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Research article

Applications of Elzaki transform to non-conformable fractional derivatives

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Abstract: This study presents the nonconformable fractional Elzaki transform (NCFET) method. We used this method to solve various fractional differential equations (FDEs) with nonconformable fractional derivatives (NCFDs). We studied and proved the basic properties and advantages of this new method. We discussed some examples and conducted a comparative study, presenting exact results through graphs and tables. The results showed that the new method worked well, was easy to use, and could correctly solve several fractional differential equations, even those with nonconformable fractional derivatives.

Keywords: nonconformable fractional derivatives; Elzaki transform; fractional ordinary differential equations
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1. Introduction

Mathematics has played an influential role in improving civilizations throughout the ages. Mathematical modeling has described and predicted real-world phenomena. Therefore, the study of numerous natural laws underscores the significance of calculus.

L'Hôpital and Leibniz exchanged a letter at the end of 1965 raising a question about the significance of taking a fractional derivative such as $d^{1/2}y/dx^{1/2}$. Its relevance has grown significantly over the past century because of its widespread applications across various scientific and engineering disciplines [1–3]. Subsequently, academics have proposed numerous definitions of fractional derivatives. These include Caputo, Riesz, Riemann–Liouville [4], conformable derivatives [5,6], beta derivatives, M-truncated derivatives [7,8], Atangana–Caputo [9,10], and Caputo–Fabrizio. Most of these lack some basic properties for the case of integer order, such as the chain rule, the product rule, and the quotient rule.

Khalil and associates [5] developed the conformable derivative (CD) as a contemporary substitute for the traditional limit definition of the function derivative. Khalil also introduced the conformable fractional derivative (CFD), which expands integer-order calculus properties and demonstrates the conformable fractional Leibniz rule. In [6], Abdeljawad introduced mathematical concepts such as the chain rule, integration by parts, and Taylor series expansion to extend the conformable operators to higher orders. Therefore, the conformable derivative exhibits almost all typical derivative characteristics. Recently, many researchers have developed several subsequent studies that cite the CFD, such as the Conformable double Laplace-Sumudu iterative method [11], the modified conformable double Laplace-Sumudu approach with applications [12], the double-conformable fractional Laplace-Elzaki decomposition method [13], the conformable fractional double Laplace transform [14–16], the Conformable double Sumudu transform [17,18], and the modified double conformable Laplace transform [19].

Martinez and colleagues discovered the nonconformable Laplace transform local fractional derivative [20]. They proved that it exists and discussed its main features. The new derivative adheres to the classical properties of integer derivatives indicated above, and its simplicity has led to its use in various situations [21,22]. Injrou and Hatem [21] introduced the nonconformable double Laplace transform to solve some fractional PDEs. In [22], Benyettou and Bouagade also discussed how to use double nonconformable Laplace and Sumudu transforms to solve fractional Fornasini-Marchesini models.

Integral transforms such as the Sumudu transform [23], the Elzaki transform [24], the Aboodh transform [25], the double Sumudu transform [26], the double Laplace-Sumudu transform [27], and the double Sumudu-Elzaki transform [28] are more advantageous due to their ability to simplify complex mathematical problems, particularly when solving differential equations with specific boundary conditions. By carefully selecting a transformation class, you can often transform the derivatives and boundary values in a complex differential equation into expressions that an algebraic equation can represent. The solution was reached by changing the resolution of the original differential equation, and to finish the process, the inverse transformation is needed [29–31]. The Elzaki transform is a changed version of the general Laplace and Sumudu transforms. It can quickly, correctly, and effectively solve a large number of linear differential equations. The Elzaki transform helped users solve integral, partial, and fractional differential equations [32,33], ordinary differential equations with variable coefficients [34], and systems with these equations [35]. The main point of this study is to find more uses for the Elzaki transform, especially in fixing various types of fractional differential equations with nonconformable derivatives.

The study's framework consists of the following components: Section 2 delineates the definitions and theories of the NCFD, highlighting its crucial properties that bolster our primary findings. Section 3 provides fundamental definitions and theories about NCET. Section 4 examines some examples to illustrate the effectiveness, convergence, and accuracy of the proposed approach. In Section 5, we demonstrate the accuracy and usefulness of the proposed method by comparing approximate and exact results using graphs and numerical tables. Section 6 discusses the results of the study, and Section 7 provides conclusions.

2. Preliminary

First, we will discuss the idea of the nonconformable fractional derivative (NCFD) and some of its most important properties, which help support our main results, as shown in the definitions below [36].

Definition 2.1 ([36]). The NCFD of the function $g:[0,\infty)\to\mathbb{R}$ of order β is denoted by $N_3^{\beta}(g)(z)$ and defined as:

$$N_3^{\beta}(g)(z)=\lim_{\delta\to 0}\frac{g\left(z+\delta z^{-\beta}\right)-g(z)}{\delta}, z>0, \beta\in(0,1]. \tag{2.1}$$

Remark 2.1. If g is β-differentiable in some $(0,\beta)$, $\beta>0$, $\lim_{z\to 0^+} g^{(\beta)}(z)$ exists, then

$$g^{(\beta)}(0)=\lim_{z\to 0^+} g^{(\beta)}(z). \tag{2.2}$$

Remark 2.2. If g is β-differentiable, then $N_3^{\beta}(g)(z)=z^{-\beta}g'(z)$, where

$$g'(z)=\lim_{\delta\to 0}\frac{g(z+\delta)-g(z)}{\delta}. \tag{2.3}$$

Remark 2.3. To denote the nonconformable fractional derivatives of the function g of order β at z, we can express $g^{(\beta)}(z)$ as $D_{\beta}(g)(z)$ or $\frac{d_{\beta}}{d_{\beta}z}(g(z))$. Furthermore, if the NCFD N_3^{β} of the function g of order β exists, we simply say that g is differentiable *N*.

In the following theorem, we will prove that the chain rule is valid for nonconformable fractional derivatives [36,37].

