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Research article

Spaces of multiplier σ-convergent vector valued sequences and uniform σ-summability

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Abstract: This study focuses on the development of novel vector-valued sequence spaces whose elements are characterized by constructing (weakly) multiplier σ-convergent series. To achieve this, the concept of invariant means is rigorously examined and utilized as a foundational tool. These newly defined spaces are proven to possess the structure of Banach spaces when equipped with their natural sup norm, thus ensuring their completeness. In addition to establishing the Banach space properties, this study delves into the inclusion relationships between these new sequence spaces and classical multiplier spaces, specifically $BMC(B)$ and $CMC(B)$, where B denotes an arbitrary Banach space. By employing the σ-convergence method, this study also culminates in a result analogous to the celebrated Hahn-Schur theorem, which traditionally establishes a connection between the weak convergence and the uniform convergence of unconditionally convergent series.

Keywords: multiplier convergence; completeness; uniform convergence; σ-convergence; summability methods
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1. Introduction and terminological excerpts

All standard sequence spaces, equipped with their respective norms, are Banach spaces, as stated in the text. A sequence space is defined as a vector subspace of either $\mathbb{R}^{\mathbb{N}}$ or $\mathbb{C}^{\mathbb{N}}$, which denote the space of all real or complex valued sequences, respectively. Specifically, the Banach spaces ℓ_∞, c, and c_0 denote the spaces of bounded, convergent and null sequences, respectively. Similarly, the spaces bs, cs, and ℓ_1 consist of sequences whose series are bounded, convergent, and absolutely convergent, respectively [12].

The Köthe–Toeplitz duals of sequence spaces have strongly connected to the theory of multiplier convergent (or bounded) series. Additionally, the duality theory also has significant implications in the fields of the topological sequence space theory and the summability theory. The alpha-, beta-, and gamma-duals, namely S^{α}, S^{β}, and S^{γ}, of a sequence space S are defined as follows:

$$\begin{aligned} S^{\alpha} &:= \left\{t = (t_k) \in \mathbb{R}^{\mathbb{N}} : ts = (t_k s_k) \in \ell_1 \text{ for all } s = (s_k) \in S\right\}, \\ S^{\beta} &:= \left\{t = (t_k) \in \mathbb{R}^{\mathbb{N}} : ts = (t_k s_k) \in cs \text{ for all } s = (s_k) \in S\right\}, \\ S^{\gamma} &:= \left\{t = (t_k) \in \mathbb{R}^{\mathbb{N}} : ts = (t_k s_k) \in bs \text{ for all } s = (s_k) \in S\right\}. \end{aligned}$$

Through this study, we denote the real normed and real Banach spaces by N and B, respectively. Additionally, the space N^* denotes the continuous dual of N. A series $\sum_k x_k$ in N is said to be either unconditionally convergent (denoted by *uc*) or unconditionally Cauchy (denoted by *uC*) if the rearranged series $\sum_k x_{\pi(k)}$ either converges or forms a Cauchy series for any permutation π of $\mathbb{N}$. In the same way, a series $\sum_k x_k$ in N is considered weakly unconditionally Cauchy (denoted by *wuC*) if the sequence $\left(\sum_{k=1}^{n} x_{\pi(k)}\right)_{n\in\mathbb{N}}$ is weakly Cauchy for every permutation π of the natural numbers. It is well-known that a series is *wuC* if $x^*(x_k) \in \ell_1$ for every $x^* \in N^*$and any *wuC* series in B is *uc* if and only if B does not include any copies of c_0, see [9, p.42–44]. If the reader is interested in particular explorations of Banach spaces, they may consult Diestel's renowned monograph [19], which is devoted to the theory of sequences and series in Banach spaces, as well as Albiac and Kalton's [9].

The behavior of the series $\sum_k v_k$ in N is significantly influenced by the form $\sum_k \mu_k v_k$. Specifically, if the series $\sum_k \mu_k v_k$ converges in N for any $\mu = (\mu_k) \in E$, then $\sum_k v_k$ is said to be E-multiplier convergent. Similarly, if the partial sums of the series $\sum_k \mu_k v_k$ form a norm Cauchy sequence in N for every $\mu = (\mu_k) \in E$, as stated in [39], then this series is called the multiplier Cauchy series. Useful characterizations of multiplier convergence for a series $\sum_k v_k$ in B can be expressed through the following numerical formulations [39]:

(i) $\sum_k v_k$ is *wuC* if and only if it is a c_0-multiplier convergent series.
(ii) $\sum_k v_k$ is *uc* if and only if it is an ℓ_∞-multiplier convergent series.
(iii) Let χ_σ be the characteristic function of σ and consider the set $M_0 = \{\chi_\sigma | \sigma \subset \mathbb{N}\}$. Then, $\sum_k v_k$ is subseries convergent if and only if it is an M_0-multiplier convergent series.

An important reference for a detailed study on the theory of multiplier convergence is [39]. In [3, 5, 6], Aizpuru et al. provided a new characterization of *wuC* and *uc* series by employing the Cesàro summability method, the almost convergence method, and a general (regular) summability matrix method. They further explored the structure of newly introduced spaces associated with series in Banach spaces to establish conditions for the completeness and barrelledness of normed spaces. Moreover, they characterized *wuC* series in terms of the continuity of linear mappings from these spaces to a normed space X and formulated new versions of the Orlicz-Pettis theorem. For some of the recent investigations into the scalar case of multiplier convergence, which involve various summability methods see [10, 24, 29]. In addition, within the scope of the topic under consideration, the multiplier convergence for vector-valued sequences one can see [11,23,27]. In [25,26], the authors introduce the vector valued multiplier spaces associated to the series of bounded linear operators $M^{\infty}_{f}(\sum_k T_k), M^{\infty}_{wf}(\sum_k T_k)$ and $M^{\infty}_{f_\lambda}(\sum_k T_k), M^{\infty}_{wf_\lambda}(\sum_k T_k)$ by means of Lorentz' almost convergence and its slight generalization, respectively. Additionally, they give some characterizations of completeness of these spaces and continuity and compactness of summing operator.

In the sequence spaces theory, the most useful application of the *Hahn–Banach Extension Theorem* may be seen as the concept of Banach limits (non-negative, normalized, and shift-invariant linear functionals) defined on ℓ_∞. This generalization of ordinary limit has many applications in various fields of mathematics. In their research paper that involved the functional characteristic and extreme points of the set of Banach limits on ℓ_∞, Semenov et al. [41] gave an impressive introduction about the recent results and developments on the theory of Banach limits and almost convergence. Banach limits are exactly an extension of the limit functional on c to ℓ_∞. In 1948, the important result due to Lorentz [31] appeared on Banach limits, which was a beautiful characterization of almost convergence. Eberlein [20] introduced the idea of the Banach–Hausdorff limit with invariance of Banach limits on regular Hausdorff transformations.

