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Research article

A common fixed point theorem and its applications in abstract convex spaces

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Abstract: First, this paper introduces a family of generalized equi-KKM mappings and the concept of quasi-abstract convexity (concavity) defined by parameter multi-valued mappings in abstract convex spaces that satisfy the H_0-condition. Then, using the Brouwer fixed-point theorem, we prove a common fixed-point theorem for this family of generalized equi-KKM mappings. Finally, we introduce two classes of generalized abstract equilibrium problems and obtain two existence theorems for their solutions, as an application of the common fixed-point theorem.

Keywords: H_0-condition; abstract convex space; the family of generalized equi-KKM mappings; common fixed points; generalized abstract equilibrium problems
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1. Introduction

The fixed point and fixed-point theorems are fundamental tools in nonlinear analysis and many research fields, such as differential equations, topology, functional analysis, optimal control, and game theory. One of the earliest fixed-point theorems was proposed by Brouwer [1] in 1910, which states that a continuous self-mapping on a non-empty bounded closed convex set in Euclidean space has at least one fixed point. Subsequently, numerous scholars extended Brouwer's fixed-point theorem to set-valued mappings, achieving significant results. These extensions include the Kakutani fixed point theorem [2], the Fan-Browder fixed point theorem [3, 4], and the Fan-Glicksberg fixed point theorem [5].

The existence of solutions to many nonlinear problems can often be addressed by determining whether certain families of sets have a nonempty intersection property. In 1929, Knaster et al. [6] first proposed the celebrated KKM theorem in finite-dimensional spaces, which is a crucial result of the intersection of nonempty sets. In 1961, Ky Fan [7] generalized the KKM theorem to the infinite dimensional spaces. These results have yielded a class of multi-valued mapping named KKM mappings, which is closely related to fixed-point theorems. Another direction for the generalization of fixed-point theorems is the extension of the convex structure of the space. In 1987, Horvath [8–10] introduced the concept of H-spaces (or C-spaces) by replacing the original linear convexity with contractibility and investigated fixed point theorems for set-valued mappings in these spaces. In 2001, Park [11] established the Fan-Browder fixed point theorems using KKM theorems in G-convex spaces. After that, Park [12] studied new versions of KKM theorems that derived the Fan-Browder type fixed point theorems in G-convex spaces. Ding [13] found generalized G-KKM theorems and then acquired a fixed-point theorem. Luo [14] obtained Ky Fan's section theorem in topological ordered spaces, as its application, a fixed-point theorem was acquired. Luo [15] established a generalized Fan-Browder fixed point theorem by using the Ky Fan inequality in topologically ordered spaces. Xiang et al. [16] demonstrated that an abstract convex space possesses the KKM property if and only if it has the strong Fan-Browder property. Subsequently, they introduced an abstract convex structure via upper semicontinuous set-valued mappings and established several generalized forms of the KKM lemma. Utilizing the general KKM lemma, they derived some extensions of minimax inequalities, which encompass several existing minimax inequalities as special cases. Since then, numerous extensions of fixed-point theorems in topological spaces equipped with these convex structures and their applications in nonlinear analysis (see references [17–19]).

In 2009, Agarwal et al. [20] introduced the concept of other set-valued mappings being generalized KKM mappings. Utilizing this concept, they established several common fixed-point theorems in locally convex Hausdorff topological linear spaces and applied them to prove Ky Fan-type minimax inequalities. In 2010, Balaj [21] employed the Brouwer fixed-point theorem to study a common fixed point theorem for a family of generalized equi-KKM set-valued mappings in topological linear spaces. As an application of this result, several important results concerning equilibrium problems and minimax inequalities were obtained. These existence results for equilibrium problems differ from the classical existence results. For more details, see [21]. Subsequently, common fixed-point theorems and their applications in variational inequalities, vector equilibrium problems, and Ky Fan-type minimax inequalities have been extensively studied, with some research achievements reported in [22–24].

Currently, the study of common fixed points has predominantly focused on Hausdorff topological spaces, while investigations within the framework of abstract convex spaces remain relatively limited. Inspired by the research above, this paper aims to introduce the concepts of generalized equi-KKM families of mappings and quasi-abstract convexity (concavity) to the setting of abstract convex spaces. Furthermore, a common fixed-point theorem is established and applied to generalized abstract equilibrium problems.

The rest of the paper is organized as follows: Section 2 provides some basic notions. Section 3 introduces the definition of a family of generalized equi-KKM mappings in an abstract convex space fulfilling the H_0-condition and establishes a common fixed-point theorem for such mappings under certain conditions. Section 4 proposes the concepts of quasi-abstract convex (concave) mappings and two classes of generalized abstract equilibrium problems as well as their solutions with respect to the abstract sequences defined by parameter multi-valued mappings. As an application of the theorem obtained in Section 3, the existence theorems for the solutions of the generalized abstract equilibrium problems are acquired under suitable conditions. This paper is summarized in Section 5.