Theorem 2.1 ([36,37]). (Chain rule) Suppose that $\beta\in(0,1]$ and *f*, *h* are two functions, where *f* is an *N*-differentiable at $z>0$, and *h* is differentiable in the range of $f(z)$. Then

$$N_3^{\beta}(f\circ h)(z)=f'(h(z))N_3^{\beta}(h(z)). \tag{2.4}$$

Proof. We will prove the rule using the principal limit in two cases.
Case i: Let *f* be constant in a neighborhood of $b>0$, then $N_3^{\beta}(f\circ h)(z)=0$.
Case ii: Let *f* be non-constant in a neighborhood of $b>0$, then $\exists\delta_0>0: f(t_1)\neq f(t_2)$ for any $t_1,t_2\in(b-z_0,b+z_0)$. Therefore, since *f* is continuous at b, for δ sufficiently small, we have

$$N_3^{\beta}\left(f\circ h\right)(b)=\lim_{\delta\to 0}\frac{f\left(h\left(b+\delta b^{-\beta}\right)\right)-f\left(h(b)\right)}{\delta}$$
$$=\lim_{\delta\to 0}\frac{f\left(h\left(b+\delta b^{-\beta}\right)\right)-f\left(h(b)\right)}{h\left(b+\delta b^{-\beta}\right)-h(b)}\frac{h\left(b+\delta b^{-\beta}\right)-h(b)}{\delta}$$
$$=\lim_{\delta\to 0}\frac{f\left(h\left(b+\delta b^{-\beta}\right)\right)-f\left(h(b)\right)}{h\left(b+\delta b^{-\beta}\right)-h(b)}\lim_{\delta\to 0}\frac{h\left(b+\delta b^{-\beta}\right)-h(b)}{\delta}.$$

Putting $\delta_1=h\left(b+\delta b^{-\beta}\right)-h(b)$ in the first limit, we get

$$N_3^{\beta}\left(f\circ h\right)(b)=\lim_{\delta_1\to 0}\frac{f\left(h(b)+\delta_1\right)-f\left(h(b)\right)}{\delta_1}\lim_{\delta\to 0}\frac{h\left(b+\delta b^{-\beta}\right)-h(b)}{\delta}$$
$$=f'\left(h(b)\right)N_3^{\beta}\left(h\right)(b).$$

The following function illustrates one of the key and fundamental properties of our work:

Definition 2.2 ([20]). Let $0<\beta\le 1,$ and $a\in\mathbb{R},$ then the definition of the fractional exponential is:

$$E_{\beta}^{n_3}\left(a,z\right)=e^{a\frac{z^{\beta+1}}{\beta+1}}. \tag{2.5}$$

In the following theorem, we will list some basic properties associated with the derivative N_3^{β} [36,37].

Theorem 2.2 ([36,37]). Suppose $0<\beta\le 1,$ and $g,h:[0,\infty)\to\mathbb{R}$ be N-differentiable at a point $z>0,$ and $a,c_1,c_2\in\mathbb{R},$ then

$$N_3^{\beta}\left(c_1\,g+c_2\,h\right)=c_1N_3^{\beta}\left(g\right)+c_2N_3^{\beta}\left(h\right),$$
$$N_3^{\beta}\left(g\,h\right)=g\,N_3^{\beta}\left(h\right)+h\,N_3^{\beta}\left(g\right),$$
$$N_3^{\beta}\left(\frac{g}{h}\right)=\frac{h\,N_3^{\beta}\left(g\right)-g\,N_3^{\beta}\left(h\right)}{h^2},$$
$$N_3^{\beta}\left(c_1\right)=0,$$
$$N_3^{\beta}\left(1\right)=0,$$
$$N_3^{\beta}\left(\frac{1}{1+\beta}z^{1+\beta}\right)=1,$$
$$N_3^{\beta}\left(E_{\beta}^{n_3}\left(a,z\right)\right)=a\,E_{\beta}^{n_3}\left(a,z\right),$$
$$N_3^{\beta}\left(\sin\left(c_1\frac{z^{1+\beta}}{1+\beta}\right)\right)=c_1\cos\left(c_1\frac{z^{1+\beta}}{1+\beta}\right),$$
$$N_3^{\beta}\left(\cos\left(c_1\frac{z^{1+\beta}}{1+\beta}\right)\right)=-c_1\sin\left(c_1\frac{z^{1+\beta}}{1+\beta}\right).$$

Proof. See [20].

Now, we will define a nonconformable fractional integral.

Definition 2.3. Let $0<\beta\le 1$, and $z_0>0$, then the definition of the nonconformable fractional integral of the function $g:[z_0,\infty)\to\mathbb{R}$ of order β is denoted by ${}_{N_3}J_{z_0}^{\beta}g(z)$ and defined as:

$$ {}_{N_3}J_{z_0}^{\beta}g(z)=\int_{z_0}^{z}g(x)\,d_{\beta}x=\int_{z_0}^{z}\frac{g(x)}{x^{-\beta}}dx. \tag{2.6}$$

The following statement bears a resemblance to one known from ordinary calculus (see [38]).

Theorem 2.3. Suppose that *h* is *N*-differentiable in (z_0,∞) with $0<\beta\le 1$. Then $\forall\, z>z_0$, we have

i) If h is differentiable, then ${}_{N_3}J_{z_0}^{\beta}\left(N_3^{\beta}h(z)\right)=h(z)-h(z_0)$.

ii) $N_3^{\beta}\left({}_{N_3}J_{z_0}^{\beta}\left(N_3^{\beta}h(z)\right)\right)=h(z)$.

Proof.

i) From (2.3), we have

$$ {}_{N_3}J_{z_0}^{\beta}\left(N_3^{\beta}h(z)\right)=\int_{z_0}^{z}N_3^{\beta}h(x)\,d_{\beta}x=\int_{z_0}^{z}\frac{N_3^{\beta}h(x)}{x^{-\beta}}dx=\int_{z_0}^{z}\frac{h'(x)x^{-\beta}}{x^{-\beta}}dx=h(z)-h(z_0).$$

ii) With the same method, we have

$$N_3^{\beta}\left({}_{N_3}J_{z_0}^{\beta}\left(N_3^{\beta}h(z)\right)\right)=z^{-\beta}\frac{d}{dz}\left[\int_{z_0}^{z}\frac{h(x)}{x^{-\beta}}dx\right]=h(z).$$

The following result clarifies another significant and essential aspect of our work:

Theorem 2.4 (Integration by parts). Suppose that u, v are *N*-differentiable functions in (z_0,∞), with $0<\beta\le 1$. Then, $\forall\, z>z_0$, and we have

$$ {}_{N_3}J_{z_0}^{\beta}\left(\left(u\,N_3^{\beta}v\right)(z)\right)=\left[uv(z)-uv(z_0)\right]-{}_{N_3}J_{z_0}^{\beta}\left(\left(u\,N_3^{\beta}v\right)(z)\right).$$

Proof. It is sufficient to apply Theorems 2.2 and 2.3.