Let F_S be the forward-shift operator on $\mathbb{R}^{\mathbb{N}}$ with $(F_S x)_n = x_{n+1}$ for every $n \in \mathbb{N}$. We say that the linear functional $\mathcal{B}$ on ℓ_∞ is a Banach limit if the following statements holds:

(i) $\mathcal{B}$ is nonnegative, i.e., $x = (x_n) \in \ell_\infty$, $\mathcal{B}(x) \geq 0$ if $x_n \geq 0$ for every $n \in \mathbb{N}$,
(ii) $\mathcal{B}(F_S x) = \mathcal{B}(x)$,
(iii) $\mathcal{B}(e) = 1$, where $e = (1, 1, 1, \dots)$

holds. All Banach limits belong to the class $\mathfrak{B}$, and this class is a closed convex set on $S_{\ell_\infty^*}$, which is the unit sphere of ℓ_∞^*, [40]. It comes from the definition that for every $\mathcal{B} \in \mathfrak{B}$, $\|B\|_{\ell_\infty^*} = 1$, i.e., Banach limits are defined on ℓ_∞ of norm-1, and the classical limit functional $\lim : c \to \mathbb{R}$ seems to be a restriction of any Banach limit $B : \ell_\infty \to \mathbb{R}$. Moreover, another name used for Banach limits is also *Banach-Mazur limits* since it is assumed that the existence was proven by Mazur. In B, the closed unit ball is denoted by $\overline{U}_B$ and the space of all vector B-valued sequences is also denoted by $\mathbb{R}^{\mathbb{N}}(B)$ (or $B^{\mathbb{N}}$).

On advanced research of recent studies related to almost summability (multiplier almost convergence), f_λ-summability (multiplier f_λ-convergence), and σ-summability (multiplier σ-convergence) associated to a formal series (an operator valued series) in normed spaces can be given by the following references: [8, 25, 28]. By employing an expanded concept of almost summability, in [24], the authors provided novel classes of sequence spaces that corresponded to a series in a Banach space. In [24], the authors also provided novel characterizations of *wuC* and *uc* series using these developed spaces. Additionally, they acquired an edition of the famous Orlicz–Pettis theorem. This theorem exactly asserts that if a series in a normed space is weakly subseries convergent, then it is also norm (strong) subseries convergent, as mentioned in [39, vii]. The reader can refer to the textbooks [17] and [34] for fundamental theorems on functional analysis and the summability theory, the papers [13, 14] on almost-conservative and almost corcive matrix transformations, and the papers [15, 16, 22] on the almost convergence, on the convergence of a series, and related topics.

2. σ-Convergence and σ-summability

In this section, we recall the concept of σ-convergent and σ-summability, which will be used in the rest of the paper. Raimi [36] introduced the concept of σ-convergence as a slight generalization of Lorentz almost convergence by means of *motion*, which can be seen as a generalization of the forward-shift operator via an injection of the set of positive integers $\mathbb{N}$ into itself. Motions have same role for linear functionals defined on ℓ_∞ with shift-invariance of the Banach limits. First, we give the notion of σ-mean.

A motion $\sigma : \mathbb{N} \to \mathbb{N}$ is a one-to-one function that does not contain any finite orbits. An invariant mean, often known as a σ-mean, is a continuous linear functional φ defined on ℓ_∞ that satisfies the following conditions:

(i) φ is non-negative,
(ii) $\varphi(x) = \varphi(x_{\sigma(n)})$,
(iii) $\varphi(e) = 1$, where $e = (1, 1, 1, \ldots)$, (cf. [36]).

Let us note that $\sigma^k(j)$ is assumed to be the k^{th} iteration of σ at j and $\sigma^k(j) \neq j$. It is said to be that the bounded sequence $x = (x_k)$ σ-converges to the generalized limit $l \in \mathbb{C}$ if $\varphi(x) = l$ for all φ. Invariant mean is a generalization of the well-known *lim* on c, which means $\varphi(x) = \lim x, \forall x \in c$ iff σ has no finite orbits and $c \subset V_\sigma \subset \ell_\infty$, [32, 33]. Let us recall that, the functional φ is 1-1 such that $\sigma^k(j) \neq j$.

We have the following:

(1) $\sigma^j(l) \neq l$ for all $j, l \in \mathbb{N}$.
(2) $\sigma^l(l) = l$, since a motion has no finite orbit and $\sigma^l(l) = (\sigma^l \circ I)(l) = \sigma^l[I(l)] = I(l)$, where I denotes the identity function.
(3) $\sigma^{i+j}(j) = \sigma^i$, since $\sigma^{i+j}(j) = (\sigma^i \circ \sigma^j)(j) = \sigma^i[\sigma^j(j)] = \sigma^i(j)$ for all $i, j \in \mathbb{N}$.

Here and after, we take $s_j = \sum_{k=1}^{j} v_k$ and $s_{\sigma^{m+n}(n)} = s_n + \sum_{k=1}^{m} v_{\sigma^k(n)}$.

Definition 2.1. Let $v = (v_k) \subseteq N$. Then, it is said that $v = (v_k)$ is σ-convergent to $v_0 \in N$, i.e., $V_\sigma - \lim_k v_k = v_0$ (respectively, weakly σ-convergent to $v_0' \in N$, i.e., $wV_\sigma - \lim_k v_k = v_0'$) if $\sum_{k=0}^{l} \dfrac{v_{\sigma^k(j)}}{l+1} \to v_0$ as $l \to \infty$ uniformly in $j \in \mathbb{N}$ (respectively, if $\sum_{k=0}^{l} \frac{v^*(v_{\sigma^k(j)})}{l+1} \to v^*(v_0')$ as $l \to \infty$, $\forall v^* \in N^*$ uniformly in $j \in \mathbb{N}$).

