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Research article

On r-fuzzy soft γ-open sets and fuzzy soft γ-continuous functions with some applications

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Abstract: In this paper, we defined and discussed a new class of fuzzy soft open (FS-open) sets, called r-fuzzy soft γ-open (r-FS-γ-open) sets in fuzzy soft topological spaces (FSTSs) based on fuzzy topologies in the sense of Šostak. The class of r-FS-γ-open sets is contained in the class of r-FS-β-open sets, and contains all r-FS-semi-open and r-FS-pre-open sets. However, we introduced the closure and interior operators with respect to the classes of r-FS-γ-closed and r-FS-γ-open sets, and studied some of their properties. Thereafter, we defined and studied some new FS-functions using r-FS-γ-open and r-FS-γ-closed sets, called FS-γ-continuous (respectively (resp. for short) FS-γ-irresolute, FS-γ-open, FS-γ-irresolute open, FS-γ-closed, and FS-γ-irresolute closed) functions. The relationships between these classes of functions were discussed with the help of some illustrative examples. We also explored and established the notions of FS-weakly (resp. FS-almost) γ-continuous functions, which are weaker forms of FS-γ-continuous functions. We showed that FS-γ-continuity $\Longrightarrow$ FS-almost γ-continuity $\Longrightarrow$ FS-weak γ-continuity, but the converse may not be true. After that, we presented some new types of FS-separation axioms, called r-FS-γ-regular and r-FS-γ-normal spaces using r-FS-γ-closed sets, and investigated some properties of them. Finally, we introduced a new type of FS-connectedness, called r-FS-γ-connected sets using r-FS-γ-closed sets.

Keywords: FS-topology; r-FS-γ-open set; FS-γ-closure operator; FS-γ-continuity; FS-weak γ-continuity; FS-γ-irresoluteness; r-FS-γ-normal space; r-FS-γ-connected set
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1. Introduction

The need for theories that cope with uncertainty emerges from daily experiences with complicated challenges requiring ambiguous facts. In 1999, the theory of soft sets (S-sets) was given by the

Russian mathematician Molodtsov [1] as a tool for modeling mathematical problems that deal with uncertainties. Molodtsov's [1] S-set is a contemporary mathematical approach to coping with these difficulties. Soft collection logic is founded on the parametrization principle, which argues that complex things must be seen from several perspectives, with each aspect providing only a partial and approximate representation of the full item. Also, Molodtsov [1] studied several applications of S-sets theory in solving different practical problems in medical science, economics, mathematics, engineering, etc. Thereafter, Maji et al. [2] focused on abstract research of S-set operators with applications in decision-making problems. Moreover, the concept of soft topological spaces (STSs) defined over an initial universe with a predetermined set of parameters was proposed by Shabir and Naz [3]; their work centered on the theoretical studies of STSs. Majumdar and Samanta [4] presented mappings on S-sets and their application in medical diagnosis. Kharal and Ahmed [5] brought up the view of soft mapping with properties; subsequently, soft continuity of soft mappings was instigated by Aygunoglu and Aygun [6]. Many works devoted to studying soft continuity and its characterizations can be found in the literature reviews provided [7–9]. Overall, many researchers have successfully generalized the theory of general topology to the soft setting; see [10–13].

The generalization of soft open sets (S-open sets) plays an effective role in a soft topology through their ability to improve on many results, or to open the door to explore and discuss several soft topological notions such as soft continuity [10, 11], soft separation and regularity axioms [12], soft connectedness [11, 13], etc. Moreover, the notions of S-α-open sets and S-β-open sets were defined and studied in STSs by the authors of [14–16]. Al-shami et al. [17] introduced and discussed the concepts of weakly S-β-open sets and weakly S-β-continuous functions. Furthermore, Kaur et al. [18] initiated a novel approach to discussing soft continuity. In addition, many researchers have contributed to the theory of S-sets in several fields such as topology and algebra, see [19, 20].

The concept of a fuzzy set (F-set) of a nonempty set Q is a mapping $\mu : Q \to I$ (where $I = [0, 1]$). This concept was first defined in 1965 by Zadeh [21]. The concept of fuzzy topological spaces (FTSs) was presented in 1968 by Chang [22]. Several authors have successfully generalized the theory of general topology to the fuzzy setting with crisp methods. According to Šostak [23], the notion of a fuzzy topology being a crisp subclass of the class of F-sets and fuzziness in the notion of openness of an F-set have not been considered, which seems to be a drawback in the process of fuzzification of a topological space. Therefore, Šostak [23] defined a novel definition of a fuzzy topology as the concept of openness of F-sets. It is an extension of a fuzzy topology introduced by Chang. Many researchers (see [24–27]) have redefined the same notion and studied FTSs being unaware of Šostak's work.

The notion of fuzzy soft sets (FS-sets) was first defined in 2001 by Maji et al. [28], which combines the S-set [1] and F-set [21]. The concept of FSTSs was introduced and many of its properties such as FS-continuity, FS-closure operators, FS-interior operators, and FS-subspaces were studied [29, 30] based on fuzzy topologies in the sense of Šostak [23]. Also, a novel approach to discussing FS-regularity axioms and FS-separation axioms using FS-sets was explored by Taha [31, 32]. The notions of r-FS-regularly-open sets, r-FS-pre-open sets, r-FS-semi-open sets, r-FS-α-open sets, and r-FS-β-open sets were introduced by the authors of [33–36]. Furthermore, Alshammari et al. [37] defined and investigated the concepts of r-FS-δ-open sets and FS-δ-continuous functions based on fuzzy topologies in the sense of Šostak [23]. Overall, Alshammari et al. [37] introduced and discussed the concepts of FS-weak (resp. FS-almost) continuity, which are weaker forms of FS-continuity [29].

We lay out the remainder of this paper as follows. Section 2 contains some basic definitions that help in understanding the obtained results. In Section 3, we display a new class of FS-open sets, called r-FS-γ-open sets in FSTSs based on fuzzy topologies in the sense of Šostak. The class of r-FS-γ-open sets is contained in the class of r-FS-β-open sets, and contains all r-FS-α-open, r-FS-semi-open, and r-FS-pre-open sets. Some properties of r-FS-γ-open sets along with their mutual relationships were specified with the help of some illustrative examples. Thereafter, we introduce the closure and interior operators with respect to the classes of r-FS-γ-closed and r-FS-γ-open sets, and study some of their properties. In Section 4, we explore and characterize some new FS-functions using r-FS-γ-open and r-FS-γ-closed sets, called FS-γ-continuous (resp. FS-γ-irresolute, FS-γ-open, FS-γ-irresolute open, FS-γ-closed, and FS-γ-irresolute closed) functions between FSTSs $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}^*_N)$. Also, the relationships between these classes of functions are discussed with the help of some illustrative examples. In Section 5, we define and discuss the notions of FS-weakly (resp. FS-almost) γ-continuous functions, which are weaker forms of FS-γ-continuous functions. We also show that FS-γ-continuity $\Longrightarrow$ FS-almost γ-continuity $\Longrightarrow$ FS-weak γ-continuity, but the converse may not be true. However, we present some new types of FS-separation axioms, called r-FS-γ-regular and r-FS-γ-normal spaces using r-FS-γ-closed sets, and investigate some properties of them. Moreover, we introduce a new type of FS-connectedness, called r-FS-γ-connected sets using r-FS-γ-closed sets. In the last section, we close this paper with conclusions and proposed future researches.

2. Preliminaries

In this study, nonempty sets will be denoted by Q, S, W, etc. Also, M is the family of each parameter for Q and $C \subseteq M$. Moreover, I^Q is the family of all F-sets on Q and for $u \in I$, $\underline{u}(q) = u$, for every $q \in Q$. The following notions will be used in the next sections.

Definition 2.1. [29, 38, 39] An FS-set t_C on Q is a function from M to I^Q, such that $t_C(m)$ is an F-set on Q, for every $m \in C$ and $t_C(m) = \underline{0}$, if $m \notin C$. On Q, $\widetilde{(Q, M)}$ is the family of all FS-sets.

Definition 2.2. [40] An FS-point m_{q_u} on Q is defined as follows:

$$m_{q_u}(k) = \begin{cases} q_u, & \text{if} \quad k = m, \\ \underline{0}, & \text{if} \quad k \in M - \{m\}, \end{cases}$$

where q_u is an F-point on Q. Moreover, we say that m_{q_u} belongs to t_C ($m_{q_u} \tilde{\in} t_C$) if $u \leq t_C(m)(q)$. On Q, $\widetilde{P_u(Q)}$ is the family of all FS-points.

Definition 2.3. [41] On Q, $m_{q_u} \in \widetilde{P_u(Q)}$ is called an S-quasi-coincident with $t_C \in \widetilde{(Q, M)}$ and is denoted by $m_{q_u} \nabla t_C$, if $u + t_C(m)(q) > 1$. An FS-set $t_C \in \widetilde{(Q, M)}$ is called an S-quasi-coincident with $h_D \in \widetilde{(Q, M)}$ and is denoted by $t_C \nabla h_D$, if there is $m \in M$ and $q \in Q$, such that $t_C(m)(q)+h_D(m)(q) > 1$. If t_C is not an S-quasi-coincident with h_D, $t_C \overline{\nabla} h_D$.

Definition 2.4. [29] A function $\mathcal{T} : M \longrightarrow I^{\widetilde{(Q,M)}}$ is called a fuzzy soft topology (FST) on Q if it satisfies the following statements, for every $m \in M$:

(i) $\mathcal{T}_m(\Phi) = \mathcal{T}_m(\widetilde{M}) = 1$.

(ii) $\mathcal{T}_m(t_C \sqcap h_D) \geq \mathcal{T}_m(t_C) \wedge \mathcal{T}_m(h_D)$, for every $t_C, h_D \in \widetilde{(Q, M)}$.

(iii) $\mathcal{T}_m(\bigsqcup_{i\in\Theta}(t_C)_i) \geq \bigwedge_{i\in\Theta} \mathcal{T}_m((t_C)_i)$, for every $(t_C)_i \in \widetilde{(Q, M)}, i \in \Theta$.

Thus, $(Q, \mathcal{T}_M)$ is called an FSTS based on fuzzy topologies in the sense of Šostak [23].

Definition 2.5. [30, 33] In an FSTS $(Q, \mathcal{T}_M)$, for each $h_C \in (\widetilde{Q, M})$, $m \in M$, and $r \in I_\circ$ (where $I_\circ = (0, 1]$), we define FS-operators $C_\mathcal{T}$ and $I_\mathcal{T} : M \times (\widetilde{Q, M}) \times I_\circ \to (\widetilde{Q, M})$ as follows:

$$C_\mathcal{T}(m, t_C, r) = \sqcap \{h_D \in (\widetilde{Q, M}) : t_C \sqsubseteq h_D,\ \mathcal{T}_m(h_D^c) \geq r\}.$$

$$I_\mathcal{T}(m, t_C, r) = \sqcup \{h_D \in (\widetilde{Q, M}) : h_D \sqsubseteq t_C,\ \mathcal{T}_m(h_D) \geq r\}.$$

Definition 2.6. [33–35] Let $(Q, \mathcal{T}_M)$ be an FSTS and $r \in I_\circ$. An FS-set t_C is said to be an r-FS-regularly-open (resp. r-FS-pre-open, r-FS-β-open, r-FS-semi-open, r-FS-α-open, and r-FS-open) set if $t_C = I_\mathcal{T}(m, C_\mathcal{T}(m, t_C, r), r)$ (resp. $t_C \sqsubseteq I_\mathcal{T}(m, C_\mathcal{T}(m, t_C, r), r)$, $t_C \sqsubseteq C_\mathcal{T}(m, I_\mathcal{T}(m, C_\mathcal{T}(m, t_C, r), r), r)$, $t_C \sqsubseteq C_\mathcal{T}(m, I_\mathcal{T}(m, t_C, r), r)$, $t_C \sqsubseteq I_\mathcal{T}(m, C_\mathcal{T}(m, I_\mathcal{T}(m, t_C, r), r), r)$, and $t_C \sqsubseteq I_\mathcal{T}(m, t_C, r)$) $\forall\ m \in M$.

Definition 2.7. [29, 30, 33] Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}^*_N)$ be FSTSs, $m \in M$, and $(n = \psi(m)) \in N$. An FS-function $\varphi_\psi : (\widetilde{Q, M}) \longrightarrow (\widetilde{S, N})$ is called

(i) FS-continuous if $\mathcal{T}_m(\varphi_\psi^{-1}(h_D)) \geq \mathcal{T}^*_n(h_D)$, for every $h_D \in (\widetilde{S, N})$, $m \in M$;

(ii) FS-open if $\mathcal{T}^*_n(\varphi_\psi(t_C)) \geq \mathcal{T}_m(t_C)$, for every $t_C \in (\widetilde{Q, M})$, $m \in M$;

(iii) FS-closed if $\mathcal{T}^*_n((\varphi_\psi(t_C))^c) \geq \mathcal{T}_m(t_C^c)$, for every $t_C \in (\widetilde{Q, M})$, $m \in M$.

