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Research article

The representation ring of a non-pointed bialgebra

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Abstract: The aim of this paper is to characterize the representation ring of a non-pointed and noncocommutative bialgebra. First, the isomorphism classes of its indecomposable modules are classified. Then the tensor product of modules is established. Finally, its representation ring is described.

Keywords: bialgebra; bound quiver; representation ring
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1. Introduction

The concept of representation rings, also known as the Green ring, was first introduced by Green [1] in the 1960's while studying the representations of finite groups. Benson and Carlson further developed this work, and applied it to the representation theory of algebras (see [2, 3]). Recently, the study of representations and representation rings of Hopf algebras and, more generally, bialgebras has received considerable attention. For example, Chen, Van Oystaeyen, and Zhang in [4] studied the representation rings of Taft algebras. Wang, Li, and Zhang in [5] determined the structure of the Green ring for finite-dimensional pointed rank one Hopf algebras of nilpotent type. The authors also provided a classification of all indecomposable modules and constructed their tensor product decomposition formulas. Subsequently, they studied the Green rings of finite-dimensional pointed rank one Hopf algebras of non-nilpotent type in [6]. Guo and Yang in [7] gave a complete list of all indecomposable modules and described all components of Auslander-Reiten quivers of a class of non-pointed tame Hopf algebras. The authors also established their projective class rings as well as the projective class rings of the category of their Yetter-Drinfeld modules. Su and Yang gave the Green rings of Δ-associative algebras (see [8]). Lin and Yang constructed and classified all string representations of a class of tame type Hopf algebras and gave the explicit description of projective class rings (see [9]).

Yang and Zhang in [10] classified the bialgebra Ore extensions of automorphism type for the 4-

dimensional Sweedler Hopf algebra and obtained a series of non-isomorphic bialgebras and Hopf algebras. The most interesting one of them, denoted by $\mathcal{A}_{24m}$, which is noncocommutative and non-pointed with dimension $24m$, arouses us to study their representations and describe its representation ring. The results may help us to understand the more general case of non-pointed bialgebras. We first classify all simple and all indecomposable modules of $\mathcal{A}_{24m}$ explicitly. Then, the decomposition formulas of the tensor products of them are established, and the description of the representation ring is given.

The outline of this paper is as follows: In Section 1, the definition of the $24m$-dimensional bialgebras $\mathcal{A}_{24m}$ is reviewed. It is shown that $\mathcal{A}_{24m}$ is a basic algebra, and its corresponding bound quiver is described. In Section 2, all indecomposable modules of $\mathcal{A}_{24m}$ are classified, and decomposition formulas of their tensor products are established. In Section 3, we focus on characterizing its representation ring of $\mathcal{A}_{24m}$.

Throughout, $\mathbb{K}$ is assumed to be an algebraically closed field containing a non-trivial $6m$-th primitive root q of unity, and $\omega = q^m$.

2. The bialgebras $\mathcal{A}_{24m}$

First of all, let us review the family of bialgebras $\mathcal{A}_{24m}$ discussed in the paper.

By definition, the algebra $\mathcal{A}_{24m}$ is generated by x, g, z subjecting to the following relations:

$$g^2 = 1,\ x^2 = 0,\ z^{6m} = 1,\ xg = -gx,\ zx = \omega gxz,\ zg = gz + 2xz.$$

If we define two $\mathbb{K}$-maps $\Delta : \mathcal{A}_{24m} \to \mathcal{A}_{24m} \otimes \mathcal{A}_{24m}$ and $\epsilon :\ \mathcal{A}_{24m} \to \mathbb{K}$ on generators by

$$\begin{aligned}
&\Delta(g) = g \otimes g,\\
&\Delta(x) = x \otimes 1 + g \otimes x,\\
&\Delta(z) = z \otimes z - 2e_1 z \otimes e_1 z + 2e_1 z \otimes xz,\\
&\epsilon(x) = 0, \epsilon(g) = \epsilon(z) = 1,
\end{aligned}$$

where $e_1 = \frac{1-g}{2}$, and extend them to the whole space $\mathcal{A}_{24m}$ in the natural way. Then, $\mathcal{A}_{24m}$ becomes a $24m$-dimensional bialgebra with a basis

$$\{x^i g^j z^k | k \in \mathbb{Z}_{6m}; i, j = 0, 1\}.$$

It is easy to see that $\mathcal{A}_{24m}$ is non-commutative, non-cocommutative and non-pointed. For the detailed construction, the readers are referred to [10, Page 20, **Case 2:** (a)(i)].

Obviously, $x\mathcal{A}_{24m}$ of $\mathcal{A}_{24m}$ is a bi-ideal; the bialgebra $\mathcal{A}_{24m}/x\mathcal{A}_{24m}$ deduced from $\mathcal{A}_{24m}$ is the group algebra $\mathbb{K}(C_2 \times C_{6m})$, which comultiplication is given by

$$\begin{aligned}
&\Delta(g) = g \otimes g,\\
&\Delta(z) = z \otimes z - 2e_1 z \otimes e_1 z = \frac{1}{2}\left(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g\right)(z \otimes z),\\
&\epsilon(g) = \epsilon(z) = 1.
\end{aligned}$$

Indeed, it is straightforward to see that $\mathbb{K}(C_2 \times C_{6m})$ is a bialgebra in this setting.

For $i \in \mathbb{Z}_{6m}$, we set

$$\epsilon_i = \frac{1}{6m}\sum_{j=0}^{6m-1} q^{-ij}z^j, \quad e_0 = \frac{1+g}{2}.$$

Lemma 2.1. *The set*

$$\{e_0\epsilon_i e_0, e_1\epsilon_i e_1 | i \in \mathbb{Z}_{6m}\}$$

is a complete set of primitive orthogonal idempotents of $\mathcal{A}_{24m}$.