We denote the space of all σ-convergent sequences and weak σ-convergent sequences in N by $V_\sigma(N)$ and by $wV_\sigma(N)$, respectively. Therefore, we have the following:

$$V_\sigma(N) := \left\{ (v_k) \in \mathbb{R}^{\mathbb{N}}(N) : V_\sigma \lim_{k\to\infty} v_k \text{ exists} \right\},$$

and

$$wV_\sigma(N) := \left\{ (v_k) \in \mathbb{R}^{\mathbb{N}}(N) : wV_\sigma \lim_{k\to\infty} v_k \text{ exists} \right\}.$$

Definition 2.2. If $v = (v_i) \subseteq N$, then $\sum_i v_i$ is σ-convergent (respectively, weakly σ-convergent) to the point $v_0 \in N$ (respectively, $v_0' \in N$), and is denoted by $V_\sigma \sum_{i=1}^{\infty} v_i = v_0$ (respectively, $wV_\sigma \sum_{i=1}^{\infty} v_i = v_0'$) if

$$\left(\sum_{i=1}^{j} v_i + \sum_{i=1}^{l} \frac{(l-i+1) v_{\sigma^i(j)}}{l+1} \right) \to v_0$$

as $l \to \infty$ uniformly in $j \in \mathbb{N}$ (respectively, if for all $v^* \in N^*$

$$\left(\sum_{i=1}^{j} v_i + \sum_{i=1}^{l} \frac{(l-i+1) v^*(v_{\sigma^i(j)})}{l+1} \right) \to v^*(v_0')$$

as $l \to \infty$ uniformly in $j \in \mathbb{N}$) holds. Therefore, $v_0 \in N$ (respectively, v_0') denotes the V_σ-sum (respectively, wV_σ-sum) of $v = (v_i)$ [10].

3. Spaces of σ-multiplier convergence

Through this section, we concern with the spaces of σ-multiplier convergence, and give the theorems related to their completeness. Prior to delving into this, recall that the spaces of sequences that form bounded and null multiplier convergent series, which are alternatively known as the spaces of the *uc* and *wuC* series, may be defined as follows:

$$B(\ell_\infty) \quad = \quad \left\{ v = (v_k) \in \mathbb{R}^{\mathbb{N}}(B) : \sum_k v_k \text{ is } \ell_\infty - \text{multiplier convergent} \right\},$$

and

$$B(c_0) \quad = \quad \left\{ v = (v_k) \in \mathbb{R}^{\mathbb{N}}(B) : \sum_k v_k \text{ is } c_0 - \text{multiplier convergent} \right\}.$$

These are also denoted by *BMC(B)* and *CMC(B)*, respectively [30]. Besides, the space $B(\mathcal{S})$ of $\mathcal{S}$-multiplier convergent series is also given as follows:

$$B(\mathcal{S}) = \left\{ v = (v_k) \in \mathbb{R}^{\mathbb{N}}(B) : \sum_k v_k \text{ is } \mathcal{S}\text{- multiplier convergent} \right\}.$$

In this text, it is assumed that the space $\mathcal{S}$ is a vector subspace of ℓ_∞ containing c_0 (i.e., $c_0 \subseteq \mathcal{S} \subseteq \ell_\infty$). All of the spaces $B(\ell_\infty)$, $B(\mathcal{S})$, and $B(c_0)$ are complete with the following:

$$\|v\| = \sup_{k\in\mathbb{N}} \left\{ \left\| \sum_{i=1}^{k} \alpha_i v_i \right\| : \alpha_i \in [-1,1],\ i \in \{1,2,\dots,k\} \right\},\ v = (v_i) \in \mathbb{R}^{\mathbb{N}}(B). \tag{3.1}$$

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be linear subspaces of ℓ_∞ such that $c_0 \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_2$. Therefore, the inclusions

$$B(\ell_\infty) \subseteq B(\mathcal{S}_2) \subseteq B(\mathcal{S}_1) \subseteq B(c_0)$$

hold [2].

We give the following definition on Grothendieck spaces to use for our main results.

Definition 3.1. Suppose that $\lambda \subseteq B^{**}$, which is the second dual of B and $\sigma(B^*, B)$ is *weak** topology induced by the duality between B^* and B. Then, B is λ-Grothendieck if every $\sigma(B^*, B)$-convergent sequence is $\sigma(B^*, \lambda)$-convergent. B is Grothendieck if $\lambda = B^{**}$ holds, [2].

Many of the proofs of the famous Orlicz–Pettis theorem (for the original proof of Pettis, [35]) enjoys some versions of the Schur lemma. In 1983, Swartz gives a version of this lemma related to the uniform convergence of unconditional convergent series in linear metric B-spaces, [38]. In 2000, Aizpuru and Pérez–Fernández proved that the uniform convergence of sequences of the *uc* series can be generalized to sequences of the *wuC* series, see [2]. Beside, in [2, 4, 7, 21], the authors also obtained some general results on the uniform convergence of the *uc* series and the *wuC* series through miscellaneous summability or non-summability methods using the following theorem as a representative result of Schur lemma.

Theorem 3.2. *If $v = (v^n)_{n\in\mathbb{N}} \in B(\ell_\infty)$ and for every $(\alpha_k)_{k\in\mathbb{N}} \in \ell_\infty$, $\lim_{n\to\infty} \sum_{k=1}^{\infty} \alpha_k v_k^n$ exists in B, then there exists $v^0 \in B(\ell_\infty)$ such that $\lim_{n\to\infty} \|v^n - v^0\| = 0$ in B, [4, 7, 21].*

By using $\mathcal{S}$ and σ-summability, we can describe the spaces $B(\mathcal{S}, V_\sigma)$ and $B_w(\mathcal{S}, V_\sigma)$ for the formal series $\sum_k v_k$ as follows:

$$B(\mathcal{S}, V_\sigma) := \left\{ v = (v_k) \in \mathbb{R}^{\mathbb{N}}(B) \middle| V_\sigma \sum_{k=1}^{\infty} \alpha_k v_k \text{ converges } \text{ for each } \alpha = (\alpha_k) \in \mathcal{S} \right\},$$

and

$$B_w(\mathcal{S}, V_\sigma) := \left\{ v = (v_k) \in \mathbb{R}^{\mathbb{N}}(B) \middle| wV_\sigma \sum_{k=1}^{\infty} \alpha_k v_k \text{ converges } \text{ for each } \alpha = (\alpha_k) \in \mathcal{S} \right\}.$$

In [10], it has been shown that $\sum_k v_k$ is a *wuC* series if and only if there exists a point $v_0 \in B$ such that $wV_\sigma \sum_{k=1}^{\infty} \alpha_k v_k = v_0$ holds for every $\alpha = (\alpha_k) \in c_0$. This leads us to the inclusion $B_w(\mathcal{S}, V_\sigma) \subseteq B(c_0)$. Therefore, one can consider both of the spaces $B(\mathcal{S}, V_\sigma)$ and $B_w(\mathcal{S}, V_\sigma)$ as normed spaces with the norm given in (3.1). Therefore, the following inclusions hold:

$$B(\ell_\infty) \subseteq B(\mathcal{S}, V_\sigma) \subseteq B_w(\mathcal{S}, V_\sigma) \subseteq B(c_0).$$

We start our main results with the following theorem, which asserts that both of the spaces $B(\mathcal{S}, V_\sigma)$ and $B_w(\mathcal{S}, V_\sigma)$ are complete. We will only show the completeness of the space $B_w(\mathcal{S}, V_\sigma)$ since the completeness of $B(\mathcal{S}, V_\sigma)$ may be proven in a similar way.

