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Research article

Algorithms and applications of the new modified decomposition method to solve initial-boundary value problems for fractional partial differential equations

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Abstract: This study aims to efficiently solve the class of initial-boundary problems for fractional partial differential equations by expanding the modified Adomian decomposition method. Three new algorithms are proposed, two for handling various fractional initial-boundary problems via the x-differential operator, and one for using the t-differential operator. The proposed methods are designed to increase computational efficiency and ensure greater accuracy in the solutions. The effectiveness of the techniques is reviewed by applying them to a variety of cases, demonstrating their ability to address a broad range of fractional calculus problems. The results emphasize the flexibility and adaptability of the developed algorithms as reliable methods.

Keywords: fractional differential equations; advanced decomposition method; initial and boundary value conditions; Riemann–Liouville derivative; Caputo derivative
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1. Introduction

The study of derivatives and integrals of fractional orders has a rich and intricate history that dates back several centuries. The concept of fractional derivatives was first introduced in the late 17th century, with the contributions of notable mathematicians such as Gottfried Wilhelm Leibniz and Leonhard Euler [1, 2]. In 1695, Leibniz raised the question of the meaning of a derivative of order $\frac{1}{2}$, which initiated interest in the concept of fractional order. Fractional calculus has gained significant momentum and continues to evolve to this day, finding applications in different domains, including engineering, biology, epidemiology, physics, and fluid dynamics [3–9]. Its increasing use in these domains underscores its growing importance in contemporary science; read [10] and the given references therewith. Fractional partial differential equations (FPDEs) are a significant extension of the traditional fractional differential equations (FDEs). Certainly, these equations modeled several complex phenomena in various fields like physics, engineering, finance, and biology [11, 12] among others. The mathematical formulation of FPDEs typically involves the use of fractional derivatives defined by various approaches such as the Grunwald–Letnikov, Caputo, and Riemann–Liouville fractional definitions. The choice of definition can impact the formulation of the problem and the methods used for finding solutions. On the other hand, the initial-value problems (IVPs), boundary-value problems (BVPs), and initial-boundary value problems (IBVPs) featuring FPDEs play vital parts in various mathematical physics models. These problems help describe complex physical and engineering systems that involve nonstandard dynamics or long-term memory effects, such as heat diffusion in heterogeneous materials [13], fluid motion affected by high viscosity [14], and biological processes with long-term interactions [15]. In fact, the implication of these problems is associated with their capability to describe phenomena that are challenging to accurately model using the traditional differential equations of integer orders. Solutions to BVPs and IVPs in FPDEs can provide more precise models for intricate interactions among variables, making them essential tools for researchers in fields like engineering physics, chemistry, biology, and finance. These problems also demand advanced analytical and numerical techniques, as exact analytical solutions are rare and challenging. Studies, therefore, rely on numerical methods such as the modified Adomian decomposition method (MADM) and the variational iteration technique (VIM), which have proven effective in yielding approximate solutions. Due to the complexity of FPDEs, traditional analytical methods may not always be applicable. Therefore, various computational and semi-analytical approaches have been devised to tackle these equations. Some of the prominent techniques include the homotopy analysis method [16], Laplace decomposition method (LDM) [17], VIM [18], Adomian decomposition method (ADM) [19, 20], and the weighted average finite difference methods [21], to mention a few; see also other relevant methods in [22, 23] and the references therein.

However, among the multitude of available techniques to tackle the governing class of FPDEs, this study has adopted the ADM as its base method. This method has gained tremendous popularity due to its reliability in handling diverse functional equations immediately after its introduction by George Adomian in 1984 [24, 25]. The method has undergone numerous enhancements aimed at increasing its accuracy, speed, and computational efficiency, as well as adapting it to a broader range of equations. These improvements have significantly accelerated the rapidity of the solution in contrast to the original ADM, marking substantial advancements in the method. In [26], new results were presented related to the Adomian series, which is used as a tool for analyzing and solving certain categories of differential and integral equations. In [18], authors have employed the ADM for tracking nonlinear FPDEs, while [27] discusses the applications of LDM on nonlinear FPDEs. In addition, the reduced differential transform coupled with the ADM was proposed on the class of FPDEs amidst the attachment of a conformable fractional operator in [28]; the authors estimated the approximate solutions for the one- and two-dimensional time-FPDEs, while the presentation of the aiming new MADM for the system of nonlinear FPDEs has been featured in [29]. Certainly, the MADM has been proven to be highly effective and computationally efficient for dissimilar problems, making it a valuable tool for scientists. It is important to note that the convergence of the Adomian series solution has been studied for different problems by many authors. In [30, 31], the convergence was investigated when the method was applied to general functional equations, while specific types of equations were considered in [32–34].

In particular, this study intends to extend the application of MADM, which was recently utilized in [35–38] for various types of IVPs and BVPs in the class of FPDEs by presenting three distinct algorithms. Algorithm 1 and Algorithm 2 will address the BVPs through the x-differential operator, while Algorithm 3 will handle the IVPs via the t-differential operator. Finally, the manuscript follows the following arrangement: Section 2 provides some fundamental definitions related to the properties of fractional operators. Section 3 explains the developed schemes for FPDEs. Section 4 applies the derived computational schemes on various examples, while Section 5 recaps with the closing note.

2. Preliminaries on fractional calculus

The current section presents certain definitions for fractional operators, including some notable operators like the Caputo and Riemann–Liouville fractional [39].

Definition 2.1. *The fractional derivatives $D^{\alpha}_{a+}y$ and $D^{\alpha}_{b-}y$, with $\alpha \in \mathbb{C}$ $(\Re(\alpha) \geq 0)$ as a fractional order, are defined by Riemann–Liouville, respectively, as follows [39]:*

$$\left(D^{\alpha}_{a+}y\right)(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_{a}^{x}(x-t)^{-\alpha+n-1}y(t)dt, \quad (x > a); n = [\Re(\alpha)] + 1), \tag{2.1}$$

and

$$\left(D^{\alpha}_{b-}y\right)(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^{n}\int_{x}^{b}(t-x)^{-\alpha+n-1}y(t)dt, \quad (x < b; n = [\Re(\alpha)] + 1). \tag{2.2}$$

Definition 2.2. *The fractional derivatives $\left({}^{C}D^{\alpha}_{a+}y\right)(x)$ and $\left({}^{C}D^{\alpha}_{b-}y\right)(x)$ with $\alpha \in \mathbb{C}$ $(\Re(\alpha) \geq 0)$ as a fractional order is defined by Caputo if $\alpha \notin \mathbb{N}_0$, respectively, as follows [39]:*

$$\left({}^{C}D^{\alpha}_{a+}y\right)(x) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-t)^{-\alpha+n-1}y^{(n)}(t)dt =: \left(I^{n-\alpha}_{a+}D^{n}y\right)(x), \tag{2.3}$$

and

$$\left({}^{C}D^{\alpha}_{b-}y\right)(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{x}^{b}(t-x)^{-\alpha+n-1}y^{(n)}(t)dt =: (-1)^{n}\left(I^{n-\alpha}_{b-}D^{n}y\right)(x). \tag{2.4}$$

Moreover, when $\alpha = n \in \mathbb{N}_0$, the aforementioned fractional derivatives by Caputo takes the following definitions:

$${}^{C}D^{\alpha}_{a+}y(x) = y^{(n)}(x) \quad \text{and} \quad {}^{C}D^{\alpha}_{b-}y(x) = (-1)^{n}y^{(n)}(x) \quad (n \in \mathbb{N}), \tag{2.5}$$

and if $\alpha = 0$, we have

$$\left({}^{C}D^{0}_{a+}y\right)(x) = \left({}^{C}D^{0}_{b-}y\right)(x) = y(x). \tag{2.6}$$

Definition 2.3. *Riemann–Liouville fractional integrals $\left(I^{\alpha}_{a+}y\right)(x)$ and $\left(I^{\alpha}_{b-}y\right)(x)$ having the fractional order $\alpha \in \mathbb{C}(\Re(\alpha) > 0)$ are respectively defined as follows:*

$$\left(I^{\alpha}_{a+}y\right)(x) := \frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{-1+\alpha}y(t)dt, \quad (x > a; \Re(\alpha) > 0), \tag{2.7}$$

and

$$\left(I^{\alpha}_{b-}y\right)(x) := \frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{-1+\alpha}y(t)dt, \quad (x < b; \Re(\alpha) > 0). \tag{2.8}$$