Proof. For $n = 0, 1, \ldots, 6m-1$, we have

$$\begin{aligned}
e_0 xz^n \epsilon_i e_0 &= xe_1 z^n \epsilon_i e_0 \\
&= x\left(z^n e_1 + \sum_{k=1 \text{ odd}}^{n} \omega^{k-1} xz^n + \sum_{l=1 \text{ even}}^{n} \omega^{l-1} gxz^n\right) e_0 \\
&= xz^n e_1 \epsilon_i e_0 \\
&= xz^n \left(\epsilon_i e_1 + \frac{1}{6m}\sum_{j=0}^{6m-1}\sum_{k=1 \text{ odd}}^{j} q^{-ij}\omega^{k-1}xz^j + \frac{1}{6m}\sum_{j=0}^{6m-1}\sum_{l=1 \text{ even}}^{j} q^{-ij}\omega^{l-1}gxz^j\right) e_0 \\
&= xz^n\epsilon_i e_1 e_0 + \frac{1}{6m}\sum_{j=0}^{6m-1}\sum_{k=1 \text{ odd}}^{j} q^{-ij}\omega^{k-1}xz^n xz^j e_0 + \frac{1}{6m}\sum_{j=0}^{6m-1}\sum_{l=1 \text{ even}}^{j} q^{-ij}\omega^{l-1}xz^n gxz^j e_0 \\
&= 0.
\end{aligned}$$

Similarly, $e_1 xz^n \epsilon_i e_1 = 0$. It follows that

$$\begin{aligned}
(e_0\epsilon_i e_0)^2 &= e_0\epsilon_i e_0 \epsilon_i e_0 \\
&= e_0\left(e_0\epsilon_i + \frac{1}{6m}\sum_{j=0}^{6m-1}\sum_{k=1 \text{ odd}}^{j} q^{-ij}\omega^{k-1}xz^j + \frac{1}{6m}\sum_{j=0}^{6m-1}\sum_{l=1 \text{ even}}^{j} q^{-ij}\omega^{l-1}gxz^j\right)\epsilon_i e_0 \\
&= e_0\epsilon_i e_0 + \frac{1}{6m}\sum_{j=0}^{6m-1}\sum_{k=1 \text{ odd}}^{j} q^{-ij}\omega^{k-1}e_0 xz^j\epsilon_i e_0 + \frac{1}{6m}\sum_{j=0}^{6m-1}\sum_{l=1 \text{ even}}^{j} q^{-ij}\omega^{l-1}e_0 gxz^j\epsilon_i e_0 \\
&= e_0\epsilon_i e_0.
\end{aligned}$$

$$\begin{aligned}
(e_1\epsilon_i e_1)^2 &= e_1\epsilon_i e_1 \epsilon_i e_1 \\
&= e_1\left(e_1\epsilon_i - \frac{1}{6m}\sum_{i=0}^{6m-1}\sum_{k=1 \text{ odd}}^{j} q^{-ij}\omega^{k-1}xz^j - \frac{1}{6m}\sum_{i=0}^{6m-1}\sum_{l=1 \text{ even}}^{j} q^{-ij}\omega^{l-1}gxz^j\right)\epsilon_i e_1 \\
&= e_1\epsilon_i e_1 - \frac{1}{6m}\sum_{i=0}^{6m-1}\sum_{k=1 \text{ odd}}^{j} q^{-ij}\omega^{k-1}e_1 xz^j\epsilon_i e_1 - \frac{1}{6m}\sum_{i=0}^{6m-1}\sum_{l=1 \text{ even}}^{j} q^{-ij}\omega^{l-1}e_1 gxz^j\epsilon_i e_1 \\
&= e_1\epsilon_i e_1.
\end{aligned}$$

Hence, $e_0\epsilon_i e_0$, $e_1\epsilon_i e_1$ are idempotents for all $i \in \mathbb{Z}_{6m}$.

Furthermore, for any $i \neq j$, we have

$$
\begin{aligned}
(e_0\epsilon_i e_0)(e_0\epsilon_j e_0) &= e_0\epsilon_i e_0\epsilon_j e_0 \\
&= e_0\left(e_0\epsilon_j + \frac{1}{6m}\sum_{j=0}^{6m-1}\sum_{k=1 \text{ odd}}^{j} q^{-ij}\omega^{k-1}xz^j + \frac{1}{6m}\sum_{j=0}^{6m-1}\sum_{l=1 \text{ even}}^{j} q^{-ij}\omega^{l-1}gxz^j\right)\epsilon_j e_0 \\
&= e_0\epsilon_i\epsilon_j e_0 + \frac{1}{6m}\sum_{j=0}^{6m-1}\sum_{k=1 \text{ odd}}^{j} q^{-ij}\omega^{k-1}e_0xz^j\epsilon_j e_0 + \frac{1}{6m}\sum_{j=0}^{6m-1}\sum_{l=1 \text{ even}}^{j} q^{-ij}\omega^{l-1}e_0gxz^j\epsilon_j e_0 \\
&= 0.
\end{aligned}
$$

In a similar way, for any $i \neq j$,

$$(e_1\epsilon_i e_1)(e_1\epsilon_j e_1) = 0, \quad (e_0\epsilon_i e_0)(e_1\epsilon_j e_1) = 0,$$

and

$$e_i\epsilon_j e_i x e_i\epsilon_j e_i = 0$$

for all $i = 0, 1$ and $j \in \mathbb{Z}_{6m}$. In other words, $\dim e_i\epsilon_j e_i\mathcal{A}_{24m}e_i\epsilon_j e_i = 1$.

Also, we have

$$\sum_{i=0}^{6m-1} e_0\epsilon_i e_0 + \sum_{i=0}^{6m-1} e_1\epsilon_i e_1 = e_0\left(\sum_{i=0}^{6m-1}\epsilon_i\right)e_0 + e_1\left(\sum_{i=0}^{6m-1}\epsilon_i\right)e_1 = e_0^2 + e_1^2 = 1.$$

Therefore, the set

$$\{e_0\epsilon_i e_0, e_1\epsilon_i e_1 | i \in \mathbb{Z}_{6m}\}$$

is indeed a complete set of primitive orthogonal idempotents of $\mathcal{A}_{24m}$. □

Lemma 2.2. *The algebra $\mathcal{A}_{24m}$ is a basic algebra.*

Proof. Recall that an algebra A is basic if and only if the algebra A/J is isomorphic to $\mathbb{K}\times\mathbb{K}\times\cdots\times\mathbb{K}$, where $J = \mathrm{rad}A$ is the radical of A. Therefore, the statement is obvious since $\mathcal{A}_{24m}/J = \mathcal{A}_{24m}/x\mathcal{A}_{24m} \cong \mathbb{K}(C_2 \times C_{6m})$, which is basic, by the previous remark.