Definition 2.8. [35, 36] Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}^*_N)$ be FSTSs. An FS-function $\varphi_\psi : (\widetilde{Q, M}) \longrightarrow (\widetilde{S, N})$ is called FS-α-continuous (resp. FS-semi-continuous, FS-pre-continuous, and FS-β-continuous) if $\varphi_\psi^{-1}(h_D)$ is an r-FS-α-open (resp. r-FS-semi-open, r-FS-pre-open, and r-FS-β-open) set, for every $h_D \in (\widetilde{S, N})$ with $\mathcal{T}^*_n(h_D) \geq r$, $m \in M$, and $(n = \psi(m)) \in N$.

The basic results and notations that we need in the sequel are found in previous studies [29, 30, 33, 35, 36].

3. On r-fuzzy soft γ-open sets

In this section, we introduce the notion of r-FS-γ-open sets in an FSTS. Some properties of r-FS-γ-open sets along with their mutual relationships are studied using some problems. The notions of FS-γ-closure operators and FS-γ-interior operators are defined and investigated.

Definition 3.1. Let $(Q, \mathcal{T}_M)$ be an FSTS and $r \in I_\circ$. An FS-set $t_C \in (\widetilde{Q, M})$ is said to be an

(i) r-FS-γ-open set if $t_C \sqsubseteq C_\mathcal{T}(m, I_\mathcal{T}(m, t_C, r), r) \sqcup I_\mathcal{T}(m, C_\mathcal{T}(m, t_C, r), r)$ for every $m \in M$;

(ii) r-FS-γ-closed set if $t_C \sqsupseteq C_\mathcal{T}(m, I_\mathcal{T}(m, t_C, r), r) \sqcap I_\mathcal{T}(m, C_\mathcal{T}(m, t_C, r), r)$ for every $m \in M$.

Remark 3.1. The complement of r-FS-γ-open sets (resp. r-FS-γ-closed sets) are r-FS-γ-closed sets (resp. r-FS-γ-open sets).

Proposition 3.1. Let $(Q, \mathcal{T}_M)$ be an FSTS and $r \in I_\circ$. Then

(i) each r-FS-semi-open set is an r-FS-γ-open set;

(ii) each r-FS-γ-open set is an r-FS-β-open set;

(iii) each r-FS-pre-open set is an r-FS-γ-open set.

Proof. (i) Let t_C be an r-FS-semi-open set. Then

$$t_C \sqsubseteq C_\mathcal{T}(m, I_\mathcal{T}(m, t_C, r), r)$$

$$\sqsubseteq C_{\mathcal{T}}(m, I_{\mathcal{T}}(m, t_C, r), r) \sqcup I_{\mathcal{T}}(m, t_C, r)$$
$$\sqsubseteq C_{\mathcal{T}}(m, I_{\mathcal{T}}(m, t_C, r), r) \sqcup I_{\mathcal{T}}(m, C_{\mathcal{T}}(m, t_C, r), r).$$

Thus, t_C is an r-FS-γ-open set.

(ii) Let t_C be an r-FS-γ-open set. Then

$$\begin{aligned} t_C &\sqsubseteq C_{\mathcal{T}}(m, I_{\mathcal{T}}(m, t_C, r), r) \sqcup I_{\mathcal{T}}(m, C_{\mathcal{T}}(m, t_C, r), r) \\ &\sqsubseteq C_{\mathcal{T}}(m, I_{\mathcal{T}}(m, C_{\mathcal{T}}(m, t_C, r), r), r) \sqcup I_{\mathcal{T}}(m, C_{\mathcal{T}}(m, t_C, r), r) \\ &\sqsubseteq C_{\mathcal{T}}(m, I_{\mathcal{T}}(m, C_{\mathcal{T}}(m, t_C, r), r), r). \end{aligned}$$

Thus, t_C is an r-FS-β-open set.

(iii) Let t_C be an r-FS-pre-open set. Then

$$\begin{aligned} t_C &\sqsubseteq I_{\mathcal{T}}(m, C_{\mathcal{T}}(m, t_C, r), r) \\ &\sqsubseteq I_{\mathcal{T}}(m, C_{\mathcal{T}}(m, t_C, r), r) \sqcup I_{\mathcal{T}}(m, t_C, r) \\ &\sqsubseteq I_{\mathcal{T}}(m, C_{\mathcal{T}}(m, t_C, r), r) \sqcup C_{\mathcal{T}}(m, I_{\mathcal{T}}(m, t_C, r), r). \end{aligned}$$

Thus, t_C is an r-FS-γ-open set.

Remark 3.2. From the previous discussions and definitions, we have the following diagram.

r-FS-open set
↓
r-FS-α-open set
↙ ↘
r-FS-pre-open set ↮ **r-FS-semi-open set**
↘ ↙
r-FS-γ-open set
↓
r-FS-β-open set

Remark 3.3. The converse of the above diagram fails as Examples 3.1–3.3 will show.

Example 3.1. Let $Q = \{q_1, q_2\}$, $M = \{m_1, m_2\}$, and define $h_M, f_M, t_M \in (\widetilde{Q, M})$ as follows: $h_M = \{(m_1, \{\frac{q_1}{0.4}, \frac{q_2}{0.3}\}), (m_2, \{\frac{q_1}{0.4}, \frac{q_2}{0.3}\})\}$, $f_M = \{(m_1, \{\frac{q_1}{0.2}, \frac{q_2}{0.6}\}), (m_2, \{\frac{q_1}{0.2}, \frac{q_2}{0.6}\})\}$, $t_M = \{(m_1, \{\frac{q_1}{0.5}, \frac{q_2}{0.7}\}), (m_2, \{\frac{q_1}{0.5}, \frac{q_2}{0.7}\})\}$. Define $\mathcal{T}_M : M \longrightarrow I^{(\widetilde{Q,M})}$ as follows:

$$\mathcal{T}_{m_1}(l_M) = \begin{cases} 1, & \text{if } l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{2}, & \text{if } l_M = h_M, \\ \frac{1}{2}, & \text{if } l_M = f_M, \\ \frac{2}{3}, & \text{if } l_M = h_M \sqcap f_M, \\ \frac{1}{2}, & \text{if } l_M = h_M \sqcup f_M, \\ 0, & \text{otherwise,} \end{cases} \qquad \mathcal{T}_{m_2}(l_M) = \begin{cases} 1, & \text{if } l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{4}, & \text{if } l_M = h_M, \\ \frac{1}{4}, & \text{if } l_M = f_M, \\ \frac{1}{2}, & \text{if } l_M = h_M \sqcap f_M, \\ \frac{1}{4}, & \text{if } l_M = h_M \sqcup f_M, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, t_M is a $\frac{1}{4}$-FS-γ-open set, but it is neither $\frac{1}{4}$-FS-pre-open nor $\frac{1}{4}$-FS-α-open.

Example 3.2. Let $Q = \{q_1, q_2\}$, $M = \{m_1, m_2\}$, and define $h_M, f_M, t_M \in (\widetilde{Q,M})$ as follows: $h_M = \{(m_1, \{\frac{q_1}{0.3}, \frac{q_2}{0.2}\}), (m_2, \{\frac{q_1}{0.3}, \frac{q_2}{0.2}\})\}$, $f_M = \{(m_1, \{\frac{q_1}{0.7}, \frac{q_2}{0.8}\}), (m_2, \{\frac{q_1}{0.7}, \frac{q_2}{0.8}\})\}$, $t_M = \{(m_1, \{\frac{q_1}{0.5}, \frac{q_2}{0.4}\}), (m_2, \{\frac{q_1}{0.5}, \frac{q_2}{0.4}\})\}$. Define $\mathcal{T}_M : M \longrightarrow I^{(\widetilde{Q,M})}$ as follows:

$$\mathcal{T}_{m_1}(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{2}, & \text{if} \quad l_M = h_M, \\ \frac{1}{4}, & \text{if} \quad l_M = f_M, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{T}_{m_2}(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{3}, & \text{if} \quad l_M = h_M, \\ \frac{1}{4}, & \text{if} \quad l_M = f_M, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, t_M is a $\frac{1}{4}$-FS-γ-open set, but it is not $\frac{1}{4}$-FS-semi-open.

Example 3.3. Let $Q = \{q_1, q_2\}$, $M = \{m_1, m_2\}$, and define $h_M, t_M \in (\widetilde{Q,M})$ as follows: $h_M = \{(m_1, \{\frac{q_1}{0.5}, \frac{q_2}{0.4}\}), (m_2, \{\frac{q_1}{0.5}, \frac{q_2}{0.4}\})\}$, $t_M = \{(m_1, \{\frac{q_1}{0.4}, \frac{q_2}{0.5}\}), (m_2, \{\frac{q_1}{0.4}, \frac{q_2}{0.5}\})\}$. Define $\mathcal{T}_M : M \longrightarrow I^{(\widetilde{Q,M})}$ as follows:

$$\mathcal{T}_{m_1}(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{3}, & \text{if} \quad l_M = h_M, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{T}_{m_2}(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{2}, & \text{if} \quad l_M = h_M, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, t_M is a $\frac{1}{3}$-FS-β-open set, but it is not $\frac{1}{3}$-FS-γ-open.

Corollary 3.1. Let t_C be an r-FS-γ-open set in an FSTS $(Q, \mathcal{T}_M)$, $m \in M$, and $r \in I_\circ$.

(i) If t_C is an r-FS-regularly-open set, then t_C is r-FS-semi-open.

(ii) If t_C is an r-FS-regularly-closed set, then t_C is r-FS-pre-open.

(iii) If $I_\mathcal{T}(m, t_C, r) = \Phi$, then t_C is r-FS-pre-open.

(iv) If $C_\mathcal{T}(m, t_C, r) = \Phi$, then t_C is r-FS-semi-open.

Proof. The proof follows by Definitions 2.6 and 3.1.

Corollary 3.2. Let t_C be an r-FS-γ-closed set in an FSTS $(Q, \mathcal{T}_M)$, $m \in M$, and $r \in I_\circ$.

(i) If t_C is an r-FS-regularly-open set, then t_C is r-FS-pre-closed.

(ii) If t_C is an r-FS-regularly-closed set, then t_C is r-FS-semi-closed.

(iii) If $I_\mathcal{T}(m, t_C, r) = \Phi$, then t_C is r-FS-semi-closed.

(iv) If $C_\mathcal{T}(m, t_C, r) = \Phi$, then t_C is r-FS-pre-closed.

Proof. The proof follows by Definitions 2.6 and 3.1.

Corollary 3.3. Let $(Q, \mathcal{T}_M)$ be an FSTS and $r \in I_\circ$. Then

(i) the union of r-FS-γ-open sets is r-FS-γ-open;

(ii) the intersection of r-FS-γ-closed sets is r-FS-γ-closed.

Proof. This is easily proved by Definition 3.1.

Definition 3.2. In an FSTS $(Q, \mathcal{T}_M)$, for each $t_C \in (\widetilde{Q,M})$, $m \in M$, and $r \in I_\circ$, we define an FS-γ-closure operator $\gamma C_\mathcal{T} : M \times (\widetilde{Q,M}) \times I_\circ \to (\widetilde{Q,M})$ as follows: $\gamma C_\mathcal{T}(m, t_C, r) = \sqcap \{h_D \in (\widetilde{Q,M}) : t_C \sqsubseteq h_D,\ h_D \text{ is } r\text{-FS-}\gamma\text{-closed}\}$.

Proposition 3.2. Let $(Q, \mathcal{T}_M)$ be an FSTS, $t_C \in (\widetilde{Q, M})$, $m \in M$, and $r \in I_\circ$. Then t_C is an r-FS-γ-closed set iff $\gamma C_\mathcal{T}(m, t_C, r) = t_C$.

Proof. The proof follows by Definition 3.2.

Theorem 3.1. In an FSTS $(Q, \mathcal{T}_M)$, for each $t_C, h_D \in (\widetilde{Q, M})$, $m \in M$, and $r \in I_\circ$, an FS-operator $\gamma C_\mathcal{T} : M \times (\widetilde{Q, M}) \times I_\circ \to (\widetilde{Q, M})$ satisfies the following properties.

(i) $\gamma C_\mathcal{T}(m, \Phi, r) = \Phi$.

(ii) $t_C \sqsubseteq \gamma C_\mathcal{T}(m, t_C, r) \sqsubseteq C_\mathcal{T}(m, t_C, r)$.

(iii) $\gamma C_\mathcal{T}(m, t_C, r) \sqsubseteq \gamma C_\mathcal{T}(m, h_D, r)$ if $t_C \sqsubseteq h_D$.

(iv) $\gamma C_\mathcal{T}(m, \gamma C_\mathcal{T}(m, t_C, r), r) = \gamma C_\mathcal{T}(m, t_C, r)$.