Property 2.1. *Some important properties for the fractional differential and integral operators are outlined in what follows:*

i)

$$I^{\alpha}I^{\beta}y(x) = I^{\alpha+\beta}y(x); \quad \alpha,\beta \geq 0\,. \tag{2.9}$$

ii)

$$I^{\alpha}\,{}^{C}D_{*}^{\alpha}y(x) = y(x) - \sum_{k=0}^{m-1} y^{(k)}\left(0^{+}\right)\frac{x^{k}}{\Gamma(1+k)}, \quad \alpha \in (m-1, m). \tag{2.10}$$

iii)

$$I^{\alpha}\,{}^{C}D_{*}^{\beta}y(x) = I^{\beta-\alpha}y(x) - \sum_{k=0}^{m-1} y^{(k)}\left(0^{+}\right)\frac{x^{k-\alpha+\beta}}{\Gamma(1+k-\alpha+\beta)}, \tag{2.11}$$

where $\alpha > \beta$, *and* $m-1 < \beta, \alpha \leq m$.

iv)

$$I^{\alpha}x^{n} = \frac{\Gamma(1+n)}{\Gamma(1+n+\alpha)}x^{n+\alpha}, \tag{2.12}$$

where $m-1 < \alpha \leq m$, $n > -1$, *and* $x > 0$.

v)

$${}^{C}D_{*}^{\alpha}x^{r} = \begin{cases} \frac{\Gamma(r+1)}{\Gamma(1+n-\alpha)}x^{r-\alpha}, & r \geq m-1, \\ 0, & r < m-1. \end{cases} \tag{2.13}$$

vi)

$${}^{C}D_{*}^{\alpha}k = 0, \quad \text{when } k \text{ represents a constant real number}\,. \tag{2.14}$$

3. Modified Adomian decomposition method

The present section dwells on the application of the three MADM-based algorithms for solving the IBVP featuring the wider class of FPDEs. To exhibit the MADM for the governing class of equations, one considers a universal linear FPDE as follows:

$$D_{t}^{\alpha}u(x,t) + Lu(x,t) + Ru(x,t) = g(x,t); \quad m-1 < \alpha < m, \quad x \in (a,b), \quad t > 0, \tag{3.1}$$

admits the prescription of the initial and boundary data, respectively, as follows

$$u(x,0) = f(x), \quad a \leq x \leq b,$$

and

$$u(a,t) = g_{0}(t),\ u(b,t) = g_{1}(t), \quad t > 0,$$

where the function $u(x,t)$ is unknown, D_{t}^{α} is the fractional differential operator defined in Caputo's sense of order α, $m \in \mathbf{N}$, L is the highest partial derivative, which might involve other fractional derivatives of order less than α and R is the remainder while $g(x,t)$ is the source function.

Thus, to derive the explicit analytical solution for the governing IBVP, which features a fractional order derivative, we will apply the MADM based on the three algorithms that follow.

3.1. MADM with respect to x-differential operator

In this subsection, we will propose two different algorithms, which are based on MADM, to solve the FPDE (3.1) with respect to the linear x-differential operator defined as follows:

$$L(\cdot) = \frac{\partial^2}{\partial x^2}(\cdot),$$

where L is a second-order linear differential operator defined in the x-variable.

Algorithm 1: First, applying the inverse operator (of the latter linear differential operator)

$$L^{-1}(\cdot) = \int_a^x \int_a^x (\cdot)\, dx\, dx, \tag{3.2}$$

on (3.1) and using its properties yields as follows:

$$u(x,t) = u(a,t) + xu'(a,t) + L^{-1}\left[g(x,t)\right] - L^{-1}\left[Ru(x,t)\right] - L^{-1}[D_t^{\alpha} u(x,t)]. \tag{3.3}$$

Note, we will further put $u'(a,t) = c$ in what follows. Next, on using the standard ADM procedure [24, 25], the latter equation becomes

$$\sum_{n=0}^{\infty} u_n(x,t) = u(a,t) + cx + L^{-1}\left[g(x,t)\right] - L^{-1}\left[\sum_{n=0}^{\infty} Ru_n(x,t)\right] - L^{-1}\left[D_t^{\alpha}\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right], \tag{3.4}$$

where the solution $u(x,t)$ in (3.4) is decomposed into a sum of infinite components $u_n(x,t)$, for $n = 0, 1, 2, 3, \ldots$, that is,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

In addition, the algorithm by MADM proceeds by adding the expression

$$-pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right], \tag{3.5}$$

in the right-hand side of (3.4) to obtain

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x,t) =& u(a,t) + cx + L^{-1}[g(x,t)] - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] \\ &+ L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] - L^{-1}\left[Ru_n(x,t)\right] - L^{-1}\left[D_t^{\alpha}\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right].\end{aligned} \tag{3.6}$$

Therefore, one obtain from (3.6) the following recursive relation:

$$\begin{aligned} u_0(x,t) &= u(a,t) + cx + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right], \\ u_1(x,t) &= L^{-1}[g(x,t)] - L^{-1}\left[Ru_0(x,t)\right] - L^{-1}\left[D_t^{\alpha} u_0(x,t)\right] - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right], \\ u_{n+1}(x,t) &= -L^{-1}\left[Ru_n(x,t)\right] - L^{-1}\left[D_t^{\alpha} u_n(x,t)\right], \quad n \geq 1. \end{aligned} \tag{3.7}$$

Lastly, one obtains the explicit values for the constants a_n, for $n = 0, 1, 2, 3, \ldots$, upon which $u_1 = 0$. In this regard, it is obvious that $u_2 = u_3 = u_4 = \cdots = 0$. Moreover, by letting $p = 1$, one obtains the solution of (3.1) as $u(x,t) = u_0(x,t)$. Additionally, one eventually substitutes the values for a_n in the determined solution, alongside making use of the last boundary datum $u(b,t) = g_1(t)$ to get hold of the presumed constant c.

Algorithm 2: Algorithm 2 commences by defining the inversion operator L^{-1}, which is based on the submission in [40] as follows:

$$L^{-1}(.) = \int_a^x dx' \int_a^{x''} (.)dx'' - \frac{x-a}{b-a} \int_a^b dx \int_a^{x''} (.)dx''. \tag{3.8}$$

Remarkably, among the advantages of this inversion operator is that its capability to admit all the prescribed boundary data unswervingly.

Therefore, applying the current inverse operator on (3.1), one gets

$$u(x,t) = u(a,t) + u(b,t)x - u(a,t)x + L^{-1}[g(x,t)] - L^{-1}\left[Ru(x,t)\right] - L^{-1}[D_t^{\alpha} u(x,t)]. \tag{3.9}$$

Accordingly, adding the expression in (3.5) to the right-hand side of (3.9), alongside using $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ gives

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x,t) =& u(a,t) + xu(b,t) - xu(a,t) + L^{-1}[g(x,t)] - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] \\ &+ L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] - L^{-1}\left[\sum_{n=0}^{\infty} Ru_n(x,t)\right] - L^{-1}\left[D_t^{\alpha}\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right],\end{aligned} \tag{3.10}$$

which then gives the overall recurrent scheme as follows:

$$\begin{aligned} u_0(x,t) &= u(a,t) + xu(b,t) - xu(a,t) + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right], \\ u_1(x,t) &= L^{-1}[g(x,t)] - L^{-1}\left[Ru_0(x,t)\right] - L^{-1}\left[D_t^{\alpha} u_0(x,t)\right] - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right], \\ u_{n+1}(x,t) &= -L^{-1}\left[Ru_n(x,t)\right] - L^{-1}\left[D_t^{\alpha} u_n(x,t)\right], \quad n \geq 1. \end{aligned} \tag{3.11}$$

Moreover, as in the above case, one computes the explicit values for a_n $(n \geq 0)$ by setting $u_1(x,t) = 0$, and subsequently $p = 1$. In addition, substituting the obtained values into $u_0(x,t)$ gives the overall closed-form solution that $u(x,t) = u_0(x,t)$.