Here, we give the detailed proof by constructing the explicit correspondence between primitive idempotents $f_{ij} = e_i\epsilon_j e_i (i \in \mathbb{Z}_2, j \in \mathbb{Z}_{6m})$ of $\mathcal{A}_{24m}$ and those of the $12m$-dimensional diagonal matrix algebra. It is convenient in describing the bound quiver of $\mathcal{A}_{24m}$ and indecomposable projective modules of $\mathcal{A}_{24m}$ in the later. It is remarked that $f_{ij} = e_i\epsilon_j e_i$, the liftings of those of $\mathbb{K}(C_2 \times C_{6m})$, are not in $\mathbb{K}(C_2 \times C_{6m})$.

Now, for $i \in \mathbb{Z}_2$ and $j \in \mathbb{Z}_{6m}$, we set

$$f_{ij} = e_i\epsilon_j e_i.$$

It is easy to see that

$$\mathrm{rad}\mathcal{A}_{24m} = \sum_{\substack{i\in\mathbb{Z}_2 \\ j\in\mathbb{Z}_{6m}}} \mathbb{K}xf_{ij}$$

of dimension $12m$, and

$$\mathrm{rad}^2\mathcal{A}_{24m} = 0.$$

The 12m-dimensional algebra

$$\mathbb{K} \times \mathbb{K} \times \cdots \times \mathbb{K}$$

can be identified with the 12m-dimensional diagonal matrix algebra

$$\mathrm{diag}\,(\mathbb{K}, \mathbb{K}, \cdots, \mathbb{K})\,.$$

Let

$$\zeta_p = \mathrm{diag}(0, \ldots, 0, \underset{p \text{ position}}{1}, 0, \ldots, 0),$$

where $p = 1, \cdots, 12m$, and only the p-th position is 1 and others are 0. Let

$$\bar{f}_{ij} = f_{ij} + \mathrm{rad}\mathcal{A}_{24m}, \text{ where } i \in \mathbb{Z}_2, j \in \mathbb{Z}_{6m}.$$

Now, define the mapping

$$\varphi : \mathcal{A}_{24m}/\mathrm{rad}\mathcal{A}_{24m} \longrightarrow \mathrm{diag}\,(\mathbb{K}, \mathbb{K}, \cdots, \mathbb{K})$$

by

$$\varphi(\bar{f}_{0j}) = \zeta_{j+1}, \text{ and } \varphi(\bar{f}_{1j}) = \zeta_{6m+1+j},$$

where $j \in \mathbb{Z}_{6m}$.

We must prove that φ is an algebra isomorphism. Indeed, we have

$$\varphi(\bar{f}_{0i})\varphi(\bar{f}_{0j}) = \zeta_{i+1}\zeta_{j+1} = \delta_{ij}\zeta_{i+1},$$

$$\varphi(\bar{f}_{0i}\bar{f}_{0j}) = \varphi(\delta_{ij}\bar{f}_{0i}) = \delta_{ij}\zeta_{i+1},$$

$$\varphi(\bar{f}_{0i})\varphi(\bar{f}_{0j}) = \varphi(\bar{f}_{0i}\bar{f}_{0j}).$$

In the same way, we can obtain

$$\varphi(\bar{f}_{1k})\varphi(\bar{f}_{1l}) = \varphi(\bar{f}_{1k}\bar{f}_{1l}).$$

Because

$$\varphi(\bar{f}_{0i})\varphi(\bar{f}_{1k}) = \zeta_{i+1}\zeta_{6m+1+k} = 0,$$

$$\varphi(\bar{f}_{0i}\bar{f}_{1k}) = \varphi(\bar{f}_{0i}\bar{f}_{1k}) = 0,$$

we have

$$\varphi(\bar{f}_{0i})\varphi(\bar{f}_{1k}) = \varphi(\bar{f}_{0i}\bar{f}_{1k}).$$

Furthermore,

$$\varphi(\sum_{i=0}^{1}\sum_{j=0}^{6m-1}\bar{f}_{ij}) = \varphi(\bar{1}) = E,$$

$$\sum_{i=0}^{1}\sum_{j=0}^{6m-1}\varphi(\bar{f}_{ij}) = \sum_{j=0}^{6m-1}\varphi(\bar{f}_{0j}) + \sum_{j=0}^{6m-1}\varphi(\bar{f}_{1j}) = E,$$

where E is the identity matrix of $12m \times 12m$.

Hence, φ is an algebra isomorphism, and the algebra $\mathcal{A}_{24m}$ is a basic algebra. □

Furthermore, we see that

$$\mathrm{rad}\mathcal{A}_{24m}/\mathrm{rad}^2\mathcal{A}_{24m} = \sum_{\substack{i\in\mathbb{Z}_2\\ j\in\mathbb{Z}_{6m}}} \mathbb{K}xf_{ij}.$$

Hence,

$$f_{0i}\left(\mathrm{rad}\mathcal{A}_{24m}/\mathrm{rad}^2\mathcal{A}_{24m}\right)f_{0j} = 0, \quad f_{1i}\left(\mathrm{rad}\mathcal{A}_{24m}/\mathrm{rad}^2\mathcal{A}_{24m}\right)f_{1j} = 0,$$

and

$$f_{0i}\left(\mathrm{rad}\mathcal{A}_{24m}/\mathrm{rad}^2\mathcal{A}_{24m}\right)f_{1j} = \begin{cases}\mathbb{K}\left\{xf_{1j}\right\}, & \text{if } i = j+m,\\ 0, & \text{otherwise,}\end{cases}$$

$$f_{1i}\left(\mathrm{rad}\mathcal{A}_{24m}/\mathrm{rad}^2\mathcal{A}_{24m}\right)f_{0j} = \begin{cases}\mathbb{K}\left\{xf_{1j}\right\}, & \text{if } i = j+4m,\\ 0, & \text{otherwise,}\end{cases}$$