3. Nonconformable fractional Elzaki transform (NCFET)

In this section, we demonstrate the first steps toward formalizing a new type of nonconformable fractional Elzaki transform. This transform can be applied to a wide range of fractional differential equations.

Definition 3.1 (Exponential order). A function $g(z)$ is considered to be of generalized exponential order β if there exist two positive constants K and a such that $\left|g(z)\right| \le K\, E_{\beta}^{n_3}(a,z)$ for sufficiently large z.

Now, we are going to define the NCFET.

Definition 3.2. Let $g(z)$ be a real function defined for $z \ge 0$, and consider $u \in \mathbb{C}$, if the integral

$$ {}_{N_3}J_0^{\beta} E_{\beta}^{n_3}\left(-\frac{1}{u}, z\right) g(z)(+\infty) = u\int_0^{+\infty} E_{\beta}^{n_3}\left(-\frac{1}{u}, z\right) g(z) d_{\beta} z = u\int_0^{+\infty} \frac{E_{\beta}^{n_3}\left(-\frac{1}{u}, z\right) g(z)}{z^{-\beta}} dz, \tag{3.1} $$

converge for the given value of u, then we define the function $G(u)$ by the following

$$ G(u) = {}_{N_3}J_0^{\beta} E_{\beta}^{n_3}\left(-\frac{1}{u}, z\right) g(z)(+\infty), \tag{3.2} $$

and we will write $G(u) = E_N(g)$.

We will refer to the operator E_N as the *N*-transformed of Elzaki and to G as the *N*-transformed of g. Consequently, we represent the *N*-transformed inverse Elzaki as follows:

$$ E_N^{-1}\left\{G(u)\right\} = g(z) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} u e^{\frac{z^{\beta+1}}{u(\beta+1)}} G(u) du. \tag{3.3} $$

Remark 3.1. According to the NCFET definition, we can conclude that the NCFET is a linear integral transformation, as illustrated below:

$$ \begin{aligned} E_N\left[c_1 g(z) + c_2 h(z)\right] &= u\int_0^{+\infty} E_{\beta}^{n_3}\left(-\frac{1}{u}, z\right)\left[c_1 g(z) + c_2 h(z)\right] d_{\beta} z \\ &= u\int_0^{+\infty} E_{\beta}^{n_3}\left(-\frac{1}{u}, z\right) c_1 g(z) d_{\beta} z + u\int_0^{+\infty} E_{\beta}^{n_3}\left(-\frac{1}{u}, z\right) c_2 h(z) d_{\beta} z \\ &= c_1 u\int_0^{+\infty} E_{\beta}^{n_3}\left(-\frac{1}{u}, z\right) g(z) d_{\beta} z + c_2 u\int_0^{+\infty} E_{\beta}^{n_3}\left(-\frac{1}{u}, z\right) h(z) d_{\beta} z \\ &= c_1 E_N\left[\, g(z)\right] + c_2 E_N\left[\, h(z)\right], \end{aligned} $$

where c_1 and c_2 are constants.

The following theorem sets the usual Laplace transform (LT) and the nonconformable fractional Laplace transform (NCFLT).

Theorem 3.1 ([20]). Let $g(z) : [0, \infty) \to \mathbb{R}$ be an *N*-transformable function and $s \in \mathbb{C}$, then

$$ L_N\left(g(z)\right) = L\left[g\left(\left((1+\beta) z\right)^{\frac{1}{1+\beta}} \right) \right], \beta \in (0,1], \tag{3.4} $$

where

$$L\left(g\left(z\right):s\right)=G\left(s\right)=\int_{0}^{\infty}e^{-sz}g\left(z\right)dz. \tag{3.5}$$

In the following theorem, we explain the NCFLT of some functions, as established in Theorem 2.1 from [20].

Theorem 3.2 ([20]). Suppose $c_1\in\mathbb{R},\ \ m\in\mathbb{Z}^{+},\ \ s\in\mathbb{C}$, and $0<\beta\le 1$. Then

1) $L_N\left[c_1\right],=\dfrac{c_1}{s},s>0,$

2) $L_N\left(\left(\dfrac{z^{\beta+1}}{\beta+1}\right)^{m}\right)=\dfrac{m!}{s^{m+1}},m>-1,$

3) $L_N\left[E_{\beta}^{n_3}\left(c_1,z\right)\right]=\dfrac{1}{\left(s-c_1\right)},s-c_1>0,$

4) $L_N\left[\sin\left(c_1\dfrac{z^{1+\beta}}{1+\beta}\right):u\right]=\dfrac{c_1}{\left(s^2+c_1{}^2\right)},$

5) $L_N\left[\cos\left(c_1\dfrac{z^{1+\beta}}{1+\beta}\right)\right]=\dfrac{s}{\left(s^2+c_1{}^2\right)},$

6) $L_N\left[\sinh\left(c_1\dfrac{z^{1+\beta}}{1+\beta}\right):u\right]=\dfrac{c_1}{\left(s^2-c_1{}^2\right)},s^2-c_1{}^2>0,$

7) $L_N\left[\cosh\left(c_1\dfrac{z^{1+\beta}}{1+\beta}\right)\right]=\dfrac{s}{\left(s^2-c_1{}^2\right)},s^2-c_1{}^2>0.$

The following theorem sets the duality between the NCFLT and the NCFET.

