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Research article

Integrability, Hirota D-operator expression, multi solitons, breather wave, and complexiton of a generalized Korteweg-de Vries–Caudrey Dodd Gibbon equation

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Abstract: In this paper, we conducted an in-depth study of a generalized Korteweg-de Vries–Caudrey Dodd Gibbon (gKdV–CDG) equation modeling specific oceanic waves. Through the Bell polynomial approach (BPA), the Hirota D-operator expression of the gKdV–CDG equation was first constructed. An integrability test of the governing model was then carried out, and consequently, multi solitons were constructed using the Hirota method. Ultimately, using symbolic computations, breather and complexiton waves were derived from the gKdV–CDG equation by serving distinct ansatzes. A few representations positioned two- and three-dimensionally were provided to characterize the nonlinear wave's physical features. Based on the results, suitable methods were suggested to assess the height and width of nonlinear waves in the ocean.

Keywords: gKdV–CDG equation; Hirota D-operator expression; integrability; multi solitons; breather and complexiton waves
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1. Introduction

As known, nonlinear partial differential (NLPD) equations have found wide applications in modeling nonlinear phenomena in vast areas of scientific disciplines, such as optics, plasma physics, and fluid mechanics. Owing to such capabilities, researchers have devoted an incredible amount of their lives to creating and exploring such models. Nowadays, scholars deal with a wide range of specific nonlinear waves for NLPD equations such as multi-soliton waves, breather waves, complexiton waves, etc. Wazwaz [1], by applying the Hirota method, extracted multi-soliton waves of KP equations. The author in [2] found breather waves of Kairat-II and Kairat-X equations using an ansatz composed of trigonometric and hyperbolic functions. Hosseini et al. [3] employed a systematic method to construct complexiton waves of a generalized KdV equation. Papers [4–7] include more details on NLPD equations and their nonlinear waves.

One of the key characteristics of NLPD equations, which is usually explored in Mathematical Physics, is integrability. Although the literature does not provide a unified definition for integrability, an integrable NLPD equation includes multi-soliton waves, bilinear Bäcklund transformation (BBT), etc. An effective method for analyzing the integrability of NLPD equations is the Painlevè method [8]. Ma et al. [9] assessed the integrability of the Sakovich equation using the Painlevè method. Chu et al. [10], in a comprehensive study, applied the Painlevè method to check the integrability of a 2D KdV equation with variable coefficient. Zhang et al. [11] explored the integrability of a variable coefficient Boussinesq equation via the Painlevè method.

According to the classical Bell polynomials theory [12] proposed in 1934, Lambert et al. [13] introduced a generalized Bell's polynomials to establish a systematic procedure for discovering the bilinear form, bilinear Bäcklund transformation, and Lax pairs for NLPD equations. The Bell polynomial approach [14–17] has extensively been used in recent decades to deal with NLPD equations. Some authors have tried to apply such an effective method to deal with a series of well-known NLPD equations. For example, Hosseini et al. [18] utilized the BPA to acquire the bilinear representation of a generalized KdV equation. Umar et al. [19] found the BBT of a 2D generalized KP equation using the BPA. Asadi et al. [20] employed the BPA to construct Lax pairs and conservation laws of a 3D extended BLMP equation.

Wazwaz in [21] proposed the Korteweg-de Vries–Caudrey Dodd Gibbon equation, i.e.,

$$u_t + c_1\left(u_{2x} + \frac{1}{5}u^2\right)_x + c_2\left(\frac{1}{15}u^3 + uu_{2x} + u_{4x}\right)_x = 0,$$

which has been composed of KdV and CDG equations, and constructed its multi solitons using the Hirota method. Later, some researchers conducted a complete study on the KdV–CDG equation and its different wave structures. For instance, Biswas et al. [22] applied the F-expansion method to acquire solitons of the KdV–CDG equation. Ma et al. [23] constructed hybrid solutions of the KdV–CDG equation using some particular operations. Hosseini et al. [24] employed systematic methods to extract solitons and complexiton of the KdV–CDG equation. Almusawa and Jhangeer [25] obtained invariant solutions of the KdV–CDG equation using the Lie method.

In this paper, we aim to conduct a detailed exploration of the following generalized Korteweg-de Vries–Caudrey Dodd Gibbon equation

$$u_t + c_1\left(u + u_{2x} + \frac{1}{5}u^2\right)_x + c_2\left(\frac{1}{15}u^3 + uu_{2x} + u_{4x}\right)_x = 0, \tag{1.1}$$

with some applications in modeling water waves in the ocean. More precisely, we:

- Examine the integrability of the gKdV–CDG equation using the Painlevè method;
- Establish the Hirota D-operator expression of the gKdV–CDG equation by employing the Bell polynomial approach;
- Obtain multi solitons of the gKdV–CDG equation through exerting the Hirota method;
- Construct breather and complexiton waves of the gKdV–CDG equation using distinct methods.

The paper's structure is as follows: In Section 2, an integrability test of the governing equation is carried out based on the Painlevè method. In Section 3, a short review of Bell's polynomials is presented, and the Hirota D-operator expression for the gKdV–CDG equation is constructed. In Section 4, multi solitons along with breather and complexiton waves to the gKdV–CDG equation are derived by serving distinct ansatzes. Additionally, in Section 4, we provide several figures positioned two- and three-dimensionally to illustrate the dynamic features of nonlinear waves. The results are summarized at the end of the paper.

2. Integrability test of the governing equation

Owing to the efforts of Weis et al. [8], the Painlevé property of the governing equation can be formally investigated. The key idea of their method is to seek the solution of Eq (1.1) as follows

$$u(x,t) = \sum_{j=0}^{\infty} u_j(x,t)\Phi^{j-\alpha}(x,t),$$

where $\Phi(x,t)$ is a singular manifold, $u_j(x,t)$ are expansion coefficients, and α is a pole order of the solution $u(x,t)$.

