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Research article

Riemann-Liouville fractional-order pantograph differential equation constrained by nonlocal and weighted pantograph integral equations

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Abstract: In this research, we investigated the Riemann-Liouville fractional-order pantograph differential equation constrained by nonlocal and weighted pantograph integral constraints. We presented novel sufficient conditions for the uniqueness of the solution. Moreover, we analyzed the continuous dependence of the solution on some functions and parameters. Additionally, we proved the Hyers-Ulam stability of the problem. To demonstrate the applicability of our results, we included several examples. The present study was located in the space $L_1[0, T]$. The techniques of Schauder's fixed point theorem and Kolmogorov's compactness criterion were the primary tools utilized in this work. These contributions offer a comprehensive framework for understanding the qualitative behavior of the fractional-order pantograph equation.

Keywords: pantograph differential equation; Schauder fixed point theorem; weighted pantograph integral equation; nonlocal conditions; Riemann-Liouville fractional derivative; continuous dependence; Hyers-Ulam stability
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1. Introduction

Fractional-order differential and integral equations play a significant role in a variety of fields, including physics, engineering, and biomedical engineering. These equations are widely applied in numerous scientific and engineering models [24–26,37,39]. In mathematical analysis, nonlocal integral conditions are often employed when analyzing differential equations, particularly when dealing with equations that involve restrictions or objectives. Existing research has mainly focused on the existence and uniqueness of solutions to such equations, which are often based on continuity or boundedness conditions. Fixed point theorems have been demonstrated to be effective techniques for analyzing the solvability of these equations as described in monographs and papers (see [7, 12, 13, 20] and the references therein).

Stability analysis is a complex and diverse field with strong theoretical foundations and numerous applications in engineering, economics, biology, physics, and other disciplines. An equation or issue can be used to model a physical process if a small change in it results in a commensurate small change in the outcome. This indicates that the equation or problem is stable.

There are various concepts of stability of differential equations, one of which is the Hyers-Ulam stability. This concept pertains to the stability of solutions to differential equations under small perturbations or approximations, specifically addressing the behavior of solutions when the equation is subject to minor errors. The Hyers-Ulam stability provides a framework for determining whether approximate solutions to a differential equation can be corrected or approximated by actual solutions that remain within a controlled deviation. Several authors investigated the Hyers-Ulam stability of differential equations [3, 23, 35, 36].

Another concept in stability theory is continuous dependency [32], which examines how mathematical solutions behave under various conditions. Hyers-Ulam stability measures the problem's resilience to interruptions, whereas continuous dependency investigates how modest parameter changes affect the problem's unique solution.

The pantograph equation is a particular type of delay differential equation derived from electrodynamics that was initially developed by investigating an electric locomotive [15, 29]. The term pantograph was first introduced in Ockendon and Taylor's research [29], which investigated the electric locomotive's catenary system. Their goal was to formulate an equation to analyze the movement of the pantograph head on an electric locomotive powered by an overhead trolley wire. The behavior of the pantograph differential equation is significant in a variety of fields of study. It has various applications, including the current-collecting system [29], cell growth models [38], the ruin problem in risk theory [14], quantum theory [34], light fusion in spiral galaxies [6], and industrial applications. Multiple studies have examined the pantograph equation with different boundary conditions or derivatives [10, 11, 18, 19]. The authors investigated the existence, uniqueness, and stability of the solution of the pantograph equation. Numerical methods for the pantograph equations were studied in [9, 17, 28] and the references therein.