(v) $\gamma C_\mathcal{T}(m, t_C \sqcup h_D, r) \sqsupseteq \gamma C_\mathcal{T}(m, t_C, r) \sqcup \gamma C_\mathcal{T}(m, h_D, r)$.

(vi) $\gamma C_\mathcal{T}(m, C_\mathcal{T}(m, t_C, r), r) = C_\mathcal{T}(m, t_C, r)$.

Proof. (i)–(iii) are easily proved by Definition 3.2.

(iv) From (ii) and (iii), we have $\gamma C_\mathcal{T}(m, t_C, r) \sqsubseteq \gamma C_\mathcal{T}(m, \gamma C_\mathcal{T}(m, t_C, r), r)$.

Now, we show that $\gamma C_\mathcal{T}(m, t_C, r) \sqsupseteq \gamma C_\mathcal{T}(m, \gamma C_\mathcal{T}(m, t_C, r), r)$. If $\gamma C_\mathcal{T}(m, t_C, r)$ does not contain $\gamma C_\mathcal{T}(m, \gamma C_\mathcal{T}(m, t_C, r), r)$, then there is $q \in Q$ and $u \in (0, 1)$ such that

$$\gamma C_\mathcal{T}(m, t_C, r)(m)(q) < u < \gamma C_\mathcal{T}(m, \gamma C_\mathcal{T}(m, t_C, r), r)(m)(q). \qquad (K)$$

Since $\gamma C_\mathcal{T}(m, t_C, r)(m)(q) < u$, by Definition 3.2, there exists h_D that is r-FS-γ-closed and $t_C \sqsubseteq h_D$ such that $\gamma C_\mathcal{T}(m, t_C, r)(m)(q) \leq h_D(m)(q) < u$. Since $t_C \sqsubseteq h_D$, then $\gamma C_\mathcal{T}(m, t_C, r) \sqsubseteq h_D$. Again, by Definition 3.2, we have $\gamma C_\mathcal{T}(m, \gamma C_\mathcal{T}(m, t_C, r), r) \sqsubseteq h_D$.

Hence, $\gamma C_\mathcal{T}(m, \gamma C_\mathcal{T}(m, t_C, r), r)(m)(q) \leq h_D(m)(q) < u$, which is a contradiction for (K). Thus, $\gamma C_\mathcal{T}(m, t_C, r) \sqsupseteq \gamma C_\mathcal{T}(m, \gamma C_\mathcal{T}(m, t_C, r), r)$, so $\gamma C_\mathcal{T}(m, \gamma C_\mathcal{T}(m, t_C, r), r) = \gamma C_\mathcal{T}(m, t_C, r)$.

(v) Since $t_C \sqsubseteq t_C \sqcup h_D$ and $h_D \sqsubseteq t_C \sqcup h_D$, hence by (iii), $\gamma C_\mathcal{T}(m, t_C, r) \sqsubseteq \gamma C_\mathcal{T}(m, t_C \sqcup h_D, r)$ and $\gamma C_\mathcal{T}(m, h_D, r) \sqsubseteq \gamma C_\mathcal{T}(m, t_C \sqcup h_D, r)$. Thus, $\gamma C_\mathcal{T}(m, t_C \sqcup h_D, r) \sqsupseteq \gamma C_\mathcal{T}(m, t_C, r) \sqcup \gamma C_\mathcal{T}(m, h_D, r)$.

(vi) From Proposition 3.2 and the fact that $C_\mathcal{T}(m, t_C, r)$ is r-FS-γ-closed, then $\gamma C_\mathcal{T}(m, C_\mathcal{T}(m, t_C, r), r) = C_\mathcal{T}(m, t_C, r)$.

Definition 3.3. In an FSTS $(Q, \mathcal{T}_M)$, for each $t_C \in (\widetilde{Q, M})$, $m \in M$, and $r \in I_\circ$, we define an FS-γ-interior operator $\gamma I_\mathcal{T} : M \times (\widetilde{Q, M}) \times I_\circ \to (\widetilde{Q, M})$ as follows: $\gamma I_\mathcal{T}(m, t_C, r) = \sqcup \{h_D \in (\widetilde{Q, M}) : h_D \sqsubseteq t_C,\ h_D \text{ is } r\text{-FS-}\gamma\text{-open}\}$.

Proposition 3.3. Let $(Q, \mathcal{T}_M)$ be an FSTS, $t_C \in (\widetilde{Q, M})$, $m \in M$, and $r \in I_\circ$. Then t_C is an r-FS-γ-open set iff $\gamma I_\mathcal{T}(m, t_C, r) = t_C$.

Proof. The proof follows by Definition 3.3.

Theorem 3.2. In an FSTS $(Q, \mathcal{T}_M)$, for each $t_C, h_D \in (\widetilde{Q, M})$, $m \in M$, and $r \in I_\circ$, an FS-operator $\gamma I_\mathcal{T} : M \times (\widetilde{Q, M}) \times I_\circ \to (\widetilde{Q, M})$ satisfies the following properties.

(i) $\gamma I_\mathcal{T}(m, \widetilde{M}, r) = \widetilde{M}$.

(ii) $I_\mathcal{T}(m, t_C, r) \sqsubseteq \gamma I_\mathcal{T}(m, t_C, r) \sqsubseteq t_C$.

(iii) $\gamma I_\mathcal{T}(m, t_C, r) \sqsubseteq \gamma I_\mathcal{T}(m, h_D, r)$ if $t_C \sqsubseteq h_D$.

(iv) $\gamma I_\mathcal{T}(m, \gamma I_\mathcal{T}(m, t_C, r), r) = \gamma I_\mathcal{T}(m, t_C, r)$.

(v) $\gamma I_\mathcal{T}(m, t_C, r) \sqcap \gamma I_\mathcal{T}(m, h_D, r) \sqsupseteq \gamma I_\mathcal{T}(m, t_C \sqcap h_D, r)$.

Proof. The proof is similar to that of Theorem 3.1.

Proposition 3.4. Let $(Q, \mathcal{T}_M)$ be an FSTS, $t_C \in (\widetilde{Q, M})$, $m \in M$, and $r \in I_\circ$. Then

(i) $\gamma I_{\mathcal{T}}(m, t_C^c, r) = (\gamma C_{\mathcal{T}}(m, t_C, r))^c$;

(ii) $\gamma C_{\mathcal{T}}(m, t_C^c, r) = (\gamma I_{\mathcal{T}}(m, t_C, r))^c$.

Proof. (i) For each $t_C \in (\widetilde{Q, M})$ and $m \in M$, we have $\gamma I_{\mathcal{T}}(m, t_C^c, r) = \sqcup\{h_D \in (\widetilde{Q, M}) : h_D \sqsubseteq t_C^c,\ h_D$ is r-FS-γ-open$\} = [\sqcap\{h_D^c \in (\widetilde{Q, M}) : t_C \sqsubseteq h_D^c,\ h_D^c$ is r-FS-γ-closed$\}]^c = (\gamma C_\tau(m, t_C, r))^c$.

(ii) This is similar to that of (i).

4. On fuzzy soft γ-continuous functions

In this section, we introduce and study some new FS-functions using r-FS-γ-open sets and r-FS-γ-closed sets, called FS-γ-continuous (resp. FS-γ-irresolute, FS-γ-open, FS-γ-irresolute open, FS-γ-closed, and FS-γ-irresolute closed) functions between FSTSs $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}_N^*)$. However, the relationships between these classes of functions are discussed.

Definition 4.1. Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}_N^*)$ be FSTSs, $t_C \in (\widetilde{Q, M})$, $h_D \in (\widetilde{S, N})$, $m \in M$, $(n = \psi(m)) \in N$, and $r \in I_\circ$. An FS-function $\varphi_\psi : (\widetilde{Q, M}) \longrightarrow (\widetilde{S, N})$ is called

(i) FS-γ-continuous if $\varphi_\psi^{-1}(h_D)$ is an r-FS-γ-open set, for every h_D with $\mathcal{T}_n^*(h_D) \geq r$;

(ii) FS-γ-open if $\varphi_\psi(t_C)$ is an r-FS-γ-open set, for every t_C with $\mathcal{T}_m(t_C) \geq r$;

(iii) FS-γ-closed if $\varphi_\psi(t_C)$ is an r-FS-γ-closed set, for every t_C with $\mathcal{T}_m(t_C^c) \geq r$;

(iv) FS-γ-irresolute if $\varphi_\psi^{-1}(h_D)$ is an r-FS-γ-open set, for every r-FS-γ-open set h_D;

(v) FS-γ-irresolute open if $\varphi_\psi(t_C)$ is an r-FS-γ-open set, for every r-FS-γ-open set t_C;

(vi) FS-γ-irresolute closed if $\varphi_\psi(t_C)$ is an r-FS-γ-closed set, for every r-FS-γ-closed set t_C.

Remark 4.1. From the previous definitions, we have the following diagram.

FS-continuity

$\downarrow$

FS-α-continuity

$\swarrow \qquad \searrow$

FS-pre-continuity $\nleftrightarrow$ **FS-semi-continuity**

$\searrow \qquad \swarrow$

FS-γ-continuity

$\downarrow$

FS-β-continuity

Remark 4.2. The converse of the above diagram fails as Examples 4.1–4.3 will show.

Example 4.1. Let $Q = \{q_1, q_2\}$, $M = \{m_1, m_2\}$, and define $h_M, f_M, t_M \in (\widetilde{Q,M})$ as follows: $h_M = \{(m_1, \{\frac{q_1}{0.4}, \frac{q_2}{0.3}\}), (m_2, \{\frac{q_1}{0.4}, \frac{q_2}{0.3}\})\}$, $f_M = \{(m_1, \{\frac{q_1}{0.2}, \frac{q_2}{0.6}\}), (m_2, \{\frac{q_1}{0.2}, \frac{q_2}{0.6}\})\}$, $t_M = \{(m_1, \{\frac{q_1}{0.5}, \frac{q_2}{0.7}\}), (m_2, \{\frac{q_1}{0.5}, \frac{q_2}{0.7}\})\}$. Define $\mathcal{T}_M, \mathcal{T}^*_M : M \longrightarrow I^{(\widetilde{Q,M})}$ as follows: $\forall m \in M$,

$$\mathcal{T}_m(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{2}, & \text{if} \quad l_M = h_M, \\ \frac{1}{2}, & \text{if} \quad l_M = f_M, \\ \frac{2}{3}, & \text{if} \quad l_M = h_M \sqcap f_M, \\ \frac{1}{2}, & \text{if} \quad l_M = h_M \sqcup f_M, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{T}^*_m(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{2}, & \text{if} \quad l_M = t_M, \\ 0, & \text{otherwise}. \end{cases}$$

Then, the identity FS-function $\varphi_\psi : (Q, \mathcal{T}_M) \longrightarrow (Q, \mathcal{T}^*_M)$ is FS-γ-continuous, but it is neither FS-pre-continuous nor FS-α-continuous.

Example 4.2. Let $Q = \{q_1, q_2\}$, $M = \{m_1, m_2\}$, and define $h_M, f_M, t_M \in (\widetilde{Q,M})$ as follows: $h_M = \{(m_1, \{\frac{q_1}{0.3}, \frac{q_2}{0.2}\}), (m_2, \{\frac{q_1}{0.3}, \frac{q_2}{0.2}\})\}$, $f_M = \{(m_1, \{\frac{q_1}{0.7}, \frac{q_2}{0.8}\}), (m_2, \{\frac{q_1}{0.7}, \frac{q_2}{0.8}\})\}$, $t_M = \{(m_1, \{\frac{q_1}{0.5}, \frac{q_2}{0.4}\}), (m_2, \{\frac{q_1}{0.5}, \frac{q_2}{0.4}\})\}$. Define $\mathcal{T}_M, \mathcal{T}^*_M : M \longrightarrow I^{(\widetilde{Q,M})}$ as follows: $\forall m \in M$,

$$\mathcal{T}_m(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{2}, & \text{if} \quad l_M = h_M, \\ \frac{1}{4}, & \text{if} \quad l_M = f_M, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{T}^*_m(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{4}, & \text{if} \quad l_M = t_M, \\ 0, & \text{otherwise}. \end{cases}$$

Then, the identity FS-function $\varphi_\psi : (Q, \mathcal{T}_M) \longrightarrow (Q, \mathcal{T}^*_M)$ is FS-γ-continuous, but it is not FS-semi-continuous.

