabling more effective modeling of diverse datasets for practical applications.

We aim to bridge this gap by introducing a new variant of the Weibull distribution, known as the generalized Kavya–Manoharan Weibull (GKMW) distribution, which is specifically designed to model a broader range of non-monotonic HRs with just three parameters. The GKMW distribution provides an improved and more flexible approach for modeling diverse lifetime data, making it a valuable tool for researchers and practitioners in survival analysis, and reliability theory. The GKMW distribution is derived from the generalized Kavya–Manoharan (GKM-G) family introduced by Mahran et al. [28]. One key characteristic of the GKMW distribution is its interpretation as a proportional reversed hazard (PRH) model. The PRH models play a crucial role in survival analysis and reliability theory, particularly when analyzing left-censored lifetime data and studying parallel systems [29]. Further details on PRH models can be found in references [30–32].

The GKMW distribution boasts several key advantages:

- It accurately captures J-shaped, decreasing, bathtub, increasing, upside-down bathtubs, modified bathtub, and reversed-J HR shapes. With its three parameters, the GKMW effectively models failure rates for both standard and modified bathtubs, a significant improvement over many distributions that require more than four parameters for precise representation.
- The GKMW distribution is particularly well-suited for non-monotonic modeling, making it applicable in diverse fields such as medicine, engineering, survival analysis, and industrial reliability.
- Our analysis demonstrates the GKMW model's superiority over seven competing lifetime

distributions through real data from three distinct fields, underscoring its practical applicability.

Our final motivation of this paper is to evaluate the performance of various frequentist estimators for the GKMW distribution across sample sizes and parameter values. Additionally, we aim to provide guidelines for selecting the most effective estimation method for the GKMW distribution, which we believe will be of interest to applied statisticians.

The remainder of this article is organized as follows: In Section 2, we introduce the GKMW distribution. The properties of the GKMW distribution are derived in Section 3. In Section 4, we detail eight estimation methods for estimating the GKMW parameters are discussed. Numerical simulations are presented in Section 5. In Section 6, we illustrate the practical application of the GKMW distribution using three real data examples. Finally, concluding remarks and some perspectives for future research are given in Section 7.

2. The GKMW distribution

In this section, we introduce the GKMW distribution, using the Weibull model as the baseline within the GKM-G family proposed by Mahran et al. [28]. The GKM-G family can be regarded as a PRH family because it is derived from the exponentiated-H (exp-H) family [33], which is one of the most commonly used generalization techniques. The cumulative distribution function (CDF) and probability density function (PDF) of the GKM-G family are defined as follows:

$$F(x; \delta, \boldsymbol{\vartheta}) = \xi^\delta [1 - e^{-G(x; \boldsymbol{\vartheta})}]^\delta, \quad x \in \mathbb{R}^+, \quad \delta > 0 \quad (1)$$

and

$$f(x; \delta, \boldsymbol{\vartheta}) = \xi^\delta \delta g(x; \boldsymbol{\vartheta}) e^{-G(x; \boldsymbol{\vartheta})} [1 - e^{-G(x; \boldsymbol{\vartheta})}]^{\delta-1}, \quad x \in \mathbb{R}^+,$$

where $\xi = e/(e-1)$, δ is a shape parameter and $\boldsymbol{\vartheta}$ refers to the baseline parameters vector.

The HR function (HRF) of the GKM-G family reduces to

$$h(x; \delta, \boldsymbol{\vartheta}) = \frac{\delta g(x; \boldsymbol{\vartheta}) e^{1-G(x; \boldsymbol{\vartheta})} [e - e^{1-G(x; \boldsymbol{\vartheta})}]^{\delta-1}}{(e-1)^\delta - [e - e^{1-G(x; \boldsymbol{\vartheta})}]^\delta}, \quad x \in \mathbb{R}^+.$$

The PDF and CDF of the Weibull distribution are $g(x; \beta, \lambda) = \beta \lambda x^{\beta-1} e^{-\lambda x^\beta}$ and $G(x; \beta, \lambda) = 1 - e^{-\lambda x^\beta}$, $\lambda, \beta > 0$. By substituting the CDF of the Weibull model in Eq (1), we derive the CDF of the GKMW distribution as follows:

$$F(x; \boldsymbol{\omega}) = \xi^\delta \left[1 - e^{-(1-e^{-\lambda x^\beta})} \right]^\delta, \quad x > 0, \quad \delta, \lambda, \beta > 0,$$

where $\boldsymbol{\omega} = (\delta, \beta, \lambda)^T$.

The PDF of the GKMW model takes the form

$$f(x; \boldsymbol{\omega}) = \xi^\delta \delta \beta \lambda x^{\beta-1} e^{-(1-e^{-\lambda x^\beta})-\lambda x^\beta} \left[1 - e^{-(1-e^{-\lambda x^\beta})} \right]^{\delta-1}, \quad x > 0, \quad (2)$$

where $\lambda > 0$ is the scale parameter and δ and β are positive shape parameters. The scale

parameter (λ) affects the spread of the GKMW distribution but does not directly influence its skewness or tail behavior. In contrast, the two shape parameters (δ and β) refine the shape of the GKMW distribution, particularly in terms of tail behavior, kurtosis and skewness. Thus, these two parameters significantly influence the asymmetry and overall shape of the distribution. Figures 1–3 illustrate the roles of the three parameters. The plots and numerical values, obtained for $\lambda = 1$ with varying δ and β , confirm the influence of all three parameters.

Therefore, a random variable with PDF (2) is denoted by $X \sim \text{GKMW}(\delta, \beta, \lambda)$.

The HRF and reversed HRF (RHRF) of the GKMW model are defined by

$$h(x; \omega) = \frac{\delta \beta \lambda x^{\beta-1} e^{e^{-\lambda x^\beta} - \lambda x^\beta} (e - e^{e^{-\lambda x^\beta}})^{\delta-1}}{(e-1)^\delta - (e - e^{e^{-\lambda x^\beta}})^\delta}, \quad x > 0$$

and

$$H(x; \omega) = \delta \beta \lambda x^{\beta-1} e^{-(1-e^{-\lambda x^\beta}) - \lambda x^\beta} \left[1 - e^{-(1-e^{-\lambda x^\beta})} \right]^{-1}, \quad x > 0.$$

The quantile function (QF) of the GKMW model reduces to

$$\vartheta(u) = \lambda^{-\frac{1}{\beta}} \left(-\ln \left\{ 1 + \log \left[1 - (\xi^{-\delta} u)^{\frac{1}{\delta}} \right] \right\} \right)^{\frac{1}{\beta}}, \quad 0 < u < 1,$$

where $\xi = e/(e-1)$.

Figures 1 and 2 display the PDF and HRF curves of the GKMW distribution for $\lambda = 1$ and various values of δ and β . Figure 2 illustrates the HRF, which can exhibit decreasing, J-shape, increasing, reversed-J shape, bathtub, modified bathtub, and unimodal forms. A key advantage of the GKMW distribution over the W distribution is its ability to model data with bathtub, modified bathtub, or unimodal failure rates, which the W cannot.

Furthermore, the QF can be employed to explore the relationships among the parameters. For the GKMW distribution, this function is useful for calculating Galton's skewness and Moors' kurtosis. Figure 3 presents Galton's skewness and Moors' kurtosis for the GKMW distribution at $\lambda = 1$ with varying δ and β values. Overall, it is evident that parameters β and λ have a significant impact on the skewness and kurtosis of the distribution.

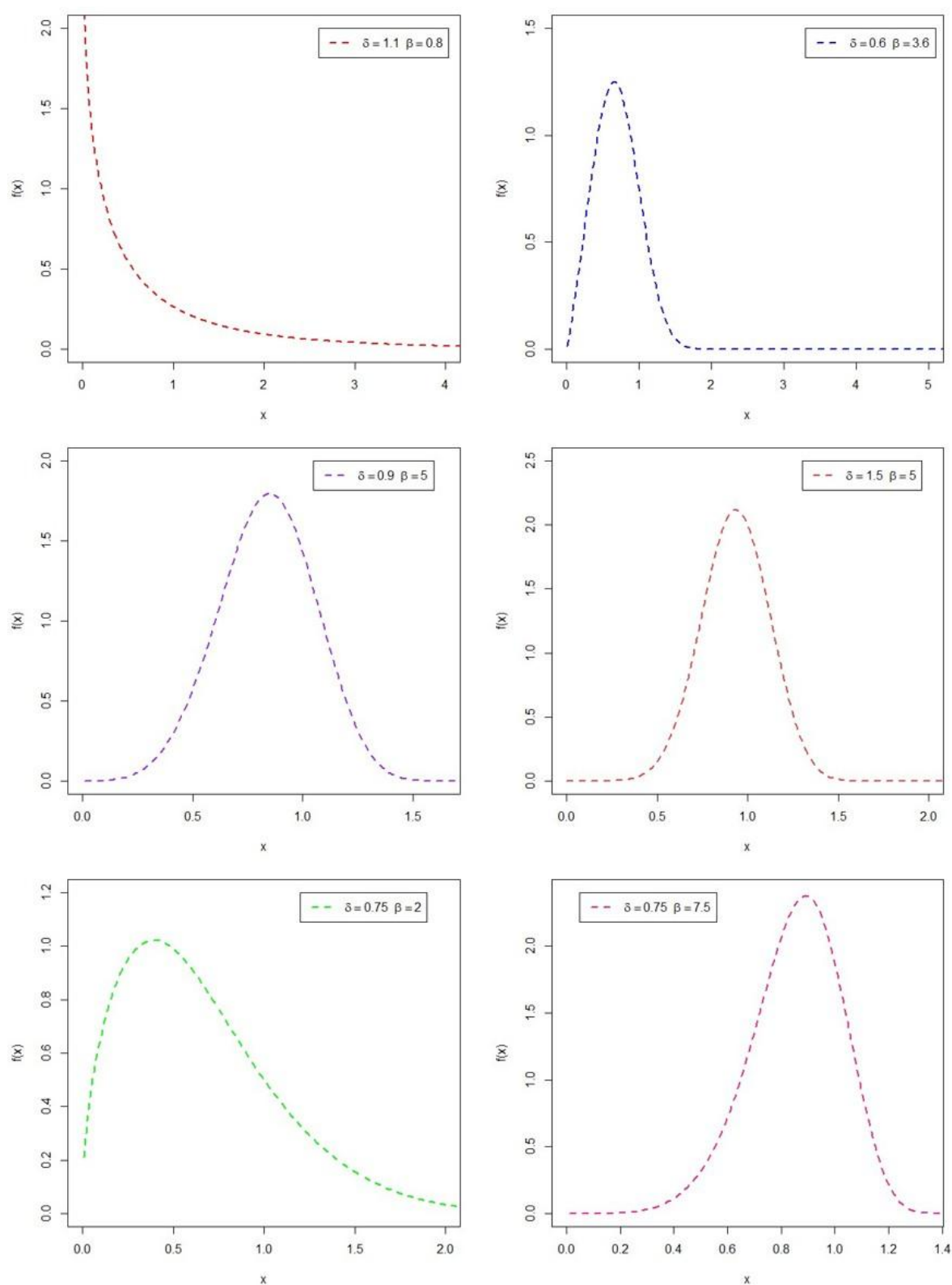


Figure 1. Plots of the GKMW density for $\lambda = 1$ and different parametric values of δ and β .

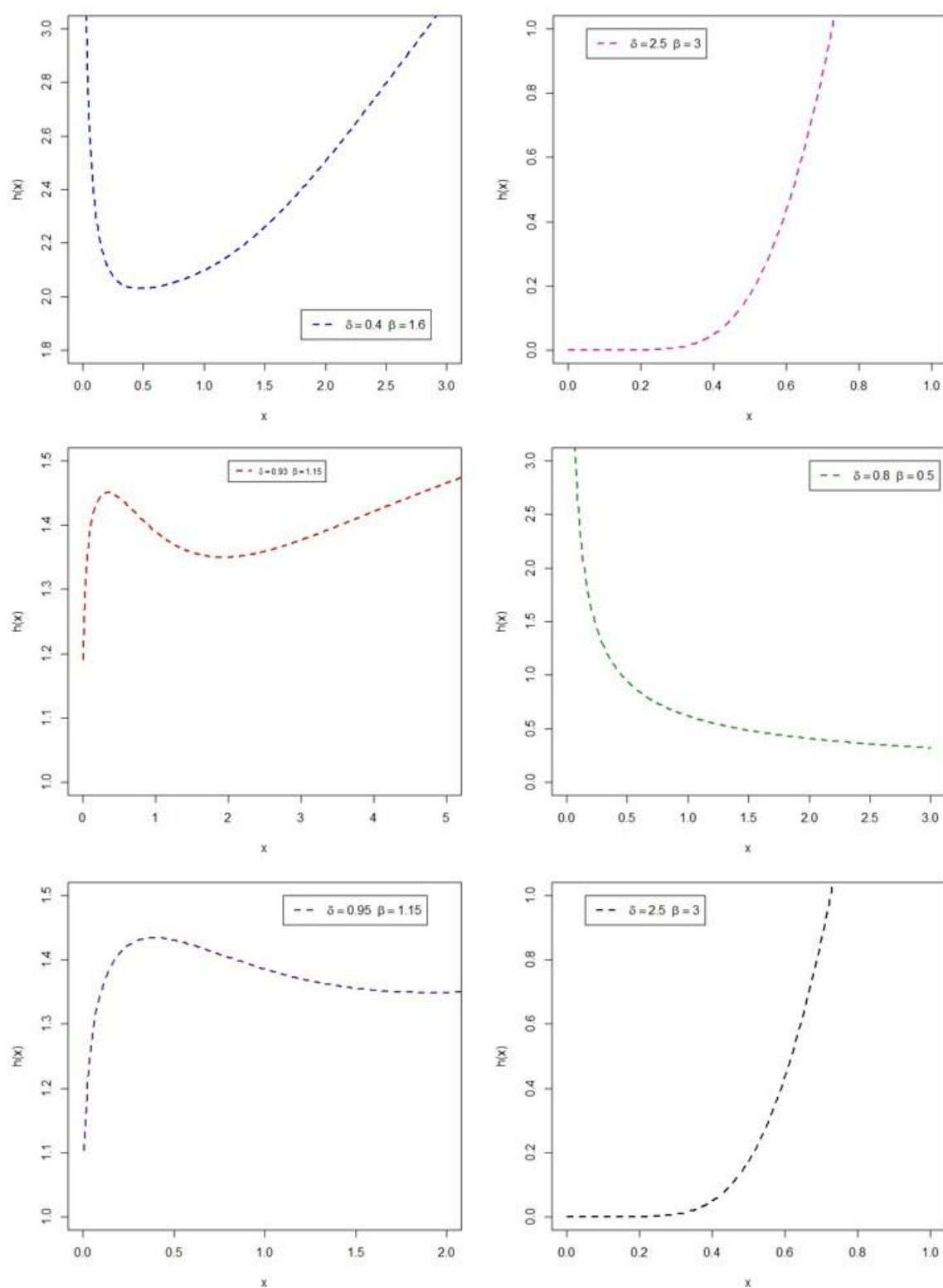


Figure 2. Plots of the GKMW HRF for $\lambda = 1$ and different parametric values of δ and β .

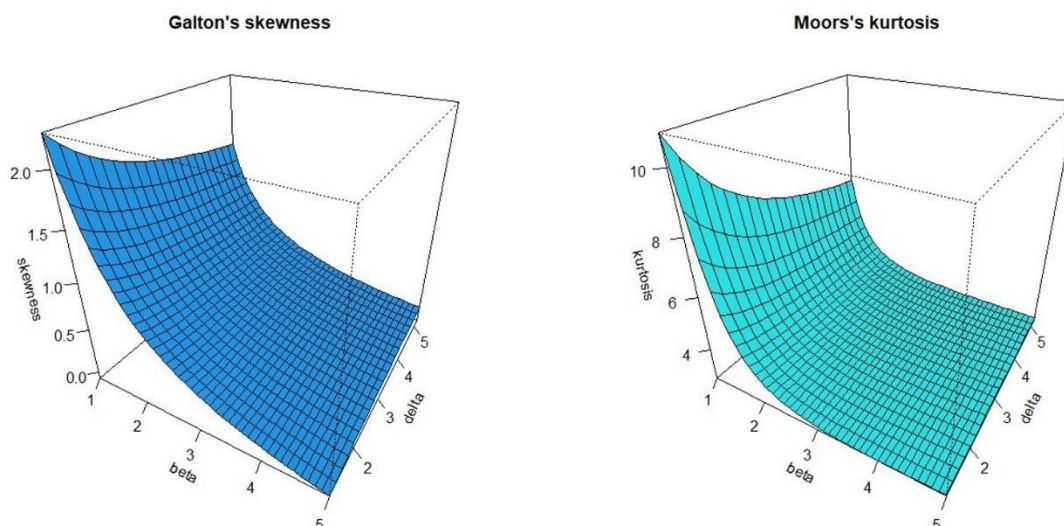


Figure 3. Galton's skewness and Moors' kurtosis for the GKMW distribution.

3. Properties

In this section, we explore several properties of the GKMW distribution.

3.1. Linear representation

Here, we present a mixture form for the GKMW density based on the linear representation of the GKM-G density introduced by Mahran et al. [28]. The GKM-G density can be expressed as follows:

$$f(x) = \sum_{k=0}^{\infty} d_k h_k(x), \quad (3)$$

where $d_k = \sum_{j=0}^{\infty} \xi^{\delta} \frac{(-1)^{j+k} j^k}{k!} \binom{\delta}{j}$ and $h_k(x) = k g(x) G(x)^{k-1}$ is the exp-G density with power parameter k . Equation (3) can be expressed using the W distribution as follows:

$$f(x) = \sum_{l=0}^{\infty} v_l g_{l+1}(x), \quad (4)$$

where $v_l = \sum_{k=0}^{\infty} \frac{d_k (-1)^l}{(l+1)} k \binom{k-1}{l}$ and $g_{l+1}(x) = \beta (l+1) \lambda x^{\beta-1} e^{-(l+1)\lambda x^{\beta}}$ denotes the W density with scale parameter $(l+1)\lambda$ and shape parameter β . Then, the GKMW PDF can be expressed as a single linear combination of Weibull densities.

3.2. Moments

Let Y be a random variable having the W distribution with PDF $g(y; \beta, \lambda) = \beta \lambda y^{\beta-1} e^{-\lambda y^\beta}$, $y > 0, \beta, \lambda > 0$, then, the r th ordinary moments of Y is

$$\mu'_{r,Y} = \Gamma\left(1 + \frac{r}{\beta}\right) \lambda^{\frac{-r}{\beta}}.$$

Therefore, we can derive the r th moment of GKMW distribution from Eq (4) as follows:

$$\mu'_r = \Gamma\left(1 + \frac{r}{\beta}\right) \sum_{l=0}^{\infty} v_l [(l+1)\lambda]^{\frac{-r}{\beta}}. \quad (5)$$

The mean of X , denoted by μ_X , follows from Eq (5) by setting $r = 1$.

Table 1 demonstrates that the summation in Eq (5) converges to the numerical integral (NI) of μ_X for various values of λ and γ as the truncated terms in this summation, say M , increase significantly. Table 2 shows that the skewness (ψ_1) and kurtosis (ψ_2) of the GKMW distribution range from -0.0874 to 11.1133 and from 2.5560 to 242.1702 , respectively. Additionally, the GKMW distribution can exhibit left-skewed, right-skewed, or symmetric properties, and can be classified as leptokurtic ($\psi_2 > 3$) or platykurtic ($\psi_2 < 3$). This versatility makes the GKMW distribution well-suited for modeling skewed data effectively.