Theorem 3.3. *$B(\mathcal{S}, V_\sigma)$ and $B_w(\mathcal{S}, V_\sigma)$ are Banach spaces with the norm given in (3.1).*

Proof. We need to show that the space $B_w(\mathcal{S}, V_\sigma)$ is a closed linear subspace of $B(c_0)$. Let $(v^n)_{n\in\mathbb{N}}$ be a sequence in $B_w(\mathcal{S}, V_\sigma)$. Then, we can find $v^0 \in B(c_0)$ that satisfies the following:

$$\lim_n \left\| v^n - v^0 \right\| = 0.$$

If $\alpha = (\alpha_k) \in S - \{0\}$ is fixed, then there exist the terms $v_n \in B$ that satisfies the following for each $n \in \mathbb{N}$:

$$\left(\sum_{k=1}^{j} \alpha_k v^*(v_k^n) + \sum_{k=1}^{l} \frac{(l-k+1)}{l+1} \alpha_{\sigma^k(j)} v^*(v_{\sigma^k(j)}^n) \right) \to v^*(v_n),$$

as $l \to \infty$ uniformly in $j \in \mathbb{N}$, for every $v^* \in B^*$.

Now, let us prove that $v = (v_n)$ is a Cauchy sequence in B. For every $\epsilon > 0$, we have to find $n_0 \in \mathbb{N}$ such that for every $p, q \geq n_0$, the following equality holds:

$$\|v^p - v^q\| \leq \frac{\epsilon}{3\|\alpha\|}.$$

If $p, q \geq n_0$ are fixed, then there exist some $v^* \in \overline{U}_{B^*}$ that satisfies the following:

$$\left\| v_p - v_q \right\| = \left| v^*(v_p) - v^*(v_q) \right|.$$

Now, one can find $m \in \mathbb{N}$ such that

$$\left| v^*(v_p) - \left(\sum_{k=1}^{j} \alpha_k v^*(v_k^p) + \sum_{k=1}^{l} \frac{(l-k+1)}{l+1} \alpha_{\sigma^k(j)} v^*(v_{\sigma^k(j)}^p) \right) \right| < \frac{\epsilon}{3}, \tag{3.2}$$

and

$$\left| v^*(v_q) - \left(\sum_{k=1}^{j} \alpha_k v^*(v_k^q) + \sum_{k=1}^{l} \frac{(l-k+1)}{l+1} \alpha_{\sigma^k(j)} v^*(v_{\sigma^k(j)}^q) \right) \right| < \frac{\epsilon}{3} \tag{3.3}$$

are satisfied uniformly in $j \in \mathbb{N}$. Therefore, for each $v^* \in B^*$,

$$\begin{aligned} \left\| v_p - v_q \right\| &= \left| v^*(v_p) - v^*(v_q) \right| \\ &\leq (3.2) + (3.3) + \left| \sum_{k=1}^{j} \alpha_k v^*(v_k^p - v_k^q) + \sum_{k=1}^{l} \frac{(l-k+1)}{l+1} \alpha_{\sigma^k(j)} v^*(v_{\sigma^k(j)}^p - v_{\sigma^k(j)}^q) \right| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \|v^p - v^q\| \, \|\alpha\| \\ &\leq \epsilon. \end{aligned}$$

Moreover, from the completeness of B, we can find $v_0 \in B$ such that the following equation holds:

$$\lim_n \|v_n - v_0\| = 0.$$

Next, let us take the fixed $v^* \in B^* - \{0\}$ and $\epsilon > 0$. There exists $n \in \mathbb{N}$ such that

$$\left\| v^n - v^0 \right\| \leq \frac{\epsilon}{3 \|\alpha\| \|v^*\|},$$

and

$$\|v_n - v_0\| \leq \frac{\epsilon}{3 \|v^*\|}.$$

Additionally, we can find k_0 such that for every $k \geq k_0$, we have

$$\left| \left(\sum_{k=1}^{j} \alpha_k v^*(v_k^n) + \sum_{k=1}^{l} \frac{(l-k+1)}{l+1} \alpha_{\sigma^k(j)} v^*(v_{\sigma^k(j)}^n) \right) - v^*(v_n) \right| < \frac{\epsilon}{3},$$

uniformly in $j \in \mathbb{N}$, for every $v^* \in B^*$. Therefore,

$$\begin{aligned} &\left| \left(\sum_{k=1}^{j} \alpha_k v^*(v_k^0) + \sum_{k=1}^{l} \frac{(l-k+1)}{l+1} \alpha_{\sigma^k(j)} v^*(v_{\sigma^k(j)}^0) \right) - v^*(v_0) \right| \\ \leq\ & \left| \sum_{k=1}^{j} \alpha_k v^*(v_k^0 - v_k^n) + \sum_{k=1}^{l} \frac{(l-k+1)}{l+1} \alpha_{\sigma^k(j)} v^*(v_{\sigma^k(j)}^0 - v_{\sigma^k(j)}^n) \right| \\ \leq\ & \left| \left(\sum_{k=1}^{j} \alpha_k v^*(v_k^n) + \sum_{k=1}^{l} \frac{(l-k+1)}{l+1} \alpha_{\sigma^k(j)} v^*(v_{\sigma^k(j)}^n) \right) - v^*(v_n) \right| + |v^*(v_n) - v^*(v_0)| \end{aligned}$$

$$\leq \left\|v^n - v^0\right\| \|\alpha\| \|v^*\| + \frac{\epsilon}{3} + \|v^*\| \|v_n - v_0\|$$
$$\leq \epsilon,$$

uniformly in $j \in \mathbb{N}$ for each $v^* \in B^*$. Therefore, the proof has been completed. That is, $v_0 \in B_w(\mathcal{S}, V_\sigma)$. □

Now, we give the definitions of linear summing and linear weak summing operators from $\mathcal{S}$ into the Banach space B.