3.2. MADM with respect to t-differential operator

The current subsection derives an efficient algorithm based on utilizing a linear t-differential operator for solving IVPs featuring the class of FPDEs.

Algorithm 3: In this algorithm, the linear direct operator and its corresponding inverse operator for the integer or fractional t-differential operators are respectively considered as follows:

$$L = \frac{\partial}{\partial t}, \qquad \text{and} \qquad L^{-1}(.) = \int_0^t (\cdot)\, dt, \tag{3.12}$$

or

$$L(.) = D_t^{\alpha}(.), \qquad \text{and} \qquad L^{-1}(.) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1}(.)ds. \tag{3.13}$$

Thus, upon deploying the inversion operator (3.12) on (3.1), one then obtains

$$u(x,t) = u(x,0) + L^{-1}[g(x,t)] - L^{-1}\left[Ru(x,t)\right] - L^{-1}[D_t^{\alpha} u(x,t)]. \tag{3.14}$$

Consequently, the MADM process requires the inclusion of the aforementioned expression into the latter equation, coupled with the usual ADM decomposition of the solution $u(x,t)$ to obtain as following:

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x,t) =& u(x,0) + L^{-1}[g(x,t)] - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] \\ &- L^{-1}\left[\sum_{n=0}^{\infty} Ru_n(x,t)\right] - L^{-1}\left[D_t^{\alpha}\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right],\end{aligned} \tag{3.15}$$

that eventually leads to the following recurrent scheme

$$\begin{aligned} u_0(x,t) &= u(x,0) + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right], \\ u_1(x,t) &= L^{-1}[g(x,t)] - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] - L^{-1}\left[Ru_0(x,t)\right] - L^{-1}\left[D_t^{\alpha} u_0(x,t)\right], \\ u_{n+1}(x,t) &= -L^{-1}\left[Ru_n(x,t)\right] - L^{-1}\left[D_t^{\alpha} u_n(x,t)\right], \quad n \ge 1. \end{aligned} \tag{3.16}$$

Accordingly, the procedure for the computation of the explicit values for a_n ($n \ge 0$) remains the same as in the above algorithm. That is, setting $u_1(x,t) = 0$, and afterward $p = 1$. In addition, substituting the obtained values into $u_0(x,t)$ gives the overall closed-form solution that $u(x,t) = u_0(x,t)$.

4. Numerical examples

The present section considers several test IBVPs for FPDEs to demonstrate the helpfulness of the proposed algorithms.

Example 4.1. *Consider the time-fractional IBVP for the diffusion-convection equation as follows [41]:*

$$D_t^{\alpha} u(x,t) + \frac{\partial^2 u(x,t)}{\partial x^2} + x\frac{\partial u(x,t)}{\partial x} = 2x^2 + 2t^{\alpha} + 2, \;\; 0 < x < 1, \;\; t > 0, \quad 0 < \alpha \le 1, \tag{4.1}$$

subject to the following initial and boundary data

$$u(x,0) = x^2; \quad u(0,t) = \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} t^{2\alpha}, \; u(1,t) = 1 + \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} t^{2\alpha}.$$

Accordingly, the governing IBVP admits the following exact solution:

$$u(x,t) = 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} t^{2\alpha} + x^2.$$

Algorithm 1: To solve (4.1) applying inverse operator L^{-1} in (3.2) on both sides such that $u'(x,t) = c$ gives

$$u(x,t) = cx + u(0,t) + L^{-1}[2x^2 + 2t^\alpha + 2] - L^{-1}[D_t^\alpha u(x,t) + x\frac{\partial u(x,t)}{\partial x}]. \tag{4.2}$$

Now using MADM As explained previously, we write

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x,t) =& cx + \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha} + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] - p\left[L^{-1}\sum_{n=0}^{\infty} a_n x^n\right] \\ &+ L^{-1}\left[2x^2 + 2t^\alpha + 2\right] - L^{-1}\left[D_t^\alpha u_{n-1}(x,t) + x\frac{\partial u_{n-1}(x,t)}{\partial x}\right].\end{aligned} \tag{4.3}$$

Thus, adopting

$$\begin{aligned}u_0(x,t) &= \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha} + cx + L^{-1}\left[a_0 + a_1 x + a_2 x^2 + \cdots\right], \\ u_1(x,t) &= -pL^{-1}\left[a_0 + a_1 x + a_2 x^2 + \cdots\right] + L^{-1}\left[2x^2 + 2t^\alpha + 2\right] - L^{-1}\left[D_t^\alpha u_0(x,t) + x\frac{\partial u_0(x,t)}{\partial x}\right].\end{aligned} \tag{4.4}$$

When expressed explicitly by using L^{-1} and property (2.13), this results

$$\begin{aligned}u_0(x,t) &= \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha} + cx + a_0\frac{x^2}{2} + a_1\frac{3}{6} + a_2\frac{x^4}{12} + \cdots, \\ u_1(x,t) &= -p\left[a_0\frac{x^2}{2} + a_1\frac{x^3}{6} + a_2\frac{x^4}{12} + \cdots\right] + \frac{x^4}{6} + t^\alpha x^2 + x^2 - \frac{cx^3}{6} - a_0\frac{x^4}{12} + a_1\frac{x^5}{40} + a_2\frac{x^6}{90} + \cdots \\ &\quad - t^\alpha x^2 - \frac{cx^3 t^{-\alpha}}{6\Gamma(1-\alpha)} - \frac{a_0 x^4 t^{-\alpha}}{24\Gamma(1-\alpha)} - \frac{a_1 x^5 t^{-\alpha}}{120\Gamma(1-\alpha)} - \frac{a_2 x^6 t^{-\alpha}}{360\Gamma(1-\alpha)}.\end{aligned}$$

But $u_1(x,t) = 0$ to calculate a_i, $i = 1, 2\cdots$ and with opposite signs simplified, we have

$$\begin{aligned}u_1(x,t) =& -p\left[a_0\frac{x^2}{2} + a_1\frac{x^3}{6} + a_2\frac{x^4}{12} + \cdots\right] + \frac{x^4}{6} + x^2 - \frac{cx^3}{6} - a_0\frac{x^4}{12} - a_1\frac{x^5}{40} - a_2\frac{x^6}{90} + \cdots \\ &- \frac{cx^3 t^{-\alpha}}{6\Gamma(1-\alpha)} - \frac{a_0 x^4 t^{-\alpha}}{24\Gamma(1-\alpha)} - \frac{a_1 x^5 t^{-\alpha}}{120\Gamma(1-\alpha)} - \frac{a_2 x^6 t^{-\alpha}}{360\Gamma(1-\alpha)} = 0.\end{aligned}$$

Now, considering that $p = 1$ and equating the coefficients of x^n it can be readily shown that $a_0 = 2$ and $a_1 = a_2 = 0$. Therefore, on substituting the values of a_0, a_1anda_2 into $u_0(x,t)$, we obtain the solution as

$$u(x,t) = u_0(x,t) = \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha} + cx + x^2. \tag{4.5}$$

At last, to find the value of c, we must first find $u(1,t)$, that is, $u(1,t) = 1 + \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}$ we have $c = 0$, So one obtains

$$u(x,t) = \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha} + x^2, \tag{4.6}$$

which gives the exact solution for the IBVP (4.1).