The Painlevé test, however, can rarely be performed directly using the above method because of numerous complications. To overcome this shortcoming, we use Kruskal's simplification [26,27], which employs a specific form for the singular manifold function as

$$\Phi(x,t) = x - \psi(t).$$

The Painlevé's test consists of three steps as follows:

***First step*:** It involves determining the leading-order terms in Eq (1.1). To achieve this, substituting

$$u(x,t) = u_0\Phi^{-\alpha},$$

into Eq (1.1) results in the values of α and $u_0(t)$ as

- 1st branch: $\alpha = 2,\ u_0 = -60\psi_x^2$;
- 2nd branch: $\alpha = 2,\ \ u_0 = -30\psi_x^2$.

***Second step*:** For the above-specified values of α and u_0, the non-negative integer values of j referred to as resonances are computed. To this end, setting

$$u(x,t) = u_0\Phi^{-2} + u_j\Phi^{j-2},$$

in Eq (1.1) gives

- resonances of 1st branch: $j = -2, -1, 5, 6, 12$;
- resonances of 2nd branch: $j = -1, 2, 3, 6, 10$.

***Third step*:** The expansion

$$u(x,t) = u_0\Phi^{-2} + \sum_{j=1}^{\infty} u_j\Phi^{j-2},$$

is substituted into Eq (1.1). For the first branch, it is found that

$$u_0 = -60,\ \ u_1 = 0,\ \ u_2 = -\frac{c_1}{c_2},\ \ u_3 = 0,$$

$$u_4 = -\frac{5c_2\psi_t + c_1^2 - 5c_1c_2}{20c_2^2},\ \ u_5 = u_5,\ \ u_6 = u_6,\ \ u_7 = 0,$$

$$u_8 = -\frac{25c_2^2\psi_t^2 + (10c_1^2c_2 - 50c_1c_2^2)\psi_t + c_1^4 - 10c_1^3c_2 + 25c_1^2c_2^2}{72000c_2^4},$$

$$u_9 = -\frac{5\psi_t^2 - 180c_2u_5\psi_t + (180c_1c_2 - 36c_1^2)u_5}{79200c_2^2},$$

$$u_{10} = \frac{30c_2u_6\psi_t + 5c_2u_{5_t} - 120c_2^2u_5^2 + (6c_1^2 - 30c_1c_2)u_6}{26400c_2^2},$$

$$u_{11} = -\frac{30c_2u_5u_6 - u_{6_t}}{4680c_2},\ \ \ u_{12} = u_{12},$$

whereas for the second branch

$$u_0 = -30,\ \ u_1 = 0,\ \ u_2 = u_2,\ \ u_3 = u_3,$$

$$u_4 = -\frac{-5\psi_t + c_2u_2^2 + 2c_1u_2 + 5c_1}{10c_2},\ \ u_5 = -\frac{u_3(c_2u_2 + c_1)}{15c_2},\ \ u_6 = u_6,$$

$$u_7 = \frac{-30c_2u_3\psi_t + 5c_2u_{2_t} + (8c_2^2u_2^2 + 16c_1c_2u_2 + 2c_1^2 + 30c_1c_2)u_3}{2400c_2^2},$$

$$u_8 = \frac{1}{27000c_2^2}\big(-75\psi_t^2 + (30c_2u_2^2 + 60c_1u_2 + 150c_1)\psi_t + 25c_2u_{3_t} + (30c_2^2u_2 + 30c_1c_2)u_3^2 +$$
$$(-3c_2^2u_2^2 - 12c_1c_2u_2 - 12c_1^2 - 30c_1c_2)u_2^2 - 60c_1^2u_2 - 75c_1^2\big),$$

$$u_9 = \frac{1}{126000c_2^3}\big(50c_2\psi_{tt} + 90(c_2^2u_2u_3 + c_1c_2u_3)\psi_t - 15(c_1c_2 + c_2^2u_2)u_{2_t} + (2c_1^3 - 90c_1^2c_2 +$$
$$20c_2^3u_3^2)u_3 - (90c_1 + 30c_1^2c_2 + 48c_1c_2^2u_2 + 16c_2^3u_2^2)u_2u_3\big),$$

$$u_{10} = u_{10}.$$

Since u_5, u_6 and u_{12} in the first branch and u_2, u_3, u_6, and u_{10} in the second branch are arbitrary functions of t, the necessary condition for the integrability of Eq (1.1) is satisfied. These show that Eq (1.1) satisfies the Painlevé test for the integrability.

3. Bell polynomials and the Hirota *D*-operator expression

As early as the 20th century, Bell [12] established his polynomial in the form

$$Y_{nt}(y)=Y_n(y_t,\dots,y_{nt})=e^{-y}\partial_t^n e^y,\ \ y=e^{\alpha t}-\alpha_0.$$

The above definition leads to the following results

$$Y_0=1,$$

$$Y_1=y_t,$$

$$Y_2=y_{2t}+y_t^2,$$

$$\vdots$$

The classical Bell polynomial, which was generalized by Lambert et al. [13], for $f(x_1,\dots,x_l)$ is

$$Y_{n_1x_1,\dots,n_lx_l}(f)=e^{-f}\partial_{x_1}^{n_1}\dots\partial_{x_l}^{n_l}e^f,$$

where the outcomes of such a generalization when $f=f(x,t)$ are

$$Y_x(f)=f_x,$$

$$Y_{2x}(f)=f_{2x}+f_x^2,$$

$$Y_{x,t}(f)=f_{x,t}+f_xf_t,$$

$$\vdots$$

Following Bell's definition, the binary Bell polynomial is

$$\mathcal{Y}_{n_1x_1,\dots,n_lx_l}(v,w)=Y_{n_1x_1,\dots,n_lx_l}(f)\Big|_{f_{r_1x_1,\dots,r_lx_l}=\begin{cases}v_{r_1x_1,\dots,r_lx_l}, & r_1+r_2+\cdots+r_l \text{ is odd,}\\ w_{r_1x_1,\dots,r_lx_l}, & r_1+r_2+\cdots+r_l \text{ is even,}\end{cases}}$$