Example 4.3. Let $Q = \{q_1, q_2\}$, $M = \{m_1, m_2\}$, and define $h_M, t_M \in (\widetilde{Q,M})$ as follows: $h_M = \{(m_1, \{\frac{q_1}{0.5}, \frac{q_2}{0.4}\}), (m_2, \{\frac{q_1}{0.5}, \frac{q_2}{0.4}\})\}$, $t_M = \{(m_1, \{\frac{q_1}{0.4}, \frac{q_2}{0.5}\}), (m_2, \{\frac{q_1}{0.4}, \frac{q_2}{0.5}\})\}$. Define $\mathcal{T}_M, \mathcal{T}^*_M : M \longrightarrow I^{(\widetilde{Q,M})}$ as follows: $\forall m \in M$,

$$\mathcal{T}_m(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{2}, & \text{if} \quad l_M = h_M, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{T}^*_m(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{3}, & \text{if} \quad l_M = t_M, \\ 0, & \text{otherwise}. \end{cases}$$

Then, the identity FS-function $\varphi_\psi : (Q, \mathcal{T}_M) \longrightarrow (Q, \mathcal{T}^*_M)$ is FS-β-continuous, but it is not FS-γ-continuous.

Theorem 4.1. Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}^*_N)$ be FSTSs, $m \in M$, $(n = \psi(m)) \in N$, and $r \in I_\circ$. An FS-function $\varphi_\psi : (\widetilde{Q,M}) \longrightarrow (\widetilde{S,N})$ is FS-γ-continuous iff for any $m_{q_u} \in \widetilde{P_u(Q)}$ and any $h_D \in (\widetilde{S,N})$ with $\mathcal{T}^*_n(h_D) \geq r$ containing $\varphi_\psi(m_{q_u})$, there exists $t_C \in (\widetilde{Q,M})$ that is an r-FS-γ-open set containing m_{q_u} with $\varphi_\psi(t_C) \sqsubseteq h_D$.

Proof. ($\Rightarrow$) Let $m_{q_u} \in \widetilde{P_u(Q)}$ and $h_D \in (\widetilde{S,N})$ with $\mathcal{T}^*_n(h_D) \geq r$ containing $\varphi_\psi(m_{q_u})$, and then $\varphi_\psi^{-1}(h_D) \sqsubseteq \gamma I_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D), r)$. Since $m_{q_u} \tilde{\in} \varphi_\psi^{-1}(h_D)$, then we obtain $m_{q_u} \tilde{\in} \gamma I_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D), r) = t_C$ (say). Hence, $t_C \in (\widetilde{Q,M})$ is an r-FS-γ-open set containing m_{q_u} with $\varphi_\psi(t_C) \sqsubseteq h_D$.

($\Leftarrow$) Let $m_{q_u} \in \widetilde{P_u(Q)}$ and $h_D \in (\widetilde{S,N})$ with $\mathcal{T}^*_n(h_D) \geq r$ such that $m_{q_u} \tilde{\in} \varphi_\psi^{-1}(h_D)$. According to the assumption there exists $t_C \in (\widetilde{Q,M})$ that is an r-FS-γ-open set containing m_{q_u}, such that $\varphi_\psi(t_C) \sqsubseteq h_D$. Hence, $m_{q_u} \tilde{\in} t_C \sqsubseteq \varphi_\psi^{-1}(h_D)$ and $m_{q_u} \tilde{\in} \gamma I_\mathcal{T}(m, \varphi_\psi^{-1}(h_D), r)$. Thus, $\varphi_\psi^{-1}(h_D) \sqsubseteq \gamma I_\mathcal{T}(m, \varphi_\psi^{-1}(h_D), r)$, so $\varphi_\psi^{-1}(h_D)$ is an r-FS-γ-open set. Thus, φ_ψ is FS-γ-continuous.

Theorem 4.2. Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}^*_N)$ be FSTSs and $\varphi_\psi : (\widetilde{Q,M}) \longrightarrow (\widetilde{S,N})$ be an FS-function. Then the following statements are equivalent for every $t_C \in (\widetilde{Q,M})$, $h_D \in (\widetilde{S,N})$, $m \in M$, $(n = \psi(m)) \in N$, and $r \in I_\circ$:

(i) φ_ψ is FS-γ-continuous.

(ii) $\varphi_\psi^{-1}(h_D)$ is r-FS-γ-closed, for every $h_D \in (\widetilde{S,N})$ with $\mathcal{T}^*_n(h^c_D) \geq r$.

(iii) $\varphi_\psi(\gamma C_\mathcal{T}(m, t_C, r)) \sqsubseteq C_{\mathcal{T}^*}(n, \varphi_\psi(t_C), r)$.

(iv) $\gamma C_\mathcal{T}(m, \varphi_\psi^{-1}(h_D), r) \sqsubseteq \varphi_\psi^{-1}(C_{\mathcal{T}^*}(n, h_D, r))$.

(v) $\varphi_\psi^{-1}(I_{\mathcal{T}^*}(n, h_D, r)) \sqsubseteq \gamma I_\mathcal{T}(m, \varphi_\psi^{-1}(h_D), r)$.

Proof. (i) $\Leftrightarrow$ (ii) The proof follows from Definition 4.1 and $\varphi_\psi^{-1}(h^c_D) = (\varphi_\psi^{-1}(h_D))^c$.
(ii) $\Rightarrow$ (iii) Let $t_C \in (\widetilde{Q,M})$; then by (ii), $\varphi_\psi^{-1}(C_{\mathcal{T}^*}(n, \varphi_\psi(t_C), r))$ is r-FS-γ-closed, hence

$$\gamma C_\mathcal{T}(m, t_C, r) \sqsubseteq \gamma C_\mathcal{T}(m, \varphi_\psi^{-1}(\varphi_\psi(t_C)), r) \sqsubseteq \gamma C_\mathcal{T}(m, \varphi_\psi^{-1}(C_{\mathcal{T}^*}(n, \varphi_\psi(t_C), r)), r) = \varphi_\psi^{-1}(C_{\mathcal{T}^*}(n, \varphi_\psi(t_C), r)).$$

Thus, $\varphi_\psi(\gamma C_\mathcal{T}(m, t_C, r)) \sqsubseteq C_{\mathcal{T}^*}(n, \varphi_\psi(t_C), r)$.
(iii) $\Rightarrow$ (iv) Let $h_D \in (\widetilde{S,N})$; hence by (iii), $\varphi_\psi(\gamma C_\mathcal{T}(m, \varphi_\psi^{-1}(h_D), r)) \sqsubseteq C_{\mathcal{T}^*}(n, \varphi_\psi(\varphi_\psi^{-1}(h_D)), r) \sqsubseteq C_{\mathcal{T}^*}(n, h_D, r)$. Thus, $\gamma C_\mathcal{T}(m, \varphi_\psi^{-1}(h_D), r) \sqsubseteq \varphi_\psi^{-1}(\varphi_\psi(\gamma C_\mathcal{T}(m, \varphi_\psi^{-1}(h_D), r))) \sqsubseteq \varphi_\psi^{-1}(C_{\mathcal{T}^*}(n, h_D, r))$.
(iv) $\Leftrightarrow$ (v) The proof follows from Proposition 3.4 and $\varphi_\psi^{-1}(h^c_D) = (\varphi_\psi^{-1}(h_D))^c$.
(v) $\Rightarrow$ (i) Let $h_D \in (\widetilde{S,N})$ with $\mathcal{T}^*_n(h_D) \geq r$. By (v), we obtain $\varphi_\psi^{-1}(h_D) = \varphi_\psi^{-1}(I_{\mathcal{T}^*}(n, h_D, r)) \sqsubseteq \gamma I_\mathcal{T}(m, \varphi_\psi^{-1}(h_D), r) \sqsubseteq \varphi_\psi^{-1}(h_D)$. Then, $\gamma I_\mathcal{T}(m, \varphi_\psi^{-1}(h_D), r) = \varphi_\psi^{-1}(h_D)$. Thus, $\varphi_\psi^{-1}(h_D)$ is r-FS-γ-open, so φ_ψ is FS-γ-continuous.

Lemma 4.1. Every FS-γ-irresolute function is FS-γ-continuous.

Proof. The proof follows from Definition 4.1.

Remark 4.3. The converse of Lemma 4.1 fails as Example 4.4 will show.

Example 4.4. Let $Q = \{q_1, q_2\}$, $M = \{m_1, m_2\}$, and define $h_M, t_M \in (\widetilde{Q,M})$ as follows: $h_M = \{(m_1, \{\frac{q_1}{0.5}, \frac{q_2}{0.5}\}), (m_2, \{\frac{q_1}{0.5}, \frac{q_2}{0.5}\})\}$, $t_M = \{(m_1, \{\frac{q_1}{0.5}, \frac{q_2}{0.4}\}), (m_2, \{\frac{q_1}{0.5}, \frac{q_2}{0.4}\})\}$. Define $\mathcal{T}_M, \mathcal{T}^*_M : M \longrightarrow I^{(\widetilde{Q,M})}$ as follows: $\forall m \in M$,

$$\mathcal{T}_m(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{2}, & \text{if} \quad l_M = t_M, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{T}^*_m(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{3}, & \text{if} \quad l_M = h_M, \\ 0, & \text{otherwise}. \end{cases}$$

Then, the identity FS-function $\varphi_\psi : (Q, \mathcal{T}_M) \longrightarrow (Q, \mathcal{T}^*_M)$ is FS-γ-continuous, but it is not FS-γ-irresolute.

Theorem 4.3. Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}^*_N)$ be FSTSs and $\varphi_\psi : (\widetilde{Q,M}) \longrightarrow (\widetilde{S,N})$ be an FS-function. Then the following statements are equivalent for every $t_C \in (\widetilde{Q,M})$, $h_D \in (\widetilde{S,N})$, $m \in M$, $(n = \psi(m)) \in N$, and $r \in I_\circ$:

(i) φ_ψ is FS-γ-irresolute.
(ii) $\varphi_\psi^{-1}(h_D)$ is r-FS-γ-closed, for every r-FS-γ-closed set h_D.
(iii) $\varphi_\psi(\gamma C_{\mathcal{T}}(m, t_C, r)) \sqsubseteq \gamma C_{\mathcal{T}^*}(n, \varphi_\psi(t_C), r)$.
(iv) $\gamma C_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D), r) \sqsubseteq \varphi_\psi^{-1}(\gamma C_{\mathcal{T}^*}(n, h_D, r))$.
(v) $\varphi_\psi^{-1}(\gamma I_{\mathcal{T}^*}(n, h_D, r)) \sqsubseteq \gamma I_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D), r)$.

Proof. (i) ⇔ (ii) The proof follows from Definition 4.1 and $\varphi_\psi^{-1}(h_D^c) = (\varphi_\psi^{-1}(h_D))^c$.

(ii) ⇒ (iii) Let $t_C \in (\widetilde{Q, M})$; then by (ii), $\varphi_\psi^{-1}(\gamma C_{\mathcal{T}^*}(n, \varphi_\psi(t_C), r))$ is r-FS-γ-closed, hence

$$\gamma C_{\mathcal{T}}(m, t_C, r) \sqsubseteq \gamma C_{\mathcal{T}}(m, \varphi_\psi^{-1}(\varphi_\psi(t_C)), r) \sqsubseteq \gamma C_{\mathcal{T}}(m, \varphi_\psi^{-1}(\gamma C_{\mathcal{T}^*}(n, \varphi_\psi(t_C), r)), r) = \varphi_\psi^{-1}(\gamma C_{\mathcal{T}^*}(n, \varphi_\psi(t_C), r)).$$

Thus, $\varphi_\psi(\gamma C_{\mathcal{T}}(m, t_C, r)) \sqsubseteq \gamma C_{\mathcal{T}^*}(n, \varphi_\psi(t_C), r)$.

(iii) ⇒ (iv) Let $h_D \in (\widetilde{S, N})$; hence by (iii), $\varphi_\psi(\gamma C_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D), r)) \sqsubseteq \gamma C_{\mathcal{T}^*}(n, \varphi_\psi(\varphi_\psi^{-1}(h_D)), r) \sqsubseteq \gamma C_{\mathcal{T}^*}(n, h_D, r)$. Thus, $\gamma C_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D), r) \sqsubseteq \varphi_\psi^{-1}(\varphi_\psi(\gamma C_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D), r))) \sqsubseteq \varphi_\psi^{-1}(\gamma C_{\mathcal{T}^*}(n, h_D, r))$.

(iv) ⇔ (v) The proof follows from Proposition 3.4 and $\varphi_\psi^{-1}(h_D^c) = (\varphi_\psi^{-1}(h_D))^c$.

(v) ⇒ (i) Let h_D be an r-FS-γ-open set. By (v),

$$\varphi_\psi^{-1}(h_D) = \varphi_\psi^{-1}(\gamma I_{\mathcal{T}^*}(n, h_D, r)) \sqsubseteq \gamma I_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D), r) \sqsubseteq \varphi_\psi^{-1}(h_D).$$

Thus, $\gamma I_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D), r) = \varphi_\psi^{-1}(h_D)$. Therefore, $\varphi_\psi^{-1}(h_D)$ is r-FS-γ-open, so φ_ψ is FS-γ-irresolute.