2. Preliminaries

In this paper, we will use 2^U to denote the set of all subsets of the space U, and $\langle U \rangle$ to denote the set of all non-empty finite subsets of U. Let $\Lambda \subset U$. The notation $|\Lambda|$ represents the number of elements of Λ. We define $\overline{\Lambda}$ as the closure of Λ, $U \setminus \Lambda$ as the set difference, $co\,\Lambda$ as the convex hull in Euclidean space, and $\operatorname{int}\Lambda$ as the interior of the subset Λ. We also assume that the abstract convex space satisfies the H_0-condition.

Definition 2.1. *([18]) A topological space U is said to be an abstract convex space with an abstract convex structure Ξ, denoted by (U, Ξ), if the family of subsets Ξ of the topological space U satisfies the following properties:*
(1) $\emptyset \in \Xi$;
(2) For any family of subsets $D \subset \Xi$, there is $\bigcap_{\Lambda \in D} \Lambda \subset \Xi$.

The elements in Ξ are called abstract convex subsets of the space U, or abstract convex sets for short.

Remark 2.1. *Let Ξ be an abstract convex structure of U. The convex hull conv is defined as*

$$conv\,(\Lambda) = \bigcap_{\Lambda \subset B} \{B : B \in \Xi\},\ \Lambda \subset U.$$

A subset $\Lambda \subset U$ is an abstract convex set if and only if it satisfies $\Lambda = conv\,(\Lambda)$.

Let $N = \{0,\ 1,\ 2,\ \ldots\ ,\ n\}$, $\Delta_N = \{e_0,\ e_1,\ \ldots\ ,\ e_n\}$ be the standard simplex of dimension n, where $\{e_0,\ e_1,\ \ldots\ ,\ e_n\}$ is the canonical basis of R^{n+1}, and for $J \subset N$, let $\Delta_J = co\left\{e_j : j \in J\right\}$ be a face of Δ_N.

Definition 2.2. *([18]) (U, Ξ) satisfies the H_0-condition if for any $\Lambda = \{\kappa_0, \kappa_1, \ldots, \kappa_n\} \in \langle U \rangle$, there exists a continuous mapping $\xi : \Delta_N \to conv\,(\Lambda)$ such that $\xi\,(\Delta_J) \subset conv\left\{\kappa_j : j \in J\right\}$ for any nonempty set $J \subset N$.*

We recall the following concepts regarding the upper and lower semicontinuity of set-valued mappings, as well as the known lemmas (see references [22] and [23] for (1) and (2) of Lemma 2.1, respectively).

Definition 2.3. *([25]) Suppose that U, V are topological spaces and $T : U \to 2^V$ is a set-valued mapping.*
(1) T is called upper semicontinuous (for short, u.s.c.) (respectively, lower semicontinuous (for short, l.s.c.)) if for each closed set B of V, the set $\{\kappa \in U : T\,(\kappa) \cap B \neq \emptyset\}$ (respectively, $\{\kappa \in U : T\,(\kappa) \subseteq B\}$) is a closed set;
(2) T is closed if its graph is a closed set in $U \times V$;
(3) T is compact if $\overline{T\,(\kappa)}$ is a compact set in V.

Lemma 2.1. *Let U and V be topological spaces, and let a set-valued mapping $T : U \to 2^V$ satisfy the following properties:*
(1) T is closed if V is regular and T is upper semicontinuous with closed values;
(2) T is l.s.c. if and only if, for any $\kappa \in U$, $\gamma \in T\,(\kappa)$, and any net $\{\kappa_t\}$ converging to κ, there exists a net $\{\gamma_t\}$ converging to γ such that $\gamma_t \in T\,(\kappa_t)$ for any t.

3. A common fixed point theorem

In this section, we first introduce the concept of a family of generalized equi-KKM mappings in abstract convex spaces and then present the existence results for common fixed points under certain conditions.

Definition 3.1. *Assume that U is a nonempty set and Z is a nonempty abstract convex subset of an abstract convex space. Let L be the family of all nonempty set-valued mappings from U to 2^Z. L is said to be a family of generalized equi-KKM mappings if, for any $\{\kappa_0, \kappa_1, \ldots, \kappa_n\} \in \langle U \rangle$, there exists $\{z_0, z_1, \ldots, z_n\} \in \langle Z \rangle$ such that $conv\{z_i : i \in J\} \subseteq \bigcup_{i \in J} T(\kappa_i)$ for any nonempty set $J \subset \{0, 1, \ldots, n\}$ and any $T \in L$.*

Remark 3.1. *If Z is a nonempty abstract convex subset of an abstract convex space and L is a family of generalized equi-KKM mappings, then it follows from Lemma 3.3 in reference [2] that the family of sets $\left\{\overline{T(\kappa)} : \kappa \in U\right\}$ has the finite intersection property for each $T \in L$.*

Theorem 3.1. *Suppose that U is a nonempty abstract convex subset of an abstract convex space and V is a nonempty set. If a compact multi-valued mapping $T : U \times V \to 2^U$ satisfies the following conditions:*

(1) For any $\gamma \in V$, $\{\kappa \in U : \kappa \in T(\kappa, \gamma)\}$ is closed;

(2) The family of multi-valued mappings $L = \{T(\kappa, \cdot)\}_{\kappa \in U}$ is generalized equi-KKM on V.