where $r_k=0,1,\dots,n_k,k=0,1,\dots,l$. It is easy to check that

$$\mathcal{Y}_x(v)=v_x,$$

$$\mathcal{Y}_{2x}(v,w)=w_{2x}+v_x^2,$$

$$\mathcal{Y}_{x,t}(v,w)=w_{x,t}+v_xv_t$$

$$\vdots$$

Theorem 1 (See [13]). For $\mathcal{Y}_{n_1x_1,\dots,n_lx_l}(v,w)$ and $D_{x_1}^{n_1}\dots D_{x_l}^{n_l}F.G$, we have

$$\mathcal{Y}_{n_1x_1,\dots,n_lx_l}(v=\ln(F/G),w=\ln(FG))=(FG)^{-1}D_{x_1}^{n_1}\dots D_{x_l}^{n_l}F.G, \tag{3.1}$$

where $n_1+n_2+\cdots+n_l\geq 1$, and

$$D_{x_1}^{n_1}\dots D_{x_l}^{n_l}F.G=\left(\partial_{x_1}-\partial_{x_1'}\right)^{n_1}\dots\left(\partial_{x_l}-\partial_{x_l'}\right)^{n_l}F(x_1,x_2,\dots,x_l)\times G(x_1',x_2',\dots,x_l')|_{x_1'=x_1,\dots,x_l'=x_l}.$$

Equation (3.1) for $F=G$ is

$$G^{-2}D_{x_1}^{n_1}\dots D_{x_l}^{n_l}G.G=\mathcal{Y}_{n_1x_1,\dots,n_lx_l}(0,2\ln(G))=\begin{cases}0, & n_1+n_2+\cdots+n_l \text{ is odd,}\\ P_{n_1x_1,\dots,n_lx_l}(q), & n_1+n_2+\cdots+n_l \text{ is even,}\end{cases}$$

where

$$P_{2x}(q) = q_{2x},$$
$$P_{x,t}(q) = q_{x,t},$$
$$P_{4x}(q) = q_{4x} + 3q_{2x}^2,$$
$$P_{6x}(q) = q_{6x} + 15q_{2x}q_{4x} + 15q_{2x}^3,$$
$$\vdots$$

Theorem 2. If $u = 30(\ln(G))_{xx}$, then the Hirota D-operator expression of Eq (1.1) is

$$(D_xD_t + c_1D_x^2 + c_1D_x^4 + c_2D_x^6)G.G = 0.$$

Proof. To prove such an assertion, we introduce

$$u = c(t)q_{2x}, \tag{3.2}$$

with $c(t)$ is a function to connect Eq (1.1) with P-polynomials. Inserting Eq (3.2) into Eq (1.1) gives

$$c'(t)q_{2x} + c(t)q_{2x,t} + c_1c(t)\left(q_{3x} + q_{5x} + \frac{2}{5}c(t)q_{2x}q_{3x}\right) + c_2c(t)\left(\frac{1}{5}c(t)^2q_{2x}^2q_{3x} + c(t)q_{3x}q_{4x} + c(t)q_{2x}q_{5x} + q_{7x}\right) = 0.$$

Integrating with respect to x yields

$$c'(t)q_x + c(t)q_{x,t} + c_1c(t)\left(q_{2x} + q_{4x} + \frac{1}{5}c(t)q_{2x}^2\right) + c_2c(t)\left(\frac{1}{15}c(t)^2q_{2x}^3 + c(t)q_{2x}q_{4x} + q_{6x}\right) = 0.$$

A direct result of comparing $\frac{1}{15}c(t)^2q_{2x}^3 + c(t)q_{2x}q_{4x} + q_{6x}$ with $P_{6x}(q)$ is $c(t) = 15$. Now, the above equation can be expressed as

$$P_{x,t}(q) + c_1P_{2x}(q) + c_1P_{4x}(q) + c_2P_{6x}(q) = 0. \tag{3.3}$$

Through the transformation

$$q = 2\ln(G) \Leftrightarrow u = 15q_{2x} = 30(\ln(G))_{xx},$$

and Eq (3.3), then the Hirota D-operator expression of Eq (1.1) is given by

$$(D_xD_t + c_1D_x^2 + c_1D_x^4 + c_2D_x^6)G.G = 0,$$

with the following bilinear representation

$$(GG_{xt} - G_xG_t) + c_1(GG_{2x} - G_x^2) + c_1(GG_{4x} - 4G_{3x}G_x + 3G_{2x}^2) + c_2(GG_{6x} - 6G_xG_{5x} + 15G_{2x}G_{4x} - 10G_{3x}^2) = 0. \tag{3.4}$$

4. The gKdV-CGD equation and its nonlinear waves

In this section, we focus on the governing model in order to establish its multi solitons, breather wave, and complexiton. The first kind of such results is supported by analyzing the three-soliton condition as a criterion for the existence of a triple-soliton wave.

4.1. Multi solitons of the governing equation

To arrive at the single-soliton wave, we assume

$$u = e^{\theta_i}, \;\; \theta_i = k_i x + \omega_i t,$$

and insert it into

$$u_t + c_1 u_x + c_1 u_{3x} + c_2 u_{5x} = 0.$$

After simplifying, we have

$$\left(\omega_i + c_1(k_i^3 + k_i) + c_2 k_i^5\right) e^{k_i x + \omega_i t} = 0.$$

A direct result of the above expression gives the dispersion relation as

$$\omega_i = -\left(c_1(k_i^3 + k_i) + c_2 k_i^5\right),$$

and so, the phase variable θ_i can be constructed as

$$\theta_i = k_i x - \left(c_1(k_i^3 + k_i) + c_2 k_i^5\right)t.$$

Now, the single-soliton wave of Eq (1.1) is

$$u = 30(\ln(G))_{xx},$$

where

$$G = 1 + e^{\theta_1}, \;\; \theta_1 = k_1 x - (c_1(k_1^3 + k_1) + c_2 k_1^5)t.$$