Theorem 3.3 (Duality between the NCFLT and the NCFET). Let $g\left(z\right):[0,\infty)\to\mathbb{R}$, be an *N*-transformable function and $0<\beta\le 1$. Then

$$E_N\left(g\left(z\right):u\right)=u\,L_N\left(g\left(z\right):s\right)_{s\to\frac{1}{u}},s,u\in\mathbb{C}. \tag{3.6}$$

Proof. From Definition 3.2, we have

$$E_N\left(g\left(z\right):u\right)=u\int_{0}^{\infty}E_{\beta}^{n_3}\left(-\frac{1}{u},z\right)g(z)d_{\beta}z=u\int_{0}^{\infty}e^{-\frac{z^{\beta+1}}{u(\beta+1)}}g(z)\,z^{\beta}dz. \tag{3.7}$$

Let's put

$$w=\frac{z^{\beta+1}}{\beta+1}\Rightarrow dw=z^{\beta}\,dz. \tag{3.8}$$

By substituting Eq (3.8) into Eq (3.7), we obtain

$$E_N\left(g(z):u\right)=u\int_0^{\infty}e^{-\frac{w}{u}}g\left(\left(\beta+1\right)w\right)^{\frac{1}{\beta+1}}dw$$
$$=u\left\{L\left[g\left(\left(\beta+1\right)w\right)^{\frac{1}{\beta+1}}\right]\right\}_{s\to\frac{1}{u}}=u\,L_N\left(g(z):s\right)_{s\to\frac{1}{u}}.$$

The following theorem describes the NCFET of a few fundamental functions.

Theorem 3.4. Suppose $c_1\in\mathbb{R}$, $m\in\mathbb{Z}^+$, $s,u\in\mathbb{C}$, and $0<\beta\leq 1$. The NCFET for some functions is given below:

1) $E_N\left[c_1\right]=c_1u^2$,

2) $E_N\left[\left(\frac{z^{\beta+1}}{\beta+1}\right)^m\right]=m!u^{m+2}, u>0,$

3) $E_N\left[E_{\beta}^{n_3}\left(c_1,z\right)\right]=\frac{u^2}{\left(1-c_1u\right)}, 1-c_1u>0.$

Proof. Here, we shall use Theorems 3.2 and 3.3 to support our conclusions.
(1) We know that:

$$E_N\left[c_1\right]=uL_N\left[c_1\right]_{s\to\frac{1}{u}}.$$

Therefore

$$E_N\left[c_1\right]=u\left[\frac{c_1}{\frac{1}{u}}\right]=c_1u^2.$$

(2) We notice that:

$$E_N\left[\left(\frac{z^{\beta+1}}{\beta+1}\right)^m\right]=u\,L_N\left[\left(\frac{z^{\beta+1}}{\beta+1}\right)^m\right]_{s\to\frac{1}{u}}.$$

Thus

$$E_N\left[\left(\frac{z^{\beta+1}}{\beta+1}\right)^m\right]=u\left[\frac{m!}{\left(\frac{1}{u}\right)^{m+1}}\right]=m!u^{m+2}.$$

(3) Similarly, from the proofs of (1) and (2), we get

$$E_N\left[E_\beta^{n_3}(c_1,z)\right]=u\,L_N\left[E_\beta^{n_3}(c_1,z)\right]_{s\to\frac{1}{u}}=u\left[\frac{1}{\left(\frac{1}{u}-c_1\right)}\right]=\frac{u^2}{(1-c_1u)},1-c_1u>0.$$

Table 1 below summarizes the NCFET for certain essential functions.

Table 1. The NCFET for certain essential functions.

Sr. No	$g(z)$	$E_N[g(z)]=G(u)$
1	c_1	c_1u^2
2	$\left(\frac{z^{\beta+1}}{\beta+1}\right)^m$	$m!u^{m+2}$
3	$E_\beta^{n_3}(c_1,z)$	$\frac{u^2}{(1-c_1u)}$
4	$\sin\left(c_1\frac{z^{1+\beta}}{1+\beta}\right)$	$\frac{c_1u^3}{(1+c_1^2u^2)}$
5	$\cos\left(c_1\frac{z^{1+\beta}}{1+\beta}\right)$	$\frac{u^2}{(1+c_1^2u^2)}$
6	$\sinh\left(c_1\frac{z^{1+\beta}}{1+\beta}\right)$	$\frac{c_1u^3}{(1-c_1^2u^2)}$
7	$\cosh\left(c_1\frac{z^{1+\beta}}{1+\beta}\right)$	$\frac{u^2}{(1-c_1^2u^2)}$

Theorem 3.5 (Derivative properties). Let $g(z):[0,\infty)\to\mathbb{R}$, be a transformable function and $0<\beta\le 1$. Then, the NCFET of the NCFDs $N_3^\beta g(z)$ and $N_3^\beta\left(N_3^\beta g(z)\right)$ can be represented as follows:

$$E_N\left[N_3^\beta g(z)\right]=\frac{1}{u}G(u)-u\,g(0), \tag{3.9}$$

$$E_N\left[N_3^\beta\left(N_3^\beta g(z)\right)\right]=\frac{1}{u^2}G(u)-g(0)-u\,N_3^\beta g(0). \tag{3.10}$$

Proof. We proceed with proving the result (3.9).

$$E_N\left[N_3^{\beta}g(z)\right]=u\int_0^{\infty}E_{\beta}^{n_3}\left(-\frac{1}{u},z\right)N_3^{\beta}g(z)\,d_{\beta}z. \tag{3.11}$$

Let's put

$$\begin{aligned}&w=E_{\beta}^{n_3}\left(-\frac{1}{u},z\right)\Rightarrow dw=-\frac{1}{u}E_{\beta}^{n_3}\left(-\frac{1}{u},z\right)dz,\\&dv=N_3^{\beta}g(z)\,d_{\beta}z\Rightarrow v=g(z).\end{aligned} \tag{3.12}$$

By substituting Eq (3.12) into Eq (3.11), we obtain

$$\begin{aligned}E_N\left[N_3^{\beta}g(z)\right]&=u\left[\left[E_{\beta}^{n_3}\left(-\frac{1}{u},z\right)g(z)\right]_0^{\infty}+\frac{1}{u}\int_0^{\infty}E_{\beta}^{n_3}\left(-\frac{1}{u},z\right)g(z)\,d_{\beta}z\right]\\&=u\left[-g(0)+\frac{1}{u^2}E_N\left[g(z)\right]\right]=\frac{1}{u}E_N\left[g(z)\right]-u\,g(0).\end{aligned}$$

We can also use the same method to illustrate the final result (3.10).