Proposition 4.1. Let $(Q, \mathcal{T}_M)$, $(W, \mathfrak{I}_H)$, and $(S, \mathcal{T}_N^*)$ be FSTSs, and $\varphi_\psi : (\widetilde{Q, M}) \longrightarrow (\widetilde{W, H})$, $\varphi_{\psi^*}^* : (\widetilde{W, H}) \longrightarrow (\widetilde{S, N})$ be two FS-functions. Then the composition $\varphi_{\psi^*}^* \circ \varphi_\psi$ is FS-γ-continuous (resp. FS-γ-irresolute) if φ_ψ is FS-γ-irresolute and $\varphi_{\psi^*}^*$ is FS-γ-continuous (resp. FS-γ-irresolute).

Proof. The proof follows from Definition 4.1.

Lemma 4.2. (i) Every FS-γ-irresolute open function is FS-γ-open.
(ii) Every FS-γ-irresolute closed function is FS-γ-closed.

Proof. The proof follows from Definition 4.1.

Remark 4.4. The converse of Lemma 4.2 fails as Example 4.5 will show.

Example 4.5. Let $Q = \{q_1, q_2\}$, $M = \{m_1, m_2\}$, and define $h_M, t_M \in (\widetilde{Q, M})$ as follows: $h_M = \{(m_1, \{\frac{q_1}{0.5}, \frac{q_2}{0.5}\}), (m_2, \{\frac{q_1}{0.5}, \frac{q_2}{0.5}\})\}$, $t_M = \{(m_1, \{\frac{q_1}{0.5}, \frac{q_2}{0.4}\}), (m_2, \{\frac{q_1}{0.5}, \frac{q_2}{0.4}\})\}$. Define $\mathcal{T}_M, \mathcal{T}_M^* : M \longrightarrow I^{(\widetilde{Q,M})}$ as follows: $\forall m \in M$,

$$\mathcal{T}_m(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{2}, & \text{if} \quad l_M = h_M, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{T}_m^*(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{2}, & \text{if} \quad l_M = t_M, \\ 0, & \text{otherwise}. \end{cases}$$

Then, the identity FS-function $\varphi_\psi : (Q, \mathcal{T}_M) \longrightarrow (Q, \mathcal{T}_M^*)$ is FS-γ-open, but it is not FS-γ-irresolute open.

Theorem 4.4. Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}_N^*)$ be FSTSs and $\varphi_\psi : (\widetilde{Q, M}) \longrightarrow (\widetilde{S, N})$ be an FS-function. Then the following statements are equivalent for every $t_C \in (\widetilde{Q, M})$, $h_D \in (\widetilde{S, N})$, $m \in M$, and $(n = \psi(m)) \in N$:

(i) φ_ψ is FS-γ-open.

(ii) $\varphi_\psi(I_{\mathcal{T}}(m, t_C, r)) \sqsubseteq \gamma I_{\mathcal{T}^*}(n, \varphi_\psi(t_C), r)$.

(iii) $I_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D), r) \sqsubseteq \varphi_\psi^{-1}(\gamma I_{\mathcal{T}^*}(n, h_D, r))$.

(iv) For every h_D and every t_C with $\mathcal{T}_m(t_C^c) \geq r$ and $\varphi_\psi^{-1}(h_D) \sqsubseteq t_C$, there exists $g_B \in \widetilde{(S, N)}$ that is r-FS-γ-closed with $h_D \sqsubseteq g_B$ such that $\varphi_\psi^{-1}(g_B) \sqsubseteq t_C$.

Proof. (i) $\Rightarrow$ (ii) Since $\varphi_\psi(I_{\mathcal{T}}(m, t_C, r)) \sqsubseteq \varphi_\psi(t_C)$, hence by (i), $\varphi_\psi(I_{\mathcal{T}}(m, t_C, r))$ is r-FS-γ-open. Then, $\varphi_\psi(I_{\mathcal{T}}(m, t_C, r)) \sqsubseteq \gamma I_{\mathcal{T}^*}(n, \varphi_\psi(t_C), r)$.

(ii) $\Rightarrow$ (iii) Set $t_C = \varphi_\psi^{-1}(h_D)$ and hence by (ii), $\varphi_\psi(I_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D), r)) \sqsubseteq \gamma I_{\mathcal{T}^*}(n, \varphi_\psi(\varphi_\psi^{-1}(h_D)), r) \sqsubseteq \gamma I_{\mathcal{T}^*}(n, h_D, r)$. Then, $I_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D), r) \sqsubseteq \varphi_\psi^{-1}(\gamma I_{\mathcal{T}^*}(n, h_D, r))$.

(iii) $\Rightarrow$ (iv) Let $h_D \in \widetilde{(S, N)}$ and $t_C \in \widetilde{(Q, M)}$ with $\mathcal{T}_m(t_C^c) \geq r$ such that $\varphi_\psi^{-1}(h_D) \sqsubseteq t_C$. Since $t_C^c \sqsubseteq \varphi_\psi^{-1}(h_D^c)$, $t_C^c = I_{\mathcal{T}}(m, t_C^c, r) \sqsubseteq I_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D^c), r)$. Hence by (iii), $t_C^c \sqsubseteq I_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D^c), r) \sqsubseteq \varphi_\psi^{-1}(\gamma I_{\mathcal{T}^*}(n, h_D^c, r))$. Thus, we have $t_C \sqsupseteq (\varphi_\psi^{-1}(\gamma I_{\mathcal{T}^*}(n, h_D^c, r)))^c = \varphi_\psi^{-1}(\gamma C_{\mathcal{T}^*}(n, h_D, r))$. Then, there exists $\gamma C_{\mathcal{T}^*}(n, h_D, r) \in \widetilde{(S, N)}$ that is r-FS-γ-closed such that $h_D \sqsubseteq \gamma C_{\mathcal{T}^*}(n, h_D, r)$ and

$$\varphi_\psi^{-1}(\gamma C_{\mathcal{T}^*}(n, h_D, r)) \sqsubseteq t_C.$$

(iv) $\Rightarrow$ (i) Let $f_A \in \widetilde{(Q, M)}$ with $\mathcal{T}_m(f_A) \geq r$. Set $h_D = (\varphi_\psi(f_A))^c$ and $t_C = f_A^c$, $\varphi_\psi^{-1}(h_D) = \varphi_\psi^{-1}((\varphi_\psi(f_A))^c) \sqsubseteq t_C$. Hence by (iv), there exists $g_B \in \widetilde{(S, N)}$ that is r-FS-γ-closed with $h_D \sqsubseteq g_B$ such that $\varphi_\psi^{-1}(g_B) \sqsubseteq t_C = f_A^c$. Thus, $\varphi_\psi(f_A) \sqsubseteq \varphi_\psi(\varphi_\psi^{-1}(g_B^c)) \sqsubseteq g_B^c$. On the other hand, since $h_D \sqsubseteq g_B$, $\varphi_\psi(f_A) = h_D^c \sqsupseteq g_B^c$. Hence, $\varphi_\psi(f_A) = g_B^c$ and $\varphi_\psi(f_A)$ is an r-FS-γ-open set. This shows that φ_ψ is an FS-γ-open function.

Theorem 4.5. Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}_N^*)$ be FSTSs and $\varphi_\psi : \widetilde{(Q, M)} \longrightarrow \widetilde{(S, N)}$ be an FS-function. Then the following statements are equivalent for every $t_C \in \widetilde{(Q, M)}$, $h_D \in \widetilde{(S, N)}$, $m \in M$, and $(n = \psi(m)) \in N$:

(i) φ_ψ is FS-γ-closed.

(ii) $\gamma C_{\mathcal{T}^*}(n, \varphi_\psi(t_C), r) \sqsubseteq \varphi_\psi(C_{\mathcal{T}}(m, t_C, r))$.

(iii) $\varphi_\psi^{-1}(\gamma C_{\mathcal{T}^*}(n, h_D, r)) \sqsubseteq C_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D), r)$.

(iv) For every h_D and every t_C with $\mathcal{T}_m(t_C) \geq r$ and $\varphi_\psi^{-1}(h_D) \sqsubseteq t_C$, there exists $g_B \in \widetilde{(S, N)}$ that is r-FS-γ-open with $h_D \sqsubseteq g_B$ such that $\varphi_\psi^{-1}(g_B) \sqsubseteq t_C$.

Proof. The proof is similar to that of Theorem 4.4.

Theorem 4.6. Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}_N^*)$ be FSTSs and $\varphi_\psi : \widetilde{(Q, M)} \longrightarrow \widetilde{(S, N)}$ be an FS-function. Then the following statements are equivalent for every $t_C \in \widetilde{(Q, M)}$, $h_D \in \widetilde{(S, N)}$, $m \in M$, and $(n = \psi(m)) \in N$:

(i) φ_ψ is FS-γ-irresolute open.

(ii) $\varphi_\psi(\gamma I_{\mathcal{T}}(m, t_C, r)) \sqsubseteq \gamma I_{\mathcal{T}^*}(n, \varphi_\psi(t_C), r)$.

(iii) $\gamma I_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D), r) \sqsubseteq \varphi_\psi^{-1}(\gamma I_{\mathcal{T}^*}(n, h_D, r))$.

(iv) For every h_D and every r-FS-γ-closed set t_C with $\varphi_\psi^{-1}(h_D) \sqsubseteq t_C$, there exists $g_B \in \widetilde{(S, N)}$ that is r-FS-γ-closed with $h_D \sqsubseteq g_B$ such that $\varphi_\psi^{-1}(g_B) \sqsubseteq t_C$.

Proof. The proof is similar to that of Theorem 4.4.

Theorem 4.7. Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}_N^*)$ be FSTSs and $\varphi_\psi : \widetilde{(Q, M)} \longrightarrow \widetilde{(S, N)}$ be an FS-function. Then the following statements are equivalent for every $t_C \in \widetilde{(Q, M)}$, $h_D \in \widetilde{(S, N)}$, $m \in M$, and $(n = \psi(m)) \in N$:

(i) φ_ψ is FS-γ-irresolute closed.

(ii) $\gamma C_{\mathcal{T}^*}(n, \varphi_\psi(t_C), r) \sqsubseteq \varphi_\psi(\gamma C_{\mathcal{T}}(m, t_C, r))$.

(iii) $\varphi_\psi^{-1}(\gamma C_{\mathcal{T}^*}(n, h_D, r)) \sqsubseteq \gamma C_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D), r)$.

(iv) For every h_D and every r-FS-γ-open set t_C with $\varphi_\psi^{-1}(h_D) \sqsubseteq t_C$, there exists $g_B \in (\widetilde{S, N})$ that is r-FS-γ-open with $h_D \sqsubseteq g_B$ such that $\varphi_\psi^{-1}(g_B) \sqsubseteq t_C$.

Proof. The proof is similar to that of Theorem 4.4.

Proposition 4.2. Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}_N^*)$ be FSTSs, and $\varphi_\psi : (\widetilde{Q, M}) \longrightarrow (\widetilde{S, N})$ be a bijective FS-function. Then φ_ψ is FS-γ-irresolute open iff φ_ψ is FS-γ-irresolute closed.

Proof. The proof follows from:
$\varphi_\psi^{-1}(\gamma C_{\mathcal{T}^*}(n, h_D, r)) \sqsubseteq \gamma C_{\mathcal{T}}(n, \varphi_\psi^{-1}(h_D), r) \Longleftrightarrow \varphi_\psi^{-1}(\gamma I_{\mathcal{T}^*}(n, h_D^c, r)) \sqsupseteq \gamma I_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D^c), r)$.

Definition 4.2. Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}_N^*)$ be FSTSs. A bijective FS-function $\varphi_\psi : (\widetilde{Q, M}) \longrightarrow (\widetilde{S, N})$ is called an FS-γ-irresolute homeomorphism if φ_ψ and φ_ψ^{-1} are FS-γ-irresolute.

The proof of the following corollary is easy and so is omitted.

Corollary 4.1. Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}_N^*)$ be FSTSs, and $\varphi_\psi : (\widetilde{Q, M}) \longrightarrow (\widetilde{S, N})$ be an FS-function and bijective. Then the following statements are equivalent for every $t_C \in (\widetilde{Q, M})$, $h_D \in (\widetilde{S, N})$, $m \in M$, $(n = \psi(m)) \in N$, and $r \in I_\circ$:

(i) φ_ψ is an FS-γ-irresolute homeomorphism.
(ii) φ_ψ is FS-γ-irresolute closed and FS-γ-irresolute.
(iii) φ_ψ is FS-γ-irresolute open and FS-γ-irresolute.
(iv) $\varphi_\psi(\gamma I_{\mathcal{T}}(m, t_C, r)) = \gamma I_{\mathcal{T}^*}(n, \varphi_\psi(t_C), r)$.
(v) $\varphi_\psi(\gamma C_{\mathcal{T}}(m, t_C, r)) = \gamma C_{\mathcal{T}^*}(n, \varphi_\psi(t_C), r)$.
(vi) $\gamma I_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D), r) = \varphi_\psi^{-1}(\gamma I_{\mathcal{T}^*}(n, h_D, r))$.
(vii) $\gamma C_{\mathcal{T}}(m, \varphi_\psi^{-1}(h_D), r) = \varphi_\psi^{-1}(\gamma C_{\mathcal{T}^*}(n, h_D, r))$.