The double-soliton wave of the governing equation can be determined by inserting

$$G = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2},$$

where

$$\theta_1 = k_1 x - (c_1(k_1^3 + k_1) + c_2 k_1^5)t,$$

$$\theta_2 = k_2 x - (c_1(k_2^3 + k_2) + c_2 k_2^5)t,$$

into Eq (1.1) and discovering the phase shift a_{12} through some systematic computations as

$$a_{12} = \frac{5c_2k_1^4 - 15c_2k_1^3k_2 + 20c_2k_1^2k_2^2 - 15c_2k_1k_2^3 + 5c_2k_2^4 + 3c_1k_1^2 - 6c_1k_1k_2 + 3c_1k_2^2}{5c_2k_1^4 + 15c_2k_1^3k_2 + 20c_2k_1^2k_2^2 + 15c_2k_1k_2^3 + 5c_2k_2^4 + 3c_1k_1^2 + 6c_1k_1k_2 + 3c_1k_2^2}.$$

Hence, the double-soliton wave of the governing equation is

$$u = 30(\ln(G))_{xx}, \;\; G = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2},$$

where

$$\theta_1 = k_1 x - (c_1(k_1^3 + k_1) + c_2 k_1^5)t,$$

$$\theta_2 = k_2 x - (c_1(k_2^3 + k_2) + c_2 k_2^5)t,$$

$$a_{12} = \frac{5c_2k_1^4 - 15c_2k_1^3k_2 + 20c_2k_1^2k_2^2 - 15c_2k_1k_2^3 + 5c_2k_2^4 + 3c_1k_1^2 - 6c_1k_1k_2 + 3c_1k_2^2}{5c_2k_1^4 + 15c_2k_1^3k_2 + 20c_2k_1^2k_2^2 + 15c_2k_1k_2^3 + 5c_2k_2^4 + 3c_1k_1^2 + 6c_1k_1k_2 + 3c_1k_2^2}.$$

In order to have the triple-soliton wave, the three-soliton condition [28–30]

$$\sum_{\mu_1,\mu_2,\mu_3=\pm1} P(\mu_1V_1+\mu_2V_2+\mu_3V_3)P(\mu_1V_1-\mu_2V_2)P(\mu_2V_2-\mu_3V_3)P(\mu_1V_1-\mu_3V_3)=$$

$$2\sum_{(\mu_1,\mu_2,\mu_3)\in S} P(\mu_1V_1+\mu_2V_2+\mu_3V_3)P(\mu_1V_1-\mu_2V_2)P(\mu_2V_2-\mu_3V_3)P(\mu_1V_1-\mu_3V_3),$$

in which

$$P=XT+c_1X^2+c_1X^4+c_2X^6,$$

$$V_i=(k_i,\omega_i), i=1,2,3,$$

$$S=\{(1,1,1),(1,1,-1),(1,-1,1),(-1,1,1)\},$$

must be zero. The results show that

$$2\sum_{(\mu_1,\mu_2,\mu_3)\in S} P(\mu_1V_1+\mu_2V_2+\mu_3V_3)P(\mu_1V_1-\mu_2V_2)P(\mu_2V_2-\mu_3V_3)P(\mu_1V_1-\mu_3V_3)=$$
$$2(e_1+e_2+e_3+e_4),$$

where

$$e_1=-81k_2^2(k_1-k_2)^2(k_2-k_3)^2k_3^2(k_2+k_3)(k_1-k_3)^2(k_1+k_3)\left(\frac{5}{3}(k_2^2-k_2k_3+k_3^2)c_2+c_1\right)k_1^2(k_1+k_2+k_3)\left(\frac{5}{3}(k_1^2-k_1k_2+k_2^2)c_2+c_1\right)\left(\frac{5}{3}(k_1^2+(k_2+k_3)k_1+k_2^2+k_2k_3+k_3^2)c_2+c_1\right)(k_1+k_2)\left(\frac{5}{3}(k_1^2-k_1k_3+k_3^2)c_2+c_1\right),$$

$$e_2=-81k_2^2(k_1-k_2)^2(k_2-k_3)k_3^2\left(\frac{5}{3}(k_1^2+(k_2-k_3)k_1+k_2^2-k_2k_3+k_3^2)c_2+c_1\right)(k_2+k_3)^2\left(\frac{5}{3}(k_2^2+k_2k_3+k_3^2)c_2+c_1\right)(k_1-k_3)(k_1+k_3)^2(k_1+k_2-k_3)\left(\frac{5}{3}(k_1^2+k_1k_3+k_3^2)c_2+c_1\right)k_1^2\left(\frac{5}{3}(k_1^2-k_1k_2+k_2^2)c_2+c_1\right)(k_1+k_2),$$

$$e_3=81k_2^2(k_1-k_2)\left(\frac{5}{3}(k_1^2+(-k_2+k_3)k_1+k_2^2-k_2k_3+k_3^2)c_2+c_1\right)\left(\frac{5}{3}(k_1^2+k_1k_2+k_2^2)c_2+c_1\right)(k_2-k_3)k_3^2(k_2+k_3)^2\left(\frac{5}{3}(k_2^2+k_2k_3+k_3^2)c_2+c_1\right)(k_1-k_2+k_3)(k_1-k_3)^2(k_1+k_3)k_1^2(k_1+k_2)^2\left(\frac{5}{3}(k_1^2-k_1k_3+k_3^2)c_2+c_1\right),$$

$$e_4=81k_2^2(k_1-k_2)\left(\frac{5}{3}(k_1^2+k_1k_2+k_2^2)c_2+c_1\right)(k_2-k_3)^2k_3^2(k_2+k_3)\left(\frac{5}{3}(k_1^2+(-k_2-k_3)k_1+k_2^2+k_2k_3+k_3^2)c_2+c_1\right)(k_1-k_3)(k_1+k_3)^2(k_1-k_2-k_3)\left(\frac{5}{3}(k_2^2-k_2k_3+k_3^2)c_2+c_1\right)\left(\frac{5}{3}(k_1^2+k_1k_3+k_3^2)c_2+c_1\right)k_1^2(k_1+k_2)^2,$$

is zero. For Eq (1.1), the triple-soliton wave exists as a result of the above condition.