Algorithm 2: We begin with applying the inverse operator (3.8) on both sides of (4.1) with $a = 0$, and $b = 1$, to obtain

$$u(x,t) = u(0,t) + xu(1,t) - xu(0,t) + L^{-1}\left[2x^2 + 2t^\alpha + 2\right] - L^{-1}\left[D_t^\alpha u(x,t) + x\frac{\partial u(x,t)}{\partial x}\right]. \tag{4.7}$$

Thus, upon implementing the MADM procedure, one further gets

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x,t) =& \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha} + x\left(1 + \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}\right) - \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}x - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] \\ &+ L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}\left[2x^2 + 2t^\alpha + 2\right] - L^{-1}\left[D_t^\alpha u_{n-1}(x,t) + x\frac{\partial u_{n-1}(x,t)}{\partial x}\right],\end{aligned} \tag{4.8}$$

that gives the following leading scheme

$$\begin{aligned}u_0(x,t) &= \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha} + x + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right], \\ u_1(x,t) &= -pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}\left[2x^2 + 2t^\alpha + 2\right] - L^{-1}\left[D_t^\alpha u_0(x,t) + x\frac{\partial u_0(x,t)}{\partial x}\right].\end{aligned}$$

Then,

$$u_0(x,t) = \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha} + x + a_0\frac{x^2}{2} + a_1\frac{x^3}{6} + a_2\frac{x^4}{12} + \cdots - x\left(a_0\frac{1}{2} + a_1\frac{1}{6} + a_2\frac{1}{12} + \cdots\right),$$

and

$$\begin{aligned}u_1(x,t) =& -p\left[a_0\frac{x^2}{2} + a_1\frac{x^3}{6} + a_2\frac{x^4}{12} - x\left(a_0\frac{1}{2} + a_1\frac{1}{6} + a_2\frac{1}{12}\right)\right] + \frac{x^4}{6} + t^\alpha x^2 + x^2 - x\left(\frac{1}{6} + t^\alpha + 1\right) \\ &- \frac{x^3}{6} - a_0\frac{x^4}{12} - a_1\frac{x^5}{40} - a_2\frac{x^6}{90} + x\left(\frac{1}{6} + a_0\frac{1}{12} + a_1\frac{1}{40} + a_2\frac{1}{90}\right) + \frac{x^3}{6}\left(a_0\frac{1}{2} + a_1\frac{1}{6} + a_2\frac{1}{12}\right) \\ &- \frac{x}{6}\left(a_0\frac{1}{2} + a_1\frac{1}{6} + a_2\frac{1}{12}\right) - t^\alpha x^2 - \frac{x^3 t^{-\alpha}}{6\Gamma(1-\alpha)} - \frac{a_0 x^4 t^{-\alpha}}{24\Gamma(1-\alpha)} - \frac{a_1 x^5 t^{-\alpha}}{120\Gamma(1-\alpha)} \\ &- \frac{a_2 x^6 t^{-\alpha}}{360\Gamma(1-\alpha)} + \frac{x^3 t^{-\alpha}}{6}(a_0\frac{1}{2} + a_1\frac{1}{6} + a_2\frac{1}{12}) + t^\alpha x + \frac{x t^{-\alpha}}{6\Gamma(1-\alpha)} + \frac{a_0 x t^{-\alpha}}{24\Gamma(1-\alpha)} \\ &+ \frac{a_1 x t^{-\alpha}}{120\Gamma(1-\alpha)} + \frac{a_2 x t^{-\alpha}}{360\Gamma(1-\alpha)} - \frac{x t^{-\alpha}}{6}\left(a_0\frac{1}{2} + a_1\frac{1}{6} + a_2\frac{1}{12}\right).\end{aligned}$$

In the same way, we set $p = 1$, and $u_1(x,t) = 0$, to accordingly obtain $a_0 = 2$, and $a_1 = a_2 = 0$, which then leads to $u(x,t) = u_0(x,t)$ as follows:

$$u(x,t) = 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha} + x^2,$$

the known exact solution.

Algorithm 3: This algorithm commences by applying the inversion operator L^{-1} alongside the property (2.10) on (4.1) to obtain

$$u(x,t) = u(x,0) + I_t^\alpha\left[2x^2 + 2t^\alpha + 2\right] - I_t^\alpha\left[x\frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2}\right]. \tag{4.9}$$

Then, on using the MADM, one obtains

$$\sum_{n=0}^{\infty} u_n(x,t) = x^2 + I_t^\alpha\left[\sum_{n=0}^{\infty} a_n x^n\right] - pI_t^\alpha\left[\sum_{n=0}^{\infty} a_n x^n\right] + I_t^\alpha\left[2x^2 + 2t^\alpha + 2\right] - I_t^\alpha\left[x\frac{\partial u_{n-1}(x,t)}{\partial x} + \frac{\partial^2 u_{n-1}(x,t)}{\partial x^2}\right].$$

Now, coupling the MADM with the reliable modification in [42], one gets the recursive relation as follows:

$$u_0(x,t) = x^2 + I_t^\alpha[2t^\alpha] + I_t^\alpha[a_0 + a_1 x + a_2 x^2 + \cdots], \tag{4.10}$$

$$u_1(x,t) = -pI_t^\alpha[a_0 + a_1 x + a_2 x^2 + \cdots] + I_t^\alpha\left[2x^2 + 2\right] - I_t^\alpha[\frac{\partial^2 u_0(x,t)}{\partial x^2} + x\frac{\partial u_0(x,t)}{\partial x}], \tag{4.11}$$

or equivalently the following after applying the inversion operator

$$\begin{aligned}
u_0(x,t) &= x^2 + \frac{2\Gamma(\alpha+1)t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^\alpha}{\Gamma(\alpha+1)}\left(a_0 + a_1 x + a_2 x^2\right),\\
u_1(x,t) &= -p\frac{t^\alpha}{\Gamma(\alpha+1)}\left(a_0 + a_1 x + a_2 x^2\right) + \frac{2x^2 t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^\alpha}{\Gamma(\alpha+1)} - \frac{2x^2 t^\alpha}{\Gamma(\alpha+1)} - \frac{2t^\alpha}{\Gamma(\alpha+1)}\\
&\quad - \frac{2a_2 t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{a_1 x t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{2x^2 a_2 t^{2\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

Finally, after eliminating similar terms and setting $p = 1$, we establish and compare the coefficients of x^n, yielding to $a_0 = a_1 = a_2 = 0$. Thus, putting these values into $u_0(x,t)$ gives the exact solution

$$u(x,t) = 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha} + x^2,$$

which is the required exact solution for the model.

Example 4.2. *Consider the time-fractional IBVP for the linear nonhomogeneous Burgers equation [43]:*

$$D_t^\alpha u + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2, \quad 0 < x < 1, t > 0, \quad 0 < \alpha \le 1, \tag{4.12}$$

with the prescribed initial and boundary data as follows:

$$u(x,0) = x^2; \quad u(0,t) = t^2,\ u(1,t) = 1 + t^2,$$

and satisfying the following actual solution

$$u(x,t) = t^2 + x^2.$$

Algorithm 1: Applying the inverse operator (3.2) to (4.12) with the assumption that $u'(0,t) = c$ yields

$$u(x,t) = cx + u(0,t) - L^{-1}\left[\frac{2t^{2-\alpha}}{\Gamma(\alpha-3)} + 2x - 2\right] + L^{-1}\left[D_t^{\alpha}u(x,t) + \frac{\partial u(x,t)}{\partial x}\right]. \tag{4.13}$$

Now, employing the MADM on the above equation gives

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x,t) =& cx + t^2 - L^{-1}\left[\frac{2t^{2-\alpha}}{\Gamma(\alpha-3)} + 2x - 2\right] - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] \\ &+ L^{-1}\left[D_t^{\alpha}u_{n-1}(x,t) + \frac{\partial u_{n-1}(x,t)}{\partial x}\right],\end{aligned} \tag{4.14}$$

which plainly gives

$$\begin{aligned}u_0(x,t) =& cx + t^2 + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] \\ =& cx + t^2 + a_0\frac{x^2}{2} + a_1\frac{x^3}{6} + a_2\frac{x^4}{12}t + \cdots \\ u_1(x,t) =& -pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] - L^{-1}\left[\frac{2t^{2-\alpha}}{\Gamma(\alpha-3)} + 2 - 2\right] + L^{-1}\left[D_t^{\alpha}u_0(x,t) + \frac{\partial u_0(x,t)}{\partial x}\right] \\ =& -p\left[a_0\frac{x^2}{2} + a_1\frac{x^3}{6} + a_2\frac{x^4}{12}\right] - \frac{t^{2-\alpha}x^2}{\Gamma(3-\alpha)} - \frac{x^3}{3} + x^2 + \frac{cx^3t^{-t}}{6\Gamma(1-\alpha)} + \frac{t^{2-t}x^2}{\Gamma(3-\alpha)} + \frac{a_0x^4t^{-t}}{24\Gamma(1-\alpha)} \\ &+ \frac{a_1x^5t^{-\alpha}}{120\Gamma(1-\alpha)} + \frac{a_2x^6t^{-\alpha}}{360\Gamma(1-\alpha)} + \frac{cx^2}{2} + \frac{a_0x^3}{6} + \frac{a_1x^4}{24} + \frac{a_2x^5}{60}.\end{aligned}$$

Accordingly, the processes gives yields $a_0 = c + 2$, and $a_1 = a_2 = 0$, upon which $u_0(x,t)$ gives the following

$$u_0(x,t) = cx + t^2 + \frac{cx^2}{2} + x^2. \tag{4.15}$$

In addition, the second boundary datum $u(1,t) = 1 + t^2$ affirms that $c = 0$, upon which the overall solution yields

$$u(x,t) = t^2 + x^2,$$

which is the same as the referenced exact solution of the examining model.