The s th incomplete moment of the GKMW model is given by

$$\varphi_s(t) = \int_{-\infty}^t x^s f(x) dx = \sum_{l=0}^{\infty} v_l [(l+1)\lambda]^{\frac{-s}{\beta}} \gamma\left(1 + \frac{s}{\beta}, (l+1)\lambda t^\beta\right), \quad (6)$$

where $\gamma(a, \omega)$ denote the lower incomplete gamma function (IGF), which is defined by $\gamma(a, \omega) = \int_0^\omega \omega^{a-1} e^{-\omega} d\omega$. The first incomplete moment, say $\varphi_1(t)$, is derived for $s = 1$ and can be utilized to construct Bonferroni and Lorenz curves, which are defined, for a given probability π , as follows: $B(\pi) = \varphi_1(q)/(\pi \mu'_1)$ and $L(\pi) = \varphi_1(q)/\mu'_1$, where μ'_1 given by (5) with $r = 1$ and $q = Q(\pi)$ is the QF of X at π .

The conditional moments of the GKMW model can be written as

$$E(X^n | X > t) = \frac{1}{S(t)} \sum_{l=0}^{\infty} v_l (l+1)\lambda \Gamma\left(1 + \frac{n}{\beta}, (l+1)\lambda t^\beta\right),$$

where $\Gamma(a, \xi)$ denotes the upper IGF defined by $\Gamma(a, \xi) = \int_\xi^\infty \xi^{a-1} e^{-\xi} d\xi$ and $S(t)$ is the survival function (SF) of the GKMW distribution.

The moment-generating function (MGF) of the W distribution has the form

$$M(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \lambda^{\frac{-j}{\beta}} \Gamma\left(1 + \frac{j}{\beta}\right). \quad (7)$$

By combining Eqs (4) and (7), the MGF of the GKMW model is expressed as follows

$$M_X(t) = \sum_{l,j=0}^{\infty} v_l \frac{t^j}{j!} [(l+1)\lambda]^{-\frac{j}{\beta}} \Gamma\left(1 + \frac{j}{\beta}\right).$$

Table 1. Generated values of μ_X based on the summation formula and the NI for various parametric values at truncated M terms.

λ	δ	β	M	Summation	NI
0.5	2	0.5	10	9.56790	
			20	9.56325	9.56325
			50	9.56325	
		1.5	10	1.65060	
			20	1.65025	1.65025
			50	1.65025	
	4	0.5	10	33.26973	
			20	16.24027	16.24025
			50	16.24025	
		1.5	10	3.46152	
			20	2.15191	2.15191
			50	2.15191	
0.9	2	0.5	10	2.95306	
			20	2.95162	2.95162
			50	2.95162	
		1.5	10	1.11548	
			20	1.11524	1.11524
			50	1.11524	
	4	0.5	10	10.26843	
			20	5.01243	5.01242
			50	5.01242	
		1.5	10	2.33929	
			20	1.45426	1.45426
			50	1.45426	
1.5	2	0.5	10	1.06310	
			20	1.06258	1.06258
			50	1.06258	
		1.5	10	0.79353	
			20	0.79336	0.79336
			50	0.79336	
	4	0.5	10	3.69664	
			20	1.80447	1.80447
			50	1.80447	
		1.5	10	1.66412	
			20	1.03453	1.03453
			50	1.03453	

Table 2. Moments of the GKMW distribution for $\lambda = 1$ with varying values of δ and β .

δ	β	μ_x	σ_x^2	ψ_1	ψ_2
0.5	0.5	0.7139	6.7945	11.1133	242.1702
	1.5	0.4887	0.2526	1.7860	7.1495
	2.8	0.5958	0.1257	0.6322	3.0474
	3.5	0.6409	0.0996	0.3478	2.6707
	5	0.7117	0.0670	-0.0309	2.5560
0.75	0.5	1.0338	9.8213	9.2802	169.9559
	1.5	0.6313	0.2893	1.5059	5.9591
	2.8	0.7128	0.1182	0.4924	2.9654
	3.5	0.7478	0.0874	0.2419	2.7303
	5	0.8014	0.0530	-0.0874	2.7295
1.5	0.5	1.8882	17.8440	6.9544	97.0234
	1.5	0.9131	0.3298	1.1791	4.8586
	2.8	0.9067	0.0981	0.3788	2.9892
	3.5	0.9156	0.0655	0.1865	2.8588
	5	0.9321	0.0345	-0.0571	2.8753
2	0.5	2.3908	22.5235	6.2248	78.4523
	1.5	1.0396	0.3379	1.0866	4.6063
	2.8	0.9823	0.0893	0.3625	3.0139
	3.5	0.9784	0.0577	0.1912	2.8969
	5	0.9785	0.0290	-0.0225	2.8929
5	0.5	4.7553	44.1219	4.5572	43.7214
	1.5	1.4588	0.3369	0.9032	4.2015
	2.8	1.2009	0.0652	0.3653	3.1052
	3.5	1.1536	0.0386	0.2431	2.9961
	5	1.1018	0.0174	0.0954	2.9332

3.3. Mean residual life and mean inactivity time

The function $\varphi_1(t)$ of X can be used to derive the mean residual life (MRL) and mean inactivity time (MIT). This function follows from Eq (6) as

$$\vartheta_1(t) = \sum_{l=0}^{\infty} v_l [(l+1)\lambda]^{-\frac{1}{\beta}} \gamma\left(1 + \frac{1}{\beta}, (l+1)\lambda t^\beta\right). \quad (8)$$

The MRL represents the expected additional lifespan for a unit, which is operational at age t and is defined by $m_X(t) = E(X - x | X > x)$, for $t > 0$. The MRL of X is

$$MRL_X(t) = \frac{[1 - \varphi_1(t)]}{S(t)} - t. \quad (9)$$

By substituting Eq (8) into Eq (9), we obtain the MRL of the GKMW distribution as follows

$$MRL_X(t) = \frac{1}{S(t)} \left\{ 1 - \sum_{l=0}^{\infty} v_l [(l+1)\lambda]^{-\frac{1}{\beta}} \gamma \left(1 + \frac{1}{\beta}, (l+1)\lambda t^{\beta} \right) \right\} - t.$$

The MIT represents the waiting time that has elapsed since the failure of an item, given that this failure occurred within the interval $(0, t)$. The MIT is defined by $MIT_X(t) = E(t - X | X \leq t)$, for $t > 0$. The MIT of X reduces to

$$MIT_X(t) = t - \frac{\varphi_1(t)}{F(t)}. \quad (10)$$

Using Eqs (8) and (10), we derive the MIT of the GKMW distribution as follows

$$MIT_X(t) = t - \frac{1}{F(t)} \sum_{l=0}^{\infty} v_l [(l+1)\lambda]^{-\frac{1}{\beta}} \gamma \left(1 + \frac{1}{\beta}, (l+1)\lambda t^{\beta} \right).$$

3.4. Order statistics

Order statistics are essential in quality control testing and reliability assessments, as they help predict the failure of future items by analyzing early failures. According to Mahran et al. [28], the PDF of i th order statistic of the GKM-G class, say $X_{(i)}$ (for $i = 1, \dots, n$), can be expressed as follows

$$f_{i:n}(x) = \sum_{k=0}^{\infty} b_k h_{k+1}(x).$$

Here, $h_{k+1}(x) = (k+1) g(x) G(x)^k$ is the exp-G density with power parameter $k+1$ and

$$b_k = \alpha \sum_{j=0}^{n-i} \sum_{m=0}^{\infty} \frac{(1+m)^k (-1)^{j+m+k}}{(k+1)! B(i, n-i+1)} \phi^{\alpha(j+i)} \binom{n-1}{j} \binom{\alpha(j+i)-1}{m}.$$

Then, the PDF of $X_{(i)}$ for the GKMW distribution reduces to

$$f_{X_{(i)}}(x) = \sum_{r=0}^{\infty} c_r \beta (r+1) \lambda x^{\beta-1} e^{-(r+1)\lambda x^{\beta}}, \quad (11)$$

where $c_r = \sum_{k=0}^{\infty} \frac{b_k (-1)^r (k+1)!}{(r+1)! (k-r)!}$. Equation (11) indicates that the PDF of the GKMW order statistics is a mixture of W densities, with a scale parameter of $(r+1)\lambda$ and a shape parameter β . Consequently, some of their mathematical properties can be derived from those of the W distribution. For instance, the q th moment of $X_{(i)}$ is given by

$$E(X_{(i)}^q) = \Gamma \left(1 + \frac{q}{\beta} \right) \sum_{r=0}^{\infty} c_r [(r+1)\lambda]^{-\frac{q}{\beta}}.$$

3.5. Probability weighted moments

Greenwood et al. [34] introduced probability weighted moments (PWMs) as a particular type of moment. PWMs are used to estimate the parameters and quantiles of distributions that can be represented in inverse form. These estimators exhibit moderate bias and low variance, making them comparable to maximum likelihood (ML) estimators.

The (j, i) th PWM of X , say $\rho_{j,i}$, is defined by

$$\rho_{j,i} = E\{X^j F(X)^i\} = \int_{-\infty}^{\infty} x^j f(x) F(x)^i dx,$$

where j and i be non-negative integers. According to Mahran et al. [28], the PWM of the GKM-G class can be expressed as

$$\rho_{j,i} = \sum_{s=0}^{\infty} d_s E(T_{s+1}^j),$$

where

$$d_s = \delta \sum_{l=0}^{\infty} \frac{(1+l)^s (-1)^{l+s}}{(s+1)!} \xi^{\delta(1+i)} \binom{\alpha(1+i)-1}{l}.$$

Using Eq (5), the PWM of GKMW model can be defined as

$$\rho_{j,i} = \sum_{s=0}^{\infty} d_s \Gamma\left(1 + \frac{j}{\beta}\right) [(s+1)\lambda]^{\frac{-j}{\beta}}.$$

3.6. Rényi and θ -entropies

Entropy measures the randomness of systems and is widely used in fields such as molecular tumor imaging, physics, and sparse kernel density estimation. The Rényi entropy of the GKM-G family [28] is

$$I_{\theta} = \frac{1}{1-\theta} \log \left[\sum_{k=0}^{\infty} \eta_k \int_{-\infty}^{\infty} g(x)^{\theta} G(x)^k dx \right], \quad (12)$$

where

$$\eta_k = \sum_{j=0}^{\infty} \frac{(\delta \xi^{\gamma})^{\theta} (\theta + j)^k}{k!} (-1)^{k+j} \binom{\theta(\gamma-1)}{j}.$$

Inserting the PDF and CDF of W distribution in Eq (12) and using binomial series, we obtain

$$g(x)^\theta G(x)^k = (\beta\lambda)^\theta x^{\theta(\beta-1)} \sum_{i=0}^{\infty} (-1)^i \binom{k}{i} e^{-(i+\theta)\lambda x^\beta}.$$

Therefore, the Rényi entropy of the GKMW distribution follows as

$$I_\theta = \frac{1}{1-\theta} \log \left[\sum_{k,i=0}^{\infty} \eta_k (-1)^i \binom{k}{i} (\beta\lambda)^\theta A \right], \quad (13)$$

where

$$A = \int_0^\infty x^{\theta(\beta-1)} e^{-(i+\theta)\lambda x^\beta} dx = \frac{1}{\beta} [(i+\theta)\lambda]^{\theta(1-\beta)-1} \Gamma\left(\frac{\theta(\beta-1)+1}{\beta}\right).$$

By substituting the quantity A from Eq (13), the Rényi entropy of the GKMW distribution simplifies to

$$I_\theta = \frac{1}{1-\theta} \log \left[\sum_{k,i=0}^{\infty} \eta_k (-1)^i \binom{k}{i} \lambda^\theta \beta^{\theta-1} [(i+\theta)\lambda]^{\theta(1-\beta)-1} \Gamma\left(\frac{\theta(\beta-1)+1}{\beta}\right) \right],$$

where $\theta > 0$ and $\theta \neq 1$.

The Shannon entropy can be seen as a special case of the Rényi entropy when θ approaches 1.

4. Estimation methods

In this section, we present eight methods for estimating the parameters of the GKMW distribution. These methods include maximum likelihood (ML), least squares (LS), weighted least squares (WLS), Cramér–von Mises (CVM), maximum product of spacings (MPS), Anderson–Darling (AD), right-tail Anderson–Darling (RTAD), and percentile (PC) estimators.

Let $x_1, \dots, x_n$ be a random sample from the GKMW distribution with parameters δ , β , and λ . Denote the ordered statistics as $x_{1:n} < x_{2:n} < \dots < x_{n:n}$.

The log-likelihood function of the GKMW model can be expressed as follows

$$\begin{aligned} \ell = n \delta \log \xi + n \log \delta + n \log \beta + n \log \lambda + (\beta - 1) \sum_{i=1}^n \log(x_i) - \sum_{i=1}^n \left(1 - e^{-\lambda x_i^\beta}\right) \\ - \lambda \sum_{i=1}^n x_i^\beta + (\delta - 1) \sum_{i=1}^n \log(k_i), \end{aligned}$$

where $k_i = 1 - e^{-\left(1 - e^{-\lambda x_i^\beta}\right)}$.

The MLEs for δ , β , and λ can be obtained by maximizing the previous equation with respect to these parameters or by solving the provided nonlinear equations:

$$\frac{\partial \ell}{\partial \delta} = \frac{n}{\delta} + n \log \xi + \sum_{i=1}^n \log(k_i) = 0,$$

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^n \log(x_i) - \lambda \sum_{i=1}^n x_i^\beta \log(x_i) - \lambda \sum_{i=1}^n x_i^\beta \log(x_i) e^{-\lambda x_i^\beta} \\ &\quad + \lambda(\delta - 1) \sum_{i=1}^n \frac{x_i^\beta \log(x_i) (1 - k_i) e^{-\lambda x_i^\beta}}{k_i} = 0 \end{aligned}$$

and

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^n x_i^\beta - \sum_{i=1}^n x_i^\beta e^{-\lambda x_i^\beta} + (\delta - 1) \sum_{i=1}^n \frac{x_i^\beta (1 - k_i) e^{-\lambda x_i^\beta}}{k_i} = 0.$$

The LS and WLS methods are employed to estimate the parameters of the beta distribution [35]. The LS estimators (LSEs) and WLS estimators (WLSEs) for the GKMW parameters can be obtained by minimizing the following:

$$V(\delta, \beta, \lambda) = \sum_{i=1}^n v_i \left[\xi^\delta k_{i:n}^\delta - \frac{i}{n+1} \right]^2,$$

where $v_i = 1$ for the LS method, $v_i = (n+1)^2(n+2)/[i(n-i+1)]$ for the WLS approach, and $k_{i:n} = 1 - e^{-\left(1 - e^{-\lambda x_{i:n}^\beta}\right)}$.

Additionally, the LSEs and WLSEs can be derived by solving the nonlinear equations (for $s = 1, 2, 3$):

$$\sum_{i=1}^n v_i \left[\xi^\delta k_{i:n}^\delta - \frac{i}{n+1} \right] \Delta_s(x_{i:n} | \delta, \beta, \lambda) = 0,$$

where

$$\Delta_1(x_{i:n} | \delta, \beta, \lambda) = \frac{\partial \ell}{\partial \delta} F(x_{i:n} | \delta, \beta, \lambda) = k_{i:n}^\delta \xi^\delta [\log(\xi) + \log(k_{i:n})], \quad (14)$$

$$\Delta_2(x_{i:n} | \delta, \beta, \lambda) = \frac{\partial \ell}{\partial \beta} F(x_{i:n} | \delta, \beta, \lambda) = \delta \lambda \xi^\delta x_{i:n}^\beta e^{-\lambda x_{i:n}^\beta} (1 - k_{i:n}) k_{i:n}^{\delta-1} \log(x_{i:n}) \quad (15)$$

and

$$\Delta_3(x_{i:n} | \delta, \beta, \lambda) = \frac{\partial \ell}{\partial \lambda} F(x_{i:n} | \delta, \beta, \lambda) = \delta \xi^\delta x_{i:n}^\beta e^{-\lambda x_{i:n}^\beta} (1 - k_{i:n}) k_{i:n}^{\delta-1}. \quad (16)$$

The CVM estimators (CVMEs) [36,37] can be derived from the difference between the estimated CDF and the empirical CDF. The CVMEs for the GKMW parameters are found by

minimizing the following function:

$$C(\delta, \beta, \lambda) = \frac{1}{12n} + \sum_{i=1}^n \left[\xi^\delta k_{i:n}^\delta - \frac{2i-1}{2n} \right]^2.$$

Further, the CVMEs follow by solving the nonlinear equations,

$$\sum_{i=1}^n \left[\xi^\delta k_{i:n}^\delta - \frac{2i-1}{2n} \right] \Delta_s(x_{i:n} | \delta, \beta, \lambda) = 0,$$

where $\Delta_s(x_{i:n} | \delta, \beta, \lambda) = 0$ are defined in (14)-(16) for $s = 1, 2, 3$.

The MPS method is used for parameter estimation in continuous univariate models as an alternative to the ML method [38,39]. The uniform spacings of a random sample of size n from the GKMW distribution can be characterized by:

$$D_i = F(x_{i:n} | \delta, \beta, \lambda) - F(x_{i-1:n} | \delta, \beta, \lambda),$$

where D_i denotes the uniform spacings, where $F(x_{0:n} | \delta, \beta, \lambda) = 0$, $F(x_{n+1:n} | \delta, \beta, \lambda) = 1$ and $\sum_{i=1}^{n+1} D_i(\delta, \beta, \lambda) = 1$. The MPS estimators (MPSEs) of the GKMW parameters can be obtained by maximizing

$$G(\delta, \beta, \lambda) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log D_i(\delta, \beta, \lambda).$$

Additionally, the MPSEs of the GKMW parameters can also be obtained by solving:

$$\frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(\delta, \beta, \lambda)} [\Delta_s(x_{i:n} | \delta, \beta, \lambda) - \Delta_s(x_{i-1:n} | \delta, \beta, \lambda)] = 0, \quad s = 1, 2, 3.$$

The AD estimators (ADEs) are another form of minimum distance estimator. The ADEs for the GKMW parameters are obtained by minimizing:

$$A(\delta, \beta, \lambda) = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) [\log F(x_{i:n} | \delta, \beta, \lambda) + \log \bar{F}(x_{n+1-i:n} | \delta, \beta, \lambda)].$$

The ADEs can also be determined by solving the corresponding nonlinear equations:

$$\sum_{i=1}^n (2i-1) \left[\frac{\Delta_s(x_{i:n} | \delta, \beta, \lambda)}{F(x_{i:n} | \delta, \beta, \lambda)} - \frac{\Delta_j(x_{n+1-i:n} | \delta, \beta, \lambda)}{S(x_{n+1-i:n} | \delta, \beta, \lambda)} \right] = 0.$$

The RTAD estimators (RTADEs) for the GKMW parameters δ , β , and λ are obtained by minimizing the following function with respect to these parameters:

$$R(\delta, \beta, \lambda) = \frac{n}{2} - 2 \sum_{i=1}^n F(x_{i:n} | \delta, \beta, \lambda) - \frac{1}{n} \sum_{i=1}^n (2i-1) \log \bar{F}(x_{n+1-i:n} | \delta, \beta, \lambda).$$

The unknown parameters of the GKMW distribution can be estimated using the PC method [40], which involves matching the sample PC points with the corresponding population PCs. An unbiased estimator of $F(x_{i:n} | \delta, \beta, \lambda)$ is given by $u_i = i/(n+1)$. The PC estimators (PCEs) for the GKMW parameters are subsequently derived by minimizing the specified function:

$$P(\delta, \beta, \lambda) = \sum_{i=1}^n \left(x_{i:n} - \frac{-1}{\lambda} \log \left\{ 1 + \log \left[1 - (u_i \xi^{-\delta})^{\frac{1}{\delta}} \right] \right\} \right)^{\frac{2}{\beta}}.$$

5. Simulation analysis

In this section, a Monte Carlo simulation analysis was conducted to assess the performance of various estimators for the unknown parameters of the GKMW distribution. The assessment centered on their average absolute biases (BIAS), average mean square errors (MSE), and average mean relative errors (MRE) of the estimates, defined as follows:

$$\text{BIAS} = \frac{1}{n} \sum_{i=1}^n |\hat{\eta} - \eta|, \quad \text{MSE} = \frac{1}{n} \sum_{i=1}^n (\hat{\eta} - \eta)^2 \quad \text{and} \quad \text{MRE} = \frac{1}{n} \sum_{i=1}^n |\hat{\eta} - \eta|/\eta.$$

We generated 5,000 samples from the GKMW distribution for different sample sizes $n = \{20, 50, 100, 300, 500\}$, selecting $\delta = (0.5, 1.5)$, $\beta = (0.25, 2)$, and $\lambda = (1.5, 3.5)$. The GKMW parameters (δ, β, λ) were estimated for each combination of parameters and sample size using eight estimators, including WLSEs, LSEs, MLEs, MPSEs, CRVMEs, ADEs, RTADEs, and PCEs. Subsequently, the MSE, BIAS, and MRE of the parameters were calculated. All computations in this section were carried out using R software Version 4.2.2.