Then $L = \{T(\kappa, \cdot)\}_{\kappa \in U}$ has a common fixed point, i.e., there exists $\kappa_0 \in U$ such that $\kappa_0 \in \bigcap_{\gamma \in V} T(\kappa_0, \gamma)$.

Proof. For each $\gamma \in V$, let $\zeta(\gamma) = \{\kappa \in U : \kappa \notin T(\kappa, \gamma)\}$. From (1) of Theorem 3.1, for any $\gamma \in V$, $\zeta(\gamma)$ is an open set. Assume that the conclusion is not true. Then for any $\kappa \in U$, we have $\kappa \in U \setminus \left[\bigcap_{\gamma \in V} T(\kappa, \gamma)\right] = \bigcup_{\gamma \in V} [U \setminus T(\kappa, \gamma)]$. This implies that for any $\kappa \in U$, there exists $\gamma \in V$ such that $\kappa \notin T(\kappa, \gamma)$. Consequently, for any $\kappa \in U$, there exists $\gamma \in V$ such that $\kappa \in \zeta(\gamma)$, and thus $U \subset \bigcup_{\gamma \in V} \zeta(\gamma)$. Since $\zeta(\gamma) = \{\kappa \in U : \kappa \notin T(\kappa, \gamma)\} \subseteq U$ for any $\gamma \in V$, we obtain $U = \bigcup_{\gamma \in V} \zeta(\gamma)$, which means that $\{\zeta(\gamma)\}_{\gamma \in V}$ is an open coverage of U. According to $T : U \times V \to 2^U$ is a compact multi-valued mapping, we acquire that the set $\overline{T(U \times V)} \subset U$ is compact and the family of sets $\{\zeta(\gamma)\}_{\gamma \in V}$ is an open coverage of $\overline{T(U \times V)}$; hence, there exists a finite subcoverage of $\{\zeta(\gamma)\}_{\gamma \in V}$. That is, there exists $\{\gamma_0, \gamma_1, \ldots, \gamma_n\} \in \langle V \rangle$ such that $\overline{T(U \times V)} \subseteq \bigcup_{i=0}^{n} \zeta(\gamma_i)$. For any $\kappa \in U \setminus \overline{T(U \times V)}$, i.e., there exists $\kappa \in U$, but $\kappa \notin \overline{T(U \times V)}$. Then $\kappa \notin T(\kappa, \gamma)$ for any $\gamma \in V$. By the definition of ζ, we have $\kappa \in \zeta(\gamma_i)$ for any $i \in N$, and hence $U \setminus \overline{T(U \times V)} \subseteq \bigcup_{i=0}^{n} \zeta(\gamma_i)$. Combining this with the previous result, we obtain $U = \bigcup_{i=0}^{n} \zeta(\gamma_i)$. By (2) of Theorem 3.1, for $\{\gamma_0, \gamma_1, \ldots, \gamma_n\} \in \langle V \rangle$, there exists $\{z_0, z_1, \ldots, z_n\} \in \langle U \rangle$ such that $conv\{z_i : i \in J\} \subseteq \bigcup_{i \in J} T(\kappa, \gamma_i)$ for any nonempty set $J \subseteq N$ and any $\kappa \in U$.

Set $W = conv\{z_0, z_1, \ldots, z_n\}$, then $\{\zeta(\gamma_i) \cap W\}_{0 \leq i \leq n}$ is the coverage of W. Consider a partition of unity on W, $\{\beta_0, \beta_1, \ldots, \beta_n\}$, subordinated to this open coverage. It satisfies the following conditions:

(1) For each $i \in N, \beta_i : W \to [0, 1]$ is continuous;

(2) $\beta_i(\kappa) > 0 \Rightarrow \kappa \in \zeta(\gamma_i) \cap W$;

(3) For each $\kappa \in W$, $\sum_{i=0}^{n} \beta_i(\kappa) = 1$.