Algorithm 2: Applying the inverse operator (3.8) with $a = b = 0$ to (4.12) unveils

$$u(x,t) = u(0,t) + xu(1,t) - xu(0,t) - L^{-1}\left[\frac{2t^{2-\alpha}}{\Gamma(\alpha-3)} + 2x - 2\right] + L^{-1}\left[D_t^{\alpha}u(x,t) + \frac{\partial u(x,t)}{\partial x}\right]. \tag{4.16}$$

Thus, upon implementing the MADM procedure, one obtains

$$\begin{aligned}\sum_{k=0}^{\infty} u_n(x,t) =& t^2 + x - L^{-1}\left[\frac{2t^{2-\alpha}}{\Gamma(\alpha-3)} + 2x - 2\right] - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] \\ &+ L^{-1}\left[D_t^{\alpha}u_{n-1}(x,t) + \frac{\partial u_{n-1}(x,t)}{\partial x}\right],\end{aligned} \tag{4.17}$$

that results in the acquisition of the following scheme

$$u_0(x,t) = t^2 + x + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right],$$

$$u_1(x,t) = -p\left[L^{-1}\sum_{n=0}^{\infty} a_n x^n\right] - L^{-1}\left[\frac{2t^{2-\alpha}}{\Gamma(\alpha-3)} + 2x - 2\right] + L^{-1}\left[D_t^{\alpha} u_0(x,t) + \frac{\partial u_0(x,t)}{\partial x}\right].$$

Accordingly, the application of the governing inverse operator on the last expressions gives

$$u_0(x,t) = t^2 + x + a_0\frac{x^2}{2} + a_1\frac{x^3}{6} + a_2\frac{x^4}{12} + \cdots - x\left(a_0\frac{1}{2} + a_1\frac{1}{6} + a_2\frac{1}{12} + \cdots\right),$$

and

$$\begin{aligned} u_1(x,t) =& -p\left[a_0\frac{x^2}{2} + a_1\frac{x^3}{6} + a_2\frac{x^4}{12} - x\left(a_0\frac{1}{2} + a_1\frac{1}{6} + a_2\frac{1}{12}\right)\right] - \frac{t^{2-\alpha}}{\Gamma(\alpha-3)}x^2 - \frac{x^3}{6} + x^2 \\ & - x\left(-\frac{2t^{2-\alpha}}{\Gamma(\alpha-3)} + \frac{2}{3}\right) - \frac{x^2}{2} + a_0\frac{x^3}{6} + a_1\frac{x^4}{24} + a_2\frac{x^5}{60} - \frac{x^2}{2}\left(a_0\frac{1}{2} + a_1\frac{1}{6} + a_2\frac{1}{12}\right) \\ & - x\left(\frac{1}{2} + a_0\frac{1}{12} + a_1\frac{1}{24} + a_2\frac{1}{40}\right) + \frac{t^{2-\alpha}}{\Gamma(\alpha-3)}x^2 + \frac{x^3 t^{-\alpha}}{6\Gamma(1-\alpha)} + \frac{a_0 x^4 t^{-\alpha}}{24\Gamma(1-\alpha)} + \frac{a_1 x^5 t^{-\alpha}}{120\Gamma(1-\alpha)} \\ & + \frac{a_2 x^6 t^{-\alpha}}{360\Gamma(1-\alpha)} - \frac{x^3 t^{-\alpha}}{12\Gamma(1-\alpha)}\left(a_0\frac{1}{2} + a_1\frac{1}{6} + a_2\frac{1}{12}\right) - x\left(\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{t^{-\alpha}}{6\Gamma(1-\alpha)} + \frac{a_0 t^{-\alpha}}{24\Gamma(1-\alpha)}\right. \\ & \left. + \frac{a_1 t^{-\alpha}}{120\Gamma(1-\alpha)} + \frac{a_2 t^{-\alpha}}{360\Gamma(1-\alpha)} - \frac{t^{-\alpha}}{12\Gamma(1-\alpha)}\left(\frac{a_0}{2} + \frac{a_1}{6} + \frac{a_2}{2}\right)\right). \end{aligned}$$

Hence, one acquires the coefficients $a_0 = 2$ and $a_1 = a_2 = 0$, such that the $u(x,t) = u_0(x,t)$ is obtained as follows:

$$u(x,t) = t^2 + x^2,$$

affirming the already said exact solution.

Algorithm 3: Applying the inverse operator L^{-1} of the current algorithm to (4.12), alongside the use of (2.10) and (2.12), reveals as follows:

$$u(x,t) = u(x,0) + I_t^{\alpha}\left[\frac{2t^{2-\alpha}}{\Gamma(\alpha-3)} + 2x - 2\right] + I_t^{\alpha}\left[\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x}\right]. \tag{4.18}$$

Further, the application of MADM with the modification in [42] further renders the latter equation as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) =& u(x,0) + I_t^{\alpha}\left[\frac{2t^{2-\alpha}}{\Gamma(\alpha-3)} + 2x - 2\right] + I_t^{\alpha}\left[\sum_{n=0}^{\infty} a_n x^n\right] - pI_t^{\alpha}\left[\sum_{n=0}^{\infty} a_n x^n\right] \\ & + I_t^{\alpha}\left[\frac{\partial^2 u_{n-1}(x,t)}{\partial x^2} - \frac{\partial u_{n-1}(x,t)}{\partial x}\right]. \end{aligned} \tag{4.19}$$

Therefore, in accordance with the described methodology, one obtains

$$u_0(x,t) = x^2 + I_t^{\alpha}\left[\frac{2t^{2-\alpha}}{\Gamma(\alpha-3)}\right] + I_t^{\alpha}\left[\sum_{n=0}^{\infty} a_n x^n\right] = x^2 + t^2 + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left(a_0 + a_1 x + a_2 x^2 \cdots\right),$$

and

$$
\begin{aligned}
u_1(x,t) =& -pI_t^\alpha\left[\sum_{n=0}^{\infty}a_nx^n\right]+I_t^\alpha\left[2x-2\right]+I_t^\alpha\left[\frac{\partial^2u_o(x,t)}{\partial x^2}-\frac{\partial u_0(x,t)}{\partial x}\right]\\
=& -p\frac{t^\alpha}{\Gamma(\alpha+1)}\left(a_0+a_1x+a_2x^2\cdots\right)+\frac{2xt^\alpha}{\Gamma(\alpha+1)}-\frac{2t^\alpha}{\Gamma(\alpha+1)}+\frac{2t^\alpha}{\Gamma(\alpha+1)}\\
&+\frac{2a_2t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{-2xt^\alpha}{\Gamma(\alpha+1)}-\frac{a_1t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{2a_2xt^{2\alpha}}{2\Gamma(\alpha+1)}.
\end{aligned}
$$

After eliminating similar terms and setting $p = 1$, we establish and compare the coefficients of x^n, yielding $a_0 = a_1 = a_2 = 0$ that eventually gives the exact solution from $u_0(x,t)$ as follows:

$$u(x,t) = t^2 + x^2,$$

confirming the trueness of the already reported actual solution of the IBVP.