The following example is a further generalization of the findings described above:

$$\begin{aligned}&E_N\left[N_3^{\beta}\left(N_3^{\beta}\left(\ldots\left(N_3^{\beta}\varphi\right)\right)\right)\right]\\&=\frac{1}{u^n}G(u)-u^{2-n}g(0)-u^{3-n}N_3^{\beta}g(0)-u^{4-n}N_3^{\beta}\left(N_3^{\beta}g(0)\right)\\&-\ldots-u\,N_3^{\beta}\left(N_3^{\beta}\left(\ldots\left(N_3^{\beta}g(0)\right)\right)\right).\end{aligned} \tag{3.13}$$

Theorem 3.6 (Convolution theorem). Let g and h be two functions, such that $g,h:[0,\infty)\to\mathbb{R}$, and $0<\beta\le 1$. Then

$$E_N\left[(g*h)(z)\right]=\frac{1}{u}G(u)H(u). \tag{3.14}$$

Proof. Using Eq (3.6), we derive

$$\begin{aligned}E_N\left[(g*h)(z)\right]&=u\,L_N\left[(g*h)(z)\right]_{s\to\frac{1}{u}}\\&=u\left[L_N\left[g(z)\right]_{s\to\frac{1}{u}}L_N\left[h(z)\right]_{s\to\frac{1}{u}}\right]\\&=u\left[\frac{1}{u}E_N\left[g(z)\right]\frac{1}{u}E_N\left[h(z)\right]\right]\\&=\frac{1}{u}G(u)H(u).\end{aligned}$$

Now, we present the conditions of boundedness and the existence of the NCFET.

If $g(z)$ is a generalized exponential order, then there exist two constants $Z, K > 0,$ and $a \in \mathbb{R}$ such that $|g(z)| \le K E_{\beta}^{n_3}(a,z)$ for all $\forall z > Z$ and $0 < \beta \le 1.$

Therefore,

$$g(z) = O E_{\beta}^{n_3}(a,z), \text{ as } z \to \infty,$$

or, equivalently,

$$\lim_{z\to\infty}\left|E_{\beta}^{n_3}\left(-\frac{1}{u},z\right)g(z)\right| = K \lim_{z\to\infty} E_{\beta}^{n_3}\left(-\left(\frac{1}{u}-a\right),z\right) = 0, \frac{1}{u} > a.$$

Therefore, $g(z)$ is an *N*-transformable function.

Theorem 3.7. Suppose that $g(z)$ is a piecewise continuous function of exponential order β defined on the interval $(0,Z)$. Then, the NCFET of $g(z)$ is well-defined for all $\frac{1}{u}$ provided that $\operatorname{Re}\left[\frac{1}{u}\right] > a.$

Proof. We deduce, from the definition of NCFET,

$$\begin{aligned} |\Psi(u)| &= \left| u\int_0^{\infty} E_{\beta}^{n_3}\left(-\frac{1}{u},z\right) g(z)\, d_{\beta} z\right| \\ &\le \left(K\int_0^{\infty} u E_{\beta}^{n_3}\left(-\left(\frac{1}{u}-a\right),z\right) d_{\beta} z\right) = \frac{Ku^2}{(1-au)}, \operatorname{Re}\left[\frac{1}{u}\right] > a. \end{aligned} \tag{3.15}$$

Thus, from Eq (3.15) we obtain

$$\lim_{z\to\infty}|\Psi(u)| = 0, \quad \text{or} \quad \lim_{z\to\infty}\Psi(u) = 0.$$

4. Applications of the NCFET method

In this section, we demonstrate the efficiency and simplicity of this transformation by solving the following classes of NCFDEs using NCFET.

Example 4.1. Consider the following differential equation of NCFD:

$$N_3^{\beta} g(z) = \mu g(z), \; z > 0, 0 < \beta \le 1, \tag{4.1}$$

with the initial condition (IC),

$$g(0) = g_0. \tag{4.2}$$

We can rewrite Eq (4.1) for $\beta = 1$ as the following of ordinary differential equation (ODE):

$$g'(z) = \mu g(z),\ g(0) = g_0, \tag{4.3}$$

and the exact solution is

$$g(z) = g_0 e^{\mu z}. \tag{4.4}$$

Solution. Applying the NCFET to Eq (4.1), we obtain

$$\frac{1}{u} G(u) - u\, g_0 = \mu G(u). \tag{4.5}$$

By simplification of Eq (4.5), we have

$$G(u) = \frac{g_0 u^2}{1 - \mu u}. \tag{4.6}$$

Taking $E_N^{-1}\{G(u)\}$ of Eq (4.6), we get

$$g(z) = g_0 E_\beta^{n_3}(\mu, z) = g_0 e^{\mu \frac{z^{\beta+1}}{\beta+1}}. \tag{4.7}$$

When $\beta \to 0,$ the solution leads to the exact answer.

Example 4.2. Consider the following differential equation of NCFD:

$$N_3^\beta \left(N_3^\beta g(z) \right) + k\, g(z) = 0,\ z > 0, 0 < \beta \le 1, \tag{4.8}$$

with the ICs

$$g(0) = g_0,\ N_3^\beta g(0) = 0. \tag{4.9}$$

We can rewrite Eq (4.8) for $\beta = 1$ as the following of ODE:

$$g''(z) + k\, g(z) = 0,\ g(0) = g_0, g'(0) = 0, \tag{4.10}$$

and the exact solution is

$$g(z) = g_0 \cos\left(\sqrt{k}\, z \right). \tag{4.11}$$

Solution. Applying the NCFET to Eq (4.7), we obtain

$$\frac{1}{u^2} G(u) - g_0 + k\, G(u) = 0. \tag{4.12}$$

By simplification of Eq (4.12), we have

$$G(u)=\frac{g_0 u^2}{1+ku^2}. \tag{4.13}$$

Taking $E_N^{-1}\{G(u)\}$ from Eq (4.13), we get

$$g(z)=g_0\cos\left(\sqrt{k}\,\frac{z^{\beta+1}}{\beta+1}\right). \tag{4.14}$$

When $\beta\to 0,$ the solution leads to an exact answer.