Define the mapping $\pi : W \to \Delta_N$ as $\pi(\kappa) = \sum_{i=0}^{n} \beta_i(\kappa) e_i$ for any $\kappa \in W$. Then π is continuous. Since U is a nonempty abstract convex subset of an abstract convex space satisfying the H_0-condition; for $\{z_0, z_1, \ldots, z_n\} \in \langle U \rangle$, there exists a continuous mapping $\xi : \Delta_N \to W$ such that $\xi(\Delta_J) \subset conv\left\{z_j : j \in J\right\}$ for any nonempty set $J \subseteq N$.

Define the mapping $h : \Delta_N \to \Delta_N$ as $h(\kappa) = (\pi \circ \xi)(\kappa)$ for any $\kappa \in \Delta_N$. Since ξ and π are continuous, the composite mapping h is also continuous. By the Brouwer fixed-point theorem, there exists $e \in \Delta_N$ such that $e = h(e) = \pi \circ \xi(e)$. Let $\kappa_0 = \xi(e)$. Then $e = \pi(\kappa_0) = \sum_{i=0}^{n} \beta_i(\kappa_0) e_i$. Define $I(\kappa_0) = \{i \in N : \beta_i(\kappa_0) > 0\}$. Then $\beta_i(\kappa_0) > 0$ for any $i \in I(\kappa_0)$. We have $\kappa \in \zeta(\gamma_i) \cap W$, which implies that $\kappa_0 \notin T(\kappa_0, \gamma_i)$. Hence, $\kappa_0 \notin \bigcup_{i \in I(\kappa_0)} T(\kappa_0, \gamma_i)$.

On the other hand, for any $i \in I(\kappa_0)$, we have $\kappa_0 = \xi(e) \in conv\{z_i : i \in I(\kappa_0)\} \subseteq \bigcup_{i \in I(\kappa_0)} T(\kappa_0, \gamma_i)$, which leads to a contradiction. □

4. Applications

In this section, we introduce several concepts and provide their equivalent descriptions. We then apply Theorem 3.1 to establish the existence of solutions for generalized abstract equilibrium problems. Our results extend Definition 2 and Theorems 1–3 from reference [20] to abstract convex spaces without an algebraic structure.

Definition 4.1. *Suppose that U is a nonempty set, V is a nonempty abstract convex subset of an abstract convex space satisfying the H_0-condition, and Φ is an abstract convex space. Let $C : U \times \Phi \to 2^{\Phi}$ and $\Gamma : U \times V \to 2^{\Phi}$ be multi-valued mappings. For any $\kappa \in U$, the parametric multi-valued mapping $C_\kappa : \Phi \to 2^{\Phi}$ is abstract convex valued.*

(1) Γ is said to be $C_\kappa(\cdot)$-quasi abstract convex if, for any $\kappa \in U$, γ_1, $\gamma_2 \in V$, and $\gamma \in conv\{\gamma_1, \gamma_2\}$, either $\Gamma(\kappa, \gamma_1) \subseteq C_\kappa(\Gamma(\kappa, \gamma))$ or $\Gamma(\kappa, \gamma_2) \subseteq C_\kappa(\Gamma(\kappa, \gamma))$;

(2) Γ is said to be $C_\kappa(\cdot)$-quasi-abstract-convex-like if, for any $\kappa \in U$, γ_1, $\gamma_2 \in V$, and $\gamma \in conv\{\gamma_1, \gamma_2\}$, either $\Gamma(\kappa, \gamma) \subseteq C_\kappa(\Gamma(\kappa, \gamma_1))$ or $\Gamma(\kappa, \gamma) \subseteq C_\kappa(\Gamma(\kappa, \gamma_2))$.

The following lemma can be proved by induction.

Lemma 4.1. *Suppose that U is a nonempty set, V is a nonempty abstract convex subset of an abstract convex space satisfying the H_0-condition, and Φ is an abstract convex space. Let $C : U \times \Phi \to 2^{\Phi}$ and $\Gamma : U \times V \to 2^{\Phi}$ be multi-valued mappings. For any $\kappa \in U$, the parametric multi-valued mapping $C_\kappa : \Phi \to 2^{\Phi}$ has abstract convex values.*

(1) Γ is said to be $C_\kappa(\cdot)$-quasi abstract convex if, for any $\kappa \in U$, $\gamma_i \in V$ for $i \in N$, and $\gamma \in conv\{\gamma_i : i \in N\}$, there exists $j \in N$ such that $\Gamma\left(\kappa, \gamma_j\right) \subseteq C_\kappa(\Gamma(\kappa, \gamma))$;

(2) Γ is said to be $C_\kappa(\cdot)$-quasi-abstract-convex-like if, for any $\kappa \in U$, $\gamma_i \in V$ for $i \in N$, and $\gamma \in conv\{\gamma_i : i \in N\}$, there exists $j \in N$ such that $\Gamma(\kappa, \gamma) \subseteq C_\kappa\left(\Gamma\left(\kappa, \gamma_j\right)\right)$.