Example 4.3. *Consider the time-fractional IBVP for the heat equation as follows [44]:*

$$D_t^\alpha u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0, \quad 0 < \alpha \leq 1, \tag{4.20}$$

subject to the initial and boundary constraints as follows:

$$u(x,0) = x^2; \quad u(0,t) = \frac{2t^\alpha}{\Gamma(\alpha+1)}, \quad u(1,t) = 1 + \frac{2t^\alpha}{\Gamma(\alpha+1)},$$

and admitting the following exact solution is

$$u(x,t) = x^2 + \frac{2t^\alpha}{\Gamma(\alpha+1)}.$$

Algorithm 1: Accordingly, solving (4.20) through this algorithm goes by applying the inverse operator in (3.2) on the governing equation, coupled with setting $u'(x,t) = c$ to obtain

$$u(x,t) = u(0,t) + cx + L^{-1}\left[D_t^\alpha u(x,t)\right]. \tag{4.21}$$

Equally, MADM yields from the latter equation as follows:

$$\sum_{k=0}^{\infty} u_n(x,t) = \frac{2t^\alpha}{\Gamma(\alpha+1)} + cx - pL^{-1}\left[\sum_{n=0}^{\infty}a_nx^n\right] + L^{-1}\left[\sum_{n=0}^{\infty}a_nx^n\right] + L^{-1}\left[D_t^\alpha u_{n-1}(x,t)\right]. \tag{4.22}$$

Thus, it obtains the following recurrent formula:

$$
\begin{aligned}
u_0(x,t) &= \frac{2t^\alpha}{\Gamma(\alpha+1)} + cx + L^{-1}\left[\sum_{n=0}^{\infty}a_nx^n\right],\\
u_1(x,t) &= -pL^{-1}\left[\sum_{n=0}^{\infty}a_nx^n\right] + L^{-1}\left[D_t^\alpha u_0(x,t)\right],
\end{aligned}
$$

which, when expressed explicitly by using L^{-1} and property (2.13), yields as follows:

$$\begin{aligned} u_0(x,t) &= \frac{2t^\alpha}{\Gamma(\alpha+1)} + cx + a_0\frac{x^2}{2} + a_1\frac{x^3}{6} + a_2\frac{x^4}{12} + \cdots, \\ u_1(x,t) &= -p\left(a_0\frac{x^2}{2} + a_1\frac{x^3}{6} + a_2\frac{x^4}{12}\right) + x^2 + \frac{t^\alpha}{\Gamma(\alpha+1)}\left(\frac{cx^3}{6} + a_0\frac{x^4}{24} + a_1\frac{x^5}{120} + a_2\frac{x^6}{360}\right). \end{aligned} \tag{4.23}$$

Hence, one obtains $a_0 = 2$ and $a_1 = a_2 = 0$, upon which the closed-form solution takes the following expression

$$u(x,t) = x^2 + \frac{2t^\alpha}{\Gamma(\alpha+1)},$$

which is after utilizing the second boundary datum to obtain $c = 0$; equally affirming the trueness of the reported exact solution.

Algorithm 2: The present inverse operator (3.8) when $a = 0, b = 1$ reveals from (4.20) as follows:

$$u(x,t) = u(0,t) + xu(1,t) - xu(0,t) + I\left[D_t^\alpha u(x,t)\right]. \tag{4.24}$$

Further, the implementing the MADM procedure gives

$$\sum_{r=0}^{\infty} u_n(x,t) = \frac{2t^\alpha}{\Gamma(\alpha+1)} + x + L^{-1}\left[D_t^\alpha u_{n-1}(x,t)\right] - pL^{-1}\sum_{n=0}^{\infty} a_n x^n + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right], \tag{4.25}$$

which subsequently gives the scheme in the following form

$$\begin{aligned} u_0(x,t) &= \frac{2t^\alpha}{\Gamma(\alpha+1)} + x + L^{-1}\left[\sum_{n=0}^{\infty} a_n x_n\right], \\ u_1(x,t) &= -pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}\left[D_t^\alpha u_0(x,t)\right], \end{aligned}$$

or equally

$$\begin{aligned} u_0(x,t) &= \frac{2t^\alpha}{\Gamma(\alpha+1)} + x + a_0\frac{x^2}{2} + a_1\frac{x^3}{6} + a_2\frac{x^4}{12} - x\left(\frac{a_0}{2} + \frac{a_1}{6} + \frac{a_2}{12}\right), \\ u_1(x,t) &= -p\left[a_0\frac{x^2}{2} + a_1\frac{x^3}{6} + a_2\frac{x^4}{12}\right] + x^2 + \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\left(\frac{x^3}{6} + \frac{a_0x^4}{24} + \frac{a_1x^5}{120} + \frac{a_2x^6}{360}\right) \\ &\quad - x\left[1 + \frac{t^{-x}}{\Gamma(1-\alpha)}\left(\frac{1}{6} + \frac{a_0}{24} + \frac{a_1}{120} + \frac{a_2}{360}\right)\right]. \end{aligned}$$

In view of that, one obtains the coefficients $a_0 = 2$ and $a_1 = a_2 = 0$ upon which the resultant solution takes the following expression:

$$u(x,t) = x^2 + \frac{2t^\alpha}{\Gamma(\alpha+1)},$$

serving as the established exact solution.

Algorithm 3: Applying the leading inverse operator in the algorithm together with the mentioned property in (2.10) to (4.20) reveals as follows:

$$u(x,t) = u(x,0) + I_t^{\alpha}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right]. \tag{4.26}$$

Next, the adopted MADM expresses the latter equation as follows:

$$\sum_{n=0}^{\infty} u_n(x,t) = x^2 + I_t^{\alpha}\left[\frac{\partial^2 u_{n-1}(x,t)}{\partial x^2}\right] - pI_t^{\alpha}\left[\sum_{n=0}^{\infty} a_n x^n\right] + I_t^{\alpha}\left[\sum_{n=0}^{\infty} a_n x^n\right], \tag{4.27}$$

upon which we iteratively consider

$$\begin{aligned} u_0(x,t) &= x^2 + I_t^{\alpha}\left[\sum_{n=0}^{\infty} a_n x^n\right], \\ u_1(x,t) &= -pI_t^{\alpha}\left[\sum_{n=0}^{\infty} a_n x^n\right] + I_t^{\alpha}\left[\frac{\partial^2 u_0(x,t)}{\partial x^2}\right], \end{aligned} \tag{4.28}$$

or equivalently after the action of the inversion operator as follows:

$$\begin{aligned} u_0(x,t) &= x^2 + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left(a_0 + a_1 x + a_2 x^2\right), \\ u_1(x,t) &= -p\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}\left(a_0 + a_1 x + a_2 x^2\right)\right] + \frac{2t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{2\alpha}a_2}{\Gamma(2\alpha+1)}. \end{aligned}$$

In the same manner, one gets $a_0 = 2$ and $a_1 = a_2 = 0$, yielding the resulting solution as follows:

$$u(x,t) = x^2 + \frac{2t^{\alpha}}{\Gamma(\alpha+1)},$$

which aligns with the established actual solution.

Example 4.4. *Consider the time-fractional IBVP for wave dispersion as follows [44]:*

$$D_t^{1+\beta}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < 1, \;\; t > 0, \quad 0 < \beta \le 1, \tag{4.29}$$

coupled with the following initial and boundary data

$$u(x,0) = x^2; \quad u(0,t) = \frac{2t^{1+\beta}}{\Gamma(\beta+2)}, \;\; u(1,t) = 1 + \frac{2t^{1+\beta}}{\Gamma(\alpha+1)},$$

that satisfies the following exact solution

$$u(x,t) = \frac{2t^{1+\beta}}{\Gamma(\beta+2)} + x^2.$$

Algorithm 1: To solve (4.29), one begins by applying the inversion operator L^{-1} in (3.2) on the main equation, and further letting $u'(x,t) = c$ to obtain

$$u(x,t) = \frac{2t^{1+\beta}}{\Gamma(\beta+2)} + cx + L^{-1}\left[D_t^{1+\beta}u(x,t)\right]. \tag{4.30}$$

Accordingly, the MADM renders the latter equation to the following

$$\sum_{n=0}^{\infty} u_n(x,t) = \frac{2t^{1+\beta}}{\Gamma(\beta+2)} + cx - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}\left[D_t^{1+\beta}u_{n-1}(x,t)\right], \tag{4.31}$$

such that the resulting scheme takes the following form

$$\begin{aligned} u_0(x,t) &= \frac{2t^{1+\beta}}{\Gamma(\beta+2)} + cx + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right], \\ u_1(x,t) &= -pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}\left[D_t^{1+\beta}u_0(x,t)\right], \end{aligned}$$

or equally after applying the inversion operator L^{-1} together with the property (2.13) mentioned the following:

$$\begin{aligned} u_0(x,t) &= \frac{2t^{1+\beta}}{\Gamma(\beta+2)} + cx + a_0\frac{x^2}{2} + a_1\frac{x^3}{6} + a_2\frac{x^4}{12} + \cdots, \\ u_1(x,t) &= -p\left(a_0\frac{x^2}{2} + a_1\frac{x^3}{6} + a_2\frac{x^4}{12}\right) + x^2 + \frac{t^{1+\beta}}{\Gamma(\beta+2)}\left(\frac{cx^3}{6} + a_0\frac{x^4}{24} + a_1\frac{x^5}{120} + a_2\frac{x^6}{360}\right). \end{aligned} \tag{4.32}$$

In the same way, one determines the related coefficients as $a_0 = 2$, and $a_1 = a_2 = 0$, and the constant $c = 0$ is determined from the second boundary data. Lastly, the net sum yields

$$u(x,t) = \frac{2t^{1+\beta}}{\Gamma(\beta+2)} + x^2,$$

which is the already stated exact solution of the model.