Definition 3.4. Let $v = (v_k)$ be an arbitrary sequence in $B(\mathcal{S}, V_\sigma)$ and $\alpha = (\alpha_k)$, as well as be the sequence of scalars in $\mathcal{S}$. We define the linear operator

$$\mathcal{L}_v : \mathcal{S} \to B$$

by

$$\mathcal{L}_v(\alpha) = V_\sigma \sum_{k=1}^{\infty} \alpha_k v_k. \tag{3.4}$$

Definition 3.5. If $v = (v_k)$ is an arbitrary sequence in $B_w(\mathcal{S}, V_\sigma)$, then, we define the linear weakly summing operator $\mathcal{L}_v^w : \mathcal{S} \to B$ as follows:

$$\mathcal{L}_v^w(\alpha) = wV_\sigma \sum_{k=1}^{\infty} \alpha_k v_k. \tag{3.5}$$

According to these definitions we give the continuity principles for the operators $\mathcal{L}_v$ and $\mathcal{L}_v^w$.

Proposition 3.6. *The linear operators $\mathcal{L}_v : \mathcal{S} \to B$ and $\mathcal{L}_v^w : \mathcal{S} \to B$ defined by (3.4) and (3.5) are continuous for the arbitrary sequences in $B(\mathcal{S}, V_\sigma)$ and $B_w(\mathcal{S}, V_\sigma)$, respectively.*

Proof. Again, to avoid routine repetition, we only show that the linear operator $\mathcal{L}_v^w$ is continuous. Now, let $v = (v_k)$ be a sequence in $B_w(\mathcal{S}, V_\sigma)$ and $\alpha = (\alpha_k) \in \mathcal{S}$. Therefore, we find $v^* \in \overline{U}_{B^*}$ that satisfies the equality:

$$\|\mathcal{L}_v^w(\alpha)\| = |v^*(\mathcal{L}_v^w(\alpha))|.$$

Additionally, we have

$$|wV_\sigma \sum_{k=1}^{\infty} \alpha_k v_k| = \lim_{l\to\infty} \left\| \left(\sum_{k=1}^{j} \alpha_k v^*(v_k) + \sum_{k=1}^{l} \frac{(l-k+1)}{l+1} \alpha_{\sigma^k(j)} v^*(v_{\sigma^k(j)}) \right) \right\|,$$

uniformly in $j \in \mathbb{N}$. If $l, j \in \mathbb{N}$, then the inequalities

$$\left\| \left(\sum_{k=1}^{j} \alpha_k v^*(v_k) + \sum_{k=1}^{l} \frac{(l-k+1)}{l+1} \alpha_{\sigma^k(j)} v^*(v_{\sigma^k(j)}) \right) \right\| \leq \|\alpha\|_\infty \|v\|.$$

holds and this completes the proof, that is, $\mathcal{L}_v^w$ is continuous. □

4. Uniform σ-convergence

Our main result establishes the existence of the uniform σ-convergence from point-wise σ-convergence in certain contexts. This result has similar implications to the Hahn–Schur Theorem, which is a fundamental theorem in functional analysis related to the uniform convergence properties of sequences in $B(\ell_\infty)$ and in $B(c_0)$. Additionally, some versions have also been employed by several authors (see, [1, 2, 18]).

Now, we give a theorem which can be considered a generalization of the uniform convergence of the *uC* series to the *wuC* series using σ-convergence method, which is exactly a version of Theorem 3.2. First, we need to give the following useful results for the proof of our main theorem.

Remark 4.1. (i) Let us consider the inclusion map $I : c_0 \to \mathcal{S}$ and the canonical base (e^k) of c_0. Thus, we have $c_0^{**} \equiv \ell_\infty$; therefore, the sequence $\alpha = (\alpha_k) \in \ell_\infty$ can be identified with the mapping $\mathcal{S}^* \to \mathbb{R}$ defined by the following:

$$s^* \longmapsto \sum_{k=1}^{\infty} \alpha_k s^*(e^k),$$

where $(e^k)_{k\in\mathbb{N}}$ is the sequence whose k^{th} position is 1 and all the others are 0. Therefore, one can identify the space ℓ_∞ with a linear subspace of $\mathcal{S}^{**}$, (see also, [4, 7, 21]).

(ii) Let the inclusions $B \subseteq \lambda \subseteq B^{**}$ hold. If B is a λ-Grothendieck and $(v_n^*) \subseteq B^*$ is w^*-convergent to some $v^* \in B^*$, then the sequence of functionals (v_n^*) is also $\sigma(B^*, \lambda)$-convergent to v^*, ([21], Lemma 2.13).

(iii) If $y = (y_k)$ is a sequence in $B(c_0)$ that satisfies $\|y\| > \epsilon$, $\forall\epsilon > 0$, then, there exists $v^* \in \overline{U}_{B^*}$ such that the following holds, ([21], Lemma 2.16):

$$\epsilon < \sum_{k=1}^{\infty} |v^*(y_k)| < \infty.$$

(iv) If $\mathcal{S}$ is an ℓ_∞-Grothendieck and $(f_n)_{n\in\mathbb{N}}$ is an arbitrary sequence in $\mathcal{S}^*$ such that weak* converging to zero, then, we have the following:

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \alpha_k f_n(e^k) = 0,$$

for each $\alpha = (\alpha_k)_{k\in\mathbb{N}} \in \ell_\infty$, ([21], Lemma 2.15 (3)).

Finally, we present our main result related to the Hahn–Schur theorem.

Theorem 4.2. *Suppose that $(v^n)_{n\in\mathbb{N}}$ is a sequence in $B(c_0)$ and $\mathcal{S}$ is an ℓ_∞-Grothendieck space. If for each $\alpha = (\alpha_k) \in \mathcal{S}$,*

$$\lim_{n\to\infty} wV_\sigma \sum_{k=1}^{\infty} \alpha_k v_k^n$$

exists, then there exist the point $v^0 \in B(c_0)$ that satisfies

$$\lim_{n\to\infty} \|v^n - v^0\| = 0$$

in $B(c_0)$.

Proof. First, suppose that $(v^n)_{n\in\mathbb{N}}$ is not a Cauchy sequence in $B(c_0)$, and let (n_j) be an increasing sequence of natural numbers for every $\epsilon > 0$ such that $\|y^j\| > \epsilon$ for each $j \in \mathbb{N}$, where $y^j = v^{n_j} - v^{n_{j+1}}$. Now, from Remark 4.1 (iii), for every $j \in \mathbb{N}$, there exists $v_j^* \in \overline{U}_{B^*}$ such that the inequality

$$\epsilon < \sum_{k=1}^{\infty} |v_j^*(y_k^j)|$$

holds. From the Proposition 3.6, we also have the continuous linear map $\mathcal{L}_{y^j}^w : \mathcal{S} \to B$ defined by the following:

$$(\alpha_k)_{k\in\mathbb{N}} \longmapsto \mathcal{L}_{y^j}^w((\alpha_k)_{k\in\mathbb{N}}) = wV_\sigma \sum_{k=1}^{\infty} \alpha_k y_k^j, \ \forall j \in \mathbb{N}.$$