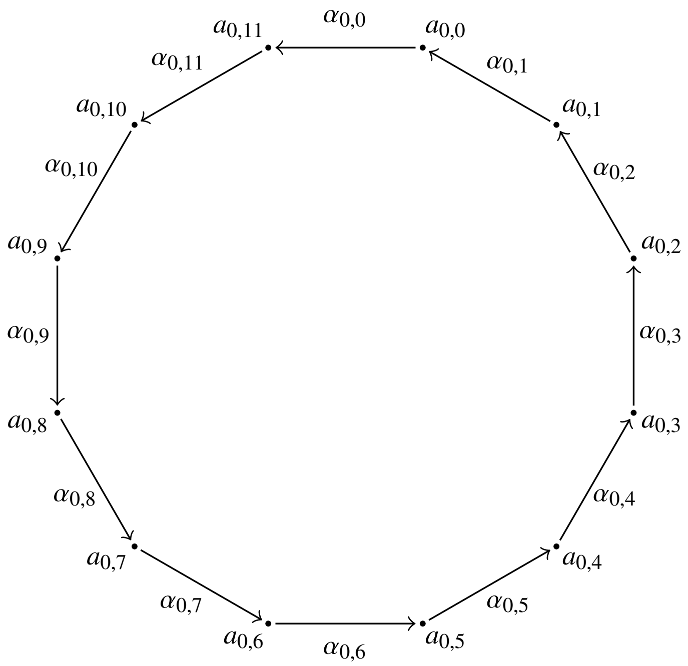
where $i, j \in \mathbb{Z}_{6m}$.

We can describe the bound quiver (Q, I) of $\mathcal{A}_{24m}$. Now, let $a_{i,j}$ be the points corresponding to the primitive idempotents f_{k_i,l_j}, where

$$k_i \equiv (j+1)\,(\mathrm{mod}\,2),\ \ l_j \equiv \left(i + \sum_{k=0}^{j+1} \frac{\left(5+3(-1)^k\right)m}{2}\right) \mathrm{mod}\, 6m, \tag{2.1}$$

and the arrows are endowed with $\alpha_{i,j+1} : a_{i,j+1} \to a_{i,j}$, for all $(i, j) \in \mathbb{Z}_m \times \mathbb{Z}_{12}$.

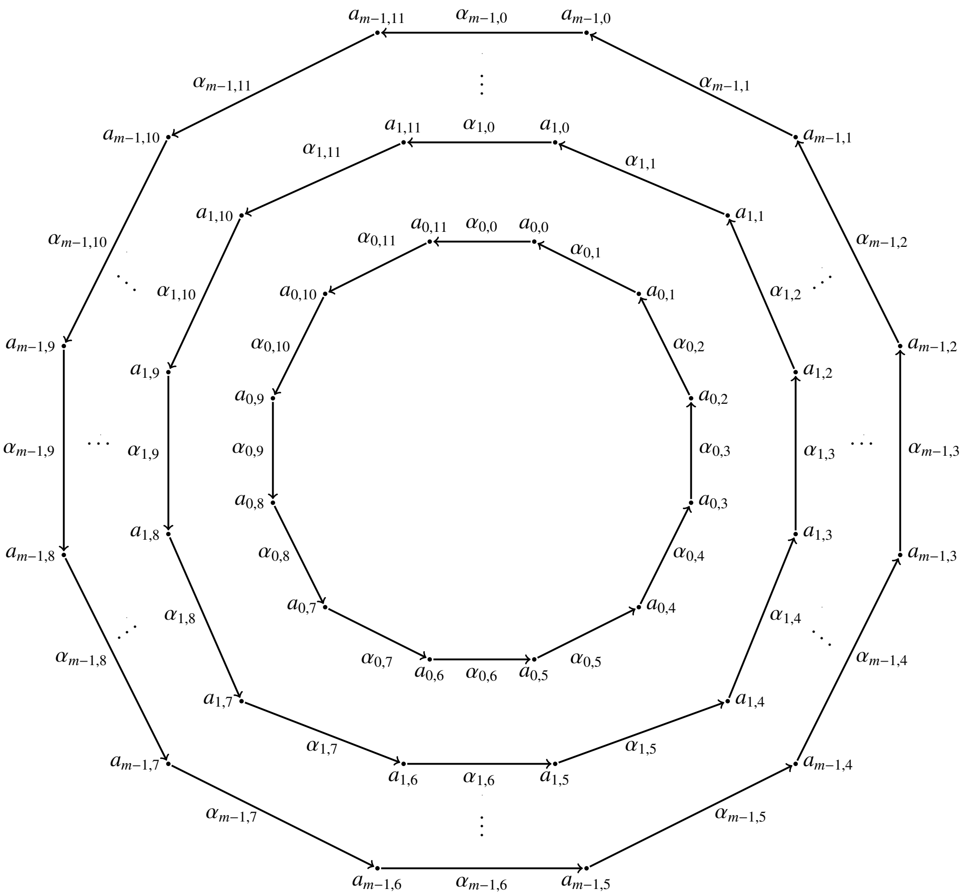
For example, for $m = 1$, the bound quiver $(Q_{\mathcal{A}_{24}}, I)$ is



with the admissible ideal

$$I = \left\{\alpha_{0,(j+1)}\alpha_{0,j} \mid j \in \mathbb{Z}_{12}\right\}.$$

In general, for $m \geq 1$, it is observed that the bound quiver of $(Q_{\mathcal{A}_{24m}}, I)$ is



with the admissible ideal

$$I = \left\{ \alpha_{i,(j+1)}\alpha_{i,j} \,|\, i \in \mathbb{Z}_m,\ j \in \mathbb{Z}_{12} \right\}.$$

3. Indecomposable $\mathcal{A}_{24m}$-modules

In this section, we mainly construct all mutually non-isomorphic indecomposable modules of $\mathcal{A}_{24m}$. Then, we consider their tensor products, respectively. Finally, we conclude that the decomposition formula of their tensor products.

By the bound quiver of algebra $(\mathcal{A}_{24m}, I)$, we see that algebra $\mathcal{A}_{24m}$ can be decomposed into a direct sum of m blocks $C_i, i \in \mathbb{Z}_m$: $\mathcal{A}_{24m} = \bigoplus_{i\in\mathbb{Z}_m} C_i$, where $C_i = \sum_{j\in\mathbb{Z}_{12}} \mathcal{A}_{24m} f_{k_i,l_j}$, and k_i, l_j are as in (2.1). Consequently, by [11, Chap. 5, Theorem 3.5], any indecomposable $\mathcal{A}_{24m}$-module is either a 1-dimensional simple module or a 2-dimensional projective $\mathcal{A}_{24m}$-module.