Assume that U is a nonempty subset of a topological linear space and $\xi : U \times U \to R$ satisfies $\xi(\kappa, \kappa) \geq 0$ for any $\kappa \in U$. The quantitative equilibrium problem, in the sense of Blum and

Oettli (1994), is to find $\kappa_0 \in U$ such that $\xi(\kappa_0, \gamma) \geq 0$ for any $\gamma \in U$. The scalar equilibrium problem has been generalized in various ways to vector equilibrium problems involving multi-valued mappings.

We will investigate two types of generalized abstract equilibrium problems, described as follows.

Suppose that U is a nonempty set, V is a nonempty, compact, abstract convex subset of an abstract convex space satisfying the H_0-condition, and Φ is an abstract convex space. Let $C : U \times \Phi \to 2^\Phi$ and $\Gamma : U \times V \to 2^\Phi$ be multi-valued mappings. For any $\kappa \in U$, the parametric multi-valued mapping $C_\kappa : \Phi \to 2^\Phi$ is abstract convex valued. Assume that $\operatorname{int} C_\kappa(v)$ is a nonempty abstract convex set for any $\kappa \in U$ and $v \in \Phi$ in the following cases. The two types of generalized abstract equilibrium problems are as follows:

(1) Find $\kappa_0 \in U$ such that for some $v_0 \in \Phi$, $\Gamma(\kappa_0, \gamma) \not\subset \operatorname{int} C_{\kappa_0}(v_0)$ holds for any $\gamma \in V$.

(2) Find $\kappa_0 \in U$ such that for some $v_0 \in \Phi$, $\Gamma(\kappa_0, \gamma) \subseteq C_{\kappa_0}(v_0)$ holds for any $\gamma \in V$.

Theorem 4.1. *Suppose that U is a nonempty set, V is a nonempty, compact, abstract convex subset of an abstract convex space satisfying the H_0-condition, and Φ is an abstract convex space. Let $\Gamma : U \times V \to 2^\Phi$ and $\zeta : U \times U \to 2^\Phi$ be multi-valued mappings, and let $C : U \times \Phi \to 2^\Phi$ satisfy that for any $\kappa \in U$, $C_\kappa = C(\kappa, \cdot) : \Phi \to 2^\Phi$ is a closed multi-valued mapping with abstract convex values. For any $v \in \Phi$, assume that (i) $\operatorname{int} C_\kappa(v) \neq \emptyset$; (ii) $C_\kappa(\Lambda) \subseteq C_\kappa(B)$ for any $\emptyset \neq \Lambda \subseteq B \subseteq \Phi$; and (iii) $C_\kappa(\operatorname{int} C_\kappa(v)) \subseteq \operatorname{int} C_\kappa(v)$ for any $(\kappa, v) \in U \times \Phi$. Assume that ζ, Γ and C satisfy the following conditions:*

(1) There exists $v^* \in \Phi$ such that $\zeta(\kappa, \kappa) \not\subset \operatorname{int} C_\kappa(v^*)$ for any $\kappa \in U$;

(2) For any $\gamma \in V$, there exists $z \in U$ such that $\zeta(\kappa, z) \subseteq \Gamma(\kappa, \gamma)$ for any $\kappa \in U$;

(3) ζ is l.s.c. on $\Delta_U = \{(\kappa, \kappa) : \kappa \in U\}$ and the multi-valued mapping $\kappa \mapsto C_\kappa(\Gamma(\kappa, \gamma))$ is closed for any $\gamma \in V$;

(4) ζ is $C_\kappa(\cdot)$-quasi-abstract-convex-like.

Then there exists $\kappa_0 \in U$ such that for some $v_0 \in \Phi$, $\Gamma(\kappa_0, \gamma) \not\subset \operatorname{int} C_{\kappa_0}(v_0)$ holds for any $\gamma \in V$.

Proof. Define a multi-valued mapping T as

$$T(\kappa, \gamma) = \{z \in U : \zeta(\kappa, z) \subseteq C_\kappa(\Gamma(\kappa, \gamma))\} \text{ for any } (\kappa, \gamma) \in U \times V.$$

Let $L = \left\{T | T : U \times V \to 2^U\right\}$ represent the family of multi-valued mappings and $M = \{\kappa \in U : \kappa \in T(\kappa, \gamma)\} = \{\kappa \in U : \zeta(\kappa, \kappa) \subseteq C_\kappa(\Gamma(\kappa, \gamma))\}$ for any $\gamma \in V$.