Tables 3–10 present the results of all simulated outcomes. Additionally, these tables display the ranking of each estimator in every row, with curly braces indicating the ranks and $\sum Ranks$ representing the cumulative sum of ranks for each column within a specific sample size. The findings in Tables 3–10 indicate that all estimation methods exhibit the property of consistency, as BIAS, MSE, and MRE decrease with increasing sample size across all parameter combinations.

The partial and total rankings of the estimators under consideration are presented in Table 11. The estimation method with the lowest overall score is regarded as the best approach. Based on Table 11, the eight estimation methods can be ranked from best to worst as follows: MPS, ML, AD, WLS, RAD, OLS, PC, and CRVM. It is important to note that, based on the results of the detailed simulation study, the MPS method, which achieved the lowest overall rank of 45.5, is considered the most effective estimation method. This lower rank indicates that the MPS method consistently produces better results, as measured by MSE, BIAS, and MRE, across sample sizes and different parameter values studied. Hence, the MPS approach, overall score of 45.5, outperforms all other approaches. Consequently, our results confirm the superiority of MPS method for estimating the GKMW parameters.

Table 3. Results for eight estimators with parameters $\delta = 0.5, \beta = 0.25$, and $\lambda = 1.5$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	BIAS	$\hat{\delta}$	0.44805{4}	0.46606{5}	0.40387{3}	0.47616{6}	0.39067{2}	2.44726{8}	0.37975{1}	0.48266{7}
		$\hat{\beta}$	0.14364{5}	0.15331{7}	0.13623{4}	0.14834{6}	0.13404{3}	0.65763{8}	0.13143{1}	0.13160{2}
		$\hat{\lambda}$	0.66572{4}	0.66928{8}	0.66508{3}	0.66627{6}	0.66622{5}	0.64430{1}	0.64661{2}	0.66871{7}
	MSE	$\hat{\delta}$	0.20075{4}	0.21722{5}	0.16311{3}	0.22673{6}	0.15262{2}	5.98910{8}	0.14421{1}	0.23296{7}
		$\hat{\beta}$	0.02063{5}	0.02350{7}	0.01856{4}	0.02200{6}	0.01797{3}	0.43248{8}	0.01727{1}	0.01732{2}
		$\hat{\lambda}$	0.44319{4}	0.44793{8}	0.44233{3}	0.44392{6}	0.44384{5}	0.41513{1}	0.41811{2}	0.44717{7}
	MRE	$\hat{\delta}$	0.59740{4}	0.62142{5}	0.53849{3}	0.63489{6}	0.52089{2}	0.88991{8}	0.50633{1}	0.64355{7}
		$\hat{\beta}$	0.28728{5}	0.30662{7}	0.27246{4}	0.29667{6}	0.26808{3}	0.32881{8}	0.26286{1}	0.26320{2}
		$\hat{\lambda}$	0.99362{4}	0.99892{8}	0.99265{3}	0.99444{6}	0.99435{5}	0.96165{1}	0.96509{2}	0.99807{7}
	$\sum RANKS$		39.0{4}	60.0{8}	30.0{2.5}	54.0{7}	30.0{2.5}	51.0{6}	12.0{1}	48.0{5}

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n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
50	BIAS	$\hat{\delta}$	0.25007{3}	0.29937{5}	0.24733{2}	0.31057{6}	0.22757{1}	2.37250{8}	0.25066{4}	0.31836{7}
		$\hat{\beta}$	0.08437{3}	0.09749{7}	0.08281{2}	0.09732{6}	0.07884{1}	0.49530{8}	0.08464{4}	0.08707{5}
		$\hat{\lambda}$	0.51229{1}	0.59472{6}	0.52254{3}	0.57940{5}	0.51959{2}	0.63231{8}	0.53069{4}	0.60133{7}
	MSE	$\hat{\delta}$	0.06254{3}	0.08962{5}	0.06117{2}	0.09645{6}	0.05179{1}	5.62873{8}	0.06283{4}	0.10135{7}
		$\hat{\beta}$	0.00712{3}	0.00951{7}	0.00686{2}	0.00947{6}	0.00622{1}	0.24532{8}	0.00716{4}	0.00758{5}
		$\hat{\lambda}$	0.26244{1}	0.35369{6}	0.27305{3}	0.33571{5}	0.26998{2}	0.39982{8}	0.28164{4}	0.36160{7}
	MRE	$\hat{\delta}$	0.33343{3}	0.39916{5}	0.32978{2}	0.41409{6}	0.30342{1}	0.86273{8}	0.33422{4}	0.42447{7}
		$\hat{\beta}$	0.16874{3}	0.19499{7}	0.16562{2}	0.19465{6}	0.15768{1}	0.24765{8}	0.16927{4}	0.17413{5}
		$\hat{\lambda}$	0.76461{1}	0.88764{6}	0.77991{3}	0.86478{5}	0.77551{2}	0.94375{8}	0.79208{4}	0.89751{7}
	$\sum RANKS$		21.0{2.5}	54.0{6}	21.0{2.5}	51.0{5}	12.0{1}	72.0{8}	36.0{4}	57.0{7}
100	BIAS	$\hat{\delta}$	0.17337{3}	0.21780{6}	0.17020{2}	0.22097{7}	0.15701{1}	2.39904{8}	0.17686{4}	0.21457{5}
		$\hat{\beta}$	0.05580{3}	0.06830{6}	0.05577{2}	0.06885{7}	0.05374{1}	0.41721{8}	0.05725{4}	0.06044{5}
		$\hat{\lambda}$	0.39296{2}	0.46927{7}	0.39857{3}	0.45336{5}	0.38016{1}	0.62525{8}	0.40082{4}	0.46343{6}
	MSE	$\hat{\delta}$	0.03006{3}	0.04744{6}	0.02897{2}	0.04883{7}	0.02465{1}	5.75540{8}	0.03128{4}	0.04604{5}
		$\hat{\beta}$	0.00311{2.5}	0.00466{6}	0.00311{2.5}	0.00474{7}	0.00289{1}	0.17406{8}	0.00328{4}	0.00365{5}
		$\hat{\lambda}$	0.15441{2}	0.22022{7}	0.15886{3}	0.20554{5}	0.14452{1}	0.39094{8}	0.16066{4}	0.21477{6}
	MRE	$\hat{\delta}$	0.23116{3}	0.29040{6}	0.22693{2}	0.29462{7}	0.20935{1}	0.87238{8}	0.23582{4}	0.28610{5}
		$\hat{\beta}$	0.11161{3}	0.13660{6}	0.11155{2}	0.13770{7}	0.10749{1}	0.20860{8}	0.11449{4}	0.12089{5}
		$\hat{\lambda}$	0.58650{2}	0.70041{7}	0.59488{3}	0.67666{5}	0.56741{1}	0.93322{8}	0.59824{4}	0.69169{6}
	$\sum RANKS$		23.5{3}	57.0{6.5}	21.5{2}	57.0{6.5}	9.0{1}	72.0{8}	36.0{4}	48.0{5}

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n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
300	BIAS	$\hat{\delta}$	0.11921{2}	0.15274{6}	0.12118{3}	0.15801{7}	0.11128{1}	2.36301{8}	0.12134{4}	0.15004{5}
		$\hat{\beta}$	0.03944{2}	0.04894{6}	0.03986{4}	0.04966{7}	0.03780{1}	0.35089{8}	0.03953{3}	0.04230{5}
		$\hat{\lambda}$	0.28268{2}	0.35098{6}	0.28501{3}	0.35253{7}	0.27401{1}	0.61586{8}	0.29546{4}	0.35009{5}
	MSE	$\hat{\delta}$	0.01421{2}	0.02333{6}	0.01469{3}	0.02497{7}	0.01238{1}	5.58381{8}	0.01472{4}	0.02251{5}
		$\hat{\beta}$	0.00156{2.5}	0.00239{6}	0.00159{4}	0.00247{7}	0.00143{1}	0.12312{8}	0.00156{2.5}	0.00179{5}
		$\hat{\lambda}$	0.07991{2}	0.12319{6}	0.08123{3}	0.12428{7}	0.07508{1}	0.37928{8}	0.08730{4}	0.12256{5}
	MRE	$\hat{\delta}$	0.15895{2}	0.20365{6}	0.16158{3}	0.21068{7}	0.14838{1}	0.85928{8}	0.16179{4}	0.20005{5}
		$\hat{\beta}$	0.07887{2}	0.09787{6}	0.07972{4}	0.09932{7}	0.07560{1}	0.17545{8}	0.07905{3}	0.08460{5}
		$\hat{\lambda}$	0.42191{2}	0.52386{6}	0.42539{3}	0.52616{7}	0.40896{1}	0.91919{8}	0.44099{4}	0.52252{5}
	$\sum RANKS$		18.5{2}	54.0{6}	30.0{3}	63.0{7}	9.0{1}	72.0{8}	32.5{4}	45.0{5}
500	BIAS	$\hat{\delta}$	0.08419{2}	0.10545{5}	0.08572{3}	0.10981{7}	0.07707{1}	2.51646{8}	0.08962{4}	0.10567{6}
		$\hat{\beta}$	0.02771{2}	0.03356{6}	0.02858{3}	0.03361{7}	0.02586{1}	0.30410{8}	0.02884{4}	0.03016{5}
		$\hat{\lambda}$	0.20338{1}	0.25009{5}	0.20668{3}	0.25785{7}	0.20635{2}	0.60619{8}	0.21717{4}	0.25239{6}
	MSE	$\hat{\delta}$	0.00709{2}	0.01112{5}	0.00735{3}	0.01206{7}	0.00594{1}	6.33269{8}	0.00803{4}	0.01117{6}
		$\hat{\beta}$	0.00077{2}	0.00113{6.5}	0.00082{3}	0.00113{6.5}	0.00067{1}	0.09247{8}	0.00083{4}	0.00091{5}
		$\hat{\lambda}$	0.04136{1}	0.06255{5}	0.04272{3}	0.06649{7}	0.04258{2}	0.36747{8}	0.04716{4}	0.06370{6}
	MRE	$\hat{\delta}$	0.11225{2}	0.14061{5}	0.11429{3}	0.14641{7}	0.10276{1}	0.91508{8}	0.11950{4}	0.14089{6}
		$\hat{\beta}$	0.05542{2}	0.06711{6}	0.05716{3}	0.06723{7}	0.05173{1}	0.15205{8}	0.05768{4}	0.06033{5}
		$\hat{\lambda}$	0.30355{1}	0.37327{5}	0.30848{3}	0.38486{7}	0.30798{2}	0.90477{8}	0.32413{4}	0.37671{6}
	$\sum RANKS$		15.0{2}	48.5{5}	27.0{3}	62.5{7}	12.0{1}	72.0{8}	36.0{4}	51.0{6}

Table 4. Results for eight estimators with parameters $\delta = 0.5, \beta = 0.25$, and $\lambda = 3.5$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	BIAS	$\hat{\delta}$	0.32426{3}	0.36749{5}	0.32579{4}	0.37178{6}	0.32305{2}	0.48308{8}	0.31426{1}	0.38908{7}
		$\hat{\beta}$	0.16279{6}	0.16255{5}	0.13270{3}	0.16800{7}	0.12181{1}	0.17744{8}	0.12182{2}	0.14583{4}
		$\hat{\lambda}$	1.02635{2}	1.49446{6}	1.25347{4}	1.89848{8}	1.11158{3}	0.98297{1}	1.27708{5}	1.62940{7}
	MSE	$\hat{\delta}$	0.10515{3}	0.13505{5}	0.10614{4}	0.13822{6}	0.10436{2}	0.23336{8}	0.09876{1}	0.15138{7}
		$\hat{\beta}$	0.02650{6}	0.02642{5}	0.01761{3}	0.02822{7}	0.01484{1.5}	0.03148{8}	0.01484{1.5}	0.02127{4}
		$\hat{\lambda}$	1.05340{2}	2.23340{6}	1.57119{4}	3.60423{8}	1.23560{3}	0.96623{1}	1.63094{5}	2.65496{7}
	MRE	$\hat{\delta}$	0.64853{3}	0.73498{5}	0.65157{4}	0.74356{6}	0.64610{2}	0.96616{8}	0.62852{1}	0.77815{7}
		$\hat{\beta}$	0.65115{6}	0.65020{5}	0.53081{3}	0.67201{7}	0.48723{1}	0.70975{8}	0.48728{2}	0.58331{4}
		$\hat{\lambda}$	0.29324{2}	0.42699{6}	0.35813{4}	0.54242{8}	0.31759{3}	0.28085{1}	0.36488{5}	0.46554{7}
	$\sum RANKS$		33.0{3.5}	48.0{5}	33.0{3.5}	63.0{8}	18.5{1}	51.0{6}	23.5{2}	54.0{7}
50	BIAS	$\hat{\delta}$	0.26539{6}	0.25563{5}	0.20593{3}	0.26669{7}	0.18851{1}	0.49483{8}	0.20458{2}	0.23977{4}
		$\hat{\beta}$	0.12742{7}	0.09473{5}	0.07337{3}	0.09606{6}	0.06554{1}	0.25000{8}	0.07268{2}	0.07792{4}
		$\hat{\lambda}$	0.31068{1}	0.87506{6}	0.65401{3}	0.93368{7}	0.63524{2}	1.70896{8}	0.70057{4}	0.77199{5}
	MSE	$\hat{\delta}$	0.07043{6}	0.06534{5}	0.04241{3}	0.07112{7}	0.03554{1}	0.24486{8}	0.04185{2}	0.05749{4}
		$\hat{\beta}$	0.01624{7}	0.00897{5}	0.00538{3}	0.00923{6}	0.00429{1}	0.06250{8}	0.00528{2}	0.00607{4}
		$\hat{\lambda}$	0.09652{1}	0.76574{6}	0.42773{3}	0.87176{7}	0.40352{2}	2.92055{8}	0.49080{4}	0.59596{5}
	MRE	$\hat{\delta}$	0.53078{6}	0.51125{5}	0.41186{3}	0.53337{7}	0.37702{1}	0.98966{8}	0.40915{2}	0.47955{4}
		$\hat{\beta}$	0.50967{7}	0.37893{5}	0.29349{3}	0.38424{6}	0.26214{1}	1.00000{8}	0.29071{2}	0.31167{4}
		$\hat{\lambda}$	0.08876{1}	0.25002{6}	0.18686{3}	0.26677{7}	0.18150{2}	0.48827{8}	0.20016{4}	0.22057{5}
	$\sum RANKS$		42.0{5}	48.0{6}	27.0{3}	60.0{7}	12.0{1}	72.0{8}	24.0{2}	39.0{4}
100	BIAS	$\hat{\delta}$	0.23814{7}	0.19152{5}	0.14325{2}	0.19335{6}	0.13433{1}	0.50000{8}	0.14782{3}	0.17393{4}
		$\hat{\beta}$	0.12253{7}	0.06875{6}	0.04895{2}	0.06860{5}	0.04607{1}	0.25000{8}	0.05166{3}	0.05613{4}
		$\hat{\lambda}$	0.13396{1}	0.59512{7}	0.43461{2}	0.59336{6}	0.45566{3}	2.33232{8}	0.48957{4}	0.49276{5}
	MSE	$\hat{\delta}$	0.05671{7}	0.03668{5}	0.02052{2}	0.03739{6}	0.01805{1}	0.25000{8}	0.02185{3}	0.03025{4}
		$\hat{\beta}$	0.01501{7}	0.00473{6}	0.00240{2}	0.00471{5}	0.00212{1}	0.06250{8}	0.00267{3}	0.00315{4}

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n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
	MRE	$\hat{\lambda}$	0.01795{1}	0.35416{7}	0.18889{2}	0.35207{6}	0.20762{3}	5.43973{8}	0.23968{4}	0.24281{5}
		$\hat{\delta}$	0.47627{7}	0.38303{5}	0.28650{2}	0.38671{6}	0.26866{1}	1.00000{8}	0.29563{3}	0.34787{4}
		$\hat{\beta}$	0.49013{7}	0.27502{6}	0.19579{2}	0.27439{5}	0.18427{1}	1.00000{8}	0.20663{3}	0.22454{4}
		$\hat{\lambda}$	0.03827{1}	0.17003{7}	0.12417{2}	0.16953{6}	0.13019{3}	0.66638{8}	0.13988{4}	0.14079{5}
		$\sum RANKS$	45.0{5}	54.0{7}	18.0{2}	51.0{6}	15.0{1}	72.0{8}	30.0{3}	39.0{4}
	BIAS	$\hat{\delta}$	0.16105{7}	0.11059{5}	0.08394{2}	0.11359{6}	0.07800{1}	0.49787{8}	0.08757{3}	0.10016{4}
		$\hat{\beta}$	0.06283{7}	0.03817{5}	0.02859{2}	0.03946{6}	0.02634{1}	0.26436{8}	0.03008{3}	0.03194{4}
		$\hat{\lambda}$	0.09811{1}	0.32542{6}	0.26408{3}	0.33776{7}	0.26030{2}	3.29688{8}	0.27467{4}	0.27699{5}
		$\hat{\delta}$	0.02594{7}	0.01223{5}	0.00705{2}	0.0129{6}	0.00608{1}	0.24788{8}	0.00767{3}	0.01003{4}
		$\hat{\beta}$	0.00395{7}	0.00146{5}	0.00082{2}	0.00156{6}	0.00069{1}	0.06989{8}	0.0009{3}	0.00102{4}
300	MSE	$\hat{\lambda}$	0.00963{1}	0.10590{6}	0.06974{3}	0.11408{7}	0.06775{2}	10.86939{8}	0.07544{4}	0.07673{5}
		$\hat{\delta}$	0.32211{7}	0.22118{5}	0.16788{2}	0.22718{6}	0.15600{1}	0.99575{8}	0.17513{3}	0.20033{4}
		$\hat{\beta}$	0.25131{7}	0.15269{5}	0.11437{2}	0.15786{6}	0.10537{1}	1.05746{8}	0.12030{3}	0.12777{4}
		$\hat{\lambda}$	0.02803{1}	0.09298{6}	0.07545{3}	0.09650{7}	0.07437{2}	0.94196{8}	0.07848{4}	0.07914{5}
		$\sum RANKS$	45.0{5}	48.0{6}	21.0{2}	57.0{7}	12.0{1}	72.0{8}	30.0{3}	39.0{4}
	MRE	$\hat{\delta}$	0.13682{7}	0.08518{5}	0.06806{2}	0.08903{6}	0.06143{1}	0.49767{8}	0.06854{3}	0.07835{4}
		$\hat{\beta}$	0.04912{7}	0.02897{5}	0.02309{2}	0.03037{6}	0.02041{1}	0.19514{8}	0.02314{3}	0.02444{4}
		$\hat{\lambda}$	0.07977{1}	0.25952{6}	0.19725{2}	0.26898{7}	0.20086{3}	3.31272{8}	0.22096{5}	0.21419{4}
		$\hat{\delta}$	0.01872{7}	0.00726{5}	0.00463{2}	0.00793{6}	0.00377{1}	0.24768{8}	0.00470{3}	0.00614{4}
		$\hat{\beta}$	0.00241{7}	0.00084{5}	0.00053{2}	0.00092{6}	0.00042{1}	0.03808{8}	0.00054{3}	0.00060{4}
500	MSE	$\hat{\lambda}$	0.00636{1}	0.06735{6}	0.03891{2}	0.07235{7}	0.04034{3}	10.97413{8}	0.04883{5}	0.04588{4}
		$\hat{\delta}$	0.27364{7}	0.17036{5}	0.13611{2}	0.17807{6}	0.12285{1}	0.99535{8}	0.13709{3}	0.15669{4}
		$\hat{\beta}$	0.19649{7}	0.11589{5}	0.09236{2}	0.12148{6}	0.08166{1}	0.78054{8}	0.09258{3}	0.09775{4}
		$\hat{\lambda}$	0.02279{1}	0.07415{6}	0.05636{2}	0.07685{7}	0.05739{3}	0.94649{8}	0.06313{5}	0.06120{4}
		$\sum RANKS$	45.0{5}	48.0{6}	18.0{2}	57.0{7}	15.0{1}	72.0{8}	33.0{3}	36.0{4}
	BIAS	$\hat{\delta}$	0.13682{7}	0.08518{5}	0.06806{2}	0.08903{6}	0.06143{1}	0.49767{8}	0.06854{3}	0.07835{4}
		$\hat{\beta}$	0.04912{7}	0.02897{5}	0.02309{2}	0.03037{6}	0.02041{1}	0.19514{8}	0.02314{3}	0.02444{4}
		$\hat{\lambda}$	0.07977{1}	0.25952{6}	0.19725{2}	0.26898{7}	0.20086{3}	3.31272{8}	0.22096{5}	0.21419{4}
		$\hat{\delta}$	0.01872{7}	0.00726{5}	0.00463{2}	0.00793{6}	0.00377{1}	0.24768{8}	0.00470{3}	0.00614{4}
		$\hat{\beta}$	0.00241{7}	0.00084{5}	0.00053{2}	0.00092{6}	0.00042{1}	0.03808{8}	0.00054{3}	0.00060{4}