Fractional pantograph equations have received significant attention due to their importance in numerous fields. This type of equation is motivated by the need to model the non-integer and memory-dependent interactions between the pantograph head and the catenary system, offering a more precise description of the system's dynamics, particularly when accounting for complex forces, vibrations, and elastic properties. Several authors have studied this type of equation; for instance, Balachandran et al. [7] considered nonlinear fractional pantograph equations with initial and nonlocal conditions and obtained some of the existence results by using the Banach and Krasnoselskii fixed point theorems. In [4], Alrabaiah et al. studied the qualitative analysis of nonlinear coupled pantograph differential equations of fractional order with integral boundary conditions. In [2], the authors introduced fractional pantograph differential equations and investigated a class of pantograph differential equations involving Riemann-Liouville derivatives with multi-point boundary conditions; they established the existence and Ulam stability of the problem. In addition, Selvam and Jacob [33] analyzed the Ulam-Hyers stability of the nonlinear pantograph fractional differential equation involving the Atangana-Baleanu derivative. Jalilian and Ghasemi [20] examined a pantograph-type fractional integro-differential equation with appropriate initial conditions. Boularesa [8] investigated sufficient conditions for the asymptotic stability of the zero solution of pantograph Caputo fractional differential equations of fractional order using Krasnoselskii's fixed point theorem in a weighted Banach space. In [27], the authors studied the existence and uniqueness of solutions, as well as the Ulam-Hyers stability, of a fractional-order pantograph differential equation involving two Caputo operators. They employed Banach's fixed point theorem and the Leray-Schauder alternative to establish the existence and uniqueness of solutions. In [3], El-Sayed and Al-Issa studied a pantograph equation of fractional orders under fractal-fractional feedback control. They proved the existence of solutions and the continuous dependence of the unique solution on some parameters; additionally, they also proved the Hyers-Ulam stability of the problem.

Inspired by recent literature, our focus is on investigating the constraint problem of the Riemann-Liouville fractional-order pantograph differential equation

$$ {}^{R}D^{\alpha}x(t) = f(t, x(t), \lambda_1 x(\gamma_1 t)), \quad a.e.\ \ t \in \ (0, T] \tag{1.1} $$

subject to the nonlocal and weighted pantograph integral constraints

$$ I^{1-\alpha}x(t)|_{t=0} = x_0 + \beta \int_0^T \ h(s, x(s), \lambda_2 x(\gamma_2 s))ds. \tag{1.2} $$

Remark 1. *We can investigate the problem under the following condition:*

$$ t^{1-\alpha}x(t)|_{t=0} = \frac{1}{\Gamma(\alpha)}(x_0 + \beta \int_0^T \ h(s, x(s), \lambda_2 x(\gamma_2 s))ds) \quad t \in (0, T]. \tag{1.3} $$

This condition is equivalent to the condition given in (1.2), as shown in ([21], Lemma 3.5).

For the mathematical formulation of the problem, ${}^{R}D^{\alpha}$ refers to the Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$ and $\gamma_i \in (0,\ 1),\ i = 1, 2.$ $x(t)$ represents the state of the system at time t, which is the unknown function. The function $f(t, x(t), \lambda_1 x(\gamma_1 t))$ is a nonlinear function involving the state variable $x(t)$ and the delayed term $\lambda_1 x(\gamma_1 t)$, where λ_1 and γ_1 are parameters. Moreover, the operator $I^{1-\alpha}$ is the fractional integral operator of order $1-\alpha$, β is a constant, and $h(s, x(s), \lambda_2 x(\gamma_2 s))$ is a nonlinear function depending on $x(t)$ and the delayed term $\lambda_2 x(\gamma_2 t)$, where λ_2 and γ_2 are parameters. The present study was based on Kolmogorov's compactness criterion [31] and Schauder's fixed point theorem [31].

Our aim in this study is to investigate the existence of solution $x \in L_1[0, T]$ of the constrained problems (1.1) and (1.2) or (1.1) and (1.3). Sufficient conditions for the uniqueness of the solution will be given. Furthermore, the continuous dependence of the unique solution on the initial data x_0, the functions f, h, and the parameters λ_i, $i = 1,\ 2$, will be proved. The Hyers-Ulam stability of the problem will be established. To further explain our findings, we provide some examples.

We outline the main contributions of this paper as follows:

- We examine the Riemann-Liouville fractional-order differential equation (1.1) of pantograph type under either of the two equivalent conditions, (1.2) and (1.3), and derive the corresponding equivalent integral equation.
- We investigate the qualitative properties of the solution of the problem, including the existence, uniqueness, and stability.

- We provide some examples to further clarify our results.

This study enhances the qualitative analysis of a fractional-order pantograph differential equation with nonlocal and weighted pantograph integral constraints. The article is structured as follows: Section 2 presents the appropriate assumptions and proves the existence of the solution of the fractional-order problem (1.1) with the constraints (1.2) or (1.3); moreover, the suitable assumptions and proofs for the uniqueness of the solution will be provided. In Section 3, we investigate the stability analysis of the problem; we test the possibility of the solution resisting disturbances through the study of the continuous dependency on the initial data x_0, the functions f and h, and the parameters λ_i, $i = 1,\ 2$. In addition, we examine the problem's resistance to interruptions through the Hyers-Ulam stability of the problem. In Section 4, we present some instances to illustrate the results and clarify the assumptions of the problem. Finally, Section 5 provides a conclusion.