5. Some applications

In this section, the notions of FS-weak γ-continuity and FS-almost γ-continuity, which are weaker forms of FS-γ-continuity and are introduced and investigated between FSTSs. Furthermore, we defined and discussed new types of FS-separation axioms, called r-FS-γ-regular spaces and FS-γ-normal spaces using r-FS-γ-closed sets. In addition, the notion of r-FS-γ-connected sets is defined and studied.

- **Fuzzy soft weak and almost γ-continuity:**

Definition 5.1. Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}_N^*)$ be FSTSs, $m \in M$, $(n = \psi(m)) \in N$, and $r \in I_\circ$. An FS-function $\varphi_\psi : (\widetilde{Q, M}) \longrightarrow (\widetilde{S, N})$ is called FS-weakly γ-continuous if

$$\varphi_\psi^{-1}(h_D) \sqsubseteq \gamma I_{\mathcal{T}}(m, \varphi_\psi^{-1}(C_{\mathcal{T}^*}(n, h_D, r)), r),$$

for each $h_D \in (\widetilde{S, N})$ with $\mathcal{T}_n^*(h_D) \geq r$.

Lemma 5.1. Every FS-γ-continuity is an FS-weak γ-continuity.

Proof. The proof follows from Definitions 4.1 and 5.1.

Remark 5.1. The converse of Lemma 5.1 fails as Example 5.1 will show.

Example 5.1. Let $Q = \{q_1, q_2, q_3\}$, $M = \{m_1, m_2\}$, and define $h_M, f_M, t_M \in \widetilde{(Q, M)}$ as follows: $h_M = \{(m_1, \{\frac{q_1}{0.4}, \frac{q_2}{0.2}, \frac{q_3}{0.4}\}), (m_2, \{\frac{q_1}{0.4}, \frac{q_2}{0.2}, \frac{q_3}{0.4}\})\}$, $f_M = \{(m_1, \{\frac{q_1}{0.5}, \frac{q_2}{0.5}, \frac{q_3}{0.4}\}), (m_2, \{\frac{q_1}{0.5}, \frac{q_2}{0.5}, \frac{q_3}{0.4}\})\}$, $t_M = \{(m_1, \{\frac{q_1}{0.3}, \frac{q_2}{0.2}, \frac{q_3}{0.6}\}), (m_2, \{\frac{q_1}{0.3}, \frac{q_2}{0.2}, \frac{q_3}{0.6}\})\}$. Define $\mathcal{T}_M, \mathcal{T}^*_M : M \longrightarrow I^{\widetilde{(Q,M)}}$ as follows: $\forall m \in M$,

$$\mathcal{T}_m(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{2}, & \text{if} \quad l_M = h_M, \\ \frac{1}{3}, & \text{if} \quad l_M = f_M, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{T}^*_m(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{4}, & \text{if} \quad l_M = t_M, \\ 0, & \text{otherwise}. \end{cases}$$

Then, the identity FS-function $\varphi_\psi : (Q, \mathcal{T}_M) \longrightarrow (Q, \mathcal{T}^*_M)$ is FS-weakly γ-continuous, but it is not FS-γ-continuous.

Theorem 5.1. Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}^*_N)$ be FSTSs, $m \in M$, $(n = \psi(m)) \in N$, and $r \in I_\circ$. An FS-function $\varphi_\psi : \widetilde{(Q, M)} \longrightarrow \widetilde{(S, N)}$ is FS-weakly γ-continuous iff for any $m_{q_u} \in \widetilde{P_u(Q)}$ and any $h_D \in \widetilde{(S, N)}$ with $\mathcal{T}^*_n(h_D) \geq r$ containing $\varphi_\psi(m_{q_u})$, there exists $t_C \in \widetilde{(Q, M)}$ that is an r-FS-γ-open set containing m_{q_u} with $\varphi_\psi(t_C) \sqsubseteq C_{\mathcal{T}^*}(n, h_D, r)$.

Proof. ($\Rightarrow$) Let $m_{q_u} \in \widetilde{P_u(Q)}$ and $h_D \in \widetilde{(S, N)}$ with $\mathcal{T}^*_n(h_D) \geq r$ containing $\varphi_\psi(m_{q_u})$, and then $\varphi^{-1}_\psi(h_D) \sqsubseteq \gamma I_{\mathcal{T}}(m, \varphi^{-1}_\psi(C_{\mathcal{T}^*}(n, h_D, r)), r)$. Since $m_{q_u} \tilde{\in} \varphi^{-1}_\psi(h_D)$, then $m_{q_u} \tilde{\in} \gamma I_{\mathcal{T}}(m, \varphi^{-1}_\psi(C_{\mathcal{T}^*}(n, h_D, r)), r) = t_C$ (say). Hence, $t_C \in \widetilde{(Q, M)}$ is an r-FS-γ-open set containing m_{q_u} with $\varphi_\psi(t_C) \sqsubseteq C_{\mathcal{T}^*}(n, h_D, r)$.

($\Leftarrow$) Let $m_{q_u} \in \widetilde{P_u(Q)}$ and $h_D \in \widetilde{(S, N)}$ with $\mathcal{T}^*_n(h_D) \geq r$ such that $m_{q_u} \tilde{\in} \varphi^{-1}_\psi(h_D)$. According to the assumption there exists $t_C \in \widetilde{(Q, M)}$ that is an r-FS-γ-open set containing m_{q_u} with $\varphi_\psi(t_C) \sqsubseteq C_{\mathcal{T}^*}(n, h_D, r)$. Hence, $m_{q_u} \tilde{\in} t_C \sqsubseteq \varphi^{-1}_\psi(C_{\mathcal{T}^*}(n, h_D, r))$ and $m_{q_u} \tilde{\in} \gamma I_{\mathcal{T}}(m, \varphi^{-1}_\psi(C_{\mathcal{T}^*}(n, h_D, r)), r)$. Thus, $\varphi^{-1}_\psi(h_D) \sqsubseteq \gamma I_{\mathcal{T}}(m, \varphi^{-1}_\psi(C_{\mathcal{T}^*}(n, h_D, r)), r)$, so φ_ψ is FS-weakly γ-continuous.

Theorem 5.2. Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}^*_N)$ be FSTSs and $\varphi_\psi : \widetilde{(Q, M)} \longrightarrow \widetilde{(S, N)}$ be an FS-function. Then the following statements are equivalent for every $h_D \in \widetilde{(S, N)}$, $m \in M$, and $(n = \psi(m)) \in N$:

(i) φ_ψ is FS-weakly γ-continuous.

(ii) $\varphi^{-1}_\psi(h_D) \sqsupseteq \gamma C_{\mathcal{T}}(m, \varphi^{-1}_\psi(I_{\mathcal{T}^*}(n, h_D, r)), r)$, if $\mathcal{T}^*_n(h^c_D) \geq r$.

(iii) $\gamma I_{\mathcal{T}}(m, \varphi^{-1}_\psi(C_{\mathcal{T}^*}(n, h_D, r)), r) \sqsupseteq \varphi^{-1}_\psi(I_{\mathcal{T}^*}(n, h_D, r))$.

(iv) $\gamma C_{\mathcal{T}}(m, \varphi^{-1}_\psi(I_{\mathcal{T}^*}(n, h_D, r)), r) \sqsubseteq \varphi^{-1}_\psi(C_{\mathcal{T}^*}(n, h_D, r))$.

Proof. (i) $\Leftrightarrow$ (ii) The proof follows from Definition 5.1, Proposition 3.4, and $\varphi^{-1}_\psi(h^c_D) = (\varphi^{-1}_\psi(h_D))^c$.

(ii) $\Rightarrow$ (iii) Let $h_D \in \widetilde{(S, N)}$. Hence, by (ii),

$$\gamma C_{\mathcal{T}}(m, \varphi^{-1}_\psi(I_{\mathcal{T}^*}(n, C_{\mathcal{T}^*}(n, h^c_D, r), r)), r) \sqsubseteq \varphi^{-1}_\psi(C_{\mathcal{T}^*}(n, h^c_D, r)).$$

Thus, $\varphi^{-1}_\psi(I_{\mathcal{T}^*}(n, h_D, r)) \sqsubseteq \gamma I_{\mathcal{T}}(m, \varphi^{-1}_\psi(C_{\mathcal{T}^*}(n, h_D, r)), r)$.

(iii) $\Leftrightarrow$ (iv) The proof follows from Proposition 3.4 and $\varphi^{-1}_\psi(h^c_D) = (\varphi^{-1}_\psi(h_D))^c$.

(iv) $\Rightarrow$ (i) Let $h_D \in \widetilde{(S, N)}$ with $\mathcal{T}^*_n(h_D) \geq r$. Hence, by (iv), $\gamma C_{\mathcal{T}}(m, \varphi^{-1}_\psi(I_{\mathcal{T}^*}(n, h^c_D, r)), r) \sqsubseteq \varphi^{-1}_\psi(C_{\mathcal{T}^*}(n, h^c_D, r)) = \varphi^{-1}_\psi(h^c_D)$. Thus, $\varphi^{-1}_\psi(h_D) \sqsubseteq \gamma I_{\mathcal{T}}(m, \varphi^{-1}_\psi(C_{\mathcal{T}^*}(n, h_D, r)), r)$, so φ_ψ is FS-weakly γ-continuous.

Definition 5.2. Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}^*_N)$ be FSTSs, $m \in M$, $(n = \psi(m)) \in N$, and $r \in I_\circ$. An FS-function $\varphi_\psi : \widetilde{(Q, M)} \longrightarrow \widetilde{(S, N)}$ is called FS-almost γ-continuous if

$$\varphi^{-1}_\psi(h_D) \sqsubseteq \gamma I_{\mathcal{T}}(m, \varphi^{-1}_\psi(I_{\mathcal{T}^*}(n, C_{\mathcal{T}^*}(n, h_D, r), r)), r),$$

for each $h_D \in \widetilde{(S,N)}$ with $\mathcal{T}^*_n(h_D) \geq r$.

Lemma 5.2. Every FS-almost γ-continuity is an FS-weak γ-continuity.

Proof. The proof follows from Definitions 5.1 and 5.2.

Remark 5.2. The converse of Lemma 5.2 fails as Example 5.2 will show.

Example 5.2. Let $Q = \{q_1, q_2, q_3\}$, $M = \{m_1, m_2\}$, and define $h_M, f_M, t_M \in \widetilde{(Q,M)}$ as follows: $h_M = \{(m_1, \{\frac{q_1}{0.6}, \frac{q_2}{0.2}, \frac{q_3}{0.4}\}), (m_2, \{\frac{q_1}{0.6}, \frac{q_2}{0.2}, \frac{q_3}{0.4}\})\}$, $f_M = \{(m_1, \{\frac{q_1}{0.3}, \frac{q_2}{0.2}, \frac{q_3}{0.5}\}), (m_2, \{\frac{q_1}{0.3}, \frac{q_2}{0.2}, \frac{q_3}{0.5}\})\}$, $t_M = \{(m_1, \{\frac{q_1}{0.3}, \frac{q_2}{0.2}, \frac{q_3}{0.4}\}), (m_2, \{\frac{q_1}{0.3}, \frac{q_2}{0.2}, \frac{q_3}{0.4}\})\}$. Define $\mathcal{T}_M, \mathcal{T}^*_M : M \longrightarrow I^{\widetilde{(Q,M)}}$ as follows: $\forall m \in M$,

$$\mathcal{T}_m(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{2}, & \text{if} \quad l_M = h_M, \\ \frac{1}{3}, & \text{if} \quad l_M = t_M, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{T}^*_m(l_M) = \begin{cases} 1, & \text{if} \quad l_M \in \{\Phi, \widetilde{M}\}, \\ \frac{1}{3}, & \text{if} \quad l_M = f_M, \\ 0, & \text{otherwise}. \end{cases}$$

Then, the identity FS-function $\varphi_\psi : (Q, \mathcal{T}_M) \longrightarrow (Q, \mathcal{T}^*_M)$ is FS-weakly γ-continuous, but it is not FS-almost γ-continuous.

Remark 5.3. From the previous discussions and definitions, we have the following diagram.

FS-γ-continuity

$\downarrow$

FS-almost γ-continuity

$\downarrow$

FS-weak γ-continuity

Theorem 5.3. Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}^*_N)$ be FSTSs, $m \in M$, $(n = \psi(m)) \in N$, and $r \in I_\circ$. An FS-function $\varphi_\psi : \widetilde{(Q,M)} \longrightarrow \widetilde{(S,N)}$ is FS-almost γ-continuous iff for any $m_{q_u} \in \widetilde{P_u(Q)}$ and any $h_D \in \widetilde{(S,N)}$ with $\mathcal{T}^*_n(h_D) \geq r$ containing $\varphi_\psi(m_{q_u})$, there exists $t_C \in \widetilde{(Q,M)}$ that is an r-FS-γ-open set containing m_{q_u} with $\varphi_\psi(t_C) \sqsubseteq I_{\mathcal{T}^*}(n, C_{\mathcal{T}^*}(n, h_D, r), r)$.

