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Research article

On fixed point theorems for ordered contractions with applications

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Abstract: The purpose of this paper is to prove some fixed point theorems for ordered contractions in partially ordered b-metric spaces. We consider several common fixed point and coincidence point results for four mappings in such spaces. The results obtained in this paper significantly extend numerous results in the existing literature. Further, we present some supportive examples to emphasize the potential value of our results. In addition, as applications, we verify the existence and uniqueness of solutions to a large number of equations.

Keywords: common fixed point; coincidence point; b-metric space; ordered contractive pair; weakly compatible

Mathematics Subject Classification: 54H25, 47H10, 54E50

1. Introduction and preliminaries

Fixed point theory is an important branch of nonlinear functional analysis. It has been widely applied in mathematics and other disciplines. As an example, it has important applications in different subjects such as quantum physics, computer science, and economics, etc. For several decades, it has been a hot topic to extend fixed point results from metric spaces to general spaces wherein b-metric space has a significant impact. The concept of b-metric space was introduced by Bakhtin [1] in 1989. The triangular inequality from the metric space is replaced by the quasi-triangular inequality in the b-metric space. Due to its simplicity, elegance, and the fact that, in general, a b-metric is not continuous (as we know, the usual metric is a continuous mapping), the study of fixed point theorems in such spaces has strong theoretical and practical significance. Moreover, [1] extended the famous Banach contraction mapping principle from metric spaces to b-metric spaces. In 1993, Czerwik [2] generalized fixed point theorems for φ-contraction and Kannan-type contraction from metric spaces to b-metric spaces. Since then, many scholars have devoted themselves to considering fixed point theorems in b-metric spaces; the reader refers to [3–5]. On the other hand, Beg and Abbas [6] obtained a common fixed point theorem for two mappings under weak contractive conditions in metric spaces. Later, Abbas et al. [7] extended the conclusion of [6] to four mappings in partially ordered metric spaces. Recently, Jiang et al. [8] obtained the common fixed point theorems for four mappings satisfying the generalized (ψ, β, L)-contractive conditions in partially ordered b-metric spaces.

Based on this, throughout this paper we acquire some conclusions for four self-mappings in complete partially ordered b-metric spaces that satisfy the generalized contractive conditions. Our results greatly weaken the conditions from [8, Theorems 1, 2] and [9, Theorem 15] by deleting the functions φ, ψ, and the constant L. Moreover, our proof process is much simpler than the counterpart of Theorem [8, Theorems 1]. Furthermore, we correct the proof errors of [8, Theorem 1]. In addition, we give some examples to illustrate the superiority of our results. Otherwise, since fixed point theory has a large number of applications for all classes of equations (see e.g., [10]), in this paper we also use our results to show the existence and uniqueness of a solution to some equations.

In this paper, $\mathbb{R}$ represents the set of all real numbers, $\mathbb{N}$ represents the set of all nonnegative integers, and $\mathbb{N}^*$ represents the set of all positive integers. First of all, let's recall some basic concepts below.

Definition 1.1. ([11]) Let X be a nonempty set, $d : X \times X \to \mathbb{R}$ a mapping, and $s \geq 1$ a constant. d is called a b-metric on X, and (X, d) is called a b-metric space if for all $x, y, z \in X$, it satisfies

(1) $d(x, y) \geq 0,\ d(x, y) = 0 \Leftrightarrow x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \leq s[d(x, z) + d(z, y)]$,

where (3) is a quasi-triangular inequality. In this case, if X is endowed with a partial order, then $(X, \preceq, d)$ is called a partially ordered b-metric space.

Remark 1.2. It is obvious that the class of b-metric space is larger than that of metric space since any metric space is a b-metric space with $s = 1$. In general, a b-metric space is not necessarily a metric space, see [12, Examples 1.2–1.5].

Definition 1.3. ([5]) Let $\{x_n\}$ be a sequence in a b-metric space (X, d) and $x \in X$. If $\lim\limits_{n\to\infty} d(x_n, x) = 0$, then $\{x_n\}$ is called to be convergent to x, denoted by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x\ (n \to \infty)$.

Definition 1.4. ([12]) Let $\{x_n\}$ be a sequence in a b-metric space (X, d). Then $\{x_n\}$ is called a b-Cauchy sequence if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$. If all b-Cauchy sequences are convergent, then (X, d) is called a complete b-metric space.

Definition 1.5. ([5]) Let (X, d) be a b-metric space. A mapping $F : X \to X$ is called continuous at $x \in X$ if $\lim\limits_{n\to\infty} d(Fx_n, Fx) = 0$ whenever $\{x_n\} \subset X$ with $\lim\limits_{n\to\infty} d(x_n, x) = 0$.

In general, b-metric is not continuous; kindly see the following example.

Example 1.6. Let $X = \mathbb{R}$ and define a mapping $d : X \times X \to \mathbb{R}$ as

$$d(x, y) = \begin{cases} |x - y|, & xy \neq 0, \\ \alpha|x - y|, & xy = 0, \end{cases}$$

where $\alpha > 1$ is a constant. It is easy to see that d is a b-metric with coefficient $s = \alpha$. Choose $x_n = 1$, $y_n = \frac{1}{n}$, then

$$d(x_n, 1) = 0, \qquad d(y_n, 0) = \frac{\alpha}{n} \to 0\,(n \to \infty).$$

Thus, $x_n \to 1\,(n \to \infty)$, $y_n \to 0\,(n \to \infty)$, but

$$d(x_n, y_n) = |1 - \frac{1}{n}| \to 1 \neq d(1, 0) = \alpha\,(n \to \infty).$$

That is to say, the b-metric d is not continuous.

Remark 1.7. In Example 1.6, we restrict $\alpha > 1$. If $\alpha = 1$, then (X, d) will become a metric space and the metric d is continuous. It shows that the value of α can affect whether the b-metric is continuous or not in the space.

Definition 1.8. ([8]) Let $(X, \preceq, d)$ be a partially ordered b-metric space, $x, y \in X$ and $\{x_n\}$ a sequence in X. Let f, g, h be self-mappings on X, (f, g) be the mapping pair with $f(X) \cup g(X) \subseteq h(X)$.

(1) The elements x, y are called comparable if $x \preceq y$ or $y \preceq x$ holds.

(2) The pair (f, g) is called partially weakly increasing with respect to h if $fx \preceq gy$, where $y = h^{-1}(fx)$ for each $x \in X$, where $h^{-1}(u) = \{v \in X : hv = u\}$ for $u \in X$.

(3) The pair (f, g) is called compatible if $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = t \in X$ implies $\lim\limits_{n\to\infty} d(fgx_n, gfx_n) = 0$.

(4) The element x is called a coincidence point of f and g, and the element y is called a point of coincidence of f and g if $y = fx = gx$.

(5) The pair (f, g) is called weakly compatible if $fx = gx$ implies $fgx = gfx$.

(6) f is called dominating if $x \preceq fx$ holds for each $x \in X$; f is called dominated if $fx \preceq x$ holds for each $x \in X$.

We give three examples as follows.

Example 1.9. Let $X = [0, +\infty)$ be endowed with the usual ordering. Define three self-mappings f, g, h as

$$f(x) = \mathrm{e}^x - 1, \qquad g(x) = x, \qquad h(x) = \ln(x + 1),$$

then $\mathrm{e}^x - 1 \leq \mathrm{e}^{\mathrm{e}^x - 1} - 1$ for all $x \in X$. Hence, the pair (f, g) is partially weakly increasing with respect to h, and the pair (f, h) is weakly compatible.

Example 1.10. Let $X = [0, +\infty)$ be endowed with the usual ordering. Define a mapping by $f(x) = \mathrm{e}^x$, then $x \leq \mathrm{e}^x$ for all $x \in X$. Hence, f is dominating.

Example 1.11. Let $X = [0, +\infty)$ be endowed with the usual ordering. Define a mapping by $f(x) = \ln(x + 1)$, then $\ln(x + 1) \leq x$ for all $x \in X$. Thus, f is dominated.

The following lemma will be used constantly in the sequel.