Now, the triple-soliton wave of Eq (1.1) is

$$u=2(\ln(G))_{xx},\ \ G=1+e^{\theta_1}+e^{\theta_2}+e^{\theta_3}+a_{12}e^{\theta_1+\theta_2}+a_{13}e^{\theta_1+\theta_3}+a_{23}e^{\theta_2+\theta_3}+a_{12}a_{13}a_{23}e^{\theta_1+\theta_2+\theta_3},$$

where

$$\theta_1=k_1x-(c_1(k_1^3+k_1)+c_2k_1^5)t,$$

$$\theta_2 = k_2 x - (c_1(k_2^3 + k_2) + c_2 k_2^5)t,$$

$$\theta_3 = k_3 x - (c_1(k_3^3 + k_3) + c_2 k_3^5)t,$$

$$a_{12} = \frac{5c_2k_1^4 - 15c_2k_1^3k_2 + 20c_2k_1^2k_2^2 - 15c_2k_1k_2^3 + 5c_2k_2^4 + 3c_1k_1^2 - 6c_1k_1k_2 + 3c_1k_2^2}{5c_2k_1^4 + 15c_2k_1^3k_2 + 20c_2k_1^2k_2^2 + 15c_2k_1k_2^3 + 5c_2k_2^4 + 3c_1k_1^2 + 6c_1k_1k_2 + 3c_1k_2^2},$$

$$a_{13} = \frac{5c_2k_1^4 - 15c_2k_1^3k_3 + 20c_2k_1^2k_3^2 - 15c_2k_1k_3^3 + 5c_2k_3^4 + 3c_1k_1^2 - 6c_1k_1k_3 + 3c_1k_3^2}{5c_2k_1^4 + 15c_2k_1^3k_3 + 20c_2k_1^2k_3^2 + 15c_2k_1k_3^3 + 5c_2k_3^4 + 3c_1k_1^2 + 6c_1k_1k_3 + 3c_1k_3^2},$$

$$a_{23} = \frac{5c_2k_2^4 - 15c_2k_2^3k_3 + 20c_2k_2^2k_3^2 - 15c_2k_2k_3^3 + 5c_2k_3^4 + 3c_1k_2^2 - 6c_1k_2k_3 + 3c_1k_3^2}{5c_2k_2^4 + 15c_2k_2^3k_3 + 20c_2k_2^2k_3^2 + 15c_2k_2k_3^3 + 5c_2k_3^4 + 3c_1k_2^2 + 6c_1k_2k_3 + 3c_1k_3^2}.$$

The single, double, and triple solitons for **(a)** $\{c_1 = 1, c_2 = 1, k_1 = 0.7\}$, **(b)** $\{c_1 = 1, c_2 = 1, k_1 = 0.7, k_2 = 1\}$, and **(c)** $\left\{c_1 = 1, c_2 = 1, k_1 = \frac{4}{5}, k_2 = 1, k_3 = \frac{9}{7}\right\}$ have been portrayed in Figure 1. Using the above parameter regimes, the height and width of such waves can be assessed.

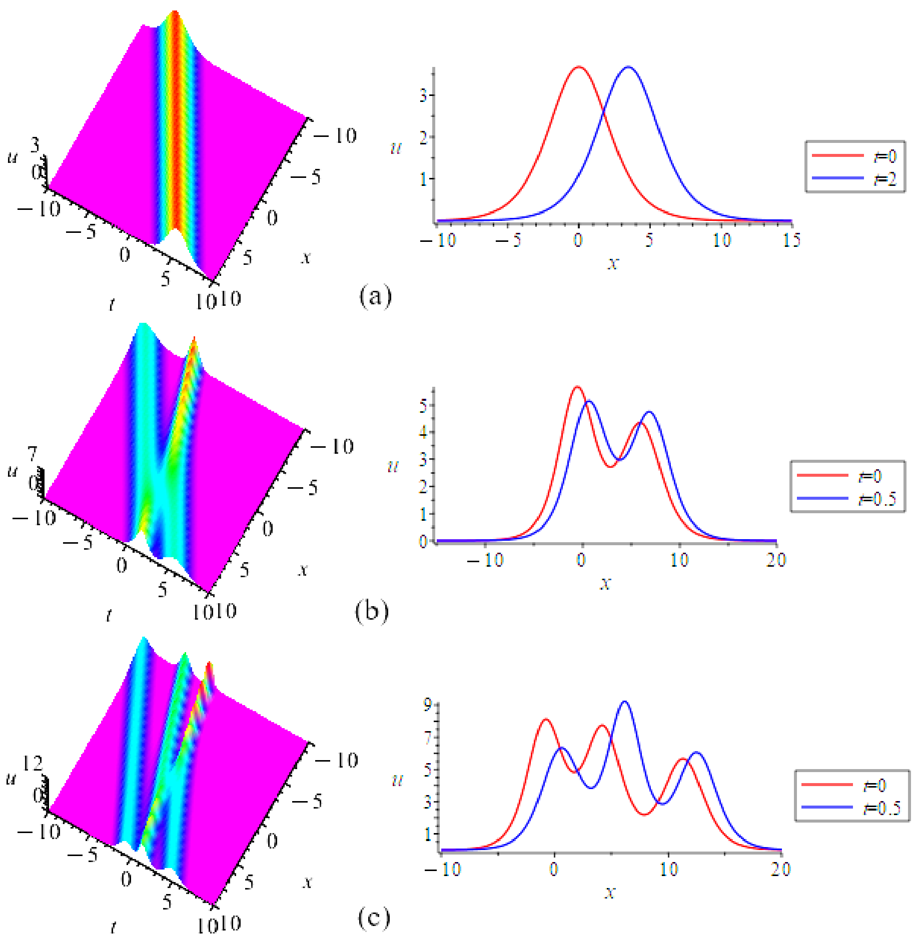

Figure 1. **(a)** Single soliton for $c_1 = 1$, $c_2 = 1$, and $k_1 = 0.7$; **(b)** Double soliton for $c_1 = 1$, $c_2 = 1$, $k_1 = 0.7$, and $k_2 = 1$; and **(c)** Triple soliton for $c_1 = 1$, $c_2 = 1$, $k_1 = \frac{4}{5}$, $k_2 = 1$, and $k_3 = \frac{9}{7}$.