Let S_{ij} be a 1-dimensional vector space spanned by v_{ij}, $i \in \mathbb{Z}_2$, $j \in \mathbb{Z}_{6m}$, we define the action of $\mathcal{A}_{24m}$ on S_{ij} as follows:

$$x \cdot v_{ij} = 0, \quad g \cdot v_{ij} = (-1)^i v_{ij}, \quad z \cdot v_{ij} = q^j \cdot v_{ij}.$$

It is straightforward to see that S_{ij} is a simple $\mathcal{A}_{24m}$-module, and $S_{ij} \cong S_{kl}$ if and only if $i = k$, $j = l$.

Lemma 3.1. *Any simple $\mathcal{A}_{24m}$-module S is isomorphic to S_{ij}, where $i \in \mathbb{Z}_2$, $j \in \mathbb{Z}_{6m}$.*

Proof. Assume that S is a simple $\mathcal{A}_{24m}$-module. By Lemma 2.2, the dimension of the simple $\mathcal{A}_{24m}$-module S is 1. Let v be the basis of S. We set

$$x \cdot v = av,\ g \cdot v = bv,\ z \cdot v = cv,$$

where $a, b, c \in \mathbb{K}$. We have

$$x^2 v = a^2 v = 0, \quad g^2 v = b^2 v = v, \quad z^{6m} v = c^{6m} v = v.$$

Hence, we have $a = 0,\ b = \pm 1$, and $c^{6m} = 1$. Consequently, we can set $c = q^j$ for some $j \in \mathbb{Z}_{6m}$ and

$$x \cdot v = 0, \quad g \cdot v = v, \quad z \cdot v = q^j v,$$

or

$$x \cdot v = 0, \quad g \cdot v = -v, \quad z \cdot v = q^j v.$$

Therefore, there exist $i \in \mathbb{Z}_2,\ j \in \mathbb{Z}_{6m}$ such that $S = S_{ij}$. □

Furthermore, the module action of the submodule $\mathcal{A}_{24m} f_{ij}$ of the left regular $\mathcal{A}_{24m}$-module is as follows:

$$\begin{aligned} &x \cdot f_{ij} = x f_{ij}, \quad x \cdot x f_{ij} = 0, \\ &g \cdot f_{ij} = (-1)^i f_{ij}, \quad g \cdot x f_{ij} = (-1)^{i+1} x f_{ij}, \\ &z \cdot f_{ij} = q^j f_{ij} + (-1)^i q^j x f_{ij}, \quad z \cdot x f_{ij} = (-1)^{i+1} \omega q^j x f_{ij}. \end{aligned}$$

Let P_{ij} be the projective cover of simple $\mathcal{A}_{24m}$-module S_{ij} for $(i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_{6m}$.

Lemma 3.2. *We have* $P_{ij} \cong \mathcal{A}_{24m} f_{ij}$.

Proof. Note that $\{f_{ij} | i \in \mathbb{Z}_2, j \in \mathbb{Z}_{6m}\}$ is the complete set of orthogonal idempotent elements of the algebra $\mathcal{A}_{24m}$. So we have

$$\mathcal{A}_{24m} = \left(\bigoplus_{j=0}^{6m-1} \mathcal{A}_{24m} f_{0j} \right) \oplus \left(\bigoplus_{j=0}^{6m-1} \mathcal{A}_{24m} f_{1j} \right).$$

Also, $\mathcal{A}_{24m} f_{ij}$ is an indecomposable projective module.

We define the map φ as follows:

$$\begin{aligned} \varphi : \mathcal{A}_{24m} f_{ij} &\to S_{ij} \\ f_{ij} &\longmapsto (-1)^i v_{ij} \\ x f_{ij} &\longmapsto 0, \end{aligned}$$

for each $(i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_{6m}$.

It is easy to see that φ is a surjective linear map. By Lemma 3.1, we have

$$\begin{aligned} &\varphi(x f_{ij}) = 0 = x \cdot v_{ij} = x \cdot \varphi(f_{ij}),\ \ \varphi(x^2 f_{ij}) = 0 = x \cdot \varphi(x f_{ij}), \\ &\varphi(g f_{ij}) = \varphi((-1)^i f_{ij}) = (-1)^i \varphi(f_{ij}) = v_{ij} = g \cdot \varphi(f_{ij}) = g \cdot (-1)^i v_{ij}, \end{aligned}$$

$$\varphi(gxf_{ij}) = \varphi((-1)^{i+1}xf_{ij}) = 0 = g \cdot \varphi(xf_{ij}),$$
$$\varphi(zf_{ij}) = \varphi(q^j f_{ij} + (-1)^i q^j xf_{ij}) = q^j(-1)^i v_{ij} = z \cdot \varphi(f_{ij}) = z \cdot (-1)^i v_{ij},$$
$$\varphi(zxf_{ij}) = \varphi((-1)^{i+1}\omega q^j xf_{ij}) = 0 = z \cdot \varphi(xf_{ij}).$$

So the map $\varphi : \mathcal{A}_{24m}f_{ij} \to S_{ij}$ can define a module epimorphism, and

$$\mathcal{A}_{24m}f_{ij}/\ker\varphi \cong S_{ij}.$$

It is easy to see that $\ker\varphi = \mathrm{rad}(\mathcal{A}_{24m}f_{ij}) = \mathbb{K}\{xf_{ij}\}$, and

$$\mathcal{A}_{24m}f_{ij}/\mathrm{rad}(\mathcal{A}_{24m}f_{ij}) \cong S_{ij}.$$

However,

$$\mathcal{A}_{24m}f_{ij}/\mathrm{rad}(\mathcal{A}_{24m}f_{ij}) \cong P_{ij}/\mathrm{rad}(P_{ij}) \cong S_{ij}$$

since P_{ij} is a projective cover of simple $\mathcal{A}_{24m}$-module S_{ij}. By the uniqueness of the projective cover of a module, we have

$$P_{ij} \cong \mathcal{A}_{24m}f_{ij},\ i \in \mathbb{Z}_2, j \in \mathbb{Z}_{6m}.$$

Now, we identify P_{ij} with $\mathcal{A}_{24m}f_{ij}$ for $(i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_{6m}$.