Lemma 1.12. *([13, 14]) Let $\{x_n\}$ be a sequence in a b-metric space (X, d). If*

$$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1}) \quad (n = 1, 2, \cdots)$$

holds, where $\lambda \in [0, 1)$ is a constant, then $\{x_n\}$ is a b-Cauchy sequence in X.

2. Main results

In this section, we give a new concept called ordered contractive pair for four mappings on partially ordered b-metric space. By using this concept, we obtain some theorems for the existence of a common fixed point and coincidence point. We also give two examples to support our results.

Definition 2.1. Let $(X, \preceq, d)$ be a partially ordered b-metric space, f, g, S, T be self-mappings on X. The pair (f, g) is called an ordered contractive pair with respect to S and T if for any comparable elements $x, y \in X$, it satisfies

$$s^{\varepsilon} d(fx, gy) \leq \max\{d(Sx, Ty), d(Sx, fx), d(Ty, gy)\}, \tag{2.1}$$

where $\varepsilon > 2$ is a constant.

Theorem 2.2. *Let $(X, \preceq, d)$ be a complete partially ordered b-metric space with coefficient $s > 1$, f, g, S, T be self-mappings on X, $\{x_n\}$, $\{y_n\}$ be sequences in X. If the following conditions are satisfied:*

(1) $fX \subseteq TX$, $gX \subseteq SX$;

(2) (f, g) *is an ordered contractive pair with respect to S and T;*

(3) f, g *are dominating, S, T are dominated;*

(4) $\{x_n\}$ *is nondecreasing, $x_n \preceq y_n$ for all $n \in \mathbb{N}^*$ and $y_n \to z$ $(n \to \infty)$ imply $x_n \preceq z$;*

(5) (i) *f or S is continuous, (f, S) are compatible, and (g, T) are weakly compatible or*

(ii) *g or T is continuous, (g, T) are compatible, and (f, S) are weakly compatible,*

then the mappings f, g, S, T possess a common fixed point in X. Moreover, the common fixed point is unique if and only if the set of common points is well ordered.

Proof. Choose $x_0 \in X$, by the condition (1), construct a sequence $\{y_n\}$ in X as follows:

$$y_{2n-1} = Tx_{2n-1} = fx_{2n-2}, \qquad y_{2n} = Sx_{2n} = gx_{2n-1},$$

where $n \in \mathbb{N}^*$. By the condition (3), it follows that

$$x_{2n-2} \preceq fx_{2n-2} = Tx_{2n-1} \preceq x_{2n-1},\ x_{2n-1} \preceq gx_{2n-1} = Sx_{2n} \preceq x_{2n}.$$

Thus, $x_n \preceq x_{n+1}$ $(n \in \mathbb{N})$.

Assume that there exists $n_0 \in \mathbb{N}$ such that $d(y_{2n_0}, y_{2n_0+1}) = 0$, i.e., $y_{2n_0} = y_{2n_0+1}$. By the condition (2), it is easy to see from (2.1) that

$$\begin{aligned}
&s^{\varepsilon} d(y_{2n_0+1}, y_{2n_0+2}) = s^{\varepsilon} d(fx_{2n_0}, gx_{2n_0+1})\\
&\leq \max\{d(Sx_{2n_0}, Tx_{2n_0+1}), d(Sx_{2n_0}, fx_{2n_0}), d(Tx_{2n_0+1}, gx_{2n_0+1})\}\\
&= \max\{d(y_{2n_0}, y_{2n_0+1}), d(y_{2n_0}, y_{2n_0+1}), d(y_{2n_0+1}, y_{2n_0+2})\}\\
&= \max\{0, 0, d(y_{2n_0+1}, y_{2n_0+2})\}\\
&= d(y_{2n_0+1}, y_{2n_0+2}),
\end{aligned}$$

then

$$d(y_{2n_0+1}, y_{2n_0+2}) \leq \frac{1}{s^{\varepsilon}} d(y_{2n_0+1}, y_{2n_0+2}),$$

so $d(y_{2n_0+1}, y_{2n_0+2}) = 0$, i.e., $y_{2n_0+1} = y_{2n_0+2}$.

Similarly, via $y_{2n_0+1} = y_{2n_0+2}$ it implies $y_{2n_0+2} = y_{2n_0+3}$. It will be seen from this that $\{y_n\}$ is a constant if $n \geq 2n_0 + 1$. Again by

$$x_{2n-2} \preceq fx_{2n-2} = y_{2n-1} = Tx_{2n-1} \preceq x_{2n-1},$$

$$x_{2n-1} \preceq gx_{2n-1} = y_{2n} = Sx_{2n} \preceq x_{2n},$$

$$x_{2n} \preceq fx_{2n} = y_{2n+1} = Tx_{2n+1} \preceq x_{2n+1},$$

it means that $y_{2n-1} \preceq x_{2n-1} \preceq y_{2n},\ y_{2n} \preceq x_{2n} \preceq y_{2n+1}$. Accordingly, if $n \geq 2n_0 + 1$, then $\{x_n\}$ is also a constant and equal to $\{y_n\}$. As a consequence, y_{2n_0} is a common fixed point of f, g, S, T.

Assume that $d(y_{2n}, y_{2n+1}) > 0$ for all $n \in \mathbb{N}$. Note that x_{2n} and x_{2n+1} are comparable; by (2.1), it establishes that

$$\begin{aligned} s^{\varepsilon} d(y_{2n+1}, y_{2n+2}) &= s^{\varepsilon} d(fx_{2n}, gx_{2n+1}) \\ &\leq \max\{d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, fx_{2n}), d(Tx_{2n+1}, gx_{2n+1})\} \\ &= \max\{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2})\} \\ &= \max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2})\}. \end{aligned} \tag{2.2}$$

If

$$d(y_{2n}, y_{2n+1}) \leq d(y_{2n+1}, y_{2n+2}),$$

then by (2.2) we have

$$d(y_{2n+1}, y_{2n+2}) \leq \frac{1}{s^{\varepsilon}} d(y_{2n+1}, y_{2n+2}),$$

which follows that $d(y_{2n+1}, y_{2n+2}) = 0$. This is a contradiction. Therefore, we get

$$d(y_{2n+1}, y_{2n+2}) < d(y_{2n}, y_{2n+1}).$$

Now by (2.2) we gain

$$s^{\varepsilon} d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n}, y_{2n+1}),$$

which implies that

$$d(y_{2n+1}, y_{2n+2}) \leq \frac{1}{s^{\varepsilon}} d(y_{2n}, y_{2n+1}). \tag{2.3}$$

Since x_{2n-1} and x_{2n} are comparable, by (2.1) it is valid that

$$\begin{aligned} s^{\varepsilon} d(y_{2n}, y_{2n+1}) &= s^{\varepsilon} d(fx_{2n}, gx_{2n-1}) \\ &\leq \max\{d(Sx_{2n}, Tx_{2n-1}), d(Sx_{2n}, fx_{2n}), d(Tx_{2n-1}, gx_{2n-1})\} \\ &= \max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\} \\ &= \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\}. \end{aligned} \tag{2.4}$$

If

$$d(y_{2n-1}, y_{2n}) \leq d(y_{2n}, y_{2n+1}),$$

then by (2.4) one has

$$d\left(y_{2n}, y_{2n+1}\right) \leq \frac{1}{s^{\varepsilon}} d\left(y_{2n}, y_{2n+1}\right),$$

thus, $d\left(y_{2n}, y_{2n+1}\right) = 0$. This is a contradiction. Consequently, we arrive at

$$d\left(y_{2n}, y_{2n+1}\right) < d\left(y_{2n-1}, y_{2n}\right),$$

by using (2.4), we deduce

$$s^{\varepsilon} d\left(y_{2n}, y_{2n+1}\right) \leq d\left(y_{2n-1}, y_{2n}\right),$$

which means that

$$d\left(y_{2n}, y_{2n+1}\right) \leq \frac{1}{s^{\varepsilon}} d\left(y_{2n-1}, y_{2n}\right). \tag{2.5}$$

Combine (2.3) and (2.5), it is obvious that

$$d\left(y_{n}, y_{n+1}\right) \leq \frac{1}{s^{\varepsilon}} d\left(y_{n-1}, y_{n}\right).$$

Making full use of Lemma 1.12, we claim that $\{y_n\}$ is a b-Cauchy sequence in X. Since $(X, \preceq, d)$ is complete, then there is a $z \in X$ such that $\lim_{n\to\infty} y_n = z$.