By our hypothesis, for every $\alpha = (\alpha_k) \in \mathcal{S}$, the following

$$\lim_{j\to\infty} \mathcal{L}_{y^j}^w((\alpha_k)_{k\in\mathbb{N}}) = \lim_{j\to\infty} wV_\sigma \sum_{k=1}^{\infty} \alpha_k y_k^j$$

exists in B, and the sequences $\left(wV_\sigma \sum_{k=1}^{\infty} \alpha_k y_k^{n_j}\right)_{j\in\mathbb{N}}$ and $\left(wV_\sigma \sum_{k=1}^{\infty} \alpha_k y_k^{n_{j+1}}\right)_{j\in\mathbb{N}}$ are the subsequences of the original sequence $\left(wV_\sigma \sum_{k=1}^{\infty} \alpha_k y_k^j\right)_{j\in\mathbb{N}}$. We conclude that the following:

$$\begin{aligned}\lim_{j\to\infty} \mathcal{L}_{y^j}^w((\alpha_k)_{k\in\mathbb{N}}) &= \lim_{j\to\infty} wV_\sigma \sum_{k=1}^{\infty} \alpha_k y_k^j \\ &= \lim_{j\to\infty} wV_\sigma \sum_{k=1}^{\infty} \alpha_k y_k^{n_j} - \lim_{j\to\infty} wV_\sigma \sum_{k=1}^{\infty} \alpha_k y_k^{n_{j+1}} \\ &= 0.\end{aligned}$$

Therefore, the composite sequence $(v_j^* \circ \mathcal{L}_{y^j}^w)_{j\in\mathbb{N}}$ is w^*-null sequence of $\mathcal{S}^*$ (i.e., it is weak* convergent to zero in $\mathcal{S}^*$). Since $\mathcal{S}$ is an ℓ_∞-Grothendieck space that satisfies the inclusions $c_0 \subseteq \mathcal{S} \subseteq \ell_\infty$, we obtain the following by (i) and (iv) in Remark 4.1:

$$\lim_{j\to\infty} \sum_{k=1}^{\infty} \alpha_k (v_j^* \circ \mathcal{L}_{y^j}^w)(e^k) = \lim_{j\to\infty} \sum_{k=1}^{\infty} \alpha_k v_j^*(y_k^j) = 0, \ \forall \alpha = (\alpha_k)_{k\in\mathbb{N}} \in \ell_\infty.$$

Therefore, we have that $\{(v_j^*(y_k^j)_{k\in\mathbb{N}}\}_{j\in\mathbb{N}} \in \ell_1$ is a weakly null sequence. On the other hand, the sequence $\{(v_j^*(y_k^j)_{k\in\mathbb{N}}\}_{j\in\mathbb{N}}$ is also null with norm in ℓ_1, since the space ℓ_1 is a Schur space, which contradicts $\sum_{k=1}^{\infty} |v_j^*(y_k^j)| < \epsilon$, for all $j \in \mathbb{N}$. □

Corollary 4.3. *Let us suppose that* $(v^n)_{n\in\mathbb{N}}$ *satisfies the conditions of Theorem 4.2, and* $\lim_{n\to\infty} V_\sigma \sum_{k=1}^{\infty} \alpha_k v_k^n$ *exists for each* $\alpha = (\alpha_k) \in \mathcal{S}$. *Then, there exists* $v^0 \in B(c_0)$ *that satisfies* $\lim_{n\to\infty} \|v^n - v^0\| = 0$ *in* $B(c_0)$.

5. An alternative formulation of the Hahn–Schur type theorem

In this section, we present an alternative formulation of our Hahn–Schur type theorem. Before doing so, we introduce several preparatory materials that will be central to the proof. Let $\mathcal{F}$ be a σ-Boolean algebra. Let $\mathcal{B}(\mathcal{F})$ be the Banach space that consists of all bounded, real-valued functions on $\mathcal{F}$ satisfies $\mathcal{F}$-measurability equipped with the supremum norm. Furthermore, let $\mathcal{B}(\mathcal{F})^*$ be the Banach space of all finitely additive measures on $\mathcal{F}$, which is equipped with the bounded variation norm.

We recall that $\mathcal{B}_S(\mathcal{F})$ denotes the space of simple functions, which is dense in $\mathcal{B}(\mathcal{F})$. The density of $\mathcal{B}_S(\mathcal{F})$ in $\mathcal{B}(\mathcal{F})$ plays a crucial role in the arguments below, thus allowing us to approximate arbitrary bounded measurable functions by simple functions.

Using these notions, we shall derive a refined version of our Hahn–Schur type theorem. Additionally, we will explore its implications in connection with three fundamental properties in the measure theory and functional analysis, namely, the Vitali-Hahn-Saks (VHS) property, the Grothendieck (G) property, and the Nikodym (N) property. These properties provide deep insights into the structure and behavior of measures and Banach spaces, which will illuminate various aspects of our main results.

Lemma 5.1. ***(Vitali-Hahn-Saks)*** *A sequence $v = (v_n)$ in $\mathcal{B}(\mathcal{F})^*$ is uniformly strongly additive if $(v_n(A))$ converges for every $A \in \mathcal{F}$ [7].*

Lemma 5.2. ***(Grothendieck)*** *A sequence $v = (v_n)$ has the same behavior with Schur sequences but in the weak-* topology. Equivalently, $\mathcal{B}(\mathcal{F})$ is a Grothendieck space [7].*

Lemma 5.3. ***(Nikodym)*** *The family $\mathcal{M} \subseteq \mathcal{B}(\mathcal{F})^*$, where $v(A) : v = (v_n) \in \mathcal{M}$ is bounded for every $A \in \mathcal{F}$, is uniformly bounded, that is, $\mathcal{B}_S(\mathcal{F})$ is barrelled [7].*

Let $\mathcal{F}$ be a Boolean subalgebra of $P(\mathbb{N})$ such that

$$\Phi(\mathbb{N}) := \{A \subseteq \mathbb{N} : \mathrm{card}(A) < \infty\} \subseteq \mathcal{F}.$$

Boolean subalgebras of this form are referred to as *natural Boolean algebras* in [2]. Recall that if $\mathcal{F}$ is a Boolean algebra, then its *Stone space* T is the totally disconnected, compact Hausdorff space that arises from the set of all ultrafilters on $\mathcal{F}$, which is equipped with the Stone topology. It is a standard result that the family $C(T)$ of the real-valued continuous functions on T can be embedded into ℓ_∞ by identifying each continuous function with its bounded extension over T.