So far, we have provided a classification of all indecomposable $\mathcal{A}_{24m}$-modules. In the following, we consider their tensor products.

Proposition 3.3. *$S_{ij} \otimes S_{kl} \cong S_{(i+k)(\mathrm{mod}\,2),(j+l)(\mathrm{mod}\,6m)} \cong S_{kl} \otimes S_{ij}$, for $i \in \mathbb{Z}_2$, $j \in \mathbb{Z}_{6m}$.*

Proof. For simple modules S_{ij}, S_{kl}, the tensor product $S_{ij} \otimes S_{kl}$ is also an $\mathcal{A}_{24m}$-module with a basis $v_{ij} \otimes v_{kl}$. By Lemma 3.1, we have

$$x \cdot \left(v_{ij} \otimes v_{kl}\right) = 0,$$
$$g \cdot \left(v_{ij} \otimes v_{kl}\right) = (-1)^{i+k}\left(v_{ij} \otimes v_{kl}\right),$$
$$z \cdot \left(v_{ij} \otimes v_{kl}\right) = q^{j+l}\left(v_{ij} \otimes v_{kl}\right).$$

Hence,

$$S_{ij} \otimes S_{kl} \cong S_{(i+k)(\mathrm{mod}\,2),(j+l)(\mathrm{mod}\,6m)} \cong S_{kl} \otimes S_{ij}, (i \in \mathbb{Z}_2, j \in \mathbb{Z}_{6m}).$$

The result follows. □

Proposition 3.4. *$S_{ij} \otimes P_{kl} \cong P_{(i+k)(\mathrm{mod}\,2),(j+l)(\mathrm{mod}\,6m)} \cong P_{kl} \otimes S_{ij}$, for $i \in \mathbb{Z}_2$, $j \in \mathbb{Z}_{6m}$.*

Proof. We consider $P_{kl} = \mathcal{A}_{24m}f_{kl}$, $(k \in \mathbb{Z}_2, l \in \mathbb{Z}_{6m})$, which is of basis $\{f_{kl},\ xf_{kl}\}$. The action of $\mathcal{A}_{24m}$ on P_{kl} is the following:

$$x \cdot f_{kl} = xf_{kl}, \quad x \cdot xf_{ij} = 0,$$
$$g \cdot f_{kl} = (-1)^k f_{kl}, \quad g \cdot xf_{kl} = (-1)^{k+1}xf_{kl},$$
$$z \cdot f_{kl} = q^l f_{kl} + (-1)^k q^l xf_{kl}, \quad z \cdot xf_{kl} = (-1)^{k+1}\omega q^l xf_{kl}.$$

The corresponding matrix representations of x, g, z are

$$A_x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_g = \begin{pmatrix} (-1)^k & 0 \\ 0 & (-1)^{k+1} \end{pmatrix}, \quad A_z = \begin{pmatrix} q^l & 0 \\ (-1)^k q^l & (-1)^{k+1}\omega q^l \end{pmatrix},$$

respectively.

On the other hand, $\{v_{ij} \otimes f_{kl},\ v_{ij} \otimes xf_{kl}\}$ form a basis of the tensor product $S_{ij} \otimes P_{kl}$. The action of $\mathcal{A}_{24m}$ on $S_{ij} \otimes P_{kl}$ is the following:

$$\begin{aligned}
x \cdot \left(v_{ij} \otimes f_{kl}\right) &= v_{ij} \otimes xf_{kl}, \\
x \cdot \left(v_{ij} \otimes xf_{ij}\right) &= 0, \\
g \cdot \left(v_{ij} \otimes f_{kl}\right) &= (-1)^{k+i} v_{ij} \otimes f_{kl}, \\
g \cdot \left(v_{ij} \otimes xf_{kl}\right) &= (-1)^{k+i+1} v_{ij} \otimes xf_{kl}, \\
z \cdot \left(v_{ij} \otimes f_{kl}\right) &= q^j v_{ij} \otimes (q^l f_{kl} + (-1)^k q^l xf_{kl}) = q^{j+l} v_{ij} \otimes f_{kl} + (-1)^k q^{j+l} v_{ij} \otimes xf_{kl}, \\
z \cdot \left(v_{ij} \otimes xf_{kl}\right) &= q^j v_{ij} \otimes (-1)^{k+1} \omega q^l xf_{kl} = (-1)^{k+1} \omega q^{j+l} v_{ij} \otimes xf_{kl}.
\end{aligned}$$

The corresponding matrix representations of x, g, z are

$$B_x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B_g = \begin{pmatrix} (-1)^{k+i} & 0 \\ 0 & (-1)^{k+i+1} \end{pmatrix}, \quad B_z = \begin{pmatrix} q^{j+l} & 0 \\ (-1)^k q^{j+l} & (-1)^{k+1} \omega q^{j+l} \end{pmatrix},$$

respectively.

Hence, we have

$$S_{ij} \otimes P_{kl} \cong P_{(i+k)(\mathrm{mod}\,2),(j+l)(\mathrm{mod}\,6m)}.$$

In a similar way, we can also prove that

$$P_{kl} \otimes S_{ij} \cong P_{(i+k)(\mathrm{mod}\,2),(j+l)(\mathrm{mod}\,6m)}.$$

Proposition 3.5.

$$P_{ij} \otimes P_{kl} \cong P_{(i+k+1)(\mathrm{mod}\,2),(j+4m+3mi+l)(\mathrm{mod}\,6m)} \oplus P_{(i+k)(\mathrm{mod}\,2),(j+l)(\mathrm{mod}\,6m)},$$

for $i \in \mathbb{Z}_2$, $j \in \mathbb{Z}_{6m}$.