Example 4.3. Consider the Bertalanffy-logistic equation of NCFD [39]:

$$N_3^{\beta}g(z)=\left[g(z)\right]^{2/3}-g(z),\ \ z>0, 0<\beta\le 1, \tag{4.15}$$

with the IC,

$$g(0)=g_0. \tag{4.16}$$

We can rewrite Eq (4.15) for $\beta=1$ as the following of ordinary Bertalanffy-logistic equation:

$$g'(z)=\left[g(z)\right]^{2/3}-g(z),\ \ g(0)=g_0, \tag{4.17}$$

and the exact solution is:

$$g(z)=\left[1+\left(g_0^{\frac{1}{3}}-1\right)e^{-\frac{z}{3}}\right]^3. \tag{4.18}$$

Solution. By using the substitute $\xi=g^{\frac{1}{3}}$ in Eq (4.15), we get

$$N_3^{\beta}\xi(z)=\frac{1}{3}\left(1-\xi(z)\right),\ \xi_0=g_0^{\frac{1}{3}}. \tag{4.19}$$

Applying the NCFET to Eq (4.19), we obtain

$$E_N\left[\xi(z)\right]=u^2-\frac{3u^2}{3+u}+\frac{3u^2\xi_0}{3+u}. \tag{4.20}$$

Taking $E_N^{-1}\{\xi(z)\}$ from Eq (4.20), we get

$$g(z)=\left[1+\left(g_0^{\frac{1}{3}}-1\right)e^{-\frac{z^{\beta+1}}{3(\beta+1)}}\right]^3. \tag{4.21}$$

When $\beta\to 0,$ the solution leads to the exact answer.

5. Numerical results

This section assesses the accuracy and effectiveness of the proposed approach by comparing the approximate and precise outcomes using graphs and numerical tables, considering various scenarios for β.

5.1. Graphical analysis

Figures 1–3 show the line plots of the approximate solutions of the proposed method from Examples 4.1–4.3 at varying values of β. Figures 4–6 display the line plots of the approximate solutions of the proposed method and the exact solutions from Examples 4.1–4.3 at $\beta = 1$.

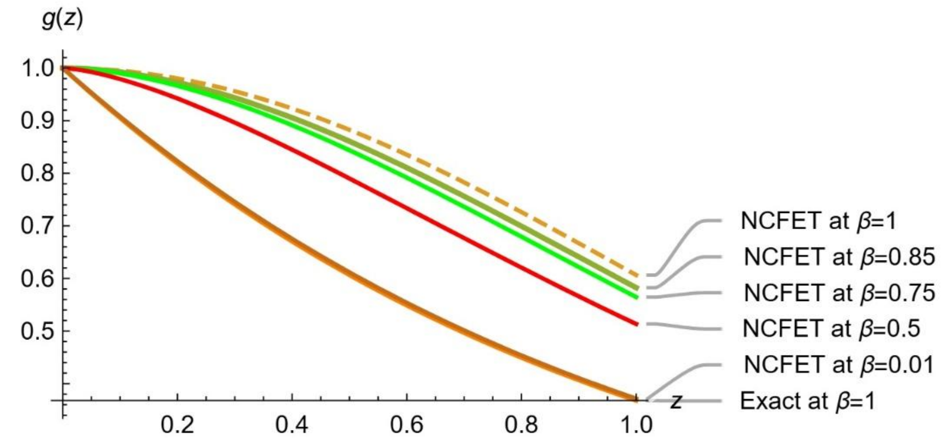

Figure 1. Example 1: approximate solution graph of $g(z)$ for Eq (4.7) at varying values of β when $\mu = -1,$ and $g_0 = 1.$

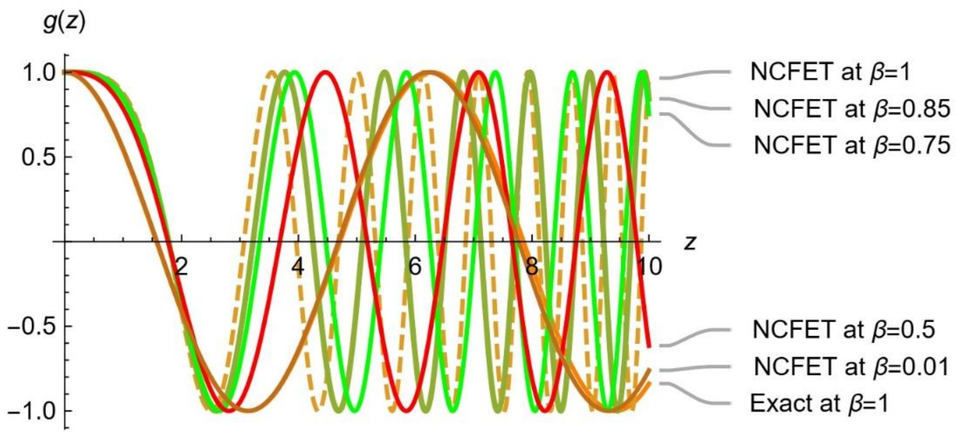

Figure 2. Example 2: approximate solution graph of $g(z)$ for Eq (4.14) at varying values of β and $k = g_0 = 1.$