Algorithm 2: Applying the inverse operator (3.8) with $a = 0, b = 1$ on (4.29) gives

$$u(x,t) = u(0,t) + xu(1,t) - xu(0,t) + L^{-1}\left[D_t^{1+\beta}u(x,t)\right]. \tag{4.33}$$

Therefore, the latter equation is re-expressed through the MADM procedure as follows:

$$\sum_{n=0}^{\infty} u_n(x,t) = \frac{2t^{1+\beta}}{\Gamma(\beta+2)} + x + L^{-1}\left[D_t^{1+\beta}u_{n-1}(x,t)\right] - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right], \tag{4.34}$$

which then leads to the following scheme

$$\begin{aligned} u_0(x,t) &= \frac{2t^{1+\beta}}{\Gamma(\beta+2)} + x + L^{-1}\left[\sum_{n=0}^{\infty} a_n x_n\right], \\ u_1(x,t) &= -pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}\left[D_t^{1+\beta}u_0(x,t)\right]. \end{aligned}$$

In addition, the above expressions are then plainly expressed after making use of the related inversion operator L^{-1} as follows:

$$\begin{aligned}
u_0(x,t) &= \frac{2t^{1+\beta}}{\Gamma(\beta+2)} + x + a_0\frac{x^2}{2} + a_1\frac{x^3}{6} + a_2\frac{x^4}{12} - x\left(\frac{a_0}{2} + \frac{a_1}{6} + \frac{a_2}{12}\right),\\
u_1(x,t) &= -p\left[a_0\frac{x^2}{2} + a_1\frac{x^3}{6} + a_2\frac{x^4}{12}\right] + x^2 + \frac{t^{-1-\beta)}}{\Gamma(-\beta)}\left(\frac{x^3}{6} + \frac{a_0x^4}{24} + \frac{a_1x^5}{120} + \frac{a_2x^6}{360}\right)\\
&\quad - x\left[1 + \frac{t^{-1-\beta}}{\Gamma(-\beta)}\left(\frac{1}{6} + \frac{a_0}{24} + \frac{a_1}{120} + \frac{a_2}{360}\right)\right].
\end{aligned}$$

Consequently, with the determination of the related coefficients as $a_0 = 2$, and $a_1 = a_2 = 0$, one obtains the following final solution:

$$u(x,t) = \frac{2t^{1+\beta}}{\Gamma(\beta+2)} + x^2,$$

serving as the already known exact solution.

Algorithm 3: Applying the t-inverse differential operator (3.12) $L^{-1}(\cdot) = \frac{1}{\Gamma(\beta+1)}\int_0^x (x-s)^\beta(\cdot)\,ds$, together with the property in (2.10) to (4.29) gives

$$u(x,t) = u(x,0) + I_t^{1+\beta}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right], \tag{4.35}$$

upon which the MADM writes the earlier equation as follows:

$$\sum_{n=0}^{\infty} u_n(x,t) = x^2 + I_t^{1+\beta}\left[\frac{\partial^2 u_{n-1}(x,t)}{\partial x^2}\right] - pI_t^{1+\beta}\left[\sum_{n=0}^{\infty} a_n x^n\right] + I_t^{1+\beta}\left[\sum_{n=0}^{\infty} a_n x^n\right], \tag{4.36}$$

that recurrently yields

$$\begin{aligned}
u_0(x,t) &= x^2 + I_t^{1+\beta}\left[\sum_{n=0}^{\infty} a_n x^n\right],\\
u_1(x,t) &= -pI_t^{1+\beta}\left[\sum_{n=0}^{\infty} a_n x^n\right] + I_t^{1+\beta}\left[\frac{\partial^2 u_0(x,t)}{\partial x^2}\right],
\end{aligned} \tag{4.37}$$

or equally after expanding, deploying the inversion operator, the following

$$\begin{aligned}
u_0(x,t) &= x^2 + \frac{t^\alpha}{\Gamma(\beta+2)}\left(a_0 + a_1x + a_2x^2\right),\\
u_1(x,t) &= -p\left[\frac{t^{1+\beta}}{\Gamma(\beta+2)}\left(a_0 + a_1x + a_2x^2\right)\right] + \frac{2t^{1+\beta}}{\Gamma(\beta+2)} + \frac{2t^{1+\beta}a_2}{\Gamma(2\beta+3)}.
\end{aligned}$$

In the same fashion, one obtains the coefficient values as $a_0 = 2$, $a_1 = 0$, and $a_2 = 2$; leading to the resulting solution, as explained as follows:

$$u(x,t) = \frac{2t^{1+\beta}}{\Gamma(\beta+2)} + x^2,$$

re-affirming the trueness of the exact solution of the fractional IBVP.

Note: Based on the previous examples, it is evident that solving the problem by the t-differential operator, utilizing the initial condition(s), only simplifies the computations and thus accelerates obtaining the solution. Therefore, the following examples will focus exclusively on the application of Algorithm 3.

Example 4.5. *Consider space-fractional IBVP [44]:*

$$\frac{\partial u(x,t)}{\partial t} = D_x^{2\alpha} u(x,t); \quad 0 < x < 1, \ \ t > 0, \quad \frac{1}{2} < \alpha \le 1, \tag{4.38}$$

along with the initial and boundary data as follows:

$$u(x,0) = \frac{2x^{2\alpha}}{\Gamma(1+2\alpha)}; \qquad u(0,t) = 2t, \quad u(1,t) = 1 + \frac{2}{\Gamma(2\alpha+1)}.$$

In addition, the above fractional IBVP admits the following actual exact solution:

$$u(x,t) = \frac{2x^2}{\Gamma(2\alpha+1)} + 2t.$$

Algorithm 3: We equally start by taking the inverse operator L^{-1}, defined in (3.12) of the leading equation, to obtain

$$u(x,t) = u(x,0) + L^{-1}\left[D_x^{2\alpha}(x,t)\right]. \tag{4.39}$$

Then, the MADM and the mentioned property in (2.13) help to get from the latter equation the following

$$u_n(x,t) = \frac{2x^{2\alpha}}{\Gamma(1+2\alpha)} - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L_x^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}\left[D_x^{2\alpha} u_{n-1}(x,t)\right]. \tag{4.40}$$

Further, the resulting relevant iterates are obtained as follows:

$$u_0(x,t) = \frac{2x^{2\alpha}}{\Gamma(1+2\alpha)} + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right],$$

$$u_1(x,t) = -pL_x^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}\left[D_x^{2\alpha} u_0(x,t)\right],$$

or after expanding the inverse operator as follows:

$$u_0(x,t) = \frac{2x^{2\alpha}}{\Gamma(1+2\alpha)} + t\left[a_0 + a_1 x + a_2 x^2\right],$$

$$u_1(x,t) = -p\left[t\left(a_0 + a_1 x + a_2 x^2\right)\right] + 2t + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)}\left[\frac{t^2 a_0}{2} + \frac{t^2 x a_1}{2} + \frac{a_2 t^2 x^2}{2}\right].$$

Accordingly, one obtains the coefficients $a_0 = 2$, and $a_1 = a_2 = 0$ such that one obtains

$$u(x,t) = \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} + 2t,$$

which is the known exact expression of the model. Notably, upon assuming the full integer order by the fractional order, that is, when $\alpha = 1$, one eventually recovers the exact solution of the corresponding integer order heat equation as follows: $u(x,t) = x^2 + 2t$.