Proof. The proof is similar to that of Theorem 5.1. □

Theorem 5.4. Let $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}^*_N)$ be FSTSs and $\varphi_\psi : \widetilde{(Q,M)} \longrightarrow \widetilde{(S,N)}$ be an FS-function. Then the following statements are equivalent for every $h_D \in \widetilde{(S,N)}$, $m \in M$, and $(n = \psi(m)) \in N$:

(i) φ_ψ is FS-almost γ-continuous.
(ii) $\varphi^{-1}_\psi(h_D)$ is r-FS-γ-open, for every r-FS-regularly-open set h_D.
(iii) $\varphi^{-1}_\psi(h_D)$ is r-FS-γ-closed, for every r-FS-regularly-closed set h_D.
(iv) $\gamma C_{\mathcal{T}}(m, \varphi^{-1}_\psi(h_D), r) \sqsubseteq \varphi^{-1}_\psi(C_{\mathcal{T}^*}(n, h_D, r))$, for every r-FS-γ-open set h_D.
(v) $\gamma C_{\mathcal{T}}(m, \varphi^{-1}_\psi(h_D), r) \sqsubseteq \varphi^{-1}_\psi(C_{\mathcal{T}^*}(n, h_D, r))$, for every r-FS-semi-open set h_D.

Proof. (i) ⇒ (ii) Let $m_{q_u} \in \widetilde{P_u(Q)}$ and h_D be an r-FS-regularly-open set such that $m_{q_u} \tilde{\in} \varphi_{\psi}^{-1}(h_D)$. Hence, by Theorem 5.3, there exists $t_C \in \widetilde{(Q, M)}$ that is r-FS-γ-open with $m_{q_u} \tilde{\in} t_C$ and $\varphi_{\psi}(t_C) \sqsubseteq I_{\mathcal{T}^*}(n, C_{\mathcal{T}^*}(n, h_D, r), r)$. Thus, $t_C \sqsubseteq \varphi_{\psi}^{-1}(I_{\mathcal{T}^*}(n, C_{\mathcal{T}^*}(n, h_D, r), r)) = \varphi_{\psi}^{-1}(h_D)$ and $m_{q_u} \tilde{\in} \gamma I_{\mathcal{T}}(n, \varphi_{\psi}^{-1}(h_D), r)$. Then, $\varphi_{\psi}^{-1}(h_D) \sqsubseteq \gamma I_{\mathcal{T}}(n, \varphi_{\psi}^{-1}(h_D), r)$, so $\varphi_{\psi}^{-1}(h_D)$ is r-FS-γ-open.
(ii) ⇒ (iii) Let h_D be r-FS-regularly-closed. Then, by (ii), $\varphi_{\psi}^{-1}(h_D^c) = (\varphi_{\psi}^{-1}(h_D))^c$ is r-FS-γ-open, hence $\varphi_{\psi}^{-1}(h_D)$ is an r-FS-γ-closed set.
(iii) ⇒ (iv) Let h_D be r-FS-γ-open and since $C_{\mathcal{T}^*}(n, h_D, r)$ is r-FS-regularly-closed, then by (iii), $\varphi_{\psi}^{-1}(C_{\mathcal{T}^*}(n, h_D, r))$ is r-FS-γ-closed. Since $\varphi_{\psi}^{-1}(h_D) \sqsubseteq \varphi_{\psi}^{-1}(C_{\mathcal{T}^*}(n, h_D, r))$, hence we have

$$\gamma C_{\mathcal{T}}(m, \varphi_{\psi}^{-1}(h_D), r) \sqsubseteq \varphi_{\psi}^{-1}(C_{\mathcal{T}^*}(n, h_D, r)).$$

(iv) ⇒ (v) The proof follows from the fact that any r-FS-semi-open set is an r-FS-γ-open set.
(v) ⇒ (iii) Let h_D be r-FS-regularly-closed, and then h_D is r-FS-semi-open, hence by (v), $\gamma C_{\mathcal{T}}(m, \varphi_{\psi}^{-1}(h_D), r) \sqsubseteq \varphi_{\psi}^{-1}(C_{\mathcal{T}^*}(n, h_D, r)) = \varphi_{\psi}^{-1}(h_D)$. Thus, $\varphi_{\psi}^{-1}(h_D)$ is an r-FS-γ-closed set.
(iii) ⇒ (i) Let $m_{q_u} \in \widetilde{P_u(Q)}$ and $h_D \in \widetilde{(S, N)}$ with $\mathcal{T}_n^*(h_D) \geq r$ such that $m_{q_u} \tilde{\in} \varphi_{\psi}^{-1}(h_D)$, and then we have $m_{q_u} \tilde{\in} \varphi_{\psi}^{-1}(I_{\mathcal{T}^*}(n, C_{\mathcal{T}^*}(n, h_D, r), r))$. Since $[I_{\mathcal{T}^*}(n, C_{\mathcal{T}^*}(n, h_D, r), r)]^c$ is r-FS-regularly-closed, by (iii), $\varphi_{\psi}^{-1}([I_{\mathcal{T}^*}(n, C_{\mathcal{T}^*}(n, h_D, r), r)]^c)$ is r-FS-γ-closed. Hence, $\varphi_{\psi}^{-1}(I_{\mathcal{T}^*}(n, C_{\mathcal{T}^*}(n, h_D, r), r))$ is r-FS-γ-open and $m_{q_u} \tilde{\in} \gamma I_{\mathcal{T}}(m, \varphi_{\psi}^{-1}(I_{\mathcal{T}^*}(n, C_{\mathcal{T}^*}(n, h_D, r), r)), r)$. So,

$$\varphi_{\psi}^{-1}(h_D) \sqsubseteq \gamma I_{\mathcal{T}}(m, \varphi_{\psi}^{-1}(I_{\mathcal{T}^*}(n, C_{\mathcal{T}^*}(n, h_D, r), r)), r).$$

Hence, φ_{ψ} is FS-almost γ-continuous.

Proposition 5.1. Let $(Q, \mathcal{T}_M)$, $(W, \Im_H)$, and $(S, \mathcal{T}_N^*)$ be FSTSs, and $\varphi_{\psi} : \widetilde{(Q, M)} \longrightarrow \widetilde{(W, H)}$, $\varphi_{\psi^*}^* : \widetilde{(W, H)} \longrightarrow \widetilde{(S, N)}$ be two FS-functions. Then the composition $\varphi_{\psi^*}^* \circ \varphi_{\psi}$ is FS-almost γ-continuous if φ_{ψ} is FS-γ-continuous (resp. FS-γ-irresolute) and $\varphi_{\psi^*}^*$ is FS-continuous (resp. FS-almost γ-continuous).

Proof. The proof follows from the previous definitions.

- ***r*-fuzzy soft γ-regular and γ-normal spaces:**

Definition 5.3. Let $t_C, h_D \in \widetilde{(Q, M)}$, $m_{q_u} \in \widetilde{P_u(Q)}$, and $r \in I_\circ$. An FSTS $(Q, \mathcal{T}_M)$ is called an (i) r-FS-γ-regular space iff $m_{q_u} \overline{\nabla}\, t_C$ for each r-FS-γ-closed set t_C, there is $g_{B_j} \in \widetilde{(Q, M)}$ with $\mathcal{T}(g_{B_j}) \geq r$ for $j \in \{1, 2\}$, such that $m_{q_u} \tilde{\in} g_{B_1}$, $t_C \sqsubseteq g_{B_2}$, and $g_{B_1} \overline{\nabla}\, g_{B_2}$;
(ii) r-FS-γ-normal space iff $t_C \overline{\nabla}\, h_D$ for each r-FS-γ-closed sets t_C and h_D, there is $g_{B_j} \in \widetilde{(Q, M)}$ with $\mathcal{T}(g_{B_j}) \geq r$ for $j \in \{1, 2\}$, such that $t_C \sqsubseteq g_{B_1}$, $h_D \sqsubseteq g_{B_2}$, and $g_{B_1} \overline{\nabla}\, g_{B_2}$.

Theorem 5.5. Let $(Q, \mathcal{T}_M)$ be an FSTS, $m_{q_u} \in \widetilde{P_u(Q)}$, $t_C, h_D \in \widetilde{(Q, M)}$, and $r \in I_\circ$. The following statements are equivalent.

(i) $(Q, \mathcal{T}_M)$ is an r-FS-γ-regular space.

(ii) If $m_{q_u} \tilde{\in}\, t_C$ for each r-FS-γ-open set t_C, there is h_D with $\mathcal{T}(h_D) \geq r$, such that $m_{q_u} \tilde{\in}\, h_D \sqsubseteq C_{\mathcal{T}}(m, h_D, r) \sqsubseteq t_C$.

(iii) If $m_{q_u} \overline{\nabla}\, t_C$ for each r-FS-γ-closed set t_C, there is $g_{B_j} \in \widetilde{(Q, M)}$ with $\mathcal{T}(g_{B_j}) \geq r$ for $j \in \{1, 2\}$, such that $m_{q_u} \tilde{\in}\, g_{B_1}$, $t_C \sqsubseteq g_{B_2}$, and $C_{\mathcal{T}}(m, g_{B_1}, r) \overline{\nabla}\, C_{\mathcal{T}}(m, g_{B_2}, r)$.

Proof. (i) ⇒ (ii) Let $m_{q_u} \tilde{\in} t_C$ for each r-FS-γ-open set t_C, then $m_{q_u} \overline{\nabla}\, t_C^c$. Since $(Q, \mathcal{T}_M)$ is r-FS-γ-regular, there is $h_D, g_B \in \widetilde{(Q, M)}$ with $\mathcal{T}(h_D) \geq r$ and $\mathcal{T}(g_B) \geq r$, such that $m_{q_u} \tilde{\in} h_D$, $t_C^c \sqsubseteq g_B$, and $h_D \overline{\nabla}\, g_B$. Thus, $m_{q_u} \tilde{\in} h_D \sqsubseteq g_B^c \sqsubseteq t_C$, so $m_{q_u} \tilde{\in} h_D \sqsubseteq C_{\mathcal{T}}(m, h_D, r) \sqsubseteq t_C$.
(ii) ⇒ (iii) Let $m_{q_u} \overline{\nabla}\, t_C$ for each r-FS-γ-closed set t_C, then $m_{q_u} \tilde{\in} t_C^c$. By (ii), there is h_D with $\mathcal{T}(h_D) \geq r$, such that $m_{q_u} \tilde{\in} h_D \sqsubseteq C_{\mathcal{T}}(m, h_D, r) \sqsubseteq t_C^c$. Since $\mathcal{T}(h_D) \geq r$, then h_D is an r-FS-γ-open set and $m_{q_u} \tilde{\in} h_D$. Again, by (ii), there is g_B with $\mathcal{T}(g_B) \geq r$ such that $m_{q_u} \tilde{\in} g_B \sqsubseteq C_{\mathcal{T}}(m, g_B, r) \sqsubseteq h_D \sqsubseteq C_{\mathcal{T}}(m, h_D, r) \sqsubseteq t_C^c$. It implies $t_C \sqsubseteq (C_{\mathcal{T}}(m, h_D, r))^c = I_{\mathcal{T}}(m, h_D^c, r) \sqsubseteq h_D^c$. Set $f_A = I_{\mathcal{T}}(m, h_D^c, r)$, and then $\mathcal{T}(f_A) \geq r$. So, $C_{\mathcal{T}}(m, f_A, r) \sqsubseteq h_D^c \sqsubseteq (C_{\mathcal{T}}(m, g_B, r))^c$, that is, $C_{\mathcal{T}}(m, f_A, r) \overline{\nabla}\, C_{\mathcal{T}}(m, g_B, r)$.
(iii) ⇒ (i) The proof is obvious.

Theorem 5.6. Let $(Q, \mathcal{T}_M)$ be an FSTS, $f_A, t_C, h_D, \in \widetilde{(Q, M)}$, and $r \in I_\circ$. The following statements are equivalent.

(i) $(Q, \mathcal{T}_M)$ is an r-FS-γ-normal space.

(ii) If $f_A \sqsubseteq t_C$ for each r-FS-γ-closed set f_A and r-FS-γ-open set t_C, there is h_D with $\mathcal{T}(h_D) \geq r$, such that $f_A \sqsubseteq h_D \sqsubseteq C_{\mathcal{T}}(m, h_D, r) \sqsubseteq t_C$.

(iii) If $f_A \overline{\nabla}\, t_C$ for each r-FS-γ-closed sets f_A and t_C, there is $g_{B_j} \in \widetilde{(Q, M)}$ with $\mathcal{T}(g_{B_j}) \geq r$ for $j \in \{1, 2\}$, such that $f_A \sqsubseteq g_{B_1}$, $t_C \sqsubseteq g_{B_2}$, and $C_{\mathcal{T}}(m, g_{B_1}, r) \overline{\nabla}\, C_{\mathcal{T}}(m, g_{B_2}, r)$.