First, we prove that M is a closed set. Assume that $\kappa \in \overline{M}$ and $\{\kappa_t\}$ is a net in M converging to κ. Because ζ is l.s.c. on Δ_U, from (3) of Theorem 4.1, for each $v \in \zeta(\kappa, \kappa)$, there exists a net $\{v_t\}$ satisfying $v_t \in \zeta(\kappa_t, \kappa_t)$ and $v_t \to v$. From $\kappa_t \in M$, we have $v_t \in C_{\kappa_t}(\Gamma(\kappa_t, \gamma))$. Since $\kappa \mapsto C_\kappa(\Gamma(\kappa, \gamma))$ is closed, we acquire $v \in C_\kappa(\Gamma(\kappa, \gamma))$. Hence, $\kappa \in M$, and M is a closed set.

Next, we prove that L is a family of generalized equi-KKM mappings with respect to γ. Suppose that $\{\gamma_0, \gamma_1, \ldots, \gamma_n\}$ is an arbitrary finite nonempty subset of V. By (2) of Theorem 4.1, there exists $\{z_0, z_1, \ldots, z_n\} \in \langle Z \rangle$ such that $\zeta(\kappa, z_i) \subseteq \Gamma(\kappa, \gamma_i)$ for any $i \in N$ and any $\kappa \in U$. Let $\emptyset \neq J \subset N$, and $z \in conv\{z_i : i \in J\}$. By (4) of Theorem 4.1, for each $\kappa \in U$, there exists $i_U \in J$ such that $\zeta(\kappa, z) \subseteq C_\kappa(\zeta(\kappa, z_{i_U}))$. Since $\zeta(\kappa, z) \subseteq C_\kappa(\zeta(\kappa, z_{i_U})) \subseteq C_\kappa(\Gamma(\kappa, \gamma_{i_U}))$, we acquire $conv\{z_i : i \in J\} \subseteq \bigcup_{i \in J} T(\kappa, \gamma_i)$ for any $\kappa \in U$. Thus, L is a family of generalized equi-KKM mappings with respect to γ.

According to Theorem 3.1, there exists $\kappa_0 \in U$ such that $\kappa_0 \in \bigcap_{\gamma \in V} T(\kappa_0, \gamma)$.

Suppose that the conclusion is not true. Then for any $v \in \Phi$, there exists $\gamma \in V$ such that $\Gamma(\kappa_0, \gamma) \subseteq \operatorname{int} C_{\kappa_0}(v)$. From $\kappa_0 \in T(\kappa_0, \gamma)$, we can obtain $\zeta(\kappa_0, \kappa_0) \subseteq C_{\kappa_0}(\Gamma(\kappa_0, \gamma)) \subseteq C_{\kappa_0}(\operatorname{int} C_{\kappa_0}(v)) \subseteq \operatorname{int} C_{\kappa_0}(v)$, which contradicts the condition (1) of Theorem 4.1. □

Theorem 4.2. *Suppose that U is a nonempty set, V is a nonempty, compact, abstract convex subset of an abstract convex space satisfying the H_0-condition, and Φ is an abstract convex space. Let $\Gamma : U \times V \to 2^{\Phi}$, $\zeta : U \times U \to 2^{\Phi}$, and $C : U \times \Phi \to 2^{\Phi}$ be multi-valued mappings. For any $\kappa \in U$, $C_\kappa = C(\kappa, \cdot) : \Phi \to 2^{\Phi}$ is a parametric multi-valued mapping with abstract convex values. Moreover, for any $\kappa \in U$ and $v \in \Phi$, (i) $C_\kappa(C_\kappa(v)) = C_\kappa(v)$; (ii) $C_\kappa(\Lambda) \subseteq C_\kappa(B)$ for any nonempty subset $\Lambda \subseteq B \subseteq \Phi$. Assume that ζ, Γ and C satisfy the following conditions:*

(1) There exists $v^* \in \Phi$ such that $\zeta(\kappa, \kappa) \subseteq C_\kappa(v^*)$ for each $\kappa \in U$;

(2) For any $\gamma \in V$, there exists $z \in U$ such that $\Gamma(\kappa, \gamma) \subseteq \zeta(\kappa, z)$ for any $\kappa \in U$;

(3) The multi-valued mapping $\kappa \mapsto C_\kappa(\zeta(\kappa, \kappa))$ is closed and $\Gamma(\cdot, \gamma)$ is l.s.c. for any $\gamma \in V$;

(4) ζ is $C_\kappa(\cdot)$-quasi abstract convex.

Then there exists $\kappa_0 \in U$ such that $\Gamma(\kappa_0, \gamma) \subseteq C_{\kappa_0}(v^*)$ for any $\gamma \in V$.

Proof. Define a multi-valued mapping T as

$$T(\kappa, \gamma) = \{z \in U : \Gamma(\kappa, \gamma) \subseteq C_\kappa(\zeta(\kappa, z))\} \text{ for any } (\kappa, \gamma) \in U \times V.$$

Let $L = \left\{T | T : U \times V \to 2^U\right\}$ represent the family of multi-valued mappings and $M = \{\kappa \in U : \kappa \in T(\kappa, \gamma)\} = \{\kappa \in U : \Gamma(\kappa, \gamma) \subseteq C_\kappa(\zeta(\kappa, \kappa))\}$ for any $\gamma \in V$.