2. Existence results

Let $L_1 = L_1(I),\ I = [0, T]$ be the class of Lebesgue integrable functions, with the standard norm

$$||x||_1 = \int_0^T |x(t)|dt.$$

In this paper, the integrals are considered in the sense of Lebesgue integration. Now consider the following assumptions:

(i) $h, f : I \times R \times R \to R$ are Carathéodory functions [3], and there exist integrable functions $a_i : I \to R,\ i = 1,\ 2$, and positive constants K and L such that

$$|f(t, x, y)| \le |a_1(t)| + L\,(|x| + |y|) \quad and \quad |h(t, x, y)| \le |a_2(t)| + K\,(|x| + |y|)$$
$$\forall\ t \in I,\ x,\ y \in R.$$

(ii)

$$\frac{T^\alpha}{\Gamma(\alpha+1)}(\beta\, K(\ 1 + \frac{\lambda_2}{\gamma_2}) + L(\ 1 + \frac{\lambda_1}{\gamma_1})) < 1.$$

Now, we have the following lemma.

Lemma 1. *The solution of the constrained problem (1.1) and (1.2) or (1.1) and (1.3) can be expressed by the fractional-order delay integral equation*

$$x(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \beta \int_0^T h(s, x(s), \lambda_2 x(\gamma_2 s))ds\ ) + I^\alpha f(t, x(t), \lambda_1 x(\gamma_1 t)). \tag{2.1}$$

Proof. Let $x \in L_1(I)$ be a solution of the constrained problems (1.1) and (1.2) or (1.1) and (1.3), and then we have

$$\frac{d}{dt} I^{1-\alpha}\ x(t) = f(t, x(t), \lambda_1 x(\gamma_1 t)).$$

By integrating the above, we obtain

$$\begin{aligned} I^{1-\alpha}\ x(t) - I^{1-\alpha}\ x(t)|_{t=0} &= \int_0^t f(s, x(s), \lambda_1 x(\gamma_1 s))ds \\ I^{1-\alpha}\ x(t) &= I^{1-\alpha}\ x(t)|_{t=0} + \int_0^t f(s, x(s), \lambda_1 x(\gamma_1 s))ds \end{aligned}$$

and from (1.2), we get

$$I^{1-\alpha}\ x(t) = x_0 + \beta \int_0^T h(s, x(s), \lambda_2 x(\gamma_2 s))ds + \int_0^t f(s, x(s), \lambda_1 x(\gamma_1 s))ds.$$

Operating with I^{α}, then

$$I\ x(t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)}\ (x_0 + \beta \int_0^T h(s, x(s), \lambda_2 x(\gamma_2 s))ds) + I^{\alpha+1}\ f(t, x(t), \lambda_1 x(\gamma_1 t)).$$

By differentiation, we obtain

$$x(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\ (x_0 + \beta \int_0^T h(s, x(s), \lambda_2 x(\gamma_2 s))ds) + I^{\alpha}\ f(t, x(t), \lambda_1 x(\gamma_1 t)).$$

Conversely, from (2.1), we have

$$I^{1-\alpha}\ x(t) = x_0 + \beta \int_0^T h(s, x(s), \lambda_2 x(\gamma_2 s))ds + I\ f(t, x(t), \lambda_1 x(\gamma_1 t)).$$

By differentiation, we get

$$\frac{d}{dt} I^{1-\alpha}\ x(t) = \frac{d}{dt}\ (x_0 + \beta \int_0^T h(s, x(s), \lambda_2 x(\gamma_2 s))ds)\ + \frac{d}{dt}\ I\ f(t, x(t), \lambda_1 x(\gamma_1 t))$$

and

$${}^{R}D^{\alpha}\ x(t) = f(t, x(t), \lambda_1 x(\gamma_1 t)),$$

and then we deduced (1.1) and also

$$I^{1-\alpha} x(t)|_{t=0} = x_0 + \beta \int_0^T h(s, x(s), \lambda_2 x(\gamma_2 s))ds.$$