Following [37], the Boolean algebra $\mathcal{F}$ is said to precisely possess the *Grothendieck property* when $C(T)$ has the Grothendieck property. Similarly, $\mathcal{F}$ is said to have the *Nikodym property* whenever the space $C_0(T)$ is a barrelled space which consists of real-valued continuous functions on T that takes finitely many values. In particular, $\mathcal{F}$ is said to satisfy the *Vitali–Hahn–Saks property* if and only if it has both the Grothendieck and Nikodym properties.

Theorem 5.4. *Let $v = (v_n)$ be a sequence in $B(c_0)$ and suppose that $\mathcal{F}$ is a natural Boolean algebra that satisfies the Vitali-Hahn-Saks property. If $\lim\limits_{n\to\infty} wV_\sigma \sum_{i\in A} v_i^n$ exists for every $A \in \mathcal{F}$, then one can find $V^0 \in B(c_0)$ such that* $\lim_{n\to\infty} \|v^n - V^0\| = 0$ *in* $B(c_0)$.

Proof. Let S denote the Stone space of $\mathcal{F}$. Then, $C(S)$ is a Grothendieck space, while $C_0(S)$ is barrelled. Moreover, $C(S)$ can be linearly and isometrically identified with a closed subspace M of

ℓ^1 that includes c_0. Naturally, this implies that M is also Grothendieck. Consider the following weakly summing operator for every $n \in \mathbb{N}$, $\mathcal{L}^w_{v^n} : \mathcal{S} \to B$ as

$$\mathcal{L}^w_{v^n}(\alpha) = wV_\sigma \sum_{k=1}^{\infty} \alpha_k v_k^n \ ,$$

and denote the corresponding restriction of $\mathcal{L}^w_{v^n}$ to M_0 by $\mathcal{L}^w_{v^n}(0)$, where M_0 represents the subspace of M that consists of finite-valued sequences. If $\beta = (\beta_k) \in M_0$, then,

$$\lim_{n\to\infty} wV_\sigma \sum_{k=1}^{\infty} \beta_k v_k^n$$

exists. Since M_0 is barrelled (as it corresponds to $C_0(S)$ in this identification), there exists a constant $H > 0$ such that $\|\mathcal{L}^w_{v^n}\| = \|\mathcal{L}^w_{v^n}(0)\| < H$ for all $n \in \mathbb{N}$. Furthermore, the density of M_0 in M implies that $\lim_{n\to\infty} wV_\sigma \sum_{k=1}^{\infty} \alpha_k v_k^n$ exists for every $\alpha = (\alpha_k) \in M$. By Theorem 4.2, we can conclude that there exists some $V_0 \in B(c_0)$ such that $\lim_{n\to\infty} \|v^n - V^0\| = 0$ in $B(c_0)$. □

Since the multiplier spaces $B(\mathcal{S}, V_\sigma)$ and $B_w(\mathcal{S}, V_\sigma)$ are complete with the norm given in (3.1), and $\mathcal{S}$ is an ℓ_∞-Grothendieck space, one can easily prove the following corollary in the light of previous results.

Corollary 5.5. *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $B_w(\mathcal{S}, V_\sigma)$ (or $B(\mathcal{S}, V_\sigma)$) and $\mathcal{S}$ be an ℓ_∞-Grothendieck space. Therefore, $(v_n)_{n\in\mathbb{N}}$ converges in $B_w(\mathcal{S}, V_\sigma)$ (or $B(\mathcal{S}, V_\sigma)$) if and only if* $\lim_{n\to\infty} wV_\sigma \sum_{k=1}^{\infty} \alpha_k v_k^n$ *(or* $\lim_{n\to\infty} V_\sigma \sum_{k=1}^{\infty} \alpha_k v_k^n$*) exists for every $\alpha = (\alpha_k)_{k\in\mathbb{N}} \in \mathcal{S}$.*

6. Conclusions

In the present paper, the spaces $B(\mathcal{S}, V_\sigma)$ and $B_w(\mathcal{S}, V_\sigma)$ were defined using σ-convergence and a subset $\mathcal{S}$ of ℓ_∞ that contains c_0, i.e., $c_0 \subseteq \mathcal{S} \subseteq \ell_\infty$. Thus, the spaces $B(\mathcal{S}, V_\sigma)$ and $B_w(\mathcal{S}, V_\sigma)$ were shown as Banach spaces with their natural supremum norm. Additionally, a classical result of the Hahn–Schur type theorem on generalization of the uniform convergence of *uc* series to the *wuC* series was given by the concept of σ-convergence as a summability method. Some versions of this type generalizations can be found in [1,4,7,21] by means of the classical concept of convergence, matrix summability methods, almost convergence, and statistically convergence, respectively.

Use of Generative-AI tools declaration

The author declares he have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest in this paper.

References

1. A. Aizpuru, F. J. Pérez-Fernández, Spaces of $\mathcal{S}$ bounded multiplier convergent series, *Acta Math. Hung.*, **87** (2000), 135–146. https://doi.org/10.1023/A:1006781218759
2. A. Aizpuru, A. Gutiérrez-Dávila, F. J. Pérez-Fernández, Boolean algebras and uniform convergence of series, *J. Math. Anal. Appl.*, **284** (2003), 89–96. https://doi.org/10.1016/s0022-247x(03)00241-5
3. A. Aizpuru, A. Gutiérrez-Dávila, A. Sala, Unconditionally Cauchy series and Cesàro summability, *J. Math. Anal. Appl.*, **324** (2006), 39–48. https://doi.org/10.1016/j.jmaa.2005.11.071
4. A. Aizpuru, F. J. García-Pacheco, C. Pérez-Eslava, Matrix summability and uniform convergence of series, *Proc. Amer. Math. Soc.*, **135** (2007), 3571–3579. https://doi.org/10.1090/s0002-9939-07-08882-x
5. A. Aizpuru, R. Armario, F. J. Pérez-Fernández, Almost summability and unconditionally Cauchy series, *Bull. Belg. Math. Soc. Simon Stevin*, **15** (2008), 635–644. https://doi.org/10.36045/bbms/1225893944
6. A. Aizpuru, C. Pérez-Eslava, J. B. Seoane-Sepúlveda, Matrix summability methods and weakly unconditionally Cauchy series, *Rocky Mountain J. Math.*, **39** (2009), 367–380. https://doi.org/10.1216/rmj-2009-39-2-367
7. A. Aizpuru, R. Armario, F. J. García-Pacheco, F. J. Pérez-Fernández, Banach limits and uniform almost summability, *J. Math. Anal. Appl.*, **379** (2011), 82–90. https://doi.org/10.1016/j.jmaa.2010.12.034
8. A. Aizpuru, R. Armario, F. J. García-Pacheco, F. J. Pérez-Fernández, Vector-valued almost convergence and classical properties in normed spaces, *Proc. Math. Sci.*, **124** (2014), 93–108. https://doi.org/10.1007/s12044-013-0160-5
9. F. Albiac, N. J. Kalton, *Topics in Banach space theory*, New York: Springer, 2006. https://doi.org/10.1007/0-387-28142-8
10. N. P. Akın, Invariant summability and unconditionally Cauchy series, *Int. J. Anal. Appl.*, **18** (2020), 663–671. https://doi.org/10.28924/2291-8639-18-2020-663
11. B. Altay, R. Kama, On Cesàro summability of vector valued multiplier spaces and operator valued series, *Positivity*, **22** (2018), 575–586. https://doi.org/10.1007/s11117-017-0528-3
12. F. Başar, *Summability theory and its applications*, 2 Eds, New York: Chapman and Hall/CRC, 2022. https://doi.org/10.1201/9781003294153
13. F. Başar, R. Çolak, Almost-conservative matrix transformations, *Turkish J. Math.*, **13** (1989), 91–100.
14. F. Başar, İ. Solak, Almost-coercive matrix transformations, *Rend. Mat. Appl.*, **11** (1991), 249–256.
15. F. Başar, Infinite matrices and almost boudedness, *Boll. Unione Mat. Ital.*, **6** (1992), 395–402.