First, we prove that M is a closed set. Assume that $\kappa \in \overline{M}$ and $\{\kappa_t\}$ is a net in M converging to κ. Since for any $\gamma \in V$, the mapping $\Gamma(\cdot, \gamma)$ is l.s.c., for each $v \in \Gamma(\kappa, \gamma)$, there exists a net $\{v_t\}$ satisfying $v_t \in \Gamma(\kappa_t, \gamma)$ and $v_t \to v$ for any t. Because $\kappa_t \in M$, we have $v_t \in C_{\kappa_t}(\zeta(\kappa_t, \kappa_t))$. Since the mapping $\kappa \mapsto C_\kappa(\zeta(\kappa, \kappa))$ is closed, it follows that $v \in C_\kappa(\zeta(\kappa, \kappa))$. Therefore, $\kappa \in M$, and hence M is a closed set.

Next, we prove that L is a family of generalized equi-KKM mappings with respect to γ. Suppose that $\{\gamma_0, \gamma_1, \ldots, \gamma_n\} \in \langle V \rangle$. By (2) of Theorem 4.2, there exists $\{z_0, z_1, \ldots, z_n\} \in \langle U \rangle$ such that $\Gamma(\kappa, \gamma_i) \subseteq \zeta(\kappa, z_i)$ for any $i \in N$ and $\kappa \in U$. Let $\emptyset \neq J \subset N$ and $z \in conv\{z_i : i \in J\}$. By (4) of Theorem 4.2, for each $\kappa \in U$, there exists $i_U \in J$ such that $\zeta(\kappa, z_{i_U}) \subseteq C_\kappa(\zeta(\kappa, \kappa))$. Then $\Gamma(\kappa, \gamma_{i_U}) \subseteq \zeta(\kappa, z_{i_U}) \subseteq C_\kappa(\zeta(\kappa, \kappa))$ and $z \in T(\kappa, \gamma_{i_U}) \subseteq \bigcup_{i \in J} T(\kappa, \gamma_i)$. Therefore, $conv\{z_i : i \in J\} \subseteq \bigcup_{i \in J} T(\kappa, \gamma_i)$, which implies that L is a family of generalized equi-KKM mappings with respect to γ.

According to Theorem 3.1, there exists $\kappa_0 \in U$ such that $\kappa_0 \in \bigcap_{\gamma \in V} T(\kappa_0, \gamma)$. Then, there exists $\kappa_0 \in U$ such that $\Gamma(\kappa_0, \gamma) \subseteq C_{\kappa_0}(\zeta(\kappa_0, \kappa_0))$ for any $\gamma \in V$. By (1) of Theorem 4.2, there exists $v^* \in \Phi$ such that $\zeta(\kappa_0, \kappa_0) \subseteq C_{\kappa_0}(v^*)$ for $\kappa_0 \in U$. Hence $\Gamma(\kappa_0, \gamma) \subseteq C_{\kappa_0}(\zeta(\kappa_0, \kappa_0)) \subseteq C_{\kappa_0}(C_{\kappa_0}(v^*)) = C_{\kappa_0}(v^*)$. □

5. Conclusions

In this paper, we introduce the concepts of quasi-abstract convexity (concavity) regarding multi-valued mappings and a family of generalized equi-KKM mappings in abstract convex spaces that satisfy the H_0-condition. These concepts are the generalizations of those given by Balaj in abstract convex spaces. Moreover, the vector equilibrium problems are extended to abstract convex spaces. Furthermore, we consider a common fixed-point theorem for a class of generalized equi-KKM mappings using the Brouwer fixed-point theorem. As applications of this result, we derive the existence theorems for two types of generalized abstract equilibrium problems. This subject is both novel and relevant today. However, the limitation of this paper is that it does not consider a Ky Fan-type minimax inequality, which could be explored in future research.

Author contributions

Shunyou Xia: conceptualization, methodology, validation, writing-original draft preparation, writing-review and editing, visualization, supervision, project administration, funding acquisition; Chongyi Zhong: conceptualization, methodology, validation, writing-review and editing, visualization, supervision, project administration, funding acquisition; Chunrong Mo: validation, supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. L. E. J. Brouwer, Über abbildung von mannigfaltigkeiten, *Math. Ann.*, **71** (1911), 598. https://doi.org/10.1007/BF01456812
2. S. Kakutani, A generalization of Brouwer's fixed point theorem, *Duke Math. J.*, **8** (1941), 457–459. https://doi.org/10.1215/S0012-7094-41-00838-4
3. F. E. Browder, The fixed point theory of multi-valued mappings in topological vector spaces, *Math. Ann.*, **177** (1968), 283–301. https://doi.org/10.1007/BF01350721
4. K. Fan, Fixed-point and minimax theorems in locally convex topological linear spaces, *Proc. Natl. Acad. Sci.*, **38** (1952), 121–126. https://doi.org/10.1073/pnas.38.2.121
5. I. L. Glicksberg, A further generalization of the Kakutani fixed theorem, with application to Nash equilibrium points, *Proc. Amer. Math. Soc.*, **3** (1952), 170–174. https://doi.org/10.1090/S0002-9939-1952-0046638-5