Now assume that the condition (i) holds. Without loss of generality, let S be continuous. Since (f, S) is compatible and

$$\lim_{n\to\infty} f x_{2n+2} = z = \lim_{n\to\infty} S x_{2n+2},$$

we speculate that

$$\lim_{n\to\infty} d\left(fS x_{2n+2}, S f x_{2n+2}\right) = 0,$$

that is,

$$\lim_{n\to\infty} d\left(f y_{2n+2}, S y_{2n+3}\right) = 0.$$

Note that S is continuous; it is clear that

$$d\left(f y_{2n+2}, S z\right) \leq s\left[d\left(f y_{2n+2}, S y_{2n+3}\right) + d\left(S y_{2n+3}, S z\right)\right] \to 0\,(n \to \infty).$$

Thus, we have

$$\lim_{n\to\infty} d\left(f y_{2n+2}, S z\right) = 0. \tag{2.6}$$

In what follows, we will prove $z = Sz$. As a matter of fact, in view of $x_{2n+1} \preceq g x_{2n+1} = S x_{2n+2}$, by (2.1), it is valid that

$$\begin{aligned} s^{\varepsilon} d\left(f y_{2n+2}, y_{2n+2}\right) &= s^{\varepsilon} d\left(fS x_{2n+2}, g x_{2n+1}\right) \\ &\leq \max\left\{d\left(SS x_{2n+2}, T x_{2n+1}\right), d\left(SS x_{2n+2}, fS x_{2n+2}\right), d\left(T x_{2n+1}, g x_{2n+1}\right)\right\} \end{aligned}$$

$$= \max\left\{d\left(Sy_{2n+2}, y_{2n+1}\right), d\left(Sy_{2n+2}, fy_{2n+2}\right), d\left(y_{2n+1}, y_{2n+2}\right)\right\}. \tag{2.7}$$

By (2.6) and the fact that S is continuous, we acquire that

$$d(Sy_{2n+2}, fy_{2n+2}) \leq s[d(Sy_{2n+2}, Sz) + d(Sz, fy_{2n+2})] \to 0\,(n \to \infty),$$

which establishes that

$$\lim_{n\to\infty} d\left(Sy_{2n+2}, fy_{2n+2}\right) = 0. \tag{2.8}$$

Since $\{y_n\}$ is a b-Cauchy sequence, then

$$\lim_{n\to\infty} d\left(y_{2n+1}, y_{2n+2}\right) = 0. \tag{2.9}$$

Taking advantage of (2.8) and (2.9), there exists $N_1 \in \mathbb{N}$ such that for any $n > N_1$, we have

$$d\left(Sy_{2n+2}, y_{2n+1}\right) \geq d\left(Sy_{2n+2}, fy_{2n+2}\right), \tag{2.10}$$

$$d\left(Sy_{2n+2}, y_{2n+1}\right) \geq d\left(y_{2n+1}, y_{2n+2}\right). \tag{2.11}$$

Utilizing (2.7) and (2.10) together with (2.11), we obtain

$$s^{\varepsilon} d\left(fy_{2n+2}, y_{2n+2}\right) \leq d\left(Sy_{2n+2}, y_{2n+1}\right)(n > N_1). \tag{2.12}$$

Making the most of the quasi-triangular inequality of b-metric, we gain

$$\begin{aligned} d\left(Sy_{2n+2}, y_{2n+1}\right) &\leq s[d\left(Sy_{2n+2}, fy_{2n+2}\right) + d\left(fy_{2n+2}, y_{2n+1}\right)] \\ &\leq sd\left(Sy_{2n+2}, fy_{2n+2}\right) + s^2 d\left(fy_{2n+2}, y_{2n+2}\right) + s^2 d\left(y_{2n+2}, y_{2n+1}\right). \end{aligned} \tag{2.13}$$

By using (2.12) and (2.13), for each $n > N_1$, it is curious that

$$s^{\varepsilon} d\left(fy_{2n+2}, y_{2n+2}\right) \leq sd\left(Sy_{2n+2}, fy_{2n+2}\right) + s^2 d\left(fy_{2n+2}, y_{2n+2}\right) + s^2 d\left(y_{2n+2}, y_{2n+1}\right).$$

By taking the upper limit as $n \to \infty$ from the above inequality together with (2.8) and (2.9), it is palpable that

$$\limsup_{n\to\infty} d\left(fy_{2n+2}, y_{2n+2}\right) \leq \frac{1}{s^{\varepsilon-2}} \limsup_{n\to\infty} d\left(fy_{2n+2}, y_{2n+2}\right), \tag{2.14}$$

then by (2.14), it is valid that

$$\lim_{n\to\infty} d\left(fy_{2n+2}, y_{2n+2}\right) = 0. \tag{2.15}$$

By virtue of

$$\begin{aligned} d\left(Sz, z\right) &\leq sd\left(Sz, fy_{2n+2}\right) + sd\left(fy_{2n+2}, z\right) \\ &\leq sd\left(Sz, fy_{2n+2}\right) + s^2 d\left(fy_{2n+2}, y_{2n+2}\right) + s^2 d\left(y_{2n+2}, z\right). \end{aligned}$$

By taking the upper limit as $n \to \infty$ from the above inequality together with (2.6) and (2.15), it leads to $d\left(Sz, z\right) = 0$, i.e., $Sz = z$.

It suffices to prove $fz = z$. Indeed, on account of $x_{2n+1} \preceq gx_{2n+1}$ and $gx_{2n+1} \to z$, then by the condition (4), $x_{2n+1} \preceq z$, in other words, z and x_{2n+1} are comparable. Using (2.1), we obtain

$$\begin{aligned} s^{\varepsilon} d(fz, y_{2n+2}) &= s^{\varepsilon} d(fz, gx_{2n+1}) \\ &\leq \max\{d(Sz, Tx_{2n+1}), d(Sz, fz), d(Tx_{2n+1}, gx_{2n+1})\} \\ &= \max\{d(z, y_{2n+1}), d(z, fz), d(y_{2n+1}, y_{2n+2})\}. \end{aligned} \tag{2.16}$$

As $\{y_n\}$ is a b-Cauchy sequence, it follows immediately that

$$\lim_{n\to\infty} d(z, y_{2n+1}) = 0, \tag{2.17}$$

$$\lim_{n\to\infty} d(y_{2n+1}, y_{2n+2}) = 0. \tag{2.18}$$

Utilizing (2.17) and (2.18), there exists $N_2 \in \mathbb{N}$ such that for any $n > N_2$, it leads to

$$d(z, fz) \geq d(z, y_{2n+1}), \tag{2.19}$$

$$d(z, fz) \geq d(y_{2n+1}, y_{2n+2}). \tag{2.20}$$

By (2.16), (2.19), and (2.20), it can be seen that

$$\begin{aligned} s^{\varepsilon} d(fz, y_{2n+2}) &\leq d(z, fz) \\ &\leq sd(z, y_{2n+2}) + sd(y_{2n+2}, fz) \quad (n > N_2). \end{aligned} \tag{2.21}$$

By taking the upper limit as $n \to \infty$ from (2.21), it is simple that

$$\limsup_{n\to\infty} d(fz, y_{2n+2}) \leq \frac{1}{s^{\varepsilon-1}} \limsup_{n\to\infty} d(fz, y_{2n+2}),$$

then

$$\lim_{n\to\infty} d(fz, y_{2n+2}) = 0. \tag{2.22}$$

Note that

$$d(fz, z) \leq sd(fz, y_{2n+2}) + sd(y_{2n+2}, z),$$

by taking the limit as $n \to \infty$ from the above inequality together with (2.22), we arrive at $d(fz, z) = 0$, i.e., $fz = z$.