4.2. Breather and complexiton waves of the governing model

To obtain the breather wave of the governing equation, we apply an ansatz as

$$G = e^{-kX} + b_0 \cos(hY) + b_1 e^{kX}, \tag{4.1}$$

where

$$X = a_1 x + a_2 t + a_3,$$

$$Y = a_4 x + a_5 t + a_6,$$

and k, b_0, h, b_1, and a_i, $i = 1,2,\ldots,6$ are unknowns. Inserting Eq (4.1) into Eq (3.4) results in

$$h^6 c_2 a_4^6 - a_4^4(15k^2 a_1^2 c_2 + c_1)h^4 + a_4(15k^4 a_1^4 a_4 c_2 + 6k^2 a_1^2 a_4 c_1 + c_1 a_4 + a_5)h^2 - k^2 a_1(c_2 a_1^5 k^4 + c_1 k^2 a_1^3 + c_1 a_1 + a_2) = 0,$$

$$h^4 c_2 a_1 a_4^5 - \frac{2}{3} a_1 a_4^3(5k^2 a_1^2 c_2 + c_1)h^2 + c_2 k^4 a_1^5 a_4 + \frac{2}{3} c_1 k^2 a_1^3 a_4 + \left(\frac{1}{3} c_1 a_4 + \frac{1}{6} a_5\right) a_1 + \frac{1}{6} a_2 a_4 = 0,$$

$$(-16h^6 c_2 a_4^6 + 4c_1 a_4^4 h^4 + (-a_4^2 c_1 - a_4 a_5)h^2) b_0^2 + 16 a_1 b_1 \left(4c_2 a_1^5 k^4 + c_1 k^2 a_1^3 + \frac{1}{4} c_1 a_1 + \frac{1}{4} a_2\right) k^2 = 0,$$

whose solution is

$$a_2 = -a_1(5a_4^4 h^4 c_2 - 10h^2 a_4^2 k^2 a_1^2 c_2 + k^4 a_1^4 c_2 - 3h^2 a_4^2 c_1 + k^2 a_1^2 c_1 + c_1),$$

$$a_5 = -h^4 a_4^5 c_2 + 10h^2 k^2 a_1^2 a_4^3 c_2 - 5k^4 a_1^4 a_4 c_2 + h^2 a_4^3 c_1 - 3k^2 a_1^2 a_4 c_1 - c_1 a_4,$$

$$b_1 = -\frac{(15h^2 a_4^2 c_2 - 5k^2 a_1^2 c_2 - 3c_1) b_0^2 a_4^2 h^2}{4(5h^2 a_4^2 c_2 - 15k^2 a_1^2 c_2 - 3c_1) k^2 a_1^2}.$$

As a consequence, the breather wave of the governing equation is

$$u = 30(\ln(G))_{xx}, \;\; G = e^{-kX} + b_0 \cos(hY) + b_1 e^{kX},$$

where

$$X = a_1 x - a_1(5a_4^4 h^4 c_2 - 10h^2 a_4^2 k^2 a_1^2 c_2 + k^4 a_1^4 c_2 - 3h^2 a_4^2 c_1 + k^2 a_1^2 c_1 + c_1)t + a_3,$$

$$Y = a_4 x - (h^4 a_4^5 c_2 - 10h^2 k^2 a_1^2 a_4^3 c_2 + 5k^4 a_1^4 a_4 c_2 - h^2 a_4^3 c_1 + 3k^2 a_1^2 a_4 c_1 + c_1 a_4)t + a_6,$$

$$b_1 = -\frac{(15h^2 a_4^2 c_2 - 5k^2 a_1^2 c_2 - 3c_1) b_0^2 a_4^2 h^2}{4(5h^2 a_4^2 c_2 - 15k^2 a_1^2 c_2 - 3c_1) k^2 a_1^2}.$$

Figure 2 represents the dynamical characteristics of the breather wave for

$$\{c_1 = 0.5, c_2 = 1, a_1 = -0.6, a_3 = 0.5, a_4 = -1, a_6 = 1, b_0 = 0.1, h = 1, k = 1\},$$

in three- and two-dimensional postures. This figure demonstrates the height and width of such a wave for the above parameter regime.

To derive the complexiton wave, the following assumptions are taken:

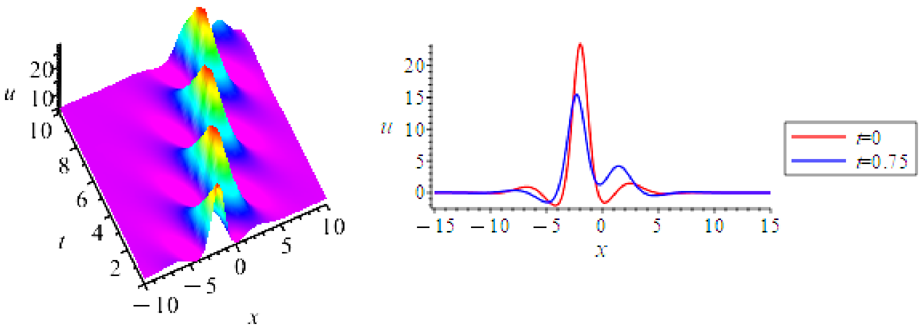

Figure 2. Breather wave when $c_1 = 0.5$, $c_2 = 1$, $a_1 = -0.6$, $a_3 = 0.5$, $a_4 = -1$, $a_6 = 1$, $b_0 = 0.1$, $h = 1$, and $k = 1$.

$$\mu = \mu_1 + i\mu_2, \ \ \nu = \nu_1 + i\nu_2, \ \ P(x,t) = xt + c_1x^2 + c_1x^4 + c_2x^6.$$