Table 5. Results for eight estimators with parameters $\delta = 0.5, \beta = 2$, and $\lambda = 1.5$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	BIAS	$\hat{\delta}$	0.33176{3}	0.36562{5}	0.33913{4}	0.37397{6}	0.32582{2}	0.38712{7}	0.32087{1}	0.38878{8}
		$\hat{\beta}$	1.12766{4}	1.27666{6}	1.07109{3}	1.35020{8}	0.96289{2}	1.28521{7}	0.94279{1}	1.17226{5}
		$\hat{\lambda}$	0.86191{4}	0.88497{5}	0.76644{3}	0.99457{8}	0.70889{1}	0.90068{6}	0.74792{2}	0.95949{7}
	MSE	$\hat{\delta}$	0.11007{3}	0.13368{5}	0.11501{4}	0.13985{6}	0.10616{2}	0.14986{7}	0.10296{1}	0.15115{8}
		$\hat{\beta}$	1.27161{4}	1.62986{6}	1.14723{3}	1.82303{8}	0.92715{2}	1.65176{7}	0.88885{1}	1.37420{5}
		$\hat{\lambda}$	0.74288{4}	0.78317{5}	0.58742{3}	0.98917{8}	0.50252{1}	0.81123{6}	0.55939{2}	0.92062{7}
	MRE	$\hat{\delta}$	0.66352{3}	0.73125{5}	0.67827{4}	0.74793{6}	0.65165{2}	0.77425{7}	0.64175{1}	0.77755{8}
		$\hat{\beta}$	0.56383{4}	0.63833{6}	0.53555{3}	0.67510{8}	0.48144{2}	0.64260{7}	0.47140{1}	0.58613{5}
		$\hat{\lambda}$	0.57461{4}	0.58998{5}	0.51096{3}	0.66305{8}	0.47259{1}	0.60045{6}	0.49861{2}	0.63966{7}
	$\sum RANKS$		33.0{4}	48.0{5}	30.0{3}	66.0{8}	15.0{2}	60.0{6.5}	12.0{1}	60.0{6.5}
50	BIAS	$\hat{\delta}$	0.21959{4}	0.24961{6}	0.21918{3}	0.26981{7}	0.19148{1}	0.27897{8}	0.20495{2}	0.24813{5}
		$\hat{\beta}$	0.62370{3}	0.76053{7}	0.62516{4}	0.77811{8}	0.54002{1}	0.73260{6}	0.57135{2}	0.64916{5}
		$\hat{\lambda}$	0.45268{4}	0.48195{5}	0.42714{2}	0.52178{7}	0.39191{1}	0.56415{8}	0.43028{3}	0.51147{6}
	MSE	$\hat{\delta}$	0.04822{4}	0.06230{6}	0.04804{3}	0.07280{7}	0.03666{1}	0.07783{8}	0.04200{2}	0.06157{5}
		$\hat{\beta}$	0.38900{3}	0.57840{7}	0.39083{4}	0.60546{8}	0.29162{1}	0.53670{6}	0.32644{2}	0.42141{5}
		$\hat{\lambda}$	0.20492{4}	0.23227{5}	0.18245{2}	0.27226{7}	0.15360{1}	0.31827{8}	0.18514{3}	0.26160{6}
	MRE	$\hat{\delta}$	0.43918{4}	0.49921{6}	0.43837{3}	0.53963{7}	0.38295{1}	0.55795{8}	0.40990{2}	0.49627{5}
		$\hat{\beta}$	0.31185{3}	0.38026{7}	0.31258{4}	0.38906{8}	0.27001{1}	0.36630{6}	0.28568{2}	0.32458{5}
		$\hat{\lambda}$	0.30179{4}	0.32130{5}	0.28476{2}	0.34786{7}	0.26128{1}	0.37610{8}	0.28685{3}	0.34098{6}
	$\sum RANKS$		33.0{4}	54.0{6}	27.0{3}	66.0{7.5}	9.0{1}	66.0{7.5}	21.0{2}	48.0{5}
100	BIAS	$\hat{\delta}$	0.14671{2}	0.18823{6}	0.15343{4}	0.19493{7}	0.13279{1}	0.21007{8}	0.15276{3}	0.17687{5}
		$\hat{\beta}$	0.39391{2}	0.53496{7}	0.42885{4}	0.55301{8}	0.35810{1}	0.52138{6}	0.41247{3}	0.44862{5}
		$\hat{\lambda}$	0.29759{3}	0.32702{5}	0.27960{2}	0.34665{6}	0.26772{1}	0.39649{8}	0.30125{4}	0.35571{7}
	MSE	$\hat{\delta}$	0.02152{2}	0.03543{6}	0.02354{4}	0.03800{7}	0.01763{1}	0.04413{8}	0.02334{3}	0.03128{5}
		$\hat{\beta}$	0.15516{2}	0.28618{7}	0.18391{4}	0.30582{8}	0.12824{1}	0.27184{6}	0.17013{3}	0.20126{5}

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n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
	MRE	$\hat{\lambda}$	0.08856{3}	0.10694{5}	0.07818{2}	0.12017{6}	0.07167{1}	0.15720{8}	0.09075{4}	0.12653{7}
		$\hat{\delta}$	0.29341{2}	0.37646{6}	0.30687{4}	0.38987{7}	0.26559{1}	0.42015{8}	0.30552{3}	0.35375{5}
		$\hat{\beta}$	0.19695{2}	0.26748{7}	0.21443{4}	0.27650{8}	0.17905{1}	0.26069{6}	0.20623{3}	0.22431{5}
		$\hat{\lambda}$	0.19839{3}	0.21801{5}	0.18640{2}	0.23110{6}	0.17848{1}	0.26433{8}	0.20083{4}	0.23714{7}
	$\sum RANKS$		21.0{2}	54.0{6}	30.0{3.5}	63.0{7}	9.0{1}	66.0{8}	30.0{3.5}	51.0{5}
300	BIAS	$\hat{\delta}$	0.08098{2}	0.10961{6}	0.08512{3}	0.11525{7}	0.07733{1}	0.12222{8}	0.08977{4}	0.10159{5}
		$\hat{\beta}$	0.21831{2}	0.30112{7}	0.23615{3}	0.31930{8}	0.20764{1}	0.28937{6}	0.24578{4}	0.26022{5}
		$\hat{\lambda}$	0.15901{2}	0.18217{5}	0.16388{3}	0.18257{6}	0.15273{1}	0.22463{8}	0.16393{4}	0.19128{7}
	MSE	$\hat{\delta}$	0.00656{2}	0.01201{6}	0.00725{3}	0.01328{7}	0.00598{1}	0.01494{8}	0.00806{4}	0.01032{5}
		$\hat{\beta}$	0.04766{2}	0.09067{7}	0.05576{3}	0.10195{8}	0.04312{1}	0.08374{6}	0.06041{4}	0.06771{5}
		$\hat{\lambda}$	0.02529{2}	0.03319{5}	0.02686{3}	0.03333{6}	0.02333{1}	0.05046{8}	0.02687{4}	0.03659{7}
	MRE	$\hat{\delta}$	0.16196{2}	0.21922{6}	0.17024{3}	0.23050{7}	0.15466{1}	0.24444{8}	0.17953{4}	0.20317{5}
		$\hat{\beta}$	0.10916{2}	0.15056{7}	0.11807{3}	0.15965{8}	0.10382{1}	0.14469{6}	0.12289{4}	0.13011{5}
		$\hat{\lambda}$	0.10601{2}	0.12145{5}	0.10925{3}	0.12171{6}	0.10182{1}	0.14975{8}	0.10929{4}	0.12752{7}
	$\sum RANKS$		18.0{2}	54.0{6}	27.0{3}	63.0{7}	9.0{1}	66.0{8}	36.0{4}	51.0{5}
500	BIAS	$\hat{\delta}$	0.06133{2}	0.08655{6}	0.06579{3}	0.08942{7}	0.05873{1}	0.09305{8}	0.06982{4}	0.07986{5}
		$\hat{\beta}$	0.16505{2}	0.23942{7}	0.17980{3}	0.24405{8}	0.15115{1}	0.21914{6}	0.18812{4}	0.20420{5}
		$\hat{\lambda}$	0.12258{3}	0.14193{5}	0.12071{2}	0.14351{6}	0.11629{1}	0.17353{8}	0.12581{4}	0.14648{7}
	MSE	$\hat{\delta}$	0.00376{2}	0.00749{6}	0.00433{3}	0.00800{7}	0.00345{1}	0.00866{8}	0.00488{4}	0.00638{5}
		$\hat{\beta}$	0.02724{2}	0.05732{7}	0.03233{3}	0.05956{8}	0.02285{1}	0.04802{6}	0.03539{4}	0.04170{5}
		$\hat{\lambda}$	0.01503{3}	0.02014{5}	0.01457{2}	0.02059{6}	0.01352{1}	0.03011{8}	0.01583{4}	0.02146{7}
	MRE	$\hat{\delta}$	0.12266{2}	0.17311{6}	0.13157{3}	0.17883{7}	0.11745{1}	0.18610{8}	0.13965{4}	0.15972{5}
		$\hat{\beta}$	0.08253{2}	0.11971{7}	0.08990{3}	0.12203{8}	0.07557{1}	0.10957{6}	0.09406{4}	0.10210{5}
		$\hat{\lambda}$	0.08172{3}	0.09462{5}	0.08047{2}	0.09567{6}	0.07753{1}	0.11568{8}	0.08387{4}	0.09765{7}
	$\sum RANKS$		21.0{2}	54.0{6}	24.0{3}	63.0{7}	9.0{1}	66.0{8}	36.0{4}	51.0{5}

Table 6. Results for eight estimators with parameters $\delta = 0.5, \beta = 2$, and $\lambda = 3.5$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	BIAS	$\hat{\delta}$	0.33110{3}	0.36615{5}	0.33585{4}	0.37061{6}	0.31868{1}	0.39913{8}	0.32453{2}	0.38922{7}
		$\hat{\beta}$	1.11460{4}	1.27786{6}	1.06641{3}	1.32616{8}	0.95015{2}	1.28377{7}	0.94504{1}	1.17382{5}
		$\hat{\lambda}$	1.55737{5}	1.59665{6}	1.34503{3}	1.91607{8}	1.19054{1}	1.53949{4}	1.23604{2}	1.70187{7}
	MSE	$\hat{\delta}$	0.10963{3}	0.13407{5}	0.11279{4}	0.13735{6}	0.10155{1}	0.15930{8}	0.10532{2}	0.15149{7}
		$\hat{\beta}$	1.24233{4}	1.63291{6}	1.13723{3}	1.75870{8}	0.90278{2}	1.64808{7}	0.89310{1}	1.37786{5}
		$\hat{\lambda}$	2.42540{5}	2.54930{6}	1.80910{3}	3.67132{8}	1.41739{1}	2.37004{4}	1.52779{2}	2.89636{7}
	MRE	$\hat{\delta}$	0.66220{3}	0.73230{5}	0.67170{4}	0.74122{6}	0.63735{1}	0.79825{8}	0.64906{2}	0.77844{7}
		$\hat{\beta}$	0.55730{4}	0.63893{6}	0.53320{3}	0.66308{8}	0.47507{2}	0.64189{7}	0.47252{1}	0.58691{5}
		$\hat{\lambda}$	0.44496{5}	0.45619{6}	0.38429{3}	0.54745{8}	0.34016{1}	0.43986{4}	0.35315{2}	0.48625{7}
	$\sum RANKS$		36.0{4}	51.0{5}	30.0{3}	66.0{8}	12.0{1}	57.0{6.5}	15.0{2}	57.0{6.5}
50	BIAS	$\hat{\delta}$	0.21390{3}	0.26182{6}	0.22139{4}	0.27439{7}	0.19140{1}	0.28679{8}	0.20785{2}	0.24663{5}
		$\hat{\beta}$	0.61150{3}	0.78181{7}	0.64730{4}	0.78486{8}	0.53103{1}	0.76841{6}	0.59128{2}	0.64931{5}
		$\hat{\lambda}$	0.78611{4}	0.86510{7}	0.76550{3}	0.92676{8}	0.68749{1}	0.80892{5}	0.73796{2}	0.81282{6}
	MSE	$\hat{\delta}$	0.04575{3}	0.06855{6}	0.04901{4}	0.07529{7}	0.03664{1}	0.08225{8}	0.04320{2}	0.06082{5}
		$\hat{\beta}$	0.37393{3}	0.61123{7}	0.41900{4}	0.61600{8}	0.28200{1}	0.59046{6}	0.34961{2}	0.42160{5}
		$\hat{\lambda}$	0.61798{4}	0.74840{7}	0.58599{3}	0.85888{8}	0.47264{1}	0.65435{5}	0.54459{2}	0.66068{6}
	MRE	$\hat{\delta}$	0.42780{3}	0.52365{6}	0.44278{4}	0.54878{7}	0.38281{1}	0.57358{8}	0.41571{2}	0.49325{5}
		$\hat{\beta}$	0.30575{3}	0.39091{7}	0.32365{4}	0.39243{8}	0.26552{1}	0.38421{6}	0.29564{2}	0.32465{5}
		$\hat{\lambda}$	0.22460{4}	0.24717{7}	0.21871{3}	0.26479{8}	0.19643{1}	0.23112{5}	0.21085{2}	0.23223{6}
	$\sum RANKS$		30.0{3}	60.0{7}	33.0{4}	69.0{8}	9.0{1}	57.0{6}	18.0{2}	48.0{5}
100	BIAS	$\hat{\delta}$	0.14038{2}	0.19164{6}	0.15667{4}	0.19596{7}	0.13391{1}	0.20535{8}	0.15291{3}	0.17192{5}
		$\hat{\beta}$	0.38434{2}	0.54048{7}	0.42196{4}	0.56208{8}	0.36139{1}	0.51354{6}	0.42009{3}	0.45263{5}
		$\hat{\lambda}$	0.47694{2}	0.58765{7}	0.51182{4}	0.60332{8}	0.45658{1}	0.51860{5}	0.49143{3}	0.53797{6}
	MSE	$\hat{\delta}$	0.01971{2}	0.03672{6}	0.02455{4}	0.03840{7}	0.01793{1}	0.04217{8}	0.02338{3}	0.02956{5}
		$\hat{\beta}$	0.14772{2}	0.29211{7}	0.17805{4}	0.31594{8}	0.13060{1}	0.26372{6}	0.17648{3}	0.20488{5}

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n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
300	MRE	$\hat{\lambda}$	0.22747{2}	0.34533{7}	0.26196{4}	0.36399{8}	0.20847{1}	0.26895{5}	0.24150{3}	0.28941{6}
		$\hat{\delta}$	0.28075{2}	0.38327{6}	0.31335{4}	0.39192{7}	0.26781{1}	0.41070{8}	0.30583{3}	0.34385{5}
		$\hat{\beta}$	0.19217{2}	0.27024{7}	0.21098{4}	0.28104{8}	0.18069{1}	0.25677{6}	0.21004{3}	0.22632{5}
		$\hat{\lambda}$	0.13627{2}	0.16790{7}	0.14623{4}	0.17238{8}	0.13045{1}	0.14817{5}	0.14041{3}	0.15371{6}
	$\sum RANKS$		18.0{2}	60.0{7}	36.0{4}	69.0{8}	9.0{1}	57.0{6}	27.0{3}	48.0{5}
	BIAS	$\hat{\delta}$	0.07851{1}	0.11388{6}	0.08746{3}	0.11416{7}	0.08299{2}	0.12225{8}	0.09006{4}	0.10489{5}
		$\hat{\beta}$	0.20925{1}	0.31575{7}	0.23592{3}	0.31907{8}	0.22303{2}	0.28419{6}	0.24085{4}	0.26648{5}
		$\hat{\lambda}$	0.26941{2}	0.33461{8}	0.27809{4}	0.33184{7}	0.24439{1}	0.27271{3}	0.28243{5}	0.29963{6}
		$\hat{\delta}$	0.00616{1}	0.01297{6}	0.00765{3}	0.01303{7}	0.00689{2}	0.01495{8}	0.00811{4}	0.01100{5}
	MSE	$\hat{\beta}$	0.04379{1}	0.09970{7}	0.05566{3}	0.10180{8}	0.04974{2}	0.08077{6}	0.05801{4}	0.07101{5}
$\hat{\lambda}$		0.07258{2}	0.11196{8}	0.07733{4}	0.11012{7}	0.05973{1}	0.07437{3}	0.07977{5}	0.08978{6}	
MRE	$\hat{\delta}$	0.15702{1}	0.22776{6}	0.17492{3}	0.22833{7}	0.16597{2}	0.24450{8}	0.18013{4}	0.20979{5}	
	$\hat{\beta}$	0.10463{1}	0.15787{7}	0.11796{3}	0.15953{8}	0.11152{2}	0.14210{6}	0.12042{4}	0.13324{5}	
	$\hat{\lambda}$	0.07697{2}	0.09560{8}	0.07945{4}	0.09481{7}	0.06983{1}	0.07792{3}	0.08069{5}	0.08561{6}	
	$\sum RANKS$		12.0{1}	63.0{7}	30.0{3}	66.0{8}	15.0{2}	51.0{6}	39.0{4}	48.0{5}
500	BIAS	$\hat{\delta}$	0.06024{1}	0.08637{7}	0.06664{3}	0.08502{6}	0.06628{2}	0.09635{8}	0.06819{4}	0.07999{5}
		$\hat{\beta}$	0.16078{1}	0.24082{7}	0.18266{3}	0.24124{8}	0.17382{2}	0.22598{6}	0.18545{4}	0.20491{5}
		$\hat{\lambda}$	0.20606{2}	0.26665{8}	0.22231{5}	0.26483{7}	0.18095{1}	0.21196{3}	0.22040{4}	0.22496{6}
		$\hat{\delta}$	0.00363{1}	0.00746{7}	0.00444{3}	0.00723{6}	0.00439{2}	0.00928{8}	0.00465{4}	0.00640{5}
	MSE	$\hat{\beta}$	0.02585{1}	0.05800{7}	0.03337{3}	0.05820{8}	0.03021{2}	0.05107{6}	0.03439{4}	0.04199{5}
		$\hat{\lambda}$	0.04246{2}	0.07110{8}	0.04942{5}	0.07014{7}	0.03274{1}	0.04493{3}	0.04858{4}	0.05061{6}
	MRE	$\hat{\delta}$	0.12048{1}	0.17273{7}	0.13328{3}	0.17003{6}	0.13257{2}	0.19270{8}	0.13637{4}	0.15997{5}
		$\hat{\beta}$	0.08039{1}	0.12041{7}	0.09133{3}	0.12062{8}	0.08691{2}	0.11299{6}	0.09272{4}	0.10245{5}
		$\hat{\lambda}$	0.05887{2}	0.07619{8}	0.06352{5}	0.07567{7}	0.05170{1}	0.06056{3}	0.06297{4}	0.06427{6}
		$\sum RANKS$		12.0{1}	66.0{8}	33.0{3}	63.0{7}	15.0{2}	51.0{6}	36.0{4}