Via the condition (1), there is a $w \in X$ such that $z = fz = Tw$. It will need to prove $Tw = gw$. Via $z = fz = Tw \preceq w$, it implies $z \preceq w$. By utilizing (2.1), we obtain

$$\begin{aligned} s^{\varepsilon} d(Tw, gw) &= s^{\varepsilon} d(fz, gw) \\ &\leq \max\{d(Sz, Tw), d(Sz, fz), d(Tw, gw)\} \\ &= \max\{0, 0, d(Tw, gw)\} \\ &= d(Tw, gw), \end{aligned}$$

which follows that $d(Tw, gw) = 0$, that is, $Tw = gw$. Because (g, T) is weakly compatible, it is curious that

$$gz = gfz = gTw = Tgw = TTw = Tfz = Tz.$$

Therefore, z is a coincidence point of g and T.

In the sequel, we need to show $z = gz$. By $x_{2n} \preceq fx_{2n}$ and $fx_{2n} \to z\,(n \to \infty)$, by the condition (4), it leads to $x_{2n} \preceq z$. Again by (2.1), it is straightforward to see that

$$\begin{aligned} s^{\varepsilon} d(y_{2n+1}, gz) &= s^{\varepsilon} d(fx_{2n}, gz) \\ &\leq \max\{d(Sx_{2n}, Tz), d(Sx_{2n}, fx_{2n}), d(Tz, gz)\} \\ &= \max\{d(y_{2n}, gz), d(y_{2n}, y_{2n+1}), 0\}. \end{aligned} \tag{2.23}$$

Owing to $\lim_{n\to\infty} d(y_{2n}, y_{2n+1}) = 0$, we claim that there exists $N_3 \in \mathbb{N}$ such that for each $n > N_3$, we have

$$d(y_{2n}, gz) \geq d(y_{2n}, y_{2n+1}). \tag{2.24}$$

By (2.23) and (2.24), it is verified that

$$s^{\varepsilon} d(y_{2n+1}, gz) \leq d(y_{2n}, gz) \leq sd(y_{2n}, y_{2n+1}) + sd(y_{2n+1}, gz) \quad (n > N_3).$$

By taking the upper limit as $n \to \infty$ from the above inequality, it can be shown that

$$\limsup_{n\to\infty} d(y_{2n+1}, gz) \leq \frac{1}{s^{\varepsilon-1}} \limsup_{n\to\infty} d(y_{2n+1}, gz).$$

Thus, one has

$$\lim_{n\to\infty} d(y_{2n+1}, gz) = 0. \tag{2.25}$$

Using the quasi-triangular inequality of b-metric, we obtain

$$d(gz, z) \leq sd(gz, y_{2n+1}) + sd(y_{2n+1}, z).$$

By taking limit as $n \to \infty$ from the above inequality together with (2.25), we declare $d(gz, z) = 0$, i.e., $gz = z$.

In summary, we have $fz = gz = Sz = Tz = z$. Hence, z is a common fixed point of f, g, S, T. We can take a similar argument when f is continuous.

Similarly, if condition (ii) holds, we can also find the common fixed point of f, g, S, T.

Finally, we assume that the set of common fixed points of f, g, S, T, denoted by Y, is well ordered. We want to prove that the common fixed point of f, g, S, T is unique. Indeed, if there exist p, $q \in Y$, then

$$fp = gp = Sp = Tp, \qquad fq = gq = Sq = Tq.$$

Since Y is well ordered, then by (2.1), it is clear that

$$\begin{aligned} s^{\varepsilon} d(p, q) &= s^{\varepsilon} d(fp, gq) \\ &\leq \max\{d(Sp, Tq), d(Sp, fp), d(Tq, gq)\} \\ &= \max\{d(p, q), 0, 0\} \\ &= d(p, q). \end{aligned}$$

This leads to $d(p, q) = 0$, that is, $p = q$.

Conversely, if the common fixed point of f, g, S, T is unique, then the set of common fixed points of f, g, S, and T is a singleton. It is clear that the set is well ordered. □

Remark 2.3. The condition $\varepsilon > 2$ from Definition 2.1 is vital, and it cannot be replaced by $\varepsilon > 1$ or $\varepsilon > 0$. Indeed, if $\varepsilon \leq 2$, then (2.15) does not hold based on the fact that (2.15) comes from (2.14).

Remark 2.4. Theorem 2.2 is an extension and improvement of Theorem 2.1 in [7]. The former extends the latter from metric space to b-metric space. In fact, the former condition is weaker than the latter. Moreover, Theorem 2.2 also improves Theorem 1 in [8]. This is because compared to Theorem 1, the conditions of Theorem 2.2 are weaker since Theorem 1 has limitations such as the functions β and ψ and the constant L, which greatly restrict its application possibility. In addition, it corrects the proof error from [8, Theorem 1]. Indeed, the 8th line of Page 8 from [8, Theorem 1] is incorrect because the authors used the false inequality $|d(x, y) - sd(x, z)| \leq sd(z, y)$. The false inequality cannot be obtained by the quasi-triangular inequality $d(x, y) \leq s[d(x, z) + d(z, y)]$. Our proof of Theorem 2.1 in this paper only uses $d(x, y) \leq s[d(x, z) + d(z, y)]$ instead of $|d(x, y) - sd(x, z)| \leq sd(z, y)$.

Remark 2.5. The results obtained in this paper are similar to those in [15], which gave the coincidence point of four mappings under different contractive conditions. However, this paper studies the existence and uniqueness of common fixed points and coincidence points of four mappings satisfying the contractive conditions. The contractive conditions of the mappings in [15, Theorem 2.1] are different from the counterpart in the present theorem, and it requires that all the mappings be continuous. In Theorem 2.2, the continuity condition was weakened, and only one of the mappings needed to be continuous. Our theorem ignores the b-closed set condition from [15, Theorem 2.2]; thereby, our result broadens the scope of the theorem from [15].

The following is an example to support the validity of Theorem 2.2.

Example 2.6. Let $X = [0, +\infty)$ with the partial order as

$$x \preceq y \Leftrightarrow \begin{cases} x \geq y + 0.01, & x \neq y, \\ x = y, & x = y. \end{cases}$$

Choose the b-metric $d(x, y)$ from Example 1.6, then $(X, \preceq, d)$ is a complete partially ordered b-metric space. Define four self-mappings f, g, S, T as

$$f(x) = \ln(x + 1), \qquad g(x) = \frac{x}{2},$$

$$S(x) = e^x - 1, \qquad T(x) = x.$$

Simple calculations show that f, g are dominating, S, T are dominated. Then the pair (f, g) is an ordered contractive pair with respect to S and T, where $\alpha = 1.0001$ and $\varepsilon = 3$. It can be seen from Figure 1 that the value on the left of (2.1) is smaller than the value on the right. As a consequence, all the conditions of Theorem 2.2 are satisfied, and f, g, S, T have a unique common fixed point $z = 0$. In fact, g, T have a common fixed point z if f, g, S, T have a common fixed point z. Hence, the fact that g, T have a common fixed point $z = 0$ implies f, g, S, T have a unique common fixed point $z = 0$.

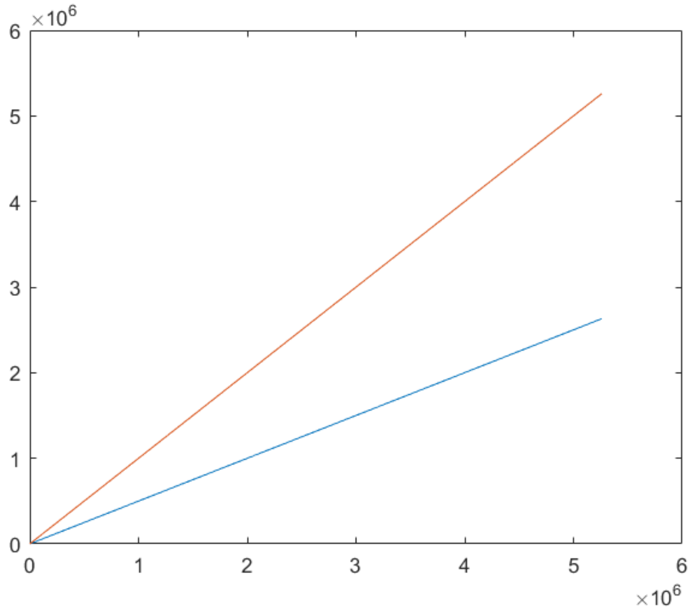

Figure 1. Left and right data of (2.1).