Proof. The proof is similar to that of Theorem 5.5.

Theorem 5.7. Let $\varphi_\psi : \widetilde{(Q, M)} \longrightarrow \widetilde{(S, N)}$ be a bijective FS-γ-irresolute and FS-open function. If $(Q, \mathcal{T}_M)$ is an r-FS-γ-regular (resp. r-FS-γ-normal) space, then $(S, \mathcal{T}_N^*)$ is an r-FS-γ-regular (resp. r-FS-γ-normal) space.

Proof. Let $n_{s_u} \overline{\nabla}\, t_C$ for each r-FS-γ-closed set $t_C \in \widetilde{(S, N)}$ and FS-γ-irresolute function φ_ψ, then $\varphi_\psi^{-1}(t_C)$ is an r-FS-γ-closed set. Set $n_{s_u} = \varphi_\psi(m_{q_u})$, and then $m_{q_u} \overline{\nabla}\, \varphi_\psi^{-1}(t_C)$. Since $(Q, \mathcal{T}_M)$ is an r-FS-γ-regular space, there is $g_{B_j} \in \widetilde{(Q, M)}$ with $\mathcal{T}(g_{B_j}) \geq r$ for $j \in \{1, 2\}$, such that $m_{q_u} \tilde{\in} g_{B_1}$, $\varphi_\psi^{-1}(t_C) \sqsubseteq g_{B_2}$, and $g_{B_1} \overline{\nabla}\, g_{B_2}$. Since φ_ψ is an FS-open and bijective function, $n_{s_u} \tilde{\in} \varphi_\psi(g_{B_1})$, $t_C = \varphi_\psi(\varphi_\psi^{-1}(t_C)) \sqsubseteq \varphi_\psi(g_{B_2})$, and $\varphi_\psi(g_{B_1}) \overline{\nabla}\, \varphi_\psi(g_{B_2})$. Hence, $(S, \mathcal{T}_N^*)$ is an r-FS-γ-regular space. The other case also follows similar lines.

Theorem 5.8. Let $\varphi_\psi : \widetilde{(Q, M)} \longrightarrow \widetilde{(S, N)}$ be an injective FS-continuous and FS-γ-irresolute closed function. If $(S, \mathcal{T}_N^*)$ is an r-FS-γ-regular (resp. r-FS-γ-normal) space, then $(Q, \mathcal{T}_M)$ is an r-FS-γ-regular (resp. r-FS-γ-normal) space.

Proof. Let $m_{q_u} \overline{\nabla}\, t_C$ for each r-FS-γ-closed set $t_C \in \widetilde{(Q, M)}$ and injective FS-γ-irresolute closed function φ_ψ, and then $\varphi_\psi(t_C)$ is an r-FS-γ-closed set and $\varphi_\psi(m_{q_u}) \overline{\nabla}\, \varphi_\psi(t_C)$. Since $(S, \mathcal{T}_N^*)$ is an r-FS-γ-regular space, there is $g_{B_j} \in \widetilde{(S, N)}$ with $\mathcal{T}^*(g_{B_j}) \geq r$ for $j \in \{1, 2\}$, such that $\varphi_\psi(m_{q_u}) \tilde{\in} g_{B_1}$, $\varphi_\psi(t_C) \sqsubseteq g_{B_2}$, and $g_{B_1} \overline{\nabla}\, g_{B_2}$. Since φ_ψ is an FS-continuous function, we have $m_{q_u} \tilde{\in} \varphi_\psi^{-1}(g_{B_1})$, $t_C \sqsubseteq \varphi_\psi^{-1}(g_{B_2})$ with $\mathcal{T}(\varphi_\psi^{-1}(g_{B_i})) \geq r$ for $i \in \{1, 2\}$, and $\varphi_\psi^{-1}(g_{B_1}) \overline{\nabla}\, \varphi_\psi^{-1}(g_{B_2})$. Hence, $(Q, \mathcal{T}_M)$ is an r-FS-γ-regular space. The other case also follows similar lines.

Theorem 5.9. Let $\varphi_\psi : \widetilde{(Q, M)} \longrightarrow \widetilde{(S, N)}$ be a surjective FS-γ-irresolute, FS-open, and FS-closed function. If $(Q, \mathcal{T}_M)$ is an r-FS-γ-regular (resp. r-FS-γ-normal) space, then $(S, \mathcal{T}_N^*)$ is an r-FS-γ-regular (resp. r-FS-γ-normal) space.

Proof. The proof is similar to that of Theorem 5.7.

- **r-fuzzy soft γ-separated and γ-connected sets:**

Definition 5.4. Let $(Q, \mathcal{T}_M)$ be an FSTS, $r \in I_\circ$, and $t_C, h_D \in (\widetilde{Q, M})$, and then we have:

(i) Two FS-sets t_C and h_D are said to be r-FS-γ-separated sets iff $h_D \ \overline{\nabla} \ \gamma C_{\mathcal{T}}(m, t_C, r)$ and $t_C \ \overline{\nabla} \ \gamma C_{\mathcal{T}}(m, h_D, r)$ for each $m \in M$.

(ii) Every FS-set which can not be expressed as the union of two r-FS-γ-separated sets is said to be an r-FS-γ-connected set.

Theorem 5.10. Let $(Q, \mathcal{T}_M)$ be an FSTS, $r \in I_\circ$, and $t_C, h_D \in (\widetilde{Q, M})$, and then we have:

(i) If t_C and h_D are r-FS-γ-separated sets and $f_A, g_B \in (\widetilde{Q, M})$ with $f_A \sqsubseteq t_C$ and $g_B \sqsubseteq h_D$, then f_A and g_B are r-FS-γ-separated sets.

(ii) If $t_C \ \overline{\nabla} \ h_D$ and either both are r-FS-γ-closed sets or both r-FS-γ-open sets, then t_C and h_D are r-FS-γ-separated sets.

(iii) If t_C and h_D are either both r-FS-γ-closed sets or both r-FS-γ-open sets, then $t_C \sqcap h_D^c$ and $h_D \sqcap t_C^c$ are r-FS-γ-separated sets.

Proof. The proofs of (i) and (ii) are obvious.

(iii) Let t_C and h_D be r-FS-γ-open sets, and since $t_C \sqcap h_D^c \sqsubseteq h_D^c$, $\gamma C_{\mathcal{T}}(m, t_C \sqcap h_D^c, r) \sqsubseteq h_D^c$. Hence, $\gamma C_{\mathcal{T}}(m, t_C \sqcap h_D^c, r) \ \overline{\nabla} \ h_D$. Thus, $\gamma C_{\mathcal{T}}(m, t_C \sqcap h_D^c, r) \ \overline{\nabla} \ (h_D \sqcap t_C^c)$.

Again, since $h_D \sqcap t_C^c \sqsubseteq t_C^c$, $\gamma C_{\mathcal{T}}(m, h_D \sqcap t_C^c, r) \sqsubseteq t_C^c$. Hence, $\gamma C_{\mathcal{T}}(m, h_D \sqcap t_C^c, r) \ \overline{\nabla} \ t_C$. Thus, $\gamma C_{\mathcal{T}}(m, h_D \sqcap t_C^c, r) \ \overline{\nabla} \ (t_C \sqcap h_D^c)$. Therefore, $t_C \sqcap h_D^c$ and $h_D \sqcap t_C^c$ are r-FS-γ-separated sets. The other case also follows similar lines.

Theorem 5.11. Two FS-sets t_C and h_D are r-FS-γ-separated sets in an FSTS $(Q, \mathcal{T}_M)$ iff there exist two r-FS-γ-open sets f_A and g_B such that $t_C \sqsubseteq f_A$, $h_D \sqsubseteq g_B$, $t_C \ \overline{\nabla} \ g_B$, and $h_D \ \overline{\nabla} \ f_A$.

Proof. ($\Rightarrow$) Let t_C and h_D be r-FS-γ-separated sets in an FSTS $(Q, \mathcal{T}_M)$, $t_C \sqsubseteq (\gamma C_{\mathcal{T}}(m, h_D, r))^c = f_A$, and $h_D \sqsubseteq (\gamma C_{\mathcal{T}}(m, t_C, r))^c = g_B$, where g_B and f_A are r-FS-γ-open sets. Thus, $g_B \ \overline{\nabla} \ \gamma C_{\mathcal{T}}(m, t_C, r)$ and $f_A \ \overline{\nabla} \ \gamma C_{\mathcal{T}}(m, h_D, r)$. Therefore, $h_D \ \overline{\nabla} \ f_A$ and $t_C \ \overline{\nabla} \ g_B$.
($\Leftarrow$) Let f_A and g_B be r-FS-γ-open sets such that $h_D \sqsubseteq g_B$, $t_C \sqsubseteq f_A$, $h_D \ \overline{\nabla} \ f_A$, and $t_C \ \overline{\nabla} \ g_B$. Then, $h_D \sqsubseteq f_A^c$ and $t_C \sqsubseteq g_B^c$. Thus, $\gamma C_{\mathcal{T}}(m, h_D, r) \sqsubseteq f_A^c$ and $\gamma C_{\mathcal{T}}(m, t_C, r) \sqsubseteq g_B^c$. Hence, $\gamma C_{\mathcal{T}}(m, h_D, r) \ \overline{\nabla} \ t_C$ and $\gamma C_{\mathcal{T}}(m, t_C, r) \ \overline{\nabla} \ h_D$. Therefore, t_C and h_D are r-FS-γ-separated sets.

Theorem 5.12. In an FSTS $(Q, \mathcal{T}_M)$, if $h_D \in (\widetilde{Q, M})$ is an r-FS-γ-connected set such that $h_D \sqsubseteq t_C \sqsubseteq \gamma C_{\mathcal{T}}(m, h_D, r)$, then t_C is an r-FS-γ-connected set.

Proof. If t_C is not an r-FS-γ-connected set, then there exists r-FS-γ-separated sets f_A^* and $g_B^* \in (\widetilde{Q, M})$ such that $t_C = f_A^* \sqcup g_B^*$. Let $f_A = h_D \sqcap f_A^*$ and $g_B = h_D \sqcap g_B^*$, and then $h_D = g_B \sqcup f_A$. Since $f_A \sqsubseteq f_A^*$ and $g_B \sqsubseteq g_B^*$, hence by Theorem 5.10, f_A and g_B are r-FS-γ-separated sets. This is a contradiction. This shows that t_C is an r-FS-γ-connected set.

6. Conclusions

In this study, a new class of FS-open sets, called r-FS-γ-open sets, has been defined in FSTSs based on fuzzy topologies in the sense of Šostak. The class of r-FS-γ-open sets is contained in the class of r-FS-β-open sets, and contains all r-FS-α-open, r-FS-semi-open, and r-FS-pre-open sets. Some characterizations of r-FS-γ-open sets along with their mutual relationships have been specified with the help of some illustrative examples. Overall, the notions of FS-γ-closure and FS-γ-interior operators have been introduced and studied. Thereafter, we defined and characterized some new FS-functions using r-FS-γ-open and r-FS-γ-closed sets, called FS-γ-continuous (resp. FS-γ-irresolute, FS-γ-open, FS-γ-irresolute open, FS-γ-closed, and FS-γ-irresolute closed) functions between FSTSs $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}^*_N)$. The relationships between these classes of functions have been discussed with the help of some illustrative examples. Moreover, the notions of FS-weakly (resp. FS-almost) γ-continuous functions, which are weaker forms of FS-γ-continuous functions, have been introduced and studied between FSTSs $(Q, \mathcal{T}_M)$ and $(S, \mathcal{T}^*_N)$. We also showed that FS-γ-continuity $\Longrightarrow$ FS-almost γ-continuity $\Longrightarrow$ FS-weak γ-continuity, but the converse may not be true. However, we defined new types of FS-separation axioms, called r-FS-γ-regular and r-FS-γ-normal spaces, and some properties have been obtained. In the end, the notion of an r-FS-γ-connected set has been introduced via r-FS-γ-closed sets. In the next articles, we intend to explore the following topics:

- Introducing r-FS-γ-compact (resp. r-FS-nearly γ-compact and r-FS-almost γ-compact) sets.
- Defining upper and lower γ-continuous (resp. weakly γ-continuous) FS-multifunctions.
- Extending these new notions given here to include FS-r-minimal spaces as defined in [34, 42].
- Finding a use for these new notions given here in the frame of fuzzy ideals as defined in [43–45].

Author contributions

Fahad Alsharari: Conceptualization, writing original draft preparation, investigation, formal analysis; Ahmed O. M. Abubaker: Conceptualization, formal analysis, investigation, writing review and editing; Islam M. Taha: Supervision, conceptualization, formal analysis, writing original draft preparation, investigation, writing review and editing. All authors have reviewed and consented to the finalized version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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