Table 7. Results for eight estimators with parameters $\delta = 1.5, \beta = 0.25$, and $\lambda = 1.5$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	BIAS	$\hat{\delta}$	1.22185{4}	1.30796{5}	1.18799{3}	1.33931{6}	1.11412{1}	1.45636{8}	1.13362{2}	1.36998{7}
		$\hat{\beta}$	0.12675{1}	0.17951{6}	0.15149{4}	0.18035{7}	0.12973{3}	0.19587{8}	0.12793{2}	0.16035{5}
		$\hat{\lambda}$	0.94495{3}	1.13870{5}	0.96082{4}	1.15670{6}	0.88794{1}	1.49717{8}	0.89346{2}	1.21562{7}
	MSE	$\hat{\delta}$	1.49293{4}	1.71077{5}	1.41132{3}	1.79374{6}	1.24126{1}	2.12099{8}	1.28510{2}	1.87684{7}
		$\hat{\beta}$	0.01607{1}	0.03223{6}	0.02295{4}	0.03253{7}	0.01683{3}	0.03837{8}	0.01637{2}	0.02571{5}
		$\hat{\lambda}$	0.89294{3}	1.29664{5}	0.92317{4}	1.33795{6}	0.78844{1}	2.24151{8}	0.79826{2}	1.47773{7}
	MRE	$\hat{\delta}$	0.81457{4}	0.87198{5}	0.79199{3}	0.89287{6}	0.74275{1}	0.97091{8}	0.75575{2}	0.91332{7}
		$\hat{\beta}$	0.50700{1}	0.71806{6}	0.60596{4}	0.72140{7}	0.51892{3}	0.78348{8}	0.51172{2}	0.64141{5}
		$\hat{\lambda}$	0.62997{3}	0.75913{5}	0.64055{4}	0.77113{6}	0.59196{1}	0.99811{8}	0.59564{2}	0.81041{7}
	$\sum RANKS$		24.0{3}	48.0{5}	33.0{4}	57.0{6.5}	15.0{1}	72.0{8}	18.0{2}	57.0{6.5}
50	BIAS	$\hat{\delta}$	0.79315{2}	0.99146{5}	0.84310{4}	1.01743{7}	0.72849{1}	1.46287{8}	0.80972{3}	0.99359{6}
		$\hat{\beta}$	0.07620{2}	0.11078{6}	0.08734{4}	0.11296{7}	0.06930{1}	0.18453{8}	0.07794{3}	0.09839{5}
		$\hat{\lambda}$	0.54797{2}	0.71063{6}	0.57525{4}	0.70987{5}	0.50083{1}	1.49746{8}	0.55093{3}	0.72034{7}
	MSE	$\hat{\delta}$	0.62908{2}	0.98300{5}	0.71082{4}	1.03516{7}	0.53070{1}	2.13998{8}	0.65565{3}	0.98721{6}
		$\hat{\beta}$	0.00581{2}	0.01227{6}	0.00763{4}	0.01276{7}	0.00480{1}	0.03405{8}	0.00607{3}	0.00968{5}
		$\hat{\lambda}$	0.30027{2}	0.50499{6}	0.33091{4}	0.50392{5}	0.25083{1}	2.24238{8}	0.30352{3}	0.51889{7}
	MRE	$\hat{\delta}$	0.52876{2}	0.66097{5}	0.56207{4}	0.67829{7}	0.48566{1}	0.97524{8}	0.53982{3}	0.66239{6}
		$\hat{\beta}$	0.30480{2}	0.44310{6}	0.34938{4}	0.45185{7}	0.27722{1}	0.73812{8}	0.31176{3}	0.39356{5}
		$\hat{\lambda}$	0.36531{2}	0.47375{6}	0.38350{4}	0.47325{5}	0.33389{1}	0.99831{8}	0.36729{3}	0.48022{7}
	$\sum RANKS$		18.0{2}	51.0{5}	36.0{4}	57.0{7}	9.0{1}	72.0{8}	27.0{3}	54.0{6}
100	BIAS	$\hat{\delta}$	0.55352{2}	0.79898{7}	0.61847{4}	0.79686{6}	0.53296{1}	1.46195{8}	0.60399{3}	0.77763{5}
		$\hat{\beta}$	0.05036{2}	0.07948{7}	0.05744{4}	0.07839{6}	0.04670{1}	0.18270{8}	0.05485{3}	0.06844{5}
		$\hat{\lambda}$	0.35861{2}	0.51616{6}	0.39537{4}	0.52199{7}	0.33928{1}	1.49165{8}	0.38740{3}	0.51469{5}
	MSE	$\hat{\delta}$	0.30638{2}	0.63837{7}	0.38250{4}	0.63499{6}	0.28404{1}	2.13729{8}	0.36481{3}	0.60470{5}
		$\hat{\beta}$	0.00254{2}	0.00632{7}	0.00330{4}	0.00615{6}	0.00218{1}	0.03338{8}	0.00301{3}	0.00468{5}

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n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
	MRE	$\hat{\lambda}$	0.12860{2}	0.26642{6}	0.15632{4}	0.27248{7}	0.11511{1}	2.22502{8}	0.15008{3}	0.26490{5}
		$\hat{\delta}$	0.36901{2}	0.53266{7}	0.41231{4}	0.53124{6}	0.35530{1}	0.97463{8}	0.40266{3}	0.51842{5}
		$\hat{\beta}$	0.20143{2}	0.31790{7}	0.22975{4}	0.31356{6}	0.18681{1}	0.73078{8}	0.21941{3}	0.27376{5}
		$\hat{\lambda}$	0.23907{2}	0.34411{6}	0.26358{4}	0.34800{7}	0.22619{1}	0.99443{8}	0.25827{3}	0.34313{5}
		$\sum RANKS$	18.0{2}	60.0{7}	36.0{4}	57.0{6}	9.0{1}	72.0{8}	27.0{3}	45.0{5}
	BIAS	$\hat{\delta}$	0.29712{1}	0.48781{7}	0.35956{3}	0.48497{6}	0.30236{2}	1.41996{8}	0.36469{4}	0.47766{5}
		$\hat{\beta}$	0.02561{2}	0.04395{7}	0.03174{4}	0.04365{6}	0.02442{1}	0.16410{8}	0.03163{3}	0.03873{5}
		$\hat{\lambda}$	0.18604{1}	0.29854{6}	0.22010{3}	0.28985{5}	0.18861{2}	1.38022{8}	0.22173{4}	0.29943{7}
		$\hat{\delta}$	0.08828{1}	0.23796{7}	0.12929{3}	0.23520{6}	0.09142{2}	2.01629{8}	0.13300{4}	0.22816{5}
		$\hat{\beta}$	0.00066{2}	0.00193{7}	0.00101{4}	0.00191{6}	0.00060{1}	0.02693{8}	0.00100{3}	0.00150{5}
300	MSE	$\hat{\lambda}$	0.03461{1}	0.08912{6}	0.04845{3}	0.08401{5}	0.03557{2}	1.90500{8}	0.04917{4}	0.08966{7}
		$\hat{\delta}$	0.19808{1}	0.32521{7}	0.23971{3}	0.32331{6}	0.20157{2}	0.94664{8}	0.24313{4}	0.31844{5}
		$\hat{\beta}$	0.10244{2}	0.17578{7}	0.12698{4}	0.17461{6}	0.09767{1}	0.65638{8}	0.12653{3}	0.15491{5}
		$\hat{\lambda}$	0.12403{1}	0.19902{6}	0.14674{3}	0.19323{5}	0.12574{2}	0.92015{8}	0.14782{4}	0.19962{7}
		$\sum RANKS$	12.0{1}	60.0{7}	30.0{3}	51.0{5.5}	15.0{2}	72.0{8}	33.0{4}	51.0{5.5}
	MRE	$\hat{\delta}$	0.25086{2}	0.38648{6}	0.28600{3}	0.38981{7}	0.07680{1}	1.39954{8}	0.29130{4}	0.36782{5}
		$\hat{\beta}$	0.02112{2}	0.03402{7}	0.02419{3}	0.03401{6}	0.01531{1}	0.15183{8}	0.02464{4}	0.02929{5}
		$\hat{\lambda}$	0.15715{2}	0.22582{6}	0.17344{3}	0.22961{7}	0.10142{1}	1.32012{8}	0.17784{4}	0.22455{5}
		$\hat{\delta}$	0.06293{2}	0.14937{6}	0.08180{3}	0.15195{7}	0.00590{1}	1.95872{8}	0.08486{4}	0.13529{5}
		$\hat{\beta}$	0.00045{2}	0.00116{6.5}	0.00059{3}	0.00116{6.5}	0.00023{1}	0.02305{8}	0.00061{4}	0.00086{5}
500	MSE	$\hat{\lambda}$	0.02470{2}	0.05100{6}	0.03008{3}	0.05272{7}	0.01029{1}	1.74272{8}	0.03163{4}	0.05042{5}
		$\hat{\delta}$	0.16724{2}	0.25766{6}	0.19067{3}	0.25987{7}	0.05120{1}	0.93303{8}	0.19420{4}	0.24521{5}
		$\hat{\beta}$	0.08448{2}	0.13606{7}	0.09677{3}	0.13603{6}	0.06123{1}	0.60731{8}	0.09858{4}	0.11714{5}
		$\hat{\lambda}$	0.10477{2}	0.15055{6}	0.11562{3}	0.15307{7}	0.06761{1}	0.88008{8}	0.11856{4}	0.14970{5}
		$\sum RANKS$	18.0{2}	56.5{6}	27.0{3}	60.5{7}	9.0{1}	72.0{8}	36.0{4}	45.0{5}

Table 8. Results for eight estimators with parameters $\delta = 1.5, \beta = 0.25$, and $\lambda = 3.5$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	BIAS	$\hat{\delta}$	1.22894{4}	1.28950{5}	1.18086{3}	1.33073{7}	1.07161{1}	1.46572{8}	1.13889{2}	1.32101{6}
		$\hat{\beta}$	0.12828{2}	0.17671{7}	0.15224{4}	0.17548{6}	0.12637{1}	0.25000{8}	0.13350{3}	0.15810{5}
		$\hat{\lambda}$	1.39638{4}	1.74017{7}	1.43119{5}	1.95764{8}	1.18946{1}	1.36218{3}	1.33183{2}	1.70681{6}
	MSE	$\hat{\delta}$	1.51029{4}	1.66282{5}	1.39442{3}	1.77085{7}	1.14834{1}	2.14833{8}	1.29707{2}	1.74507{6}
		$\hat{\beta}$	0.01645{2}	0.03123{7}	0.02318{4}	0.03079{6}	0.01597{1}	0.06250{8}	0.01782{3}	0.02500{5}
		$\hat{\lambda}$	1.94988{4}	3.02819{7}	2.04831{5}	3.83234{8}	1.41481{1}	1.85553{3}	1.77378{2}	2.91320{6}
	MRE	$\hat{\delta}$	0.81929{4}	0.85967{5}	0.78724{3}	0.88716{7}	0.71440{1}	0.97715{8}	0.75926{2}	0.88067{6}
		$\hat{\beta}$	0.51311{2}	0.70683{7}	0.60895{4}	0.70191{6}	0.50547{1}	1.00000{8}	0.53399{3}	0.63242{5}
		$\hat{\lambda}$	0.39897{4}	0.49719{7}	0.40891{5}	0.55932{8}	0.33985{1}	0.38919{3}	0.38052{2}	0.48766{6}
	$\sum RANKS$		30.0{3}	57.0{6.5}	36.0{4}	63.0{8}	9.0{1}	57.0{6.5}	21.0{2}	51.0{5}
50	BIAS	$\hat{\delta}$	0.77632{2}	1.00262{6}	0.83470{4}	0.99986{5}	0.74758{1}	1.49055{8}	0.80751{3}	1.00385{7}
		$\hat{\beta}$	0.07277{2}	0.11010{6}	0.08714{4}	0.11377{7}	0.06972{1}	0.25000{8}	0.07820{3}	0.09997{5}
		$\hat{\lambda}$	0.64433{2}	0.84802{5}	0.70844{4}	0.93484{7}	0.61040{1}	3.02012{8}	0.67029{3}	0.85822{6}
	MSE	$\hat{\delta}$	0.60268{2}	1.00524{6}	0.69673{4}	0.99972{5}	0.55887{1}	2.22174{8}	0.65208{3}	1.00772{7}
		$\hat{\beta}$	0.00530{2}	0.01212{6}	0.00759{4}	0.01294{7}	0.00486{1}	0.06250{8}	0.00612{3}	0.00999{5}
		$\hat{\lambda}$	0.41517{2}	0.71914{5}	0.50188{4}	0.87393{7}	0.37259{1}	9.12110{8}	0.44929{3}	0.73654{6}
	MRE	$\hat{\delta}$	0.51755{2}	0.66841{6}	0.55647{4}	0.66657{5}	0.49838{1}	0.99370{8}	0.53834{3}	0.66924{7}
		$\hat{\beta}$	0.29110{2}	0.44040{6}	0.34855{4}	0.45506{7}	0.27886{1}	1.00000{8}	0.31281{3}	0.39987{5}
		$\hat{\lambda}$	0.18410{2}	0.24229{5}	0.20241{4}	0.26710{7}	0.17440{1}	0.86289{8}	0.19151{3}	0.24521{6}
	$\sum RANKS$		18.0{2}	51.0{5}	36.0{4}	57.0{7}	9.0{1}	72.0{8}	27.0{3}	54.0{6}
100	BIAS	$\hat{\delta}$	0.52877{2}	0.75982{5}	0.61324{4}	0.79300{7}	0.50211{1}	1.49311{8}	0.59239{3}	0.76087{6}
		$\hat{\beta}$	0.04745{2}	0.07544{6}	0.05512{4}	0.07800{7}	0.04411{1}	0.24115{8}	0.05318{3}	0.06564{5}
		$\hat{\lambda}$	0.41302{2}	0.54237{5}	0.44237{4}	0.55690{7}	0.40469{1}	3.40952{8}	0.43465{3}	0.54540{6}
	MSE	$\hat{\delta}$	0.27959{2}	0.57732{5}	0.37606{4}	0.62885{7}	0.25211{1}	2.22937{8}	0.35092{3}	0.57892{6}
		$\hat{\beta}$	0.00225{2}	0.00569{6}	0.00304{4}	0.00608{7}	0.00195{1}	0.05816{8}	0.00283{3}	0.00431{5}

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n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
	MRE	$\hat{\lambda}$	0.17058{2}	0.29416{5}	0.19569{4}	0.31014{7}	0.16378{1}	11.62479{8}	0.18892{3}	0.29747{6}
		$\hat{\delta}$	0.35251{2}	0.50654{5}	0.40883{4}	0.52867{7}	0.33474{1}	0.99540{8}	0.39493{3}	0.50724{6}
		$\hat{\beta}$	0.18979{2}	0.30177{6}	0.22048{4}	0.31201{7}	0.17643{1}	0.96461{8}	0.21274{3}	0.26254{5}
		$\hat{\lambda}$	0.11800{2}	0.15496{5}	0.12639{4}	0.15912{7}	0.11563{1}	0.97415{8}	0.12419{3}	0.15583{6}
	$\sum RANKS$		18.0{2}	48.0{5}	36.0{4}	63.0{7}	9.0{1}	72.0{8}	27.0{3}	51.0{6}
300	BIAS	$\hat{\delta}$	0.30459{2}	0.47883{6}	0.34599{3}	0.49318{7}	0.26509{1}	1.49412{8}	0.36119{4}	0.47006{5}
		$\hat{\beta}$	0.02610{2}	0.04267{6}	0.03044{3}	0.04300{7}	0.02472{1}	0.19207{8}	0.03073{4}	0.03788{5}
		$\hat{\lambda}$	0.22180{2}	0.28352{5}	0.23322{3}	0.29399{6}	0.21278{1}	3.31702{8}	0.24050{4}	0.29989{7}
	MSE	$\hat{\delta}$	0.09277{2}	0.22927{6}	0.11971{3}	0.24323{7}	0.07027{1}	2.23239{8}	0.13046{4}	0.22096{5}
		$\hat{\beta}$	0.00068{2}	0.00182{6}	0.00093{3}	0.00185{7}	0.00061{1}	0.03689{8}	0.00094{4}	0.00143{5}
		$\hat{\lambda}$	0.04919{2}	0.08038{5}	0.05439{3}	0.08643{6}	0.04528{1}	11.00263{8}	0.05784{4}	0.08994{7}
	MRE	$\hat{\delta}$	0.20306{2}	0.31922{6}	0.23066{3}	0.32879{7}	0.17673{1}	0.99608{8}	0.24079{4}	0.31337{5}
		$\hat{\beta}$	0.10439{2}	0.17067{6}	0.12174{3}	0.17200{7}	0.09888{1}	0.76827{8}	0.12292{4}	0.15151{5}
		$\hat{\lambda}$	0.06337{2}	0.08101{5}	0.06663{3}	0.08400{6}	0.06080{1}	0.94772{8}	0.06872{4}	0.08568{7}
	$\sum RANKS$		18.0{2}	51.0{5.5}	27.0{3}	60.0{7}	9.0{1}	72.0{8}	36.0{4}	51.0{5.5}
500	BIAS	$\hat{\delta}$	0.23897{2}	0.38214{7}	0.25570{3}	0.37845{6}	0.16914{1}	1.48632{8}	0.28414{4}	0.37467{5}
		$\hat{\beta}$	0.01978{2}	0.03282{7}	0.02240{3}	0.03281{6}	0.01819{1}	0.18313{8}	0.02426{4}	0.03009{5}
		$\hat{\lambda}$	0.16576{2}	0.21315{6}	0.17902{3}	0.20973{5}	0.16236{1}	3.09819{8}	0.18473{4}	0.23445{7}
	MSE	$\hat{\delta}$	0.05711{2}	0.14603{7}	0.06538{3}	0.14322{6}	0.02861{1}	2.20913{8}	0.08074{4}	0.14037{5}
		$\hat{\beta}$	0.00039{2}	0.00108{6.5}	0.00050{3}	0.00108{6.5}	0.00033{1}	0.03354{8}	0.00059{4}	0.00091{5}
		$\hat{\lambda}$	0.02748{2}	0.04543{6}	0.03205{3}	0.04399{5}	0.02636{1}	9.59881{8}	0.03413{4}	0.05496{7}
	MRE	$\hat{\delta}$	0.15931{2}	0.25476{7}	0.17047{3}	0.25230{6}	0.11276{1}	0.99088{8}	0.18943{4}	0.24978{5}
		$\hat{\beta}$	0.07914{2}	0.13129{7}	0.08961{3}	0.13125{6}	0.07275{1}	0.73250{8}	0.09705{4}	0.12037{5}
		$\hat{\lambda}$	0.04736{2}	0.06090{6}	0.05115{3}	0.05992{5}	0.04639{1}	0.88520{8}	0.05278{4}	0.06698{7}
	$\sum RANKS$		18.0{2}	59.5{7}	27.0{3}	51.5{6}	9.0{1}	72.0{8}	36.0{4}	51.0{5}