In the sequel, we will give a theorem for the existence of a coincidence point of four mappings.

Theorem 2.7. *Let* $(X, \preceq, d)$ *be a complete partially ordered b-metric space with coefficient* $s > 1$ *and* f, g, S, T *be continuous self-mappings on* X*. If the following conditions are satisfied:*

(1) $fX \subseteq TX$, $gX \subseteq SX$;

(2) *The pairs* (f, S) *and* (g, T) *are compatible;*

(3) *The pairs* (f, g) *and* (g, f) *are partially weakly increasing with respect to* T *and* S*, respectively;*

(4) (2.1) *holds for some* $\varepsilon > 0$ *when* Sx *and* Ty *are comparable,*

then (f, S) *and* (g, T) *have a coincidence point* $z \in X$*. Moreover,* z *is the coincidence point of* f, g, S, T *if* Sz *and* Tz *are comparable.*

Proof. Choose $x_0 \in X$, by the condition (1), construct a sequence $\{y_n\}$ in X as follows:

$$y_{2n-2} = fx_{2n-2} = Tx_{2n-1}, \qquad y_{2n-1} = gx_{2n-1} = Sx_{2n},$$

where $n \in \mathbb{N}^*$. Then $y_n \preceq y_{n+1}$ $(n \in \mathbb{N})$. Indeed, by the construction, one the one hand, it establishes $x_{2n+1} \in T^{-1}(fx_{2n})$. By the condition (3), it follows that

$$Tx_{2n+1} = fx_{2n} \preceq gx_{2n+1} = Sx_{2n+2}.$$

On the other hand, $x_{2n+2} \in S^{-1}(gx_{2n+1})$, as the condition (3), it satisfies

$$Sx_{2n+2} = gx_{2n+1} \preceq fx_{2n+2} = Tx_{2n+3}.$$

Accordingly, we obtain

$$Tx_1 \preceq Sx_2 \preceq Tx_3 \preceq \cdots \preceq Tx_{2n+1} \preceq Sx_{2n+2} \preceq Tx_{2n+3} \preceq \cdots.$$

In other words, we obtain

$$y_0 \preceq y_1 \preceq y_2 \preceq \cdots \preceq y_{2n} \preceq y_{2n+1} \preceq y_{2n+2} \preceq \cdots .$$

Next, we will finish the proof of this theorem.

First, we will prove

$$d(y_{n+1}, y_{n+2}) \leq \lambda d(y_n, y_{n+1}) \quad (n \in \mathbb{N}), \tag{2.26}$$

where $\lambda \in [0, 1)$ is a constant.

Assume that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Due to the fact that $Sx_{2n} = y_{2n-1}$ and $Tx_{2n+1} = y_{2n}$ are comparable, by the condition (4), (2.1) implies

$$\begin{aligned} s^{\varepsilon} d(y_{2n}, y_{2n+1}) &= s^{\varepsilon} d(fx_{2n}, gx_{2n+1}) \\ &\leq \max\{d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, fx_{2n}), d(Tx_{2n+1}, gx_{2n+1})\} \\ &= \max\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} \\ &= \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\}. \end{aligned} \tag{2.27}$$

If

$$d(y_{2n-1}, y_{2n}) \leq d(y_{2n}, y_{2n+1}),$$

then by (2.27) it follows that

$$s^{\varepsilon} d(y_{2n}, y_{2n+1}) \leq d(y_{2n}, y_{2n+1}),$$

so $d(y_{2n}, y_{2n+1}) = 0$, that is, $y_{2n} = y_{2n+1}$. This is a contradiction with the assumption. Thus, we have

$$s^{\varepsilon} d(y_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n}),$$

which means that

$$d(y_{2n}, y_{2n+1}) \leq \frac{1}{s^{\varepsilon}} d(y_{2n-1}, y_{2n}). \tag{2.28}$$

Thanks to the fact that $Sx_{2n+2} = y_{2n+1}$ and $Tx_{2n+1} = y_{2n}$ are comparable, then by (2.1) it is not hard to verify that

$$\begin{aligned} s^{\varepsilon} d(y_{2n+1}, y_{2n+2}) &= s^{\varepsilon} d(fx_{2n+2}, gx_{2n+1}) \\ &\leq \max\{d(Sx_{2n+2}, Tx_{2n+1}), d(Sx_{2n+2}, fx_{2n+2}), d(Tx_{2n+1}, gx_{2n+1})\} \\ &= \max\{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1})\} \\ &= \max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2})\}. \end{aligned} \tag{2.29}$$

If

$$d(y_{2n}, y_{2n+1}) \leq d(y_{2n+1}, y_{2n+2}),$$

then by (2.29), one has

$$s^{\varepsilon} d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n+1}, y_{2n+2}),$$

which leads to $d(y_{2n+1}, y_{2n+2}) = 0$, i.e., $y_{2n+1} = y_{2n+2}$. This is a contradiction with the assumption. Hence, we claim

$$s^{\varepsilon} d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n}, y_{2n+1}),$$

which educes that

$$d(y_{2n+1}, y_{2n+2}) \le \frac{1}{s^{\varepsilon}} d(y_{2n}, y_{2n+1}). \tag{2.30}$$

Put $\lambda = \frac{1}{s^{\varepsilon}} \in [0,1)$; then by (2.28) and (2.30), (2.26) holds.

Assume that there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} = y_{n_0+1}$. Next, first we will prove it by two cases that $\{y_n\}$ is a constant sequence as $n > n_0$. At this time, (2.26) holds, too.

Case 1. If $n_0 = 2k-1$, then $y_{2k-1} = y_{2k}$, thereby $y_{2k} = y_{2k+1}$. Actually, notice that $Sx_{2k} = y_{2k-1}$ and $Tx_{2k+1} = y_{2k}$ are comparable, then by (2.27), it is clear that

$$s^{\varepsilon} d(y_{2k}, y_{2k+1}) \le \max\{d(y_{2k-1}, y_{2k}), d(y_{2k}, y_{2k+1})\} = d(y_{2k}, y_{2k+1}).$$

So then $d(y_{2k}, y_{2k+1}) = 0$, that is, $y_{2k} = y_{2k+1}$.

Case 2. If $n_0 = 2k$, then $y_{2k} = y_{2k+1}$, accordingly, $y_{2k+1} = y_{2k+2}$. Virtually, since $Sx_{2k+2} = y_{2k+1}$ and $Tx_{2k+1} = y_{2k}$ are comparable; thus, by using (2.29), it is valid that

$$s^{\varepsilon} d(y_{2k+1}, y_{2k+2}) \le \max\{d(y_{2k}, y_{2k+1}), d(y_{2k+1}, y_{2k+2})\} = d(y_{2k+1}, y_{2k+2}).$$

Hence, $d(y_{2k+1}, y_{2k+2}) = 0$, i.e., $y_{2k+1} = y_{2k+2}$.

Secondly, we will prove that the mappings f, g, S, T have a coincidence point. In fact, making full use of (2.26) and Lemma 1.12, we claim that $\{y_n\}$ is a b-Cauchy sequence. Since $(X, \preceq, d)$ is complete, then there is a $z \in X$ satisfying $\lim_{n\to\infty} y_n = z$. Thereupon, one has

$$\lim_{n\to\infty} d(Sx_{2n}, z) = \lim_{n\to\infty} d(fx_{2n}, z) = \lim_{n\to\infty} d(Tx_{2n+1}, z) = \lim_{n\to\infty} d(gx_{2n+1}, z) = 0.$$

By the condition (2), it implies

$$\lim_{n\to\infty} d(Sfx_{2n}, fSx_{2n}) = \lim_{n\to\infty} d(Tgx_{2n+1}, gTx_{2n+1}) = 0.$$

Note that f, g, S, T are continuous; it is easy to see that

$$\lim_{n\to\infty} d(Sfx_{2n}, Sz) = \lim_{n\to\infty} d(fSx_{2n}, fz) = 0,$$

$$\lim_{n\to\infty} d(Tgx_{2n+1}, Tz) = \lim_{n\to\infty} d(gTx_{2n+1}, gz) = 0.$$

Notice

$$\begin{aligned} d(Sz, fz) &\le sd(Sz, Sfx_{2n}) + sd(Sfx_{2n}, fz) \\ &\le sd(Sz, Sfx_{2n}) + s^2 d(Sfx_{2n}, fSx_{2n}) + s^2 d(fSx_{2n}, fz), \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} d(Tz, gz) &\le sd(Tz, Tgx_{2n+1}) + sd(Tgx_{2n+1}, gz) \\ &\le sd(Tz, Tgx_{2n+1}) + s^2 d(Tgx_{2n+1}, gTx_{2n+1}) + s^2 d(gTx_{2n+1}, gz). \end{aligned} \tag{2.32}$$

By taking the limits as $n \to \infty$ from (2.31) and (2.32), we have

$$d(Sz, fz) = 0, \qquad d(Tz, gz) = 0,$$

that is, $Sz = fz$, $Tz = gz$. Consequently, z is a coincidence point of (f, S) and (g, T).