Example 4.6. *Consider the space–time fractional IBVP [44]:*

$$D_t^{\beta} u(x,t) = D_x^{2\alpha} u(x,t), \quad 0 < x < 1, \;\; t > 0, \quad \frac{1}{2} < \alpha \le 1, \quad 0 < \beta \le 1, \tag{4.41}$$

subject to

$$u(x,0) = \frac{2x^{2\alpha}}{\Gamma(1+2\alpha)}; \quad u(0,t) = \frac{2t^{2\beta}}{\Gamma(2\beta+1)}, \quad u(1,t) = \frac{2}{\Gamma(2\alpha+1)} + \frac{2t^{\beta}}{\Gamma(1+\beta)},$$

and satisfying the following exact solution

$$u(x,t) = \frac{2x^{2\alpha}}{\Gamma(2\alpha)} + \frac{2t^{\beta}}{\Gamma(1+\beta)}.$$

Algorithm 3: Applying the inverse operator $L^{-1}(.) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1}(.)ds$ on (4.41), one gets as follows:

$$u(x,t) = u(x,0) + I_t^{\beta}\left[D_x^{2\alpha} u(x,t)\right],$$

such that MADM expresses the later equation as follows:

$$\sum_{n=0}^{\infty} u_n(x,t) = \frac{2x^{2\alpha}}{\Gamma(1+2\alpha)} - pI_t^{\beta}\left[\sum_{n=0}^{\infty} a_n x^n\right] + I_t^{\beta}\left[\sum_{n=0}^{\infty} a_n x^n\right] + I_t^{\beta}\left[D_x^{2\alpha} u_{n-1}(x,t)\right]. \tag{4.42}$$

Equally, the relevant iterates are accordingly obtained as follows:

$$\begin{aligned} u_0(x,t) &= \frac{2x^{2\alpha}}{\Gamma(1+2\alpha)} + I_t^{\beta}\left[\sum_{n=0}^{\infty} a_n x^n\right] \\ &= \frac{2x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{\beta}}{\Gamma(\beta+1)}\left(a_0 + a_1 x + a_2 x^2\right), \end{aligned}$$

and

$$\begin{aligned} u_1(x,t) &= -pI_t^{\beta}\left[\sum_{n=0}^{\infty} a_n x^n\right] + I_t^{\beta}\left[D_x^{2\alpha} u_o(x,t)\right] \\ &= -p\frac{t^{\beta}}{\Gamma(\beta+1)}\left(a_0 + a_1 x + a_2 x^2\right) + \frac{2t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)}\left(\frac{a_0^{-2\alpha}}{\Gamma(1-2\alpha)} + \frac{a_1 x^{1-2\alpha}}{\Gamma(2-2\alpha)} + \frac{2a_2 x^{2-2\alpha}}{\Gamma(3-2\alpha)}\right). \end{aligned}$$

What is more, the method equally yields the related coefficients as follows: a_0, a_1 and a_2, which means that the exact solution of the IBVP is obtained as follows:

$$u(x,t) = \frac{2t^{\beta}}{\Gamma(1+\beta)} + \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)}.$$

Remarkably, considering $\alpha = \beta = 1$, the obtained fractional exact solution corresponds to that of an integer order heat equation as follows: $u(x,t) = x^2 + 2t$. Conversely, when $\alpha = 1$, and $\beta = 2$, the fractional solution reduces to $u(x,t) = x^2 + t^2$, which is the exact solution of the analogous integer order wave propagation equation.

Example 4.7. *Consider the space-time fractional IBVP as follows [44]:*

$$D_t^{2\beta}u(x,t) = D_x^{2\alpha}u(x,t), \quad 0 < x < 1, \; t > 0, \quad \frac{1}{2} < \alpha, \beta \leq 1, \tag{4.43}$$

subject to

$$u(x,0) = \frac{2x^{2\alpha}}{\Gamma(1+2\alpha)}; \quad u(0,t) = \frac{2t^{2\beta}}{\Gamma(2\beta+1)}, \quad u(1,t) = \frac{2}{\Gamma(2\alpha+1)} + \frac{2t^2\beta}{\Gamma(1+2\beta)},$$

and admitting the exact solution as follows

$$u(x,t) = \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2t^{2\beta}}{\Gamma(1+2\beta)}.$$

Algorithm 3: Accordingly, by taking the inverse operator I_t^{β}, defined as $L^{-1}(.) = \frac{1}{\Gamma(2\beta)}\int_0^x (x-s)^{2\beta-1}ds$ of (4.43), one obtains as follows

$$u(x,t) = u(x,0) + I_t^{2\beta}\left[D_x^{2\alpha}u(x,t)\right].$$

Next, the MADM gives

$$\sum_{n=0}^{\infty} u_n(x,t) = \frac{2x^{2\alpha}}{\Gamma(1+2\alpha)} - pI_t^{2\beta}\left[\sum_{n=0}^{\infty} a_n x^n\right] + I_t^{2\beta}\left[\sum_{n=0}^{\infty} a_n x^n\right] + I_t^{2\beta}\left[D_x^{2\alpha}u_{n-1}(x,t)\right], \tag{4.44}$$

such that the resulting scheme looks as follows:

$$\begin{aligned}
u_0(x,t) &= \frac{2x^{2\alpha}}{\Gamma(1+2\alpha)} + I_t^{2\beta}\left[\sum_{n=0}^{\infty} a_n x^n\right] \\
&= \frac{2x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)}\left(a_0 + a_1 x + a_2 x^2\right), \\
u_1(x,t) &= -pI_t^{2\beta}\left[\sum_{n=0}^{\infty} a_n x^n\right] + I_t^{2\beta}\left[D_x^{2\alpha}u_o(x,t)\right] \\
&= -p\frac{t^{2\beta}}{\Gamma(2\beta+1)}\left(a_0 + a_1 x + a_2 x^2\right) + \frac{2t^{2\beta}}{\Gamma(2\beta+1)} \\
&\quad + \frac{t^{2\beta}}{\Gamma(2\beta+1)}\left(\frac{a_0 x^{-2\alpha}}{\Gamma(1-2\alpha)} + \frac{a_1 x^{1-2\alpha}}{\Gamma(2-2\alpha)} + \frac{2a_2 x^{2-2\alpha}}{\Gamma(3-2\alpha)}\right).
\end{aligned}$$

In the like manner, we obtain the coefficients as $a_0 = 2$, and $a_1 = a_2 = 0$, such that the resulting exact solution is obtained as follows:

$$u(x,t) = \frac{2t^{2\beta}}{\Gamma(1+2\beta)} + \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)}.$$

Notably, one can observe from the above solution, after setting $\alpha = \beta = 1$ to obtain the solution of the corresponding integer order wave equation.

5. Conclusions

This study makes an original mathematical contribution by developing three modified algorithms based on the Adomian Decomposition Method (MADM) for solving FPDEs, particularly those involving initial and boundary conditions. The procedures for each algorithm were explained extensively. Two of the algorithms were designed using the x-differential operators for boundary conditions, while the third algorithm utilized the t-differential operator for initial conditions. Through practical examples, it was confirmed that all three algorithms provided accurate and reliable solutions. Moreover, the proposed methods were found to drastically reduce the number of iterations due to relying solely on u_0 and u_1, unlike the standard ADM, which typically requires numerous iterations. As a result, the computational workload is considerably reduced. For more complex calculations, Maple software can be used to perform the computations. It was also observed that the third algorithm, which relies on initial conditions only, significantly simplified the calculations. Furthermore, the proven effectiveness of the proposed algorithms suggests future applications for nonlinear FPDEs and systems of FPDEs, offering great potential for fields like engineering, physics, and biology. Certainly, the proposed algorithms could help in developing solutions for advanced problems in these areas.

Author contributions

M. Al-Mazmumy: Conceptualization; N. AL-Yazidi: Formal analysis, writing–original draft; N. AL-Yazidi, M. Al-Mazmumy, and M. Alsulami: Methodology, investigation, writing review & editing; N. AL-Yazidi and M. Al-Mazmumy: Software. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

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