Table 9. Results for eight estimators with parameters $\delta = 1.5, \beta = 2$, and $\lambda = 1.5$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	BIAS	$\hat{\delta}$	1.21790{4}	1.30887{6}	1.21865{5}	1.34306{7}	1.11438{1}	1.21283{3}	1.17647{2}	1.35914{8}
		$\hat{\beta}$	1.05826{2}	1.39100{7}	1.23608{5}	1.44930{8}	1.03052{1}	1.18538{4}	1.06741{3}	1.27677{6}
		$\hat{\lambda}$	0.93963{2}	1.12156{6}	0.98145{4}	1.15279{7}	0.87281{1}	1.02784{5}	0.94055{3}	1.17198{8}
	MSE	$\hat{\delta}$	1.48329{4}	1.71313{6}	1.48511{5}	1.80380{7}	1.24185{1}	1.47095{3}	1.38408{2}	1.84725{8}
		$\hat{\beta}$	1.11992{2}	1.93489{7}	1.52788{5}	2.10047{8}	1.06198{1}	1.40512{4}	1.13937{3}	1.63014{6}
		$\hat{\lambda}$	0.88290{2}	1.25789{6}	0.96325{4}	1.32892{7}	0.76181{1}	1.05646{5}	0.88463{3}	1.37353{8}
	MRE	$\hat{\delta}$	0.81194{4}	0.87258{6}	0.81243{5}	0.89537{7}	0.74292{1}	0.80855{3}	0.78431{2}	0.90609{8}
		$\hat{\beta}$	0.52913{2}	0.69550{7}	0.61804{5}	0.72465{8}	0.51526{1}	0.59269{4}	0.53371{3}	0.63839{6}
		$\hat{\lambda}$	0.62642{2}	0.74771{6}	0.65430{4}	0.76853{7}	0.58188{1}	0.68523{5}	0.62703{3}	0.78132{8}
	$\sum RANKS$		24.0{2.5}	57.0{6}	42.0{5}	66.0{7.5}	9.0{1}	36.0{4}	24.0{2.5}	66.0{7.5}
50	BIAS	$\hat{\delta}$	0.79539{3}	1.00353{6}	0.85786{4}	1.01445{7}	0.75478{1}	0.91041{5}	0.79011{2}	1.01556{8}
		$\hat{\beta}$	0.59966{2}	0.90678{7}	0.70921{4}	0.91942{8}	0.56750{1}	0.72809{5}	0.62593{3}	0.77242{6}
		$\hat{\lambda}$	0.53020{2}	0.72106{6}	0.59163{4}	0.72648{7}	0.52005{1}	0.64409{5}	0.54187{3}	0.75003{8}
	MSE	$\hat{\delta}$	0.63265{3}	1.00707{6}	0.73592{4}	1.02910{7}	0.56970{1}	0.82884{5}	0.62427{2}	1.03136{8}
		$\hat{\beta}$	0.35960{2}	0.82226{7}	0.50298{4}	0.84533{8}	0.32206{1}	0.53012{5}	0.39178{3}	0.59663{6}
		$\hat{\lambda}$	0.28111{2}	0.51992{6}	0.35002{4}	0.52777{7}	0.27045{1}	0.41485{5}	0.29363{3}	0.56255{8}
	MRE	$\hat{\delta}$	0.53026{3}	0.66902{6}	0.57190{4}	0.67630{7}	0.50319{1}	0.60694{5}	0.52674{2}	0.67704{8}
		$\hat{\beta}$	0.29983{2}	0.45339{7}	0.35461{4}	0.45971{8}	0.28375{1}	0.36405{5}	0.31296{3}	0.38621{6}
		$\hat{\lambda}$	0.35347{2}	0.48070{6}	0.39442{4}	0.48432{7}	0.34670{1}	0.42939{5}	0.36125{3}	0.50002{8}
	$\sum RANKS$		21.0{2}	57.0{6}	36.0{4}	66.0{7.5}	9.0{1}	45.0{5}	24.0{3}	66.0{7.5}
100	BIAS	$\hat{\delta}$	0.56242{2}	0.77628{6}	0.61241{4}	0.80807{8}	0.54109{1}	0.68672{5}	0.59950{3}	0.77986{7}
		$\hat{\beta}$	0.39983{2}	0.61330{7}	0.43840{4}	0.63189{8}	0.38098{1}	0.49069{5}	0.43820{3}	0.56198{6}
		$\hat{\lambda}$	0.36312{2}	0.49911{6}	0.38759{4}	0.52894{8}	0.35611{1}	0.45134{5}	0.38358{3}	0.52830{7}
	MSE	$\hat{\delta}$	0.31631{2}	0.60261{6}	0.37505{4}	0.65298{8}	0.29278{1}	0.47158{5}	0.35940{3}	0.60819{7}
		$\hat{\beta}$	0.15986{2}	0.37613{7}	0.19220{4}	0.39929{8}	0.14515{1}	0.24078{5}	0.19202{3}	0.31582{6}

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n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
		$\hat{\lambda}$	0.13185{2}	0.24911{6}	0.15023{4}	0.27978{8}	0.12682{1}	0.20370{5}	0.14714{3}	0.27910{7}
		$\hat{\delta}$	0.37494{2}	0.51752{6}	0.40827{4}	0.53871{8}	0.36073{1}	0.45781{5}	0.39966{3}	0.51991{7}
		MRE $\hat{\beta}$	0.19991{2}	0.30665{7}	0.21920{4}	0.31595{8}	0.19049{1}	0.24535{5}	0.21910{3}	0.28099{6}
		$\hat{\lambda}$	0.24208{2}	0.33274{6}	0.25839{4}	0.35263{8}	0.23741{1}	0.30089{5}	0.25572{3}	0.35220{7}
		$\sum RANKS$	18.0{2}	57.0{6}	36.0{4}	72.0{8}	9.0{1}	45.0{5}	27.0{3}	60.0{7}
		$\hat{\delta}$	0.32323{2}	0.49569{8}	0.36255{4}	0.48956{7}	0.30782{1}	0.41508{5}	0.35994{3}	0.48713{6}
		BIAS $\hat{\beta}$	0.22461{2}	0.35416{7}	0.25156{3}	0.35426{8}	0.19092{1}	0.27075{5}	0.25279{4}	0.31981{6}
		$\hat{\lambda}$	0.20246{2}	0.30157{7}	0.21812{3}	0.29793{6}	0.17845{1}	0.25460{5}	0.22575{4}	0.30237{8}
		MSE $\hat{\delta}$	0.10448{2}	0.24571{8}	0.13144{4}	0.23967{7}	0.09475{1}	0.17229{5}	0.12955{3}	0.23729{6}
		$\hat{\beta}$	0.05045{2}	0.12543{7}	0.06328{3}	0.12550{8}	0.03645{1}	0.07331{5}	0.06390{4}	0.10228{6}
300		$\hat{\lambda}$	0.04099{2}	0.09095{7}	0.04757{3}	0.08876{6}	0.03184{1}	0.06482{5}	0.05096{4}	0.09143{8}
		MRE $\hat{\delta}$	0.21549{2}	0.33046{8}	0.24170{4}	0.32637{7}	0.20521{1}	0.27672{5}	0.23996{3}	0.32475{6}
		$\hat{\beta}$	0.11231{2}	0.17708{7}	0.12578{3}	0.17713{8}	0.09546{1}	0.13538{5}	0.12640{4}	0.15991{6}
		$\hat{\lambda}$	0.13497{2}	0.20105{7}	0.14541{3}	0.19862{6}	0.11897{1}	0.16974{5}	0.15050{4}	0.20158{8}
		$\sum RANKS$	18.0{2}	66.0{8}	30.0{3}	63.0{7}	9.0{1}	45.0{5}	33.0{4}	60.0{6}
		$\hat{\delta}$	0.24699{2}	0.39408{8}	0.27937{3}	0.38856{7}	0.06292{1}	0.32464{5}	0.29210{4}	0.36675{6}
		BIAS $\hat{\beta}$	0.16520{2}	0.27112{8}	0.18661{3}	0.26750{7}	0.10938{1}	0.20723{5}	0.20130{4}	0.23228{6}
		$\hat{\lambda}$	0.15305{2}	0.23378{8}	0.16744{3}	0.22923{7}	0.10184{1}	0.20077{5}	0.17960{4}	0.22513{6}
		MSE $\hat{\delta}$	0.06100{2}	0.15530{8}	0.07805{3}	0.15098{7}	0.00396{1}	0.10539{5}	0.08532{4}	0.13450{6}
		$\hat{\beta}$	0.02729{2}	0.07350{8}	0.03482{3}	0.07155{7}	0.01196{1}	0.04295{5}	0.04052{4}	0.05395{6}
500		$\hat{\lambda}$	0.02343{2}	0.05465{8}	0.02804{3}	0.05255{7}	0.01037{1}	0.04031{5}	0.03226{4}	0.05068{6}
		MRE $\hat{\delta}$	0.16466{2}	0.26272{8}	0.18625{3}	0.25904{7}	0.04195{1}	0.21643{5}	0.19473{4}	0.24450{6}
		$\hat{\beta}$	0.08260{2}	0.13556{8}	0.09331{3}	0.13375{7}	0.05469{1}	0.10362{5}	0.10065{4}	0.11614{6}
		$\hat{\lambda}$	0.10203{2}	0.15585{8}	0.11163{3}	0.15282{7}	0.06789{1}	0.13385{5}	0.11973{4}	0.15008{6}
		$\sum RANKS$	18.0{2}	72.0{8}	27.0{3}	63.0{7}	9.0{1}	45.0{5}	36.0{4}	54.0{6}

Table 10. Results for eight estimators with parameters $\delta = 1.5, \beta = 2$, and $\lambda = 3.5$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	BIAS	$\hat{\delta}$	1.23462{5}	1.31283{7}	1.20528{3}	1.31253{6}	1.10360{1}	1.22863{4}	1.14760{2}	1.36649{8}
		$\hat{\beta}$	1.05009{3}	1.40142{7}	1.24705{5}	1.42459{8}	1.02087{1}	1.20712{4}	1.04620{2}	1.28075{6}
		$\hat{\lambda}$	1.50774{5}	1.68716{6}	1.45155{4}	2.02470{8}	1.20515{1}	1.41364{3}	1.31014{2}	1.72834{7}
	MSE	$\hat{\delta}$	1.52429{5}	1.72353{7}	1.45270{3}	1.72274{6}	1.21793{1}	1.50952{4}	1.31698{2}	1.86730{8}
		$\hat{\beta}$	1.10269{3}	1.96399{7}	1.55514{5}	2.02944{8}	1.04218{1}	1.45713{4}	1.09454{2}	1.64033{6}
		$\hat{\lambda}$	2.27328{5}	2.84649{6}	2.10700{4}	4.09940{8}	1.45239{1}	1.99839{3}	1.71647{2}	2.98715{7}
	MRE	$\hat{\delta}$	0.82308{5}	0.87522{7}	0.80352{3}	0.87502{6}	0.73573{1}	0.81908{4}	0.76507{2}	0.91099{8}
		$\hat{\beta}$	0.52504{3}	0.70071{7}	0.62353{5}	0.71229{8}	0.51044{1}	0.60356{4}	0.52310{2}	0.64038{6}
		$\hat{\lambda}$	0.43078{5}	0.48204{6}	0.41473{4}	0.57848{8}	0.34433{1}	0.40390{3}	0.37433{2}	0.49381{7}
	$\sum RANKS$		39.0{5}	60.0{6}	36.0{4}	66.0{8}	9.0{1}	33.0{3}	18.0{2}	63.0{7}
50	BIAS	$\hat{\delta}$	0.80023{2}	0.99949{7}	0.86165{4}	1.00938{8}	0.75322{1}	0.92477{5}	0.80210{3}	0.97979{6}
		$\hat{\beta}$	0.60978{2}	0.90738{8}	0.71608{4}	0.89419{7}	0.56524{1}	0.73706{5}	0.62063{3}	0.75250{6}
		$\hat{\lambda}$	0.68845{3}	0.85525{7}	0.71642{4}	0.92531{8}	0.62309{1}	0.72238{5}	0.66644{2}	0.85091{6}
	MSE	$\hat{\delta}$	0.64037{2}	0.99899{7}	0.74245{4}	1.01885{8}	0.56734{1}	0.85521{5}	0.64336{3}	0.95999{6}
		$\hat{\beta}$	0.37183{2}	0.82334{8}	0.51277{4}	0.79957{7}	0.31950{1}	0.54325{5}	0.38519{3}	0.56626{6}
		$\hat{\lambda}$	0.47397{3}	0.73146{7}	0.51326{4}	0.85620{8}	0.38824{1}	0.52183{5}	0.44414{2}	0.72404{6}
	MRE	$\hat{\delta}$	0.53349{2}	0.66633{7}	0.57444{4}	0.67292{8}	0.50215{1}	0.61652{5}	0.53473{3}	0.65319{6}
		$\hat{\beta}$	0.30489{2}	0.45369{8}	0.35804{4}	0.44709{7}	0.28262{1}	0.36853{5}	0.31032{3}	0.37625{6}
		$\hat{\lambda}$	0.19670{3}	0.24436{7}	0.20469{4}	0.26437{8}	0.17803{1}	0.20639{5}	0.19041{2}	0.24312{6}
	$\sum RANKS$		21.0{2}	66.0{7}	36.0{4}	69.0{8}	9.0{1}	45.0{5}	24.0{3}	54.0{6}
100	BIAS	$\hat{\delta}$	0.55451{2}	0.78491{8}	0.64174{4}	0.77990{6}	0.53134{1}	0.70296{5}	0.62483{3}	0.78258{7}
		$\hat{\beta}$	0.39869{2}	0.61053{7}	0.46633{4}	0.61217{8}	0.37452{1}	0.50410{5}	0.46542{3}	0.55297{6}
		$\hat{\lambda}$	0.42451{2}	0.53326{6}	0.45485{3}	0.57349{8}	0.40630{1}	0.49665{5}	0.45643{4}	0.56196{7}
	MSE	$\hat{\delta}$	0.30748{2}	0.61608{8}	0.41183{4}	0.60824{6}	0.28232{1}	0.49416{5}	0.39041{3}	0.61243{7}
		$\hat{\beta}$	0.15895{2}	0.37274{7}	0.21747{4}	0.37475{8}	0.14027{1}	0.25412{5}	0.21662{3}	0.30578{6}

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n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
		$\hat{\lambda}$	0.18021{2}	0.28437{6}	0.20689{3}	0.32889{8}	0.16508{1}	0.24667{5}	0.20833{4}	0.31580{7}
		$\hat{\delta}$	0.36967{2}	0.52327{8}	0.42783{4}	0.51993{6}	0.35423{1}	0.46864{5}	0.41655{3}	0.52172{7}
	MRE	$\hat{\beta}$	0.19934{2}	0.30526{7}	0.23317{4}	0.30608{8}	0.18726{1}	0.25205{5}	0.23271{3}	0.27649{6}
		$\hat{\lambda}$	0.12129{2}	0.15236{6}	0.12996{3}	0.16385{8}	0.11609{1}	0.14190{5}	0.13041{4}	0.16056{7}
		$\sum RANKS$	18.0{2}	63.0{7}	33.0{4}	66.0{8}	9.0{1}	45.0{5}	30.0{3}	60.0{6}
300	BIAS	$\hat{\delta}$	0.31588{2}	0.49339{7}	0.36347{3}	0.49549{8}	0.30500{1}	0.41165{5}	0.38186{4}	0.48256{6}
		$\hat{\beta}$	0.21678{2}	0.35354{8}	0.25102{3}	0.35267{7}	0.19375{1}	0.26734{5}	0.26120{4}	0.31289{6}
		$\hat{\lambda}$	0.21739{1}	0.29760{7}	0.23903{3}	0.29268{6}	0.22287{2}	0.26049{5}	0.24469{4}	0.31120{8}
	MSE	$\hat{\delta}$	0.09978{2}	0.24344{7}	0.13211{3}	0.24551{8}	0.09302{1}	0.16946{5}	0.14581{4}	0.23287{6}
		$\hat{\beta}$	0.04700{2}	0.12499{8}	0.06301{3}	0.12438{7}	0.03754{1}	0.07147{5}	0.06823{4}	0.09790{6}
		$\hat{\lambda}$	0.04726{1}	0.08857{7}	0.05713{3}	0.08566{6}	0.04967{2}	0.06786{5}	0.05987{4}	0.09684{8}
	MRE	$\hat{\delta}$	0.21059{2}	0.32893{7}	0.24231{3}	0.33033{8}	0.20333{1}	0.27443{5}	0.25457{4}	0.32171{6}
		$\hat{\beta}$	0.10839{2}	0.17677{8}	0.12551{3}	0.17633{7}	0.09688{1}	0.13367{5}	0.13060{4}	0.15645{6}
		$\hat{\lambda}$	0.06211{1}	0.08503{7}	0.06829{3}	0.08362{6}	0.06368{2}	0.07443{5}	0.06991{4}	0.08891{8}
		$\sum RANKS$	15.0{2}	66.0{8}	27.0{3}	63.0{7}	12.0{1}	45.0{5}	36.0{4}	60.0{6}
500	BIAS	$\hat{\delta}$	0.24597{2}	0.38259{7}	0.28932{3}	0.38364{8}	0.11464{1}	0.32482{5}	0.29036{4}	0.37746{6}
		$\hat{\beta}$	0.16657{2}	0.26821{7}	0.19443{3}	0.26895{8}	0.12803{1}	0.21010{5}	0.19701{4}	0.24031{6}
		$\hat{\lambda}$	0.16887{1}	0.21270{6}	0.18628{4}	0.22333{7}	0.17062{2}	0.20334{5}	0.18523{3}	0.22863{8}
	MSE	$\hat{\delta}$	0.06050{2}	0.14637{7}	0.08371{3}	0.14718{8}	0.01314{1}	0.10551{5}	0.08431{4}	0.14247{6}
		$\hat{\beta}$	0.02775{2}	0.07194{7}	0.03780{3}	0.07233{8}	0.01639{1}	0.04414{5}	0.03881{4}	0.05775{6}
		$\hat{\lambda}$	0.02852{1}	0.04524{6}	0.03470{4}	0.04988{7}	0.02911{2}	0.04135{5}	0.03431{3}	0.05227{8}
	MRE	$\hat{\delta}$	0.16398{2}	0.25506{7}	0.19288{3}	0.25576{8}	0.07643{1}	0.21655{5}	0.19357{4}	0.25164{6}
		$\hat{\beta}$	0.08329{2}	0.13411{7}	0.09722{3}	0.13448{8}	0.06401{1}	0.10505{5}	0.09851{4}	0.12015{6}
		$\hat{\lambda}$	0.04825{1}	0.06077{6}	0.05322{4}	0.06381{7}	0.04875{2}	0.05810{5}	0.05292{3}	0.06532{8}
		$\sum RANKS$	15.0{2}	60.0{6.5}	30.0{3}	69.0{8}	12.0{1}	45.0{5}	33.0{4}	60.0{6.5}

Table 11. Partial and overall rankings of all estimation methods for various combinations of η .