Due to $P(\mu,\nu) = 0$, we obtain a nonlinear system of algebraic equations as

$$6c_2\mu_1^5\mu_2 - 20c_2\mu_1^3\mu_2^3 + 6c_2\mu_1\mu_2^5 + 4c_1\mu_1^3\mu_2 - 4c_1\mu_1\mu_2^3 + 2c_1\mu_1\mu_2 + \nu_2\mu_1 + \nu_1\mu_2 = 0,$$

$$c_2\mu_1^6 - 15c_2\mu_1^4\mu_2^2 + 15c_2\mu_1^2\mu_2^4 - c_2\mu_2^6 + c_1\mu_1^4 - 6c_1\mu_1^2\mu_2^2 + c_1\mu_2^4 + c_1\mu_1^2 - c_1\mu_2^2 + \nu_1\mu_1 - \nu_2\mu_2 = 0.$$

The solution set for the above system is

$$\nu_1 = -\mu_1(c_2\mu_1^4 - 10c_2\mu_1^2\mu_2^2 + 5c_2\mu_2^4 + c_1\mu_1^2 - 3c_1\mu_2^2 + c_1),$$

$$\nu_2 = -5c_2\mu_1^4\mu_2 + 10c_2\mu_1^2\mu_2^3 - c_2\mu_2^5 - 3c_1\mu_1^2\mu_2 + c_1\mu_2^3 - c_1\mu_2.$$

Additionally, the phase shift can be found as

$$a_{12} = -\frac{p(2i\mu_2,2i\nu_2)}{p(2\mu_1,2\nu_1)} = -\frac{-64c_2\mu_2^6+16c_1\mu_2^4-4c_1\mu_2^2-4(-5c_2\mu_1^4\mu_2+10c_2\mu_1^2\mu_2^3-c_2\mu_2^5-3c_1\mu_1^2\mu_2+c_1\mu_2^3-c_1\mu_2)\mu_2}{64c_2\mu_1^6+16c_1\mu_1^4+4c_1\mu_1^2-4\mu_1^2(c_2\mu_1^4-10c_2\mu_1^2\mu_2^2+5c_2\mu_2^4+c_1\mu_1^2-3c_1\mu_2^2+c_1)}.$$

Now, the complexiton wave of the governing equation is

$$u = 30(\ln(G))_{xx},$$

where

$$G = 1 + 2e^{\vartheta_1}\cos(\vartheta_2) + a_{12}e^{2\vartheta_1},$$

$$\vartheta_i = \mu_i x + \nu_i t, \ \ i = 1,2,$$

$$\nu_1 = -\mu_1(c_2\mu_1^4 - 10c_2\mu_1^2\mu_2^2 + 5c_2\mu_2^4 + c_1\mu_1^2 - 3c_1\mu_2^2 + c_1),$$

$$\nu_2 = -5c_2\mu_1^4\mu_2 + 10c_2\mu_1^2\mu_2^3 - c_2\mu_2^5 - 3c_1\mu_1^2\mu_2 + c_1\mu_2^3 - c_1\mu_2,$$

$$a_{12} = -\frac{-64c_2\mu_2^6+16c_1\mu_2^4-4c_1\mu_2^2-4(-5c_2\mu_1^4\mu_2+10c_2\mu_1^2\mu_2^3-c_2\mu_2^5-3c_1\mu_1^2\mu_2+c_1\mu_2^3-c_1\mu_2)\mu_2}{64c_2\mu_1^6+16c_1\mu_1^4+4c_1\mu_1^2-4\mu_1^2(c_2\mu_1^4-10c_2\mu_1^2\mu_2^2+5c_2\mu_2^4+c_1\mu_1^2-3c_1\mu_2^2+c_1)}.$$

The dynamical feature of the complexiton wave positioned three- and two-dimensionally for

$$\{c_1 = 1, c_2 = 1.75, \mu_1 = 1, \mu_2 = -1.5\},$$

has been given in Figure 3. Through the above parameter regime, the height and width of such a wave can be analyzed. More details about the origin of the complexiton wave can be found in [31].

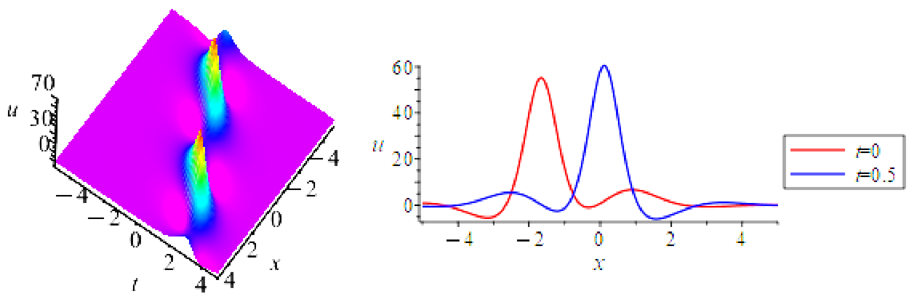

Figure 3. Complexiton wave when $c_1 = 1$, $c_2 = 1.75$, $\mu_1 = 1$, and $\mu_2 = -1.5$.

4.3. Other nonlinear waves of the governing equation

Other nonlinear waves of the governing equation can be derived by serving the following ansatzes [32]:

1) $u = A\,\mathrm{SN}^2(x - wt, k)$,
2) $u = A\,\mathrm{CN}^2(x - wt, k)$,
3) $u = A\,\mathrm{DN}^2(x - wt, k)$,

where A and w are unknowns.