16. F. Başar, M. Kirişçi, Almost convergence and generalized difference matrix, *Comput. Math. Appl.*, **61** (2011), 602–611. https://doi.org/10.1016/j.camwa.2010.12.006

17. J. Boos, P. Cass, *Classical and modern methods in summability*, New York: Oxford University Press, 2000. https://doi.org/10.1093/oso/9780198501657.001.0001

18. Q. Y. Bu, C. X. Wu, Unconditionally convergent series of operators on Banach spaces, *J. Math. Anal. Appl.*, **207** (1997), 291–299. https://doi.org/10.1006/jmaa.1997.5218

19. J. Diestel, *Sequences and series in Banach spaces*, New York: Springer, 1984. https://doi.org/10.1007/978-1-4612-5200-9

20. W. F. Eberlein, Banach-Hausdorff limits, *Proc. Amer. Math. Soc.*, **1** (1950), 662–665. https://doi.org/10.2307/2032296

21. F. J. García-Pacheco, R. Kama, M. Murillo-Arcila, Vector-valued spaces of multiplier statistical convergent series and uniform convergence, *Results Math.*, **77** (2022), 43. https://doi.org/10.1007/s00025-021-01582-4

22. R. Kama, B. Altay, F. Başar, On the domains of backward difference matrix and the space of convergence of a series, *Bull. Allahabad Math. Soc.*, **33** (2018), 139–153.

23. M. Karakuş, On certain vector valued multiplier spaces and series of operators, *J. Math. Anal.*, **10** (2019), 1–11.

24. M. Karakuş, F. Başar, A generalization of almost convergence, completeness of some normed spaces with *wuC* series and a version of Orlicz-Pettis theorem, *RACSAM*, **113** (2019), 3461–3475. https://doi.org/10.1007/s13398-019-00701-9

25. M. Karakuş, F. Başar, Vector valued multiplier spaces of f_λ-summability, completeness through $c_0(X)$-multiplier convergence and continuity and compactness of summing operators, *RACSAM*, **114** (2020), 169. https://doi.org/10.1007/s13398-020-00898-0

26. M. Karakuş, F. Başar, Operator valued series, almost summability of vector valued multipliers and (weak) compactness of summing operator, *J. Math. Anal. Appl.*, **484** (2020), 123651. https://doi.org/10.1016/j.jmaa.2019.123651

27. M. Karakuş, F. Başar, On some classical properties of normed spaces via generalized vector valued almost convergence, *Math. Slovaca*, **72** (2022), 1551–1566. https://doi.org/10.1515/ms-2022-0106

28. M. Karakuş, F. Başar, Vector valued closed subspaces and characterizations of normed spaces through σ-summability, *Indian J. Math.*, **66** (2024), 85–105.

29. M. Karakuş, F. Başar, Characterizations of unconditionally convergent and weakly unconditionally Cauchy series via w_p^R-summability, Orlicz–Pettis type theorems and compact summing operator, *Filomat*, **36** (2022), 6347–6358. https://doi.org/10.2298/fil2218347k

30. R. L. Li, Q. Y. Bu, Locally convex spaces containing no copy of c_0, *J. Math. Anal. Appl.*, **172** (1993), 205–211. https://doi.org/10.1006/jmaa.1993.1017

31. G. G. Lorentz, A contribution to the theory of divergent sequences, *Acta Math.*, **80** (1948), 167–190. https://doi.org/10.1007/BF02393648

32. M. Mursaleen, On some new invariant matrix methods of summability, *Q. J. Math.*, **34** (1983), 77–86. https://doi.org/10.1093/qmath/34.1.77

33. M. Mursaleen, O. H. H. Edely, On the invariant mean and statistical convergence, *Appl. Math. Lett.*, **22** (2009), 1700–1704. https://doi.org/10.1016/j.aml.2009.06.005

34. M. Mursaleen, F. Başar, *Sequence spaces: topics in modern summability theory*, Boca Raton: CRC Press, 2020. https://doi.org/10.1201/9781003015116

35. B. J. Pettis, On integration in vector spaces, *Trans. Amer. Math. Soc.*, **44** (1938), 277–304.

36. R. A. Raimi, Invariant means and invariant matrix methods of summability, *Duke Math. J.*, **30** (1963), 81–94. https://doi.org/10.1215/s0012-7094-63-03009-6

37. W. Schachermayer, On some classical measure-theoretic theorems for non-sigma-complete Boolean algebras, *Diss. Math.*, **214** (1982), 1–33.

38. C. Swartz, The schur lemma for bounded multiplier convergent series, *Math. Ann.*, **263** (1983), 283–288. https://doi.org/10.1007/bf01457131

39. C. Swartz, *Multiplier convergent series*, Singapore: World Scientific Publishing, 2008. https://doi.org/10.1142/9789812833884

40. E. M. Semenov, F. A. Sukochev, Invariant Banach limits and applications, *J. Funct. Anal.*, **259** (2010), 1517–1541. https://doi.org/10.1016/j.jfa.2010.05.011

41. E. M. Semenov, F. A. Sukochev, A. Usachev, The main classes of invariant Banach limits, *Izv. Math.*, **83** (2019), 124–150. https://doi.org/10.1070/im8704