η^T	n	MLE	OLSE	WLSE	CRVME	MPS	PCE	ADE	RADE
$(\delta = 0.5, \beta = 0.25, \lambda = 1.5)$	20	4	8	2.5	7	2.5	6	1	5
	50	2.5	6	2.5	5	1	8	4	7
	100	3	6.5	2	6.5	1	8	4	5
	300	2	6	3	7	1	8	4	5
	500	2	5	3	7	1	8	4	6
$(\delta = 0.5, \beta = 0.25, \lambda = 3.5)$	20	3.5	5	3.5	8	1	6	2	7
	50	5	6	3	7	1	8	2	4
	100	5	7	2	6	1	8	3	4
	300	5	6	2	7	1	8	3	4
	500	5	6	2	7	1	8	3	4
$(\delta = 0.5, \beta = 2, \lambda = 1.5)$	20	4	5	3	8	2	6.5	1	6.5
	50	4	6	3	7.5	1	7.5	2	5
	100	2	6	3.5	7	1	8	3.5	5
	300	2	6	3	7	1	8	4	5
	500	2	6	3	7	1	8	4	5
$(\delta = 0.5, \beta = 2, \lambda = 3.5)$	20	4	5	3	8	1	6.5	2	6.5
	50	3	7	4	8	1	6	2	5
	100	2	7	4	8	1	6	3	5
	300	1	7	3	8	2	6	4	5
	500	1	8	3	7	2	6	4	5
$(\delta = 1.5, \beta = 0.25, \lambda = 1.5)$	20	3	5	4	6.5	1	8	2	6.5
	50	2	5	4	7	1	8	3	6
	100	2	7	4	6	1	8	3	5
	300	1	7	3	5.5	2	8	4	5.5
	500	2	6	3	7	1	8	4	5
$(\delta = 1.5, \beta = 0.25, \lambda = 3.5)$	20	3	6.5	4	8	1	6.5	2	5
	50	2	5	4	7	1	8	3	6
	100	2	5	4	7	1	8	3	6
	300	2	5.5	3	7	1	8	4	5.5
	500	2	7	3	6	1	8	4	5
$(\delta = 1.5, \beta = 2, \lambda = 1.5)$	20	2.5	6	5	7.5	1	4	2.5	7.5
	50	2	6	4	7.5	1	5	3	7.5
	100	2	6	4	8	1	5	3	7
	300	2	8	3	7	1	5	4	6
	500	2	8	3	7	1	5	4	6

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η^T	n	MLE	OLSE	WLSE	CRVME	MPS	PCE	ADE	RADE
$(\delta = 1.5, \beta = 2, \lambda = 3.5)$	20	5	6	4	8	1	3	2	7
	50	2	7	4	8	1	5	3	6
	100	2	7	4	8	1	5	3	6
	300	2	8	3	7	1	5	4	6
	500	2	6.5	3	8	1	5	4	6.5
$\sum Ranks$		106.5	252	131	286	45.5	270	124	225
Overall Rank		2	6	4	8	1	7	3	5

6. Real-life data modeling

In this section, we analyze three real datasets to demonstrate the flexibility of the proposed GKMW model. The first dataset comprises 63 observations of gauge lengths of 10 mm from Kundu and Raqab [41]. The second dataset is uncensored and comes from Murty et al. [42], representing the failure times (in weeks) of 50 components that were put into use at a certain time. The third dataset details the distances from the transect line for 68 stakes detected while walking along a length of 1000 m and searching 20 m on each side of the line [43]. The three datasets are provided in Appendix A. We compare the fits of the GKMW distribution with several other competitive models, as presented in Table 12.

Table 12. The list of competitive distributions.

Distribution	Abbreviation	Author
Modified beta Weibull	MBW	Khan [7]
Beta Weibull	BW	Lee and Famoye [1]
Odd log-logistic exponentiated Weibull	OLLEW	Afify et al. [11]
Exponentiated generalized Weibull	EGW	Cordeiro et al. [5]
Lindley Weibull	LiW	Cordeiro et al. [12]
Exponentiated Weibull	EW	Mudholkar and Srivastava [13]
Transmuted Weibull	TW	Aryal and Tsokos [4]

For model comparison, we employ four widely recognized statistics: Akaike information criterion (AIC), consistent AIC (CAIC), Bayesian information criterion (BIC), and Hannan–Quinn information criterion (HQIC), as well as Cramér–von Mises (W^*) statistics, Anderson–Darling (A^*), minus log-likelihood ($-\mathcal{L}$), and the Kolmogorov–Smirnov (KS) distance along with its associated p – value. Smaller values for these statistics indicate a better fit. Visual comparisons of the TTT, HRF, PDF, CDF, SF, and probability-probability (PP) plots for the GKMW model are also provided for the three datasets.

Tables 13–15 present the estimated parameters obtained through ML estimation, along with their corresponding standard errors (SE) (in parentheses). The goodness-of-fit measures for the fitted models are provided in Tables 16–18. The findings from these tables demonstrate the superiority of the GKMW model compared to other distributions for the three analyzed datasets.

Table 13. ML estimates and SE from the gauge lengths dataset for the fitted distributions.

Distribution	ML estimates and SE				
GKMW	$\hat{\delta}= 45.2721$ (108.9500)	$\hat{\beta}= 1.5646$ (0.9374)	$\hat{\lambda}= 0.6627$ (1.1158)		
MBW	$\hat{\delta}= 0.1328$ (0.2897)	$\hat{\beta}= 0.5224$ (0.3417)	$\hat{a}= 236.8925$ (1389.4352)	$\hat{b}= 3.9570$ (7.5948)	$\hat{c}= 0.4084$ (2.4858)
BW	$\hat{\delta}=1.5535$ (5.6168)	$\hat{\beta}= 0.9162$ (2.0369)	$\hat{a}= 102.4980$ (517.5903)	$\hat{b}= 2.0925$ (8.0543)	
OLLEW	$\hat{\delta}= 69.5586$ (306.6782)	$\hat{\beta}=3.4425$ (6.6654)	$\hat{\gamma}= 0.0641$ (0.0384)	$\hat{\theta}= 19.5547$ (27.8608)	
EGW	$\hat{\delta}= 3.7852$ (181.1960)	$\hat{a}= 5.6583$ (393.8165)	$\hat{b}= 37.1571$ (79.3795)	$\hat{c}= 1.4540$ (0.7599)	
LiW	$\hat{\delta}= 0.1238$ (0.5147)	$\hat{\beta}= 5.0487$ (0.4560)	$\hat{\theta}= 90.5958$ (1882.5304)		
EW	$\hat{\delta}= 0.8180$ (1.1200)	$\hat{\beta}= 1.4532$ (0.7583)	$\hat{\theta}= 37.2311$ (79.4533)		
TW	$\hat{\delta}= 3.6164$ (0.1515)	$\hat{\beta}= 5.4807$ (0.5021)	$\hat{\lambda}=0.7453$ (0.2633)		

Table 14. ML estimates and SE from the failure times dataset for the fitted distributions.

Distribution	ML estimates and SE				
GKMW	$\hat{\delta}= 0.4582$ (0.1995)	$\hat{\beta}= 1.3987$ (0.4033)	$\hat{\lambda}= 0.0184$ (0.0272)		
MBW	$\hat{\delta}=4.3285$ (54.3129)	$\hat{\beta}=0.3702$ (0.6122)	$\hat{a}= 1.6702$ (1.1828)	$\hat{b}= 22.2114$ (265.3049)	$\hat{c}=0.0342$ (0.1894)
BW	$\hat{\delta}= 0.0252$ (0.0813)	$\hat{\beta}=1.663$ (0.4550)	$\hat{a}=0.5592$ (0.3169)	$\hat{b}= 3.5694$ (12.7630)	
OLLEW	$\hat{\delta}= 72.9308$ (0.3032)	$\hat{\beta}= 3.2596$ (0.2568)	$\hat{\gamma}= 0.0769$ (0.0084)	$\hat{\theta}= 2.2419$ (0.2703)	
EGW	$\hat{\delta}= 2.0863$ (37.6697)	$\hat{a}= 0.1545$ (3.0671)	$\hat{b}= 0.5983$ (0.3183)	$\hat{c}=1.1009$ (0.3915)	
LiW	$\hat{\delta}= 0.2790$ (0.7219)	$\hat{\beta}= 0.7193$ (0.1332)	$\hat{\theta}= 0.9699$ (1.4562)		
EW	$\hat{\delta}=0.0687$ (0.0978)	$\hat{\beta}= 1.1011$ (0.3874)	$\hat{\theta}= 0.5982$ (0.3150)		
TW	$\hat{\delta}= 6.9739$ (5.0869)	$\hat{\beta}= 0.8004$ (0.1739)	$\hat{\lambda}= 0.0010$ (0.9657)		

Table 15. The ML estimates and SE from the distance dataset for the fitted distributions.

Distribution	ML estimates and SE				
GKMW	$\hat{\delta}= 0.4596$ (0.0924)	$\hat{\beta}= 2.1497$ (0.2449)	$\hat{\lambda}= 0.0055$ (0.0038)		
MBW	$\hat{\delta}= 22.8768$ (52.4569)	$\hat{\beta}= 1.3294$ (2.0900)	$\hat{\alpha}= 0.7716$ (1.3573)	$\hat{b}= 26.1991$ (169.7519)	$\hat{c}=0.1360$ (0.9503)
BW	$\hat{\delta}=0.0948$ (0.2660)	$\hat{\beta}=1.7636$ (0.7832)	$\hat{\alpha}=0.5664$ (0.3403)	$\hat{b}= 1.3142$ (5.5844)	
OLLEW	$\hat{\delta}=20.1065$ (0.3340)	$\hat{\beta}= 5.3078$ (0.2855)	$\hat{\gamma}= 0.0921$ (0.0099)	$\hat{\theta}=1.6927$ (0.1816)	
EGW	$\hat{\delta}= 2.3954$ (16.6021)	$\hat{\alpha}= 0.1067$ (1.2706)	$\hat{b}= 0.5795$ (0.3050)	$\hat{c}=1.7274$ (0.6189)	
LiW	$\hat{\delta}= 0.2309$ (0.4284)	$\hat{\beta}=1.1040$ (0.2085)	$\hat{\theta}= 1.0343$ (1.7142)		
EW	$\hat{\delta}= 0.0237$ (0.0359)	$\hat{\beta}= 1.7264$ (0.5159)	$\hat{\theta}=0.5797$ (0.2574)		
TW	$\hat{\delta}= 6.2398$ (2.4596)	$\hat{\beta}= 1.2250$ (0.2312)	$\hat{\lambda}=0.0010$ (0.7996)		

Table 16. Findings from the gauge lengths dataset for the fitted distributions.

Distribution	AIC	CAIC	BIC	HQIC	W^*	A^*	$-\mathcal{L}$	KS	$p - value$
GKMW	118.5520	118.9588	124.9814	121.0807	0.0601	0.3216	56.2760	0.0795	0.821305
MBW	122.6182	123.6708	133.3338	126.8327	0.0615	0.3272	56.3091	0.0800	0.815108
BW	120.6346	121.3242	129.2071	124.0062	0.0612	0.3268	56.3173	0.0796	0.820005
OLLEW	123.9248	124.6144	132.4973	127.2964	0.0866	0.5041	57.9624	0.0916	0.665628
EGW	120.6216	121.3112	129.1941	123.9932	0.0619	0.3287	56.3108	0.0813	0.799515
LiW	129.9178	130.3246	136.3472	132.4465	0.1285	0.8922	61.9589	0.0876	0.718911
EW	118.6216	119.0284	125.0510	121.1503	0.0619	0.3288	56.3108	0.0813	0.799320
TW	127.1226	127.5294	133.5520	129.6513	0.1100	0.7623	60.5613	0.0835	0.772281

Table 17. Findings from the failure times dataset for the fitted distributions.

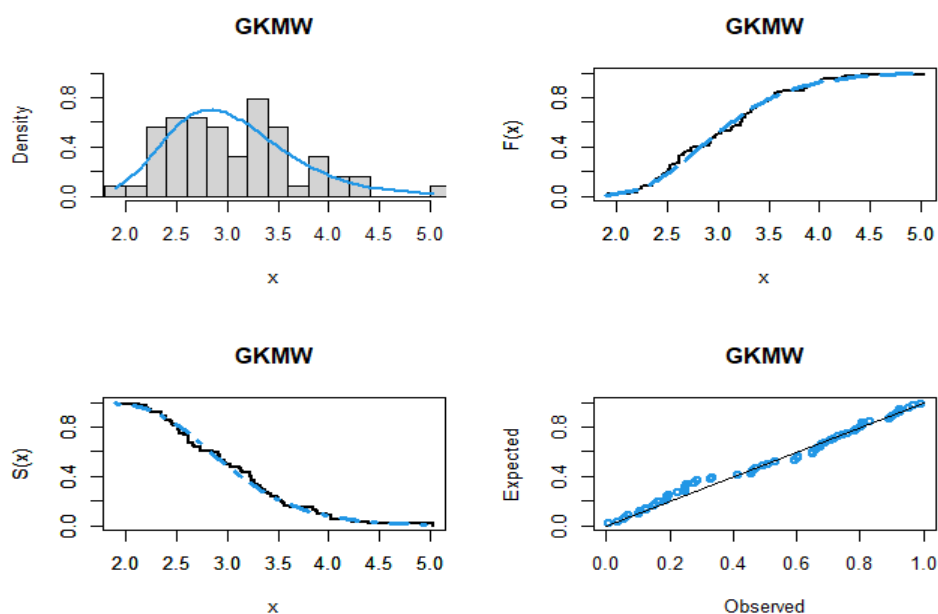
Distribution	AIC	CAIC	BIC	HQIC	W^*	A^*	$-\mathcal{L}$	KS	$p - value$
GKMW	306.4025	306.9242	312.1386	308.5868	0.0575	0.2948	150.2012	0.0934	0.775179
MBW	310.5294	311.8931	320.0895	314.1700	0.0582	0.2974	150.2647	0.0948	0.759432
BW	308.4788	309.3677	316.1269	311.3913	0.0587	0.2990	150.2394	0.0957	0.750141
OLLEW	309.0636	309.9525	316.7117	311.9760	0.0757	0.3810	150.5318	0.0999	0.700003
EGW	308.5187	309.4076	316.1668	311.4311	0.0599	0.3044	150.2593	0.0965	0.740273
LiW	306.7964	307.3181	312.5325	308.9807	0.0709	0.3539	150.3982	0.1018	0.677787
EW	306.5187	307.0404	312.2548	308.7030	0.0599	0.3044	150.2593	0.0965	0.740475
TW	307.3553	307.8771	313.0914	309.5396	0.0857	0.4275	150.6777	0.1119	0.558858

Table 18. Findings from the distance dataset for the fitted distributions.

Distribution	AIC	CAIC	BIC	HQIC	W^*	A^*	$-\mathcal{L}$	KS	$p - value$
GKMW	377.1478	377.5228	383.8063	379.7861	0.0385	0.2474	185.5739	0.0804	0.771339
MBW	381.2396	382.2073	392.3371	385.6368	0.0420	0.2673	185.6198	0.0843	0.719002
BW	379.3222	379.9571	388.2002	382.8399	0.0398	0.2547	185.6611	0.0818	0.753500
OLLEW	379.0712	379.7061	387.9492	382.5890	0.0421	0.2739	185.5356	0.0807	0.767901
EGW	379.3276	379.9625	388.2057	382.8454	0.0399	0.2553	185.6638	0.0820	0.751070
LiW	377.8265	378.2015	384.4850	380.4648	0.0402	0.2658	185.9133	0.0820	0.749931
EW	377.3276	377.7026	383.9862	379.9659	0.0399	0.2553	185.6638	0.0821	0.749379
TW	378.3411	378.7161	384.9997	380.9794	0.0487	0.3192	186.1706	0.0887	0.659154

Figures 4–6 illustrate the fitted PDF, CDF, SF, and PP plots of the GKMW distribution for the three datasets, respectively. These figures reinforce the results shown in Tables 16–18, indicating that the proposed distribution offers a close fit for all datasets.

The GKMW distribution effectively models a wide range of data behaviors, including skewness and heavy tails, providing a better fit for the three datasets compared to other models. Its parameterization enables more accurate estimation and captures underlying patterns that may be missed by more restrictive models. Additionally, the GKMW model yields the lowest goodness-of-fit values and the highest p-values, confirming its superior fit.

**Figure 4.** Fitted PDF, CDF, SF, and PP plots of the GKMW distribution for the gauge lengths dataset.

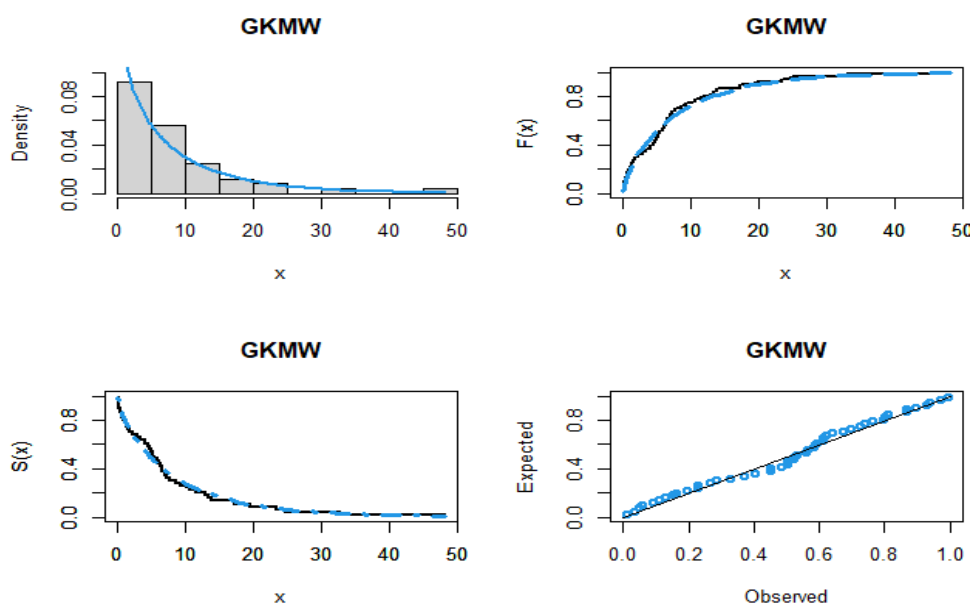


Figure 5. Fitted PDF, CDF, SE, and PP plots of the GKMW distribution for the failure times dataset.

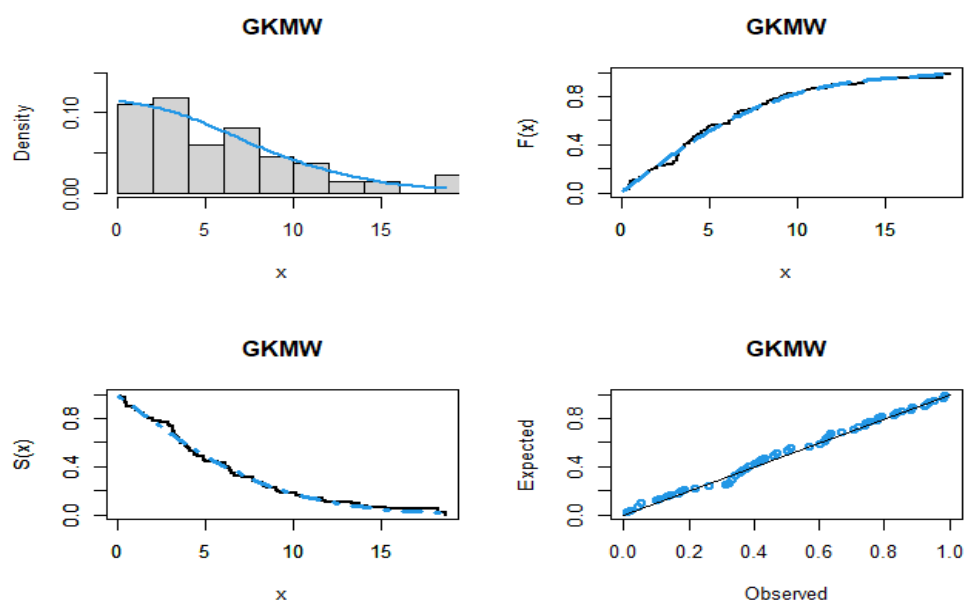


Figure 6. Fitted PDF, CDF, SE, and PP plots of the GKMW distribution for the distance dataset.