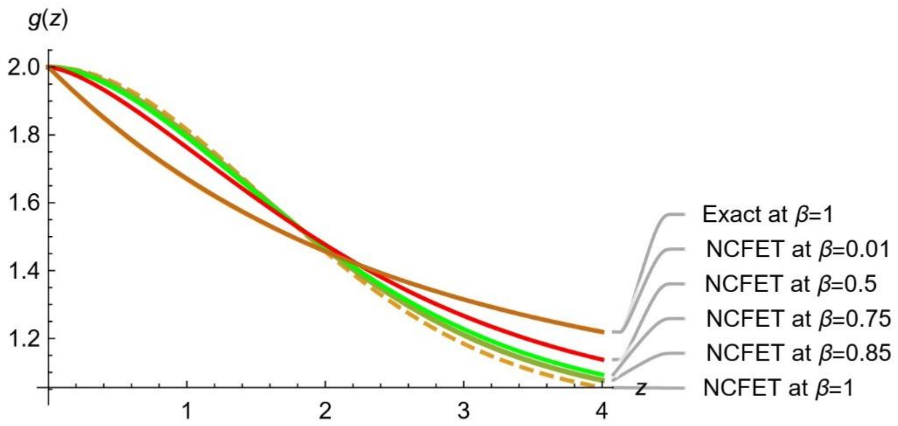

Figure 3. Example 3: approximate solution graph of $g(z)$ for Eq (4.21) at varying values of β and $g_0 = 2$.

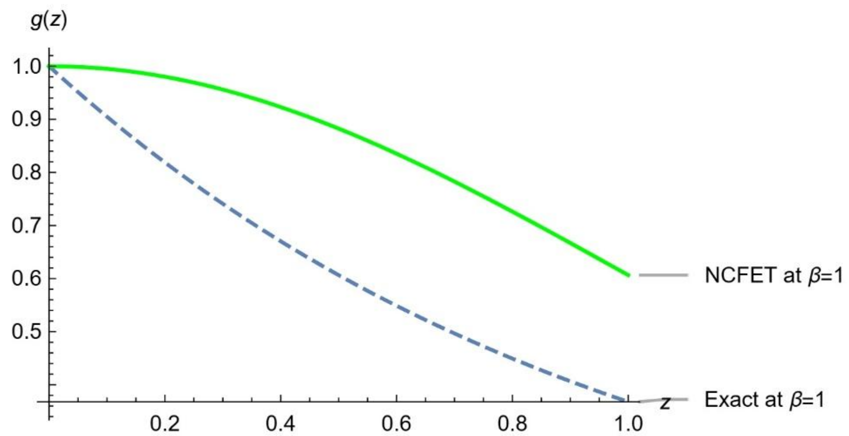

Figure 4. Example 1: approximate and exact solutions graph of $g(z)$ for Eqs (4.3) and (4.7) at $\beta = 1,$ $\mu = -1,$ and $g_0 = 1.$

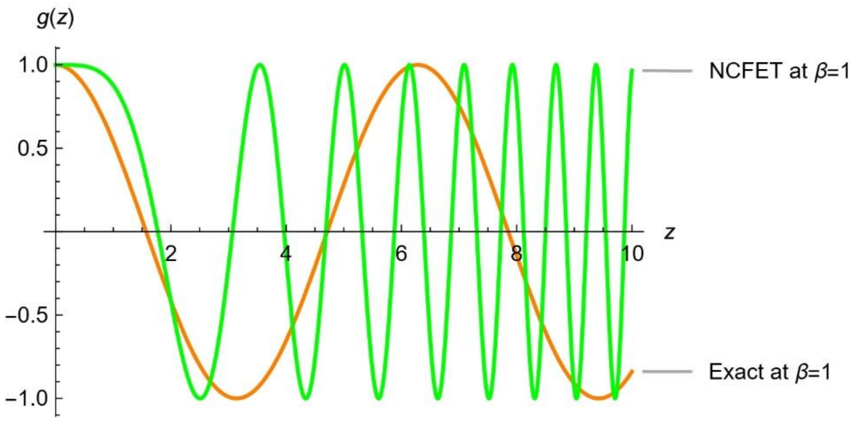

Figure 5. Example 2: approximate and exact solutions graph of $g(z)$ for Eqs (4.11) and (4.14) at $\beta = 1,$ and $k = g_0 = 1.$

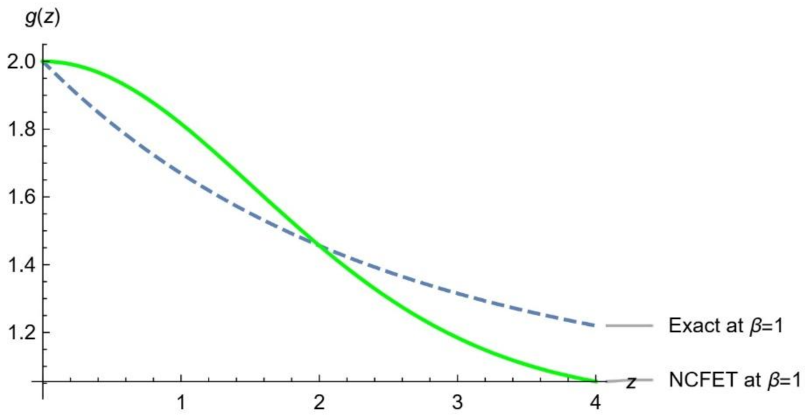

Figure 6. Example 3: approximate and exact solutions graph of $g(z)$ for Eqs (4.18) and (4.21) at $\beta = 1,$ and $g_0 = 2.$

5.2. Tabular results

Tables 2–7 show a comparison of the exact and approximate answers for each problem. These comparisons give us valuable information about how the solution works in different fractional situations for β, which allows us to fully examine its performance.