6. B. Kanster, C. Kuratowski, S. Mazurkiewicz, Ein beweis des fixpunktsatzes für n-dimensionale simplexe, *Fund. Math.*, **14** (1929), 132–137.
7. K. Fan, A generalization of Tychonoff's fixed-point theorem, *Math. Ann.*, **142** (1961), 305–310. https://doi.org/10.1007/BF01353421
8. C. D. Horvath, Some results on multi-valued mappings and inequalities without convexity, In: *Nonlinear and convex analysis*, New York: CRC Press, 1987, 99–106.
9. C. D. Horvath, Contractibility and generalized convexity, *J. Math. Anal. Appl.*, **156** (1991), 341–357. https://doi.org/10.1016/0022-247X(91)90402-L
10. C. D. Horvath, Extension and selection theorems in topological spaces with a generalized convexity structure, *Annales de la Faculté des sciences de Toulouse: Mathématiques*, **2** (1993), 253–269. https://doi.org/10.5802/AFST.766
11. S. Park, New topological versions of the Fan-Browder fixed point theorem, *Nonlinear Anal.-Theor.*, **47** (2001), 595–606. https://doi.org/10.1016/S0362-546X(01)00204-8
12. S. Park, The KKM, matching, and fixed point theorems in generalized convex spaces, *Nonlinear Funct. Anal. Appl.*, **11** (2006), 139–154.
13. X. P. Ding, Generalized G-KKM theorems in generalized convex spaces and their applications, *J. Math. Anal. Appl.*, **266** (2002), 21–37. https://doi.org/10.1006/jmaa.2000.7207
14. Q. Luo, KKM and Nash equilibria type theorems in topological ordered spaces, *J. Math. Anal. Appl.*, **264** (2001), 262–269. https://doi.org/10.1006/jmaa.2001.7624
15. Q. Luo, Ky Fan's section theorem and its applications in topological ordered spaces, *Appl. Math. Lett.*, **17** (2004), 1113–1119. https://doi.org/10.1016/j.aml.2003.12.003
16. S. Xiang, S. Xia, J. Chen, KKM lemmas and minimax inequality theorems in abstract convexity spaces, *Fixed Point Theory Appl.*, **2013** (2013), 209. https://doi.org/10.1186/1687-1812-2013-209
17. Q. Luo, The applications of the Fan-Browder fixed point theorem in topological ordered spaces, *Appl. Math. Lett.*, **19** (2006), 1265–1271. https://doi.org/10.1016/j.aml.2006.01.016
18. S. W. Xiang, S. Y. Xia, A further characteristic of abstract convexity structures on topological spaces, *J. Math. Anal. Appl.*, **335** (2007), 716–723. https://doi.org/10.1016/j.jmaa.2007.01.101
19. C. R. Mo, Y. L. Yang The unified description of abstract convexity structures, *Axioms*, **13** (2024), 506. https://doi.org/10.3390/axioms13080506
20. R. Agarwal, M. Balaj, D. O'Regan, Common fixed point theorems and minimax inequalities in locally convex Hausdorff topological vector spaces, *Appl. Anal.*, **88** (2009), 1691–1699. https://doi.org/10.1080/00036810903331874
21. M. Balaj, A common fixed point theorem with applications to vector equilibrium problems, *Appl. Math. Lett.*, **23** (2010), 241–245. https://doi.org/10.1016/j.aml.2009.09.019
22. R. Agarwal, M. Balaj, D. O'Regan, Common fixed point theorems in topological vector spaces via intersection theorems, *J. Optim. Theory Appl.*, **173** (2017), 443–458. https://doi.org/10.1007/s10957-017-1082-7
23. M. Balaj, M. A. Khamsi, Common fixed point theorems for set-valued mappings in normed spaces, *RACSAM*, **113** (2019), 1893–1905. https://doi.org/10.1007/s13398-018-0588-7

24. M. Balaj, E. Jorquera, M. Khamsi, Common fixed points of set-valued mappings in hyperconvex metric spaces, *J. Fixed Point Theory Appl.*, **20** (2018), 22. https://doi.org/10.1007/s11784-018-0493-x

25. J. Yu, *An introduction to game theory and nonlinear analysis (Chinese)*, Beijing: Science Press, 2011.

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