Finally, we assume that Sz and Tz are comparable. We will prove $fz = gz$. As a matter of fact, by virtue of (2.1), it follows that

$$\begin{aligned} & s^{\varepsilon} d(fz, gz) \\ \leq & \max\{d(Sz, Tz), d(Sz, fz), d(Tz, gz)\} \\ = & d(Sz, Tz) = d(fz, gz), \end{aligned}$$

that is,

$$d(fz, gz) \leq \frac{1}{s^{\varepsilon}} d(fz, gz),$$

hence, $d(fz, gz) = 0$. Therefore, $fz = gz = Sz = Tz$, that is to say, z is a coincidence point of f, g, S, T. □

Remark 2.8. Based on the main results of [8, 9], Theorem 2.7 has a sharp improvement. For one thing, it deletes the functions β and ψ and the constant L from [8, Theorem 2]. For another thing, it changes [9, Theorem 15] from metric space to b-metric space; further, it also deletes the functions β and ψ from [9, Theorem 15]. Therefore, Theorem 2.7 has a possibility for greater applications in the future.

The following corollary is a distinct outcome of Theorem 2.7, which is different from the main results from [15].

Corollary 2.9. *Let $(X, \preceq, d)$ be a complete partially ordered b-metric space with coefficient $s > 1$ and f, g, S be continuous self-mappings on X. If the following conditions are satisfied:*

(1) *$fX \cup gX \subseteq SX$;*

(2) *The pairs (f, S) and (g, S) are compatible;*

(3) *The pairs (f, g) and (g, f) are partially weakly increasing with respect to S;*

(4) *The following inequality*

$$s^{\varepsilon} d(fx, gy) \leq \max\{d(Sx, Sy), d(Sx, fx), d(Sy, gy)\}$$

holds for all $x, y \in X$ satisfying Sx and Sy are comparable, where $\varepsilon > 0$ is a constant, then f, g, S have a coincidence point in X.

Finally, we give an example to illustrate Theorem 2.7.

Example 2.10. Let $X = \{1, 2, 3\}$ with the usual partial order. Act a mapping $d : X \times X \to \mathbb{R}$ as

$$d(x, y) = |x - y|^2,$$

then $(X, \preceq, d)$ is a complete partially ordered b-metric space with coefficient $s = 2$. Define self-mappings f, g, S, T on X by

$$f(x) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \qquad g(x) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix},$$

$$T(x) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \qquad S(x) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix}.$$

After careful circulations, we know f, g, S, T satisfy all the conditions of Theorem 2.7 with $\varepsilon = 2$. Therefore, f, g, S, T have a coincidence point $z = 1$.

3. Applications

In this section, we will give some applications in the existence and uniqueness of solutions to several equations.

First of all, we consider the following equation:

$$\begin{cases} \dfrac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \lambda(t, x(t)),\ t \in [a, b], \\ x(a) = x(b) = 0, \end{cases} \tag{3.1}$$

where $\lambda : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Theorem 3.1. *For Eq* (3.1)*, if the following condition holds:*

$$|\lambda(t, x(t)) - \lambda(t, y(t))| \leq k|x(t) - y(t)|, \tag{3.2}$$

where $k > 0$ is a constant, $\lambda(t, 0) = 0$, and $k(b-a)^2 \leq 2^{3-\frac{\varepsilon}{2}}$ with $\varepsilon > 2$, then Eq (3.1) *has a unique solution in* $[a, b]$.

Proof. Equation (3.1) is equivalent to the equation below:

$$x(t) = \int_a^b \Omega(t, r)\lambda(r, x(r))\,\mathrm{d}r, \tag{3.3}$$

where

$$\Omega(t, r) = \begin{cases} \dfrac{r-a}{a-b}(b-t),\ a \leq r \leq t, \\ \dfrac{t-a}{a-b}(b-r),\ t \leq r \leq b. \end{cases} \tag{3.4}$$

We use reverse thinking to prove (3.3). That is to say, we utilize (3.3) to show (3.1). Indeed, by using (3.3) and (3.4), we have

$$\begin{aligned} x(t) &= \int_a^b \Omega(t, r)\lambda(r, x(r))\,\mathrm{d}r \\ &= \int_a^t \Omega(t, r)\lambda(r, x(r))\,\mathrm{d}r + \int_t^b \Omega(t, r)\lambda(r, x(r))\,\mathrm{d}r \\ &= \int_a^t \frac{r-a}{a-b}(b-t)\lambda(r, x(r))\,\mathrm{d}r + \int_t^b \frac{t-a}{a-b}(b-r)\lambda(r, x(r))\,\mathrm{d}r \\ &= \frac{b-t}{a-b}\int_a^t (r-a)\lambda(r, x(r))\,\mathrm{d}r + \frac{t-a}{a-b}\int_t^b (b-r)\lambda(r, x(r))\,\mathrm{d}r, \end{aligned} \tag{3.5}$$

then

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{a-b}\int_a^t (r-a)\lambda(r, x(r))\,\mathrm{d}r + \frac{b-t}{a-b}(t-a)\lambda(t, x(t))$$

$$
\begin{aligned}
&+\frac{1}{a-b}\int_t^b (b-r)\lambda(r,x(r))\,\mathrm{d}r-\frac{t-a}{a-b}(b-t)\lambda(t,x(t))\\
&=-\frac{1}{a-b}\int_a^t (r-a)\lambda(r,x(r))\,\mathrm{d}r+\frac{1}{a-b}\int_t^b (b-r)\lambda(r,x(r))\,\mathrm{d}r,
\end{aligned}
$$

which infers that

$$
\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\frac{1}{a-b}(t-a)\lambda(t,x(t))-\frac{1}{a-b}(b-t)\lambda(t,x(t))=\lambda(t,x(t)).
$$

By (3.5), $x(a)=x(b)=0$. As a consequence, (3.1) holds.

Let $X=C[a,b]$ with partial order as

$$
x\preceq y \Leftrightarrow \max_{t\in[a,b]}|x(t)|\geq \max_{t\in[a,b]}|y(t)|.
$$

Define a mapping $d: X\times X\to[0,+\infty)$ as

$$
d(x,y)=\max_{t\in[a,b]}|x(t)-y(t)|^2,
$$

then (X,d) is a complete b-metric space with $s=2$.

In order to find the solution of (3.3), we need to look for the fixed point of the following mapping:

$$
fx(t)=\int_a^b \Omega(t,r)\lambda(r,x(r))\,\mathrm{d}r,\quad \forall\, t\in[a,b].
$$

Since λ is continuous, then f is a self-mapping on X. Via (3.2) it is not hard to verify that

$$
\begin{aligned}
|fx-gy|^2&=\left|\int_a^b \Omega(t,r)(\lambda(r,x(r))-\lambda(r,y(r)))\,\mathrm{d}r\right|^2\\
&\leq\left(\int_a^b |\Omega(t,r)||\lambda(r,x(r))-\lambda(r,y(r))|\,\mathrm{d}r\right)^2\\
&\leq\left(\int_a^b |\Omega(t,r)|k|x(r)-y(r)|\,\mathrm{d}r\right)^2\\
&\leq k^2\left(\int_a^b |\Omega(t,r)|\max_{r\in[a,b]}|x(r)-y(r)|\,\mathrm{d}r\right)^2\\
&=k^2\left(\int_a^b |\Omega(t,r)|\,\mathrm{d}r\right)^2\max_{r\in[a,b]}|x(r)-y(r)|^2\\
&=\frac{1}{4}k^2[(t-a)(b-t)]^2d(x,y)\\
&\leq\frac{1}{64}k^2(b-a)^4d(x,y)\\
&\leq\frac{1}{2^6}\left(2^{3-\frac{\varepsilon}{2}}\right)^2d(x,y)\\
&=\frac{1}{2^\varepsilon}d(x,y),
\end{aligned}
$$

where $g = f$.