Now consider problems (1.1) and (1.3), and then

$$\frac{d}{dt} I^{1-\alpha}\ x(t) = f(t, x(t), \lambda_1 x(\gamma_1 t)),$$

with

$$t^{1-\alpha}\ x(t)|_{t=0} = \frac{1}{\Gamma(\alpha)}\ (x_0 + \beta \int_0^T h(s, x(s), \lambda_2 x(\gamma_2 s))ds).$$

Integrating the preceding gives

$$I^{1-\alpha}\ x(t) - C = \int_0^t f(s, x(s), \lambda_1 x(\gamma_1 s))ds,$$

$$I^{1-\alpha}\ x(t) = C + I\ f(t, x(t), \lambda_1 x(\gamma_1 t)).$$

Operating with I^α on both sides yields

$$I\ x(t) = \frac{Ct^\alpha}{\Gamma(\alpha+1)} + I^{\alpha+1}\ f(t, x(t), \lambda_1 x(\gamma_1 t)).$$

Differentiate the above, and we have

$$x(t) = \frac{Ct^{\alpha-1}}{\Gamma(\alpha)} + I^\alpha\ f(t, x(t), \lambda_1 x(\gamma_1 t))$$

and

$$t^{1-\alpha}\ x(t) = \frac{C}{\Gamma(\alpha)} + t^{1-\alpha}\ I^\alpha\ f(t, x(t), \lambda_1 x(\gamma_1 t)).$$

From this, we arrive at

$$t^{1-\alpha}\ x(t)|_{t=0} = \frac{C}{\Gamma(\alpha)}.$$

This leads to

$$\frac{1}{\Gamma(\alpha)}(x_0 + \beta \int_0^T\ h(s, x(s), \lambda_2 x(\gamma_2 s))ds) = \frac{C}{\Gamma(\alpha)}.$$

As a result, we obtain (2.1). Conversely, let $x \in L_1(I)$ be a solution of (2.1). Then we have

$$t^{1-\alpha}\ x(t) = \frac{1}{\Gamma(\alpha)}(x_0 + \beta \int_0^T\ h(s, x(s), \lambda_2 x(\gamma_2 s))ds) + t^{1-\alpha}\ I^\alpha\ f(t, x(t), \lambda_1 x(\gamma_1 t)),$$

$$t^{1-\alpha}\ x(t)|_{t=0} = \frac{1}{\Gamma(\alpha)}(x_0 + \beta \int_0^T\ h(s, x(s), \lambda_2 x(\gamma_2 s))ds)$$

and

$$I^{1-\alpha}\ x(t) = \frac{1}{\Gamma(\alpha)}(x_0 + \beta \int_0^T\ h(s, x(s), \lambda_2 x(\gamma_2 s))ds)\ +\ I^{1-\alpha} I^\alpha\ f(t, x(t), \lambda_1 x(\gamma_1 t)).$$

Consequently, we get

$$\frac{d}{dt} I^{1-\alpha}\ x(t) = f(t, x(t), \lambda_1 x(\gamma_1 t)).$$

Now, consider the following existence theorem.

Theorem 1. *Let the assumptions* (i) *and* (ii) *be satisfied. Then there exists at least one solution* $x \in L_1(I)$ *of the problems (1.1) and (1.2) or (1.1) and (1.3).*

Proof. Let the set Q_r be defined by

$$Q_r = \{x \in L_1(I) : \|x\|_1 \le r\},\quad r = \frac{\frac{T^\alpha}{\Gamma(\alpha+1)}(|x_0| + \beta\ \|a_2\| + \|a_1\|)}{1 - \frac{T^\alpha}{\Gamma(\alpha+1)}(\beta\ K(1 + \frac{\lambda_2}{\gamma_2}) + L\ (1 + \frac{\lambda_1}{\gamma_1}))}.$$

Define the operator F by

$$Fx(t) \quad = \quad \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \beta \int_0^T h(s, x(s), \lambda_2 x(\gamma_2 s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f(s, x(s), \lambda_1 x(\gamma_1 s))ds.$$

Now, let $x \in Q_r$, and then

$$
\begin{aligned}
|Fx(