By setting the first ansatz in Eq (1.1), we have a nonlinear system as

$$\left((k^2+1)c_2 - \frac{1}{20}c_1\right)(30k^2 + A) = 0,$$

$$(60k^2 + A)(30k^2 + A) = 0,$$

$$10(8k^4 + 52k^2 + A + 8)c_2 - 20k^2c_1 - 5w - 15c_1 = 0,$$

for which the solution set is

$$A = -30k^2,\ \ w = 16k^4c_2 - 4k^2c_1 + 44k^2c_2 - 3c_1 + 16c_2.$$

As a result, the Jacobi elliptic solution of the governing equation is

$$u = -30k^2\,\mathrm{SN}^2(x - (16k^4c_2 - 4k^2c_1 + 44k^2c_2 - 3c_1 + 16c_2)t, k).$$

Considering $k = 1$, it gives the following bright soliton

$$u = -30\tanh^2(x - (76c_2 - 7c_1)t).$$

Additionally, by substituting the second ansatz in Eq (1.1), we arrive at the following nonlinear system

$$(-30k^2 + A)\big((-20k^2 + A - 20)c_2 + c_1\big) = 0,$$

$$(-30k^2 + A)(-60k^2 + A) = 0,$$

$$(A^2 + (-20k^2 - 30)A + 80k^4 + 520k^2 + 80)c_2 - 20c_1k^2 + 2c_1A - 5w - 15c_1 = 0,$$

whose solution is

$$A = 30k^2, \;\; w = 76k^4c_2 + 8c_1k^2 - 76k^2c_2 - 3c_1 + 16c_2.$$

Accordingly, the Jacobi elliptic solution of the governing equation is

$$u = 30k^2 \,\mathrm{CN}^2(x - (76k^4c_2 + 8c_1k^2 - 76k^2c_2 - 3c_1 + 16c_2)t, k).$$

Assuming $k = 1$, we yield the following soliton wave

$$u = 30\,\mathrm{sech}^2(x - (5c_1 + 16c_2)t).$$

Now, setting the third ansatz in Eq (1.1) leads to a nonlinear system as

$$(A - 30)(A - 60) = 0,$$

$$(A - 30)k^4(-20k^2c_2 + (A - 20)c_2 + c_1) = 0,$$

$$80k^4c_2 + \big((-30A + 520)c_2 - 20c_1\big)k^2 + (A^2 - 20A + 80)c_2 + 2c_1A - 5w - 15c_1 = 0,$$

with the following solution set

$$A = 30, w = 16k^4c_2 - 4c_1k^2 - 76k^2c_2 + 9c_1 + 76c_2.$$

As a consequence, the Jacobi elliptic solution of the governing equation is

$$u = 30\,\mathrm{DN}^2(x - (16k^4c_2 - 4c_1k^2 - 76k^2c_2 + 9c_1 + 76c_2)t, k).$$

Letting $k = 1$, it can obtain the following soliton wave

$$u = 30\,\mathrm{sech}^2(x - (5c_1 + 16c_2)t).$$

The first continuous wave and its corresponding bright wave have been depicted in three- and two-dimensional postures in Figure 4 when **(a)** $\{c_1 = 1, c_2 = 1, k = 0.1\}$ and **(b)** $\{c_1 = 1, c_2 = 1\}$. This figure illustrates the height and width of such waves for the above parameter regimes.

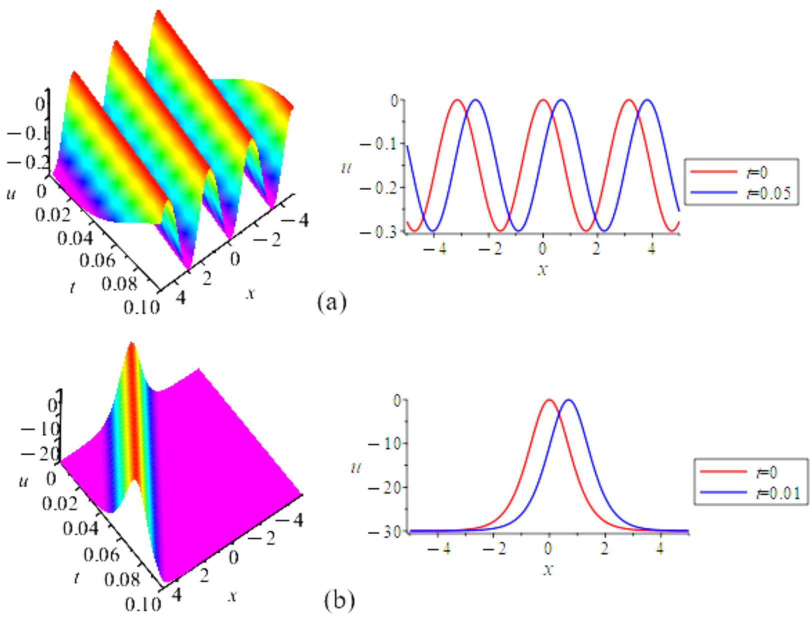

Figure 4. (a) The first continuous wave for $c_1 = 1$, $c_2 = 1$, and $k = 0.1$; and **(b)** the first bright wave for $c_1 = 1$ and $c_2 = 1$.

5. Conclusions

In this paper, we presented an in-depth study of specific oceanic waves based on a generalized Korteweg-de Vries–Caudrey Dodd Gibbon equation. As a starting point, we constructed the Hirota D-operator expression of the gKdV–CDG equation by using the Bell polynomial approach. Based on the Painlevè analysis, the governing model was tested for integrability, and multi solitons were formally retrieved. As a result of symbolic computations, breather and complexiton waves were derived from the gKdV–CDG equation based on distinct ansatzes. For the dynamical features of nonlinear waves, a few representations positioned two- and three-dimensionally have been provided. Our findings suggest useful ways for assessing the height and width of nonlinear waves in the ocean. In light of the fact that some other waves are missing in this paper, future work can be devoted to finding these waves.

Author contributions

Kamyar Hosseini: Supervision, Conceptualization, Writing–original draft; Farzaneh Alizadeh: Writing–original draft, Methodology, Investigation; Sekson Sirisubtawee: Supervision, Methodology, Investigation, Writing–original draft, Funding acquisition; Chaiyod Kamthorncharoen: Methodology, Investigation, Writing–review & editing; Samad Kheybari: Writing – review & editing; Kaushik Dehingia: Writing–review & editing.

All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have used Artificial Intelligence (AI) tools (Grammarly and Wordtune) to improve their writing quality.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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