Since $\lambda(t,0) = 0$, then by (3.2), it is easy to see that

$$\begin{aligned}
|fx| &= \left|\int_a^b \Omega(t,r)\lambda(r,x(r))\,\mathrm{d}r\right| \\
&\le \int_a^b |\Omega(t,r)||\lambda(r,x(r)) - 0|\,\mathrm{d}r \\
&= \int_a^b |\Omega(t,r)||\lambda(r,x(r)) - \lambda(r,0)|\,\mathrm{d}r \\
&\le \int_a^b |\Omega(t,r)|k|x(r) - 0|\,\mathrm{d}r \\
&\le k\int_a^b |\Omega(t,r)| \max_{r\in[a,b]} |x(r)|\,\mathrm{d}r \\
&= k\int_a^b |\Omega(t,r)|\,\mathrm{d}r \max_{r\in[a,b]} |x(r)| \\
&= \frac{1}{2}k(t-a)(b-t) \max_{r\in[a,b]} |x(r)| \\
&\le \frac{1}{8}k(b-a)^2 \max_{r\in[a,b]} |x(r)| \\
&\le \max_{r\in[a,b]} |x(r)|,
\end{aligned}$$

so $x \preceq fx$, that is, f, g are dominating. Let $S = T = I$ be an identity mapping on X. Accordingly, S, T are dominated. Consequently, all the conditions of Theorem 2.2 are satisfied. Hence, by Theorem 2.2, f has a unique fixed point in X. Thus, Eq (3.3) has a unique solution in $[a,b]$. In other words, Eq (3.1) has a unique solution in $[a,b]$. □

We next consider the following equation:

$$x(t) = \gamma\int_a^b \lambda(t,r,x(r))\,\mathrm{d}r, \tag{3.6}$$

where γ is a constant and $\lambda : [a,b]\times[a,b]\times\mathbb{R} \to \mathbb{R}$ is a continuous function.

Theorem 3.2. *For Eq* (3.6)*, if the following condition is satisfied:*

$$|\lambda(s,t,x(t)) - \lambda(s,t,y(t))| \le k|x(t) - y(t)|, \tag{3.7}$$

where $k > 0$ is a constant, $\lambda(s,t,0) = 0$, and $k\gamma(b-a)s^{\frac{\varepsilon}{2}} \le 1$ with $\varepsilon > 2$, then Eq (3.6) *has a unique solution in* $[a,b]$.

Proof. Let $X = C[a,b]$ with partial order as

$$x \preceq y \Leftrightarrow \max_{t\in[a,b]} |x(t)| \ge \max_{t\in[a,b]} |y(t)|.$$

Define a mapping $d : X\times X \to \mathbb{R}$ as

$$d(x,y) = \max_{t\in[a,b]} |x(t) - y(t)|^2.$$

then (X, d) is a complete b-metric space with $s = 2$.

In order to find the solution of (3.6), we need to look for the fixed point of the following mapping:

$$fx(t) = \gamma \int_a^b \lambda(t, r, x(r))\,\mathrm{d}r, \quad \forall\, t \in [a, b].$$

Since $fx(t)$ is continuous, then f is a self-mapping on X. It is not hard to verify from (3.7) that

$$\begin{aligned}
|fx - gy|^2 &= \left| \int_a^b \gamma(\lambda(t, r, x(r)) - \lambda(t, r, y(r)))\,\mathrm{d}r \right|^2 \\
&\leq |\gamma|^2 \left(\int_a^b |\lambda(t, r, x(r)) - \lambda(t, r, y(r))|\,\mathrm{d}r \right)^2 \\
&\leq |\gamma|^2 \left(\int_a^b k|x(r) - y(r)|\,\mathrm{d}r \right)^2 \\
&\leq |k\gamma|^2 \left(\int_a^b \max_{r\in[a,b]} |x(r) - y(r)|\,\mathrm{d}r \right)^2 \\
&= (k\gamma)^2 (b-a)^2 \max_{r\in[a,b]} |x(r) - y(r)|^2 \\
&\leq \frac{1}{s^{\varepsilon}} d(x, y),
\end{aligned}$$

where $g = f$.

Since $\lambda(s, t, 0) = 0$, then by (3.7), it follows that

$$\begin{aligned}
|fx| &= \left| \int_a^b \gamma(\lambda(t, r, x(r)) - 0)\,\mathrm{d}r \right| \\
&\leq |\gamma| \int_a^b |\lambda(t, r, x(r)) - 0|\,\mathrm{d}r \\
&= |\gamma| \int_a^b |\lambda(t, r, x(r)) - \lambda(t, r, 0)|\,\mathrm{d}r \\
&\leq |\gamma| \int_a^b k|x(r) - 0|\,\mathrm{d}r \\
&\leq k|\gamma| \int_a^b \max_{r\in[a,b]} |x(r)|\,\mathrm{d}r \\
&= k|\gamma|(b-a) \max_{r\in[a,b]} |x(r)| \\
&\leq \max_{r\in[a,b]} |x(r)|,
\end{aligned}$$

so $x \leq fx$, that is, f, g are dominating. Let $S = T = I$ be an identity mapping on X. It is easy to see that S, T are dominated. Thus, all the conditions of Theorem 2.2 are satisfied. Therefore, by Theorem 2.2, f has a unique solution in X, i.e., Eq (3.6) has a unique solution in $[a, b]$. □

Subsequently, we consider Riemann–Liouville type nonlinear fractional-order differential equations as follows:

$$ {}_a^R D_t^{\mu} x(t) = \lambda(t, x(t)),\ t \in [a, b],\ 1 \leq \mu < 2, \tag{3.8}$$

where $\lambda : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Theorem 3.3. *For Eq* (3.8)*, if the following condition holds:*

$$|\lambda(t, x(t)) - \lambda(t, y(t))| \leq k|x(t) - y(t)|, \tag{3.9}$$

where $k > 0$ *is a constant,* $\lambda(t, 0) = 0$ *and* $k(b-a)^{\mu} s^{\varepsilon+1} \leq \Gamma(\mu+1)$ *with* $\varepsilon > 2$*, then Eq* (3.8) *has a unique solution in* $[a,b]$*.*

Proof. By using integration by substitutions and parts, after careful computations, Eq (3.8) is equivalent to the equation below:

$$x(t) = \frac{1}{\Gamma(\mu)} \int_a^t (t-r)^{\mu-1} \lambda(r, x(r)) \, \mathrm{d}r, \tag{3.10}$$

where $\Gamma(\mu)$ is a Gamma function.

Let $X = C[a,b]$ with partial order as

$$x \leq y \Leftrightarrow \max_{t\in[a,b]} |x(t)| \geq \max_{t\in[a,b]} |y(t)|,$$

and define a mapping $d : X \times X \to \mathbb{R}$ as

$$d(x,y) = \begin{cases} \max\limits_{t\in[a,b]} |x(t) - y(t)|, & xy \neq 0, \\ \alpha \max\limits_{t\in[a,b]} |x(t) - y(t)|, & xy = 0, \end{cases}$$

where $\alpha \geq 1$ is a constant, then (X, d) is a complete b-metric space with $s = \alpha$.