Proof. Considering the exact sequence

$$P_{ij} \xrightarrow{\psi} S_{ij} \longrightarrow 0,$$

where ψ is defined by $\psi(f_{ij}) = (-1)^i v_{ij},\ \psi(xf_{ij}) = 0$. We have $\mathrm{Ker}\psi = \mathbb{K}\{xf_{ij}\} = \Omega\left(S_{ij}\right)$, and

$$0 \longrightarrow \Omega(S_{ij}) \longrightarrow P_{ij} \longrightarrow S_{ij} \longrightarrow 0.$$

On the other hand, for a Hopf algebra H, and any H-module M, if P is a projective H-module, then $P \otimes M$ or $M \otimes P$ is also a projective H-module (see, for example, [12]). Therefore, we obtain the following splitting short exact sequence:

$$0 \longrightarrow \Omega(S_{ij}) \otimes P_{kl} \longrightarrow P_{ij} \otimes P_{kl} \longrightarrow S_{ij} \otimes P_{kl} \longrightarrow 0.$$

Hence, we obtain

$$P_{ij} \otimes P_{kl} = \Omega(S_{ij}) \otimes P_{kl} \oplus S_{ij} \otimes P_{kl}.$$

Note that q is the $6m$-th primitive root of the unit, and ω is the 6-th primitive root of the unit; we have $q^{3m} = -1$. Also, without loss of generality, we assume that $q^m = \omega$. Accordingly, the action of the algebra $\mathcal{A}_{24m}$ on $\Omega\left(S_{ij}\right)$ is as follows:

$$\begin{aligned} x \cdot xf_{ij} &= 0, \\ g \cdot xf_{ij} &= (-1)^{i+1} xf_{ij}, \\ z \cdot xf_{ij} &= (-1)^{i+1} q^j \omega xf_{ij} \\ &= q^{3m(i+1)} q^j q^m xf_{ij} \\ &= q^{3mi+j+4m} xf_{ij}. \end{aligned}$$

Reviewing the construction of simple $\mathcal{A}_{24m}$-module S_{ij}, one sees that

$$\Omega(S_{ij}) \cong S_{(i+1)\bmod 2,(j+4m+3mi)\bmod 6m}.$$

Therefore,

$$P_{ij} \otimes P_{kl} \cong (S_{(i+1)\bmod 2,(j+4m+3mi)\bmod 6m} \otimes P_{kl}) \oplus (S_{ij} \otimes P_{kl}).$$

By Proposition 3.4, we obtain

$$P_{ij} \otimes P_{kl} \cong P_{(i+k+1)(\bmod 2),(j+4m+3mi+l)(\bmod 6m)} \oplus P_{(i+k)(\bmod 2),(j+l)(\bmod 6m)}.$$

4. The representation ring of $\mathcal{A}_{24m}$

In this section, we mainly focus on establishing the representation ring of the algebra $\mathcal{A}_{24m}$. Let us first review the definition of the projective class ring.

Assuming that H is a Hopf algebra, we denote by $[V]$, the isomorphic class of a finite-dimensional H-module V. Let $F(H)$ be the free abelian group generated by $[V]$, we define a multiplication on $F(H)$ as $[M][N] = [M \otimes N]$. Then, $F(H)$ is an associative ring with the identity element $[\mathbb{K}]$. The quotient ring $r(H) := F(H)/I$ is called the Green ring of H, where I is the ideal generated by the relation $[M \otimes N] - [M][N]$ and $[M \oplus N] - [M] - [N]$. It is clear that $\{[V] \mid V \in \mathrm{ind}(H)\}$ forms a $\mathbb{Z}$-basis for $r(H)$, where $\mathrm{ind}(H)$ is the set consisting of isomorphism classes of all finite-dimensional indecomposable H-modules.

The subring $\mathcal{P}(H)$ of $r(H)$ generated by the projective modules and simple modules of H is called the projective class ring of H.

It is noted that the representation ring of $\mathcal{A}_{24m}$ is the same as its projective class ring of $\mathcal{A}_{24m}$.

Now, let us determine the representation ring $r(\mathcal{A}_{24m})$ of the algebra $\mathcal{A}_{24m}$.

Lemma 4.1. *Let $t = [S_{01}], f = [P_{01}], u = [S_{11}], h = [P_{11}]$, then the following equations hold:*

(1) $t^{6m} = 1$, $u^{6m} = 1$.
(2) $tf = ft$, $tu = ut$, $th = ht$, $uh = hu$, $uf = fu$.
(3) $t^2 = u^2$, $fh = u^{4m+1}h + th$, $hf = t^{m+1}f + uf$.
(4) $f^2 = u^{4m+1}f + tf$, $h^2 = t^{m+1}h + uh$.

Proof. (1) By Proposition 3.3, we obtain

$$[S_{01}]^{6m} = [S_{01} \otimes S_{01} \otimes \cdots \otimes S_{01}] = [S_{0,6m(\bmod 6m)}] = [S_{00}] = 1.$$

Therefore, $t^{6m} = 1$. Similarly, $u^{6m} = 1$.

(2) By Proposition 3.4, $S_{ij} \otimes P_{kl} = P_{kl} \otimes S_{ij}$, so, $tf = [S_{01}][P_{01}] = [S_{01} \otimes P_{01}] = [P_{01} \otimes S_{01}] = [P_{01}][S_{01}] = ft$, thus, we have $tf = ft$. Similarly, we obtain

$$tu = ut, th = ht, uh = hu, uf = fu.$$

(3) By Proposition 3.3, we obtain

$$[S_{01}]^2 = [S_{01} \otimes S_{01}] = [S_{02}];\ [S_{11}]^2 = [S_{11} \otimes S_{11}] = [S_{02}].$$

Thus, $t^2 = u^2$. By Proposition 3.5, we obtain

$$[P_{01} \otimes P_{11}] = [S_{1,4m+1} \otimes P_{11} \oplus S_{01} \otimes P_{11}] = u^{4m+1}h + th.$$

$$[P_{11} \otimes P_{01}] = [S_{0,m+1} \otimes P_{01} \oplus S_{11} \otimes P_{01}] = t^{m+1}f + uf.$$

Thus, $fh = u^{4m+1}h + th$, $hf = t^{m+1}f + uf$.

(4) By Proposition 3.5, we obtain

$$\begin{aligned} f^2 &= [P_{01}]^2 = [P_{01} \otimes P_{01}] \\ &= [S_{1,4m+1} \otimes P_{01} \oplus S_{01} \otimes P_{01}] \\ &= u^{4m+1}f + tf. \end{aligned}$$

$$\begin{aligned} h^2 &= [P_{11}]^2 = [P_{11} \otimes P_{11}] \\ &= [S_{0,m+1} \otimes P_{11} \oplus S_{11} \otimes P_{11}] \\ &= t^{m+1}h + uh. \end{aligned}$$

The lemma follows.