Additionally, Figures 7–9 display the histograms of the three datasets along with the fitted densities for the GKMW distribution and other competing distributions. The GKMW distribution consistently outperforms the other Weibull extensions across all three datasets. Moreover, the PP plots for these datasets, shown in Figures 10–12, further illustrate that the GKMW distribution provides a superior fit compared to the other distributions analyzed.

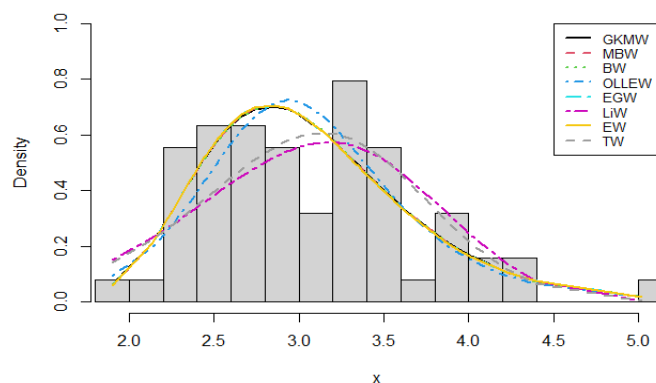


Figure 7. The fitted GKMW PDF alongside the PDFs of other fitted distributions for the gauge lengths dataset.

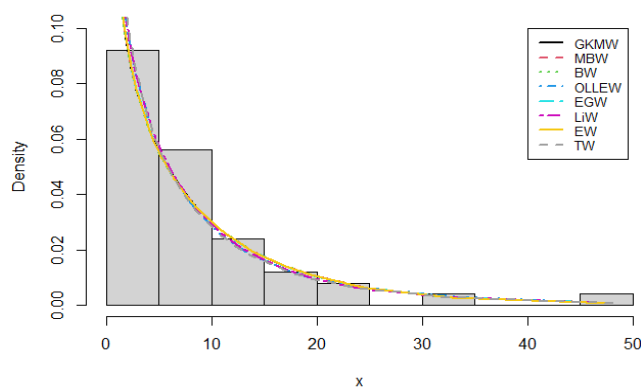


Figure 8. The fitted GKMW PDF alongside the PDFs of other fitted distributions for the failure times dataset.

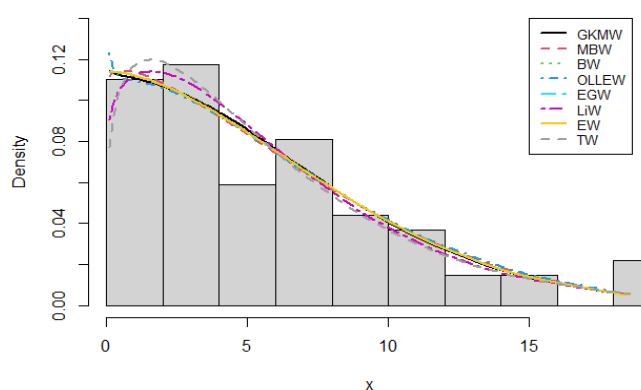


Figure 9. The fitted GKMW PDF alongside the PDFs of other fitted distributions for the distance dataset.

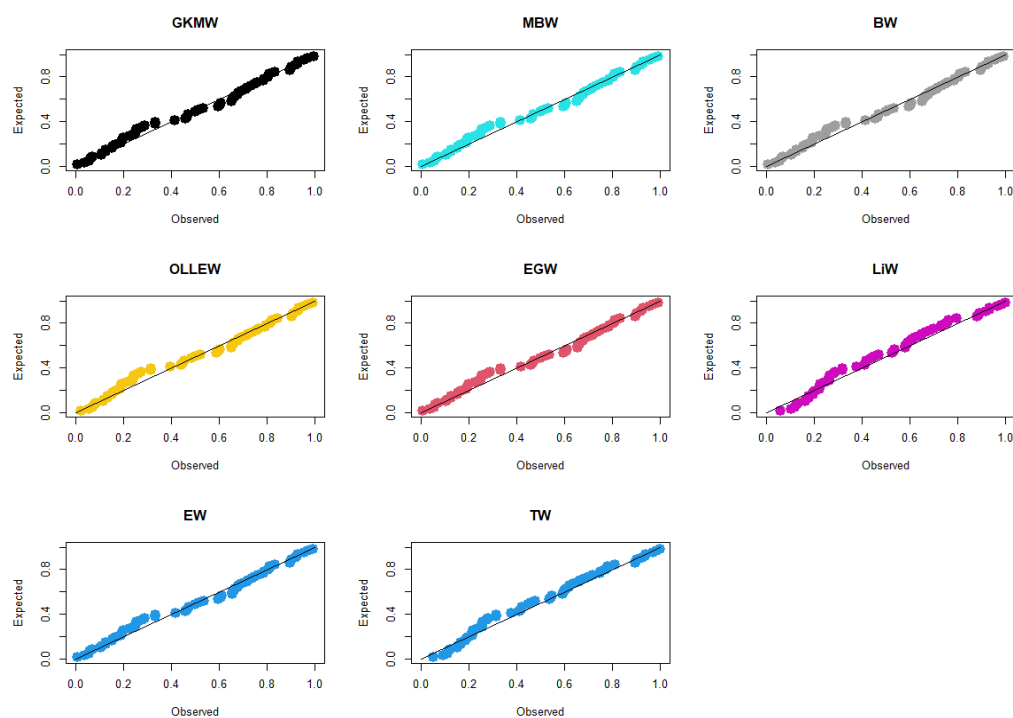


Figure 10. The PP plots comparing the GKMW distribution with other distributions for the gauge lengths dataset.

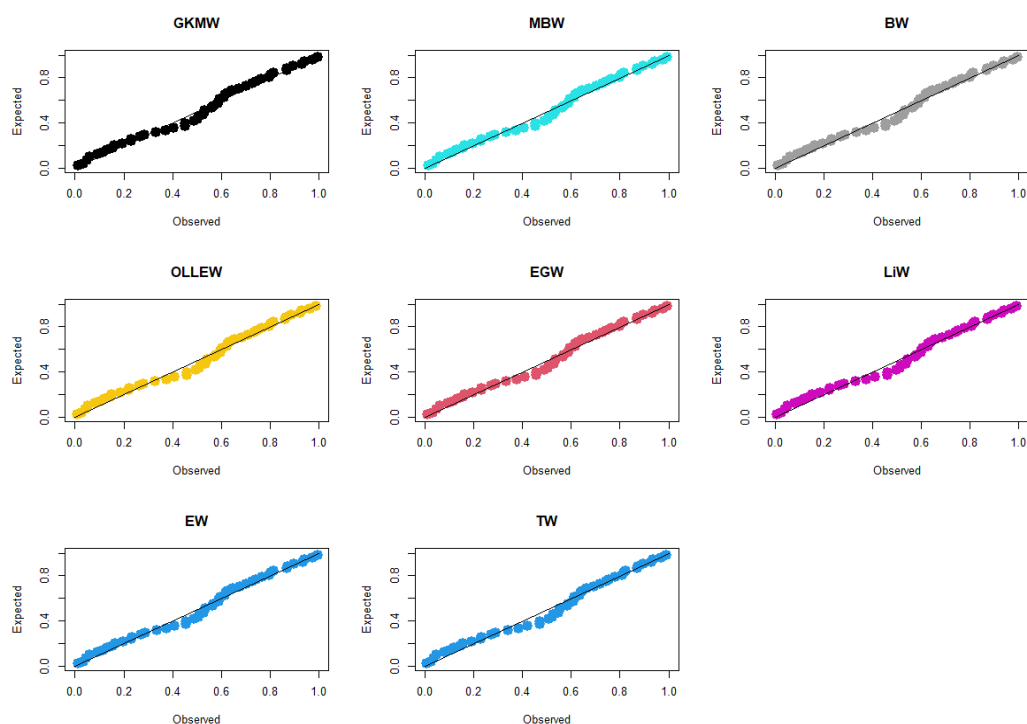


Figure 11. The PP plots comparing the GKMW distribution with other distributions for the failure times dataset.

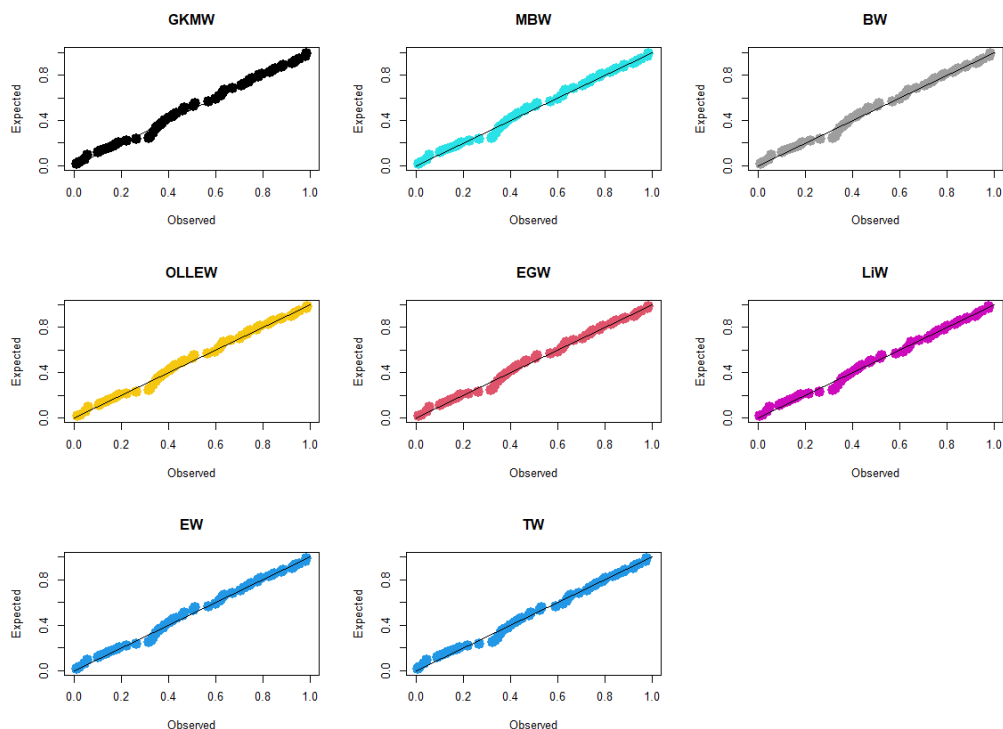


Figure 12. The PP plots comparing the GKMW distribution with other distributions for the distance dataset.

The TTT and HRF plots of the GKMW distribution for the gauge lengths, failure time, and transect line datasets are presented in Figures 13–15. The TTT plots reveal concave shapes for the gauge lengths and distances datasets, indicating increasing HRFs, while it appears convex for the failure time dataset, suggesting a decreasing HRF. The GKMW model can accommodate both increasing and decreasing HRF, making it well-suited for modeling all datasets.

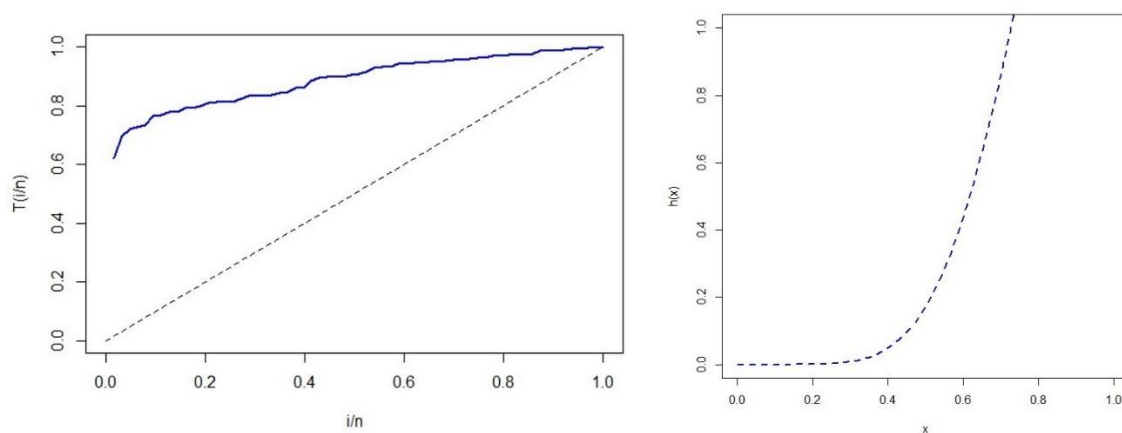


Figure 13. TTT plot for the gauge lengths dataset and the GKMW HRF plot for the same dataset.

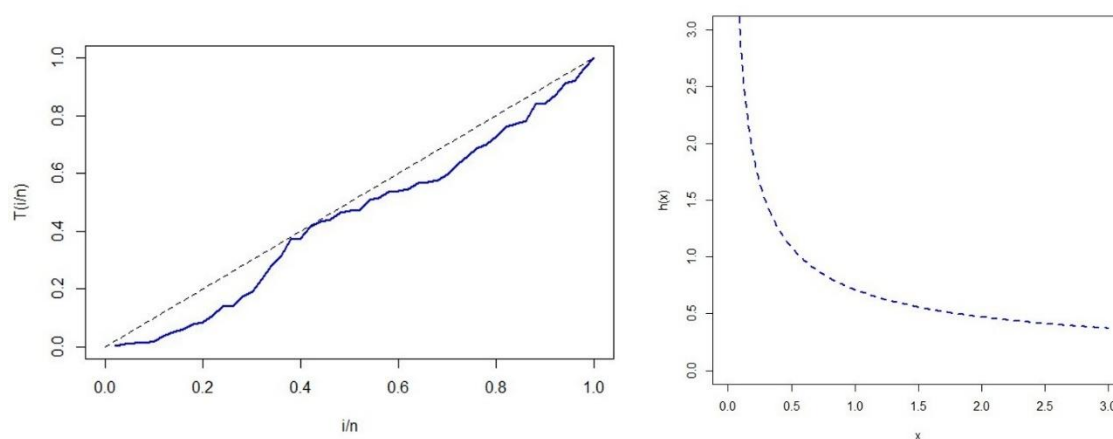


Figure 14. TTT plot for the failure times dataset and the GKMW HRF plot for the same dataset.

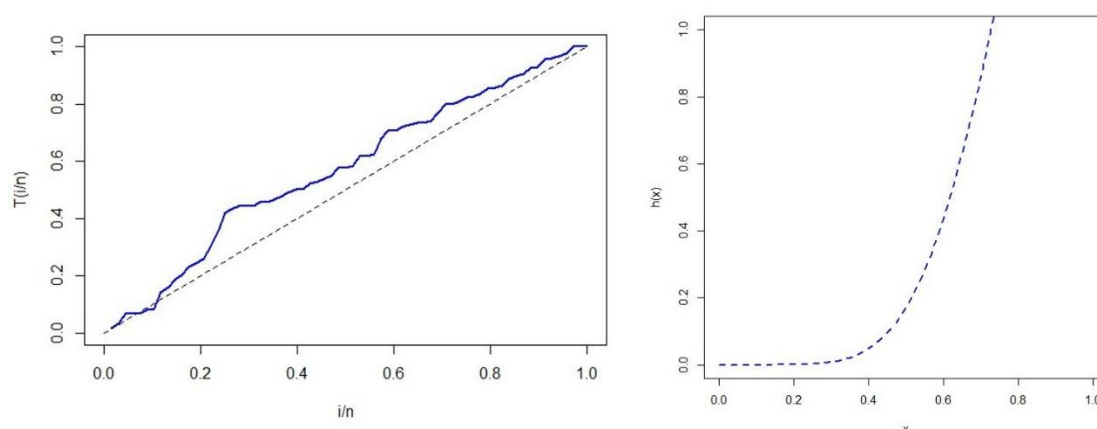


Figure 15. TTT plot for the distance dataset and the GKMW HRF plot for the same dataset.

7. Conclusions future perspectives

In this paper, we present the extended Kavya–Manoharan Weibull (GKMW) distribution, a novel extension of the Weibull distribution that provides a versatile and adaptable way to model diverse types of data. The proposed model is notable for its ability to support a wide range of distribution shapes, including symmetric, right-skewed, reversed-J, and left-skewed densities, making it adaptable to a variety of real-world datasets. Furthermore, it can model both non-monotonic and monotonic failure rates, increasing its applicability in a variety of statistical settings.

The mathematical properties of the GKMW model are investigated. Additionally, its parameters are estimated using eight alternative estimation techniques. Simulation studies show that the

maximum product of the spacing estimation method outperforms all other estimators for reliably calculating GKMW parameters. This finding has significant implications for increasing the precision of statistical modeling in real-world applications.

The GKMW distribution is applied to three real-life datasets and outperforms existing Weibull distributions, highlighting its potential for improved data processing. The GKMW model's practical importance stems from its capacity to improve modeling flexibility and accuracy, especially in fields such as survival analysis, where it can provide more reliable insights into failure rates and data behavior.

In future work, we will focus on expanding the GKMW distribution's applications beyond survival analysis, including its use with large-scale datasets and refining computational methods for parameter estimation. Key areas for future research include:

- Improving parameter estimation techniques, such as maximum likelihood or Bayesian methods for censored data, to enhance the GKMW model's robustness and accuracy.
- Exploring non-parametric or semi-parametric versions of the model for broader applicability.
- Applying Bayesian methods for both parameter estimation and model comparison, offering a promising extension to the GKMW framework.
- Developing a discrete version of the GKMW model to facilitate its use in modeling count data in diverse applied fields.

Author contributions

A.Z.A.: Conceptualization, Methodology, Software, Project administration, Writing-original draft preparation, Writing-review and editing; R.A. and A.S.A.: Validation, Formal analysis, Investigation, Resources, Writing-review and editing; H.A.M.: Conceptualization, Methodology, Software, Writing-original draft preparation, Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests.

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Appendix A. The three datasets used to evaluate the performance of the proposed GKMW model.

Gauge lengths dataset									
1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397	2.445
2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614	2.616	2.618
2.624	2.659	2.675	2.738	2.740	2.856	2.917	2.928	2.937	2.937
2.996	3.125	2.977	3.030	3.139	3.145	3.220	3.223	3.235	3.243
3.264	3.272	3.294	3.332	3.346	3.377	3.408	3.435	3.493	3.501
3.537	3.554	3.562	3.628	3.852	3.871	3.886	3.971	4.024	4.027
4.225	4.395	5.020							

Failure times dataset									
0.013	0.065	0.111	0.111	0.163	0.309	0.426	0.535	0.684	0.747
0.997	1.284	1.304	1.647	1.829	2.336	2.838	3.269	3.977	3.981
4.520	4.789	4.849	5.202	5.291	5.349	5.911	6.018	6.427	6.456
6.572	7.023	7.087	7.291	7.787	8.596	9.388	10.261	10.713	11.658
13.006	13.388	13.842	17.152	17.283	19.418	23.471	24.777	32.795	48.105

Distance dataset									
2.0	0.5	10.4	3.6	0.9	1.0	3.4	2.9	8.2	6.5
5.7	3.0	4.0	0.1	11.8	14.2	2.4	1.6	13.3	6.5
8.3	4.9	1.5	18.6	0.4	0.4	0.2	11.6	3.2	7.1
10.7	3.9	6.1	6.4	3.8	15.2	3.5	3.1	7.9	18.2
10.1	4.4	1.3	13.7	6.3	3.6	9.0	7.7	4.9	9.1
3.3	8.5	6.1	0.4	9.3	0.5	1.2	1.7	4.5	3.1
3.1	6.6	4.4	5.0	3.2	7.7	18.2	4.1		



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