In order to find the solution of (3.10), we need to look for the fixed point of the following mapping:

$$fx(t) = \frac{1}{\Gamma(\mu)} \int_a^t (t-r)^{\mu-1} \lambda(r, x(r)) \, \mathrm{d}r.$$

Owing to $fx(t)$ being continuous, f is thus a self-mapping on X. Hence, by (3.9) we have

$$\begin{aligned} \alpha|fx - gy| &= \alpha \left| \frac{1}{\Gamma(\mu)} \int_a^t (t-r)^{\mu-1} (\lambda(r, x(r)) - \lambda(r, y(r))) \, \mathrm{d}r \right| \\ &\leq \frac{\alpha}{\Gamma(\mu)} \int_a^t (t-r)^{\mu-1} |\lambda(r, x(r)) - \lambda(r, y(r))| \, \mathrm{d}r \\ &\leq \frac{\alpha}{\Gamma(\mu)} \int_a^t (t-r)^{\mu-1} k|x(r) - y(r)| \, \mathrm{d}r \\ &\leq \frac{\alpha}{\Gamma(\mu)} \int_a^t (t-r)^{\mu-1} k \max_{r\in[a,b]} |x(r) - y(r)| \, \mathrm{d}r \\ &\leq \frac{\alpha k}{\Gamma(\mu)} \max_{r\in[a,b]} |x(r) - y(r)| \int_a^t (t-r)^{\mu-1} \, \mathrm{d}r \\ &= \frac{\alpha k (t-a)^{\mu}}{\mu \Gamma(\mu)} d(x,y) \\ &\leq \frac{\alpha k (b-a)^{\mu}}{\Gamma(\mu+1)} d(x,y) \end{aligned}$$

$$\leq \frac{1}{\alpha^{\varepsilon}} d(x,y),$$

where $g = f$.

Since $\lambda(t,0) = 0$, then by (3.9), it establishes that

$$\begin{aligned}
|fx| &= \left| \frac{1}{\Gamma(\mu)} \int_a^t (t-r)^{\mu-1} (\lambda(r,x(r)) - 0)\,\mathrm{d}r \right| \\
&\leq \frac{1}{\Gamma(\mu)} \int_a^t (t-r)^{\mu-1} |\lambda(r,x(r)) - \lambda(r,0)|\,\mathrm{d}r \\
&\leq \frac{1}{\Gamma(\mu)} \int_a^t (t-r)^{\mu-1} k|x(r) - 0|\,\mathrm{d}r \\
&\leq \frac{1}{\Gamma(\mu)} \int_a^t (t-r)^{\mu-1} k \max_{r\in[a,b]} |x(r)|\,\mathrm{d}r \\
&\leq \frac{k}{\Gamma(\mu)} \max_{r\in[a,b]} |x(r)| \int_a^t (t-r)^{\mu-1}\,\mathrm{d}r \\
&= \frac{k(t-a)^{\mu}}{\mu\Gamma(\mu)} \max_{r\in[a,b]} |x(r)| \\
&\leq \frac{k(b-a)^{\mu}}{\Gamma(\mu+1)} \max_{r\in[a,b]} |x(r)| \\
&\leq \max_{r\in[a,b]} |x(r)|,
\end{aligned}$$

thus, $x \preceq fx$, that is, f, g are dominating. Let $S = T = I$ be an identity mapping on X. As a result, S, T are dominated. Then all the conditions of Theorem 2.2 are satisfied. Therefore, by Theorem 2.2, f has a unique solution in X. Thus, Eq (3.10) has a unique solution in $[a,b]$. In other words, Eq (3.8) has a unique solution in $[a,b]$. □

In the end, we consider the equation as below:

$$\begin{cases} \dfrac{\mathrm{d}^2 x}{\mathrm{d}t^2} + w^2 x = G(t,x(t)),\ t \in [0,1], \\ x(0) = 0,\quad x'(0) = 0, \end{cases} \tag{3.11}$$

where $G : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $w \neq 0$.

Theorem 3.4. *For Eq* (3.11)*, if it satisfies*

$$|G(t,x(t)) - G(t,y(t))| \leq k|x(t) - y(t)|, \tag{3.12}$$

where $k > 0$ *is a constant,* $G(t,0) = 0$ *and* $ks^{\frac{\varepsilon}{2}} \leq 1$ *with* $\varepsilon > 2$*, then Eq* (3.11) *has a unique solution in* $[0,1]$.

Proof. Eq (3.11) is equivalent to the equation below:

$$x(t) = \int_0^t \Omega(t,r) G(r,x(r))\,\mathrm{d}r, \tag{3.13}$$

where

$$\Omega(t,r) = \frac{1}{w}\sin(w(t-r)).$$

Let $X = C[0,1]$ with partial order as

$$x \preceq y \Leftrightarrow \max_{t\in[0,1]} |x(t)| \geq \max_{t\in[0,1]} |y(t)|.$$

Define a mapping $d : X \times X \to [0,+\infty)$ as

$$d(x,y) = \max_{t\in[0,1]} |x(t) - y(t)|^2,$$

then (X,d) is a complete b-metric space with $s = 2$.

In order to find the solution of (3.13), we need to look for the fixed point of the following mapping:

$$fx(t) = \int_0^t \Omega(t,r)G(r,x(r))\,\mathrm{d}r, \quad \forall\, t \in [0,1].$$

Since G is continuous, then f is a self-mapping on X. By utilizing (3.12), we arrive at

$$\begin{aligned}
|fx - gy|^2 &= \left|\int_0^t \Omega(t,r)(G(r,x(r)) - G(r,y(r)))\,\mathrm{d}r\right|^2 \\
&\leq \left(\int_0^t |\Omega(t,r)||G(r,x(r)) - G(r,y(r))|\,\mathrm{d}r\right)^2 \\
&\leq \left(k\int_0^t |\Omega(t,r)||x(r) - y(r)|\,\mathrm{d}r\right)^2 \\
&\leq k^2\left(\int_0^t |\Omega(t,r)| \max_{r\in[0,1]} |x(r) - y(r)|\,\mathrm{d}r\right)^2 \\
&= k^2\left(\int_0^t |\Omega(t,r)|\,\mathrm{d}r\right)^2 \max_{r\in[0,1]} |x(r) - y(r)|^2 \\
&\leq k^2 d(x,y) \\
&\leq \frac{1}{s^{\varepsilon}} d(x,y),
\end{aligned}$$

where $g = f$.

Owing to $G(t,0) = 0$, then by (3.12), it is valid that

$$\begin{aligned}
|fx| &= \left|\int_0^t \Omega(t,r)G(r,x(r))\,\mathrm{d}r\right| \\
&\leq \int_0^t |\Omega(t,r)||G(r,x(r)) - 0|\,\mathrm{d}r \\
&= \int_0^t |\Omega(t,r)||G(r,x(r)) - G(r,0)|\,\mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&\leq k \int_0^t |\Omega(t,r)||x(r) - 0| \, \mathrm{d}r \\
&\leq k \int_0^t |\Omega(t,r)| \max_{r\in[0,1]} |x(r)| \, \mathrm{d}r \\
&\leq k \max_{r\in[0,1]} |x(r)| \\
&\leq \max_{r\in[0,1]} |x(r)|,
\end{aligned}
$$

which implies $x \leq fx$. Thereby, f, g are dominating. Let $S = T = I$ be an identity mapping on X. Accordingly, S, T are dominated. As a result, all the conditions of Theorem 2.2 are satisfied. Hence, by Theorem 2.2, f has a unique fixed point in X. That is to say, Eq (3.13) has a unique solution in $[0, 1]$. In other words, Eq (3.11) has a unique solution in $[0, 1]$. □

4. Conclusions

Fixed point theory is one of the most important branches of nonlinear analysis. The development of fixed point theory for contractive mappings is of great importance. Stimulated by this fact, in this paper, the ordered contractive pair for four mappings defined on a partially ordered b-metric space $(X, \preceq, d)$ is considered. The mapping pair (f, g) is called an ordered contractive pair with respect to the mappings S and T if (2.1) is satisfied for every comparable elements $x, y \in X$. By using this concept, under suitable conditions, we establish the existence of common fixed point and coincidence point for the four mappings f, g, S, T. Moreover, we claim that the common fixed point is unique if and only if the set of common points is well ordered (see Theorem 2.2). In addition, we give two examples to illustrate the superiority of our results (see Examples 2.6 and 2.10). We also use our results to cope with the existence and uniqueness of solution to several equations. We make a conclusion that our results will be helpful for researchers in this field for further study and substantial development.

Author contributions

All authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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