Corollary 4.2. $\{t^i f^j; ut^{i_1} f^{j_1}; u^k h^l; tu^{k_1} h^{l_1}\}$ *is a* $\mathbb{Z}$*-basis of* $r(\mathcal{A}_{24m})$*, where* $i, i_1, k, k_1 \in \mathbb{Z}_{6m}$*,* $j, j_1, l, l_1 \in \mathbb{Z}_2$*, and* i, k *are even,* i_1, k_1 *are odd.*

Proof. Note that $r(\mathcal{A}_{24m})$ has a $\mathbb{Z}$-basis $\left\{[S_{ij}], [P_{ij}] \mid i \in \mathbb{Z}_2, j \in \mathbb{Z}_{6m}\right\}$. Hence, the rank of $r(\mathcal{A}_{24m})$ is $24m$. By Lemma 4.1, $r(\mathcal{A}_{24m})$ is spanned by the set $\{t^i f^j; ut^{i_1} f^{j_1}; u^k h^l; tu^{k_1} h^{l_1}\}$, which consists of $24m$ elements. Hence, the set is a $\mathbb{Z}$-basis of $r(\mathcal{A}_{24m})$.

Now, we can obtain the main result of the paper.

Theorem 4.3. *The representation ring* $r(\mathcal{A}_{24m})$ *is isomorphic to the quotient ring* $\mathbb{Z}\langle \alpha, \beta, \gamma, \eta \rangle / \mathcal{I}$*, where* $\mathcal{I}$ *is the ideal generated by the relations*

$$\alpha^{6m} - 1,\ \beta^2 - \gamma^{4m+1}\beta - \alpha\beta,\ \alpha^2 - \gamma^2,\ \eta^2 - \alpha^{m+1}\eta - \gamma\eta,\ \beta\eta - \gamma^{4m+1}\eta - \alpha\eta,\ \eta\beta - \alpha^{m+1}\beta - \gamma\beta.$$

Proof. Let

$$\pi : \mathbb{Z}\langle \alpha, \beta, \gamma, \eta \rangle \longrightarrow \mathbb{Z}\langle \alpha, \beta, \gamma, \eta \rangle / \mathcal{I}$$

be the natural epimorphism. It is straightforward to check that $\mathbb{Z}\langle \alpha, \beta, \gamma, \eta \rangle / \mathcal{I}$ is spanned by

$$\left\{\alpha^i \beta^j; \gamma\alpha^{i_1}\beta^{j_1}; \gamma^k \eta^l; \alpha\gamma^{k_1}\eta^{l_1}\right\},$$

where $i, i_1, k, k_1 \in \mathbb{Z}_{6m}$, $j, j_1, l, l_1 \in \mathbb{Z}_2$, and i, k are even, i_1, k_1 are odd. This means that the rank of $\mathbb{Z}\langle\alpha, \beta, \gamma, \eta\rangle/\mathcal{I}$ is equal or less than $24m$.

By Corollary 4.2, $r(\mathcal{A}_{24m})$ is generated by t, f, u, and h. So, there is a uniqueness ring epimorphism

$$\Phi : \mathbb{Z}\langle\alpha, \beta, \gamma, \eta\rangle \longrightarrow r(\mathcal{A}_{24m})$$

satisfying

$$\Phi(\alpha) = t, \quad \Phi(\beta) = f, \quad \Phi(\gamma) = u, \Phi(\eta) = h.$$

By Lemma 4.1, we see that

$$t^{6m} = 1,\ tf = ft,\ tu = ut,\ th = ht,\ uh = hu,\ uf = fu,$$

$$t^2 = u^2,\ f^2 = u^{4m+1}f + tf,\ h^2 = t^{m+1}h + uh,$$
$$fh = u^{4m+1}h + th,\ hf = t^{m+1}f + uf.$$

It follows that

$$\Phi(\alpha^{6m} - 1) = 0,\ \Phi(\alpha\beta - \beta\alpha) = 0,$$
$$\Phi(\alpha\gamma - \gamma\alpha) = 0,\ \Phi(\alpha\eta - \eta\alpha) = 0,\ \Phi(\gamma\eta - \eta\gamma) = 0,$$
$$\Phi(\gamma\beta - \beta\gamma) = 0,\ \Phi(\alpha^2 - \beta^2) = 0,$$
$$\Phi(\eta^2 - \alpha^{m+1}\eta - \gamma\eta) = 0,\ \Phi(\beta^2 - \gamma^{4m+1}\beta - \alpha\beta) = 0,$$
$$\Phi(\beta\eta - \gamma^{4m+1}\eta - \alpha\eta) = 0,\ \Phi(\eta\beta - \alpha^{m+1}\beta - \gamma\beta) = 0.$$

Hence, $\Phi(\mathcal{I}) = 0$, and Φ induces an epimorphism

$$\overline{\Phi} : \mathbb{Z}\langle\alpha, \beta, \gamma, \eta\rangle/\mathcal{I} \longrightarrow r(\mathcal{A}_{24m})$$

satisfying $\overline{\Phi}(\overline{v}) = \Phi(v)$, for any $v \in \mathbb{Z}\langle\alpha, \beta, \gamma, \eta\rangle$, where $\overline{v} = \pi(v) = v + \mathcal{I}$.

Comparing the ranks of $\mathbb{Z}\langle\alpha, \beta, \gamma, \eta\rangle/\mathcal{I}$ with $r(\mathcal{A}_{24m})$, we get that $\overline{\Phi}$ is indeed a ring isomorphism.

The result is obtained.

5. Conclusions

In this paper, all indecomposable modules of $\mathcal{A}_{24m}$ are constructed and classified. Furthermore, we obtain the decomposition formulas for the tensor products of indecomposable modules as well as the representation ring of the bialgebra. These results may help us to understand the general representation theory of a non-pointed bialgebra.

Author Contributions

Shilin Yang contributed the creative ideas and proof techniques for this paper, including the construction of the article structure and the revision of content; Huaqing Gong consulted the relevant background of the paper and wrote the article. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

All authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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