Table 2. Comparison of the approximate solution's behaviour and exact solution when β takes the values 0.01, 0.03, 0.05, and 1 for Eq (4.7), $\mu = -1,$ and $g_0 = 1.$

z	NCFET at $\beta = 0.05$	NCFET at $\beta = 0.03$	NCFET at $\beta = 0.01$	NCFET at $\beta = 1$	Exact at $\beta = 1$
0	1	1	1	1	1
0.25	0.800795	0.792287	0.783396	0.969233	0.778801
0.5	0.631303	0.621607	0.611629	0.882497	0.606531
0.75	0.494561	0.485829	0.476904	0.75484	0.472367
1	0.385821	0.378752	0.371540	0.606531	0.367879

Table 3. Comparison of the approximate solution's behaviour and exact solution when β takes the values 0.5, 0.75, 0.85, and 1 for Eq (4.7), $\mu = -1,$ and $g_0 = 1.$

z	NCFET at $\beta = 0.5$	NCFET at $\beta = 0.75$	NCFET at $\beta = 0.85$	NCFET at $\beta = 1$	Exact at $\beta = 1$
0	1	1	1	1	1
0.25	0.920044	0.950747	0.959260	0.969233	0.778801
0.5	0.790016	0.843760	0.860758	0.882497	0.606531
0.75	0.648552	0.707939	0.727994	0.75484	0.472367
1	0.513417	0.564718	0.582433	0.606531	0.367879

Table 4. Comparison of the approximate solution's behaviour and exact solution when β takes the values 0.01, 0.03, 0.05, and 1 for Eq (4.14), and $k = g_0 = 1$.

z	NCFET at $\beta = 0.05$	NCFET at $\beta = 0.03$	NCFET at $\beta = 0.01$	NCFET at $\beta = 1$	Exact at $\beta = 1$
0	1	1	1	1	1
0.25	0.975426	0.973017	0.970351	0.999512	0.968912
0.5	0.896066	0.889088	0.881565	0.992198	0.877583
0.75	0.762204	0.7505523	0.738171	0.960709	0.731689
1	0.579745	0.564579	0.548607	0.877583	0.540302

Table 5. Comparison of the approximate solution's behaviour and exact solution when β takes the values 0.5, 0.75, 0.85, and 1 for Eq (4.14), and $k = g_0 = 1$.

z	NCFET at $\beta = 0.5$	NCFET at $\beta = 0.75$	NCFET at $\beta = 0.85$	NCFET at $\beta = 1$	Exact at $\beta = 1$
0	1	1	1	1	1
0.25	0.99653	0.998725	0.999135	0.999512	0.968912
0.5	0.972351	0.985604	0.98878	0.992198	0.877583
0.75	0.907706	0.940941	0.950031	0.960709	0.731689
1	0.785887	0.841129	0.857431	0.877583	0.540302

Table 6. Comparison of the approximate solution's behaviour and exact solution when β takes the values 0.01, 0.03, 0.05, and 1 for Eq (4.21), and $g_0 = 2$.

z	NCFET at $\beta = 0.05$	NCFET at $\beta = 0.03$	NCFET at $\beta = 0.01$	NCFET at $\beta = 1$	Exact at $\beta = 1$
0	2	2	2	2	2
0.25	1.912947	1.908984	1.904818	1.987201	1.902655
0.5	1.829159	1.824016	1.818677	1.949909	1.815930
0.75	1.752081	1.746781	1.741308	1.891262	1.738505
1	1.681853	1.676925	1.671843	1.815930	1.669242

Table 7. Comparison of the approximate solution's behaviour and exact solution when β takes the values 0.5, 0.75, 0.85, and 1 for Eq (4.21), and $g_0 = 2$.

z	NCFET at $\beta = 0.5$	NCFET at $\beta = 0.75$	NCFET at $\beta = 0.85$	NCFET at $\beta = 1$	Exact at $\beta = 1$
0	2	2	2	2	2
0.25	1.966281	1.979406	1.983006	1.987201	1.902655
0.5	1.907922	1.932624	1.940260	1.949909	1.815930
0.75	1.838205	1.868385	1.878265	1.891262	1.738505
1	1.763354	1.792913	1.802775	1.815930	1.669242

6. Discussion

This section discusses the accuracy and usefulness of the proposed approach based on the graphs and tabular results from the previous section. Figures 1–3 show that the current method's solutions get closer to each other as β approaches one, but they do not get closer to the exact solutions. We also note that when the fractional value of β approaches zero, the solutions closely approximate the exact ones. Furthermore, we can see in Figures 4–6 that the NCFET solutions do not represent the exact solutions in the classical case, i.e., $\beta = 1$. Based on Tables 2–7, the NCFET method is more accurate and comes closer to exact solutions when $\beta = 0.05, 0.03, 0.01$. Finally, the test case's figures and tables show how effective, reliable, and feasible the suggested approach is. They also show that our method quickly finds the exact solutions by looking at how it works in various fractional situations.

The previous analysis reveals the following observations regarding the comparison between nonconformable fractional derivatives and other fractional derivatives:

- Scholars have primarily written about nonconformable fractional derivatives. For instance, they discussed the single and double Laplace transforms [20–22]. They did not compare the approximate solutions of these techniques to the exact solutions. So, we fixed these issues with this method. We suggested and showed that the answers to nonconformable fractional derivatives can be very close to the real ones if you look at how they behave in various fractional situations.
- It was shown that the approximate solutions, graphs, and numerical results of nonconformable fractional derivatives in the classical case ($\beta = 1$) do not converge to the exact solutions found in the studied examples. This differs from other fractional derivatives (see [14,18,40,41]).
- The findings show that the way nonconformable fractional derivatives behave as solutions changes for different fractional cases. For example, as β approaches zero, the approximate solution for the nonconformable fractional derivative converges to the exact solution. On the other hand, for the other fractional derivatives [11,17,42], the approximate solution gets closer and closer to the exact solution as the fractional values get closer to the classical case, that is, $\beta = 1$.

7. Conclusions

The main goal of this work was to apply the basic ideas of the classical Elzaki transform to the nonconformable fractional Elzaki transform. We have successfully constructed some of these theorems and relations using the nonconformable derivative definition. We discussed and proved new results related to derivatives, boundedness, existence, and the convolution theorem. In addition, we offered 2D graphical representations for obtaining solutions with varying values of β. This study's findings suggest that the results from the fractional case align with those from the ordinary case. The results of this study show that NCFET is a beneficial and easy way to solve fractional ODEs with nonconformable derivatives. We plan to use NCFET in the future to solve more FDEs involving nonconformable derivatives that arise in applied sciences and engineering.

Author contributions

S.A.A. and T.M.E: Conceptualization; S.A.A: formal analysis; A.A.H: investigation; S.A.A. and T.M.E.: Methodology; S.A.A. and T.M.E: Supervision; H.M.B: validation; M.S.H: visualization; S.A.A: Writing – original draft; S.A.A. and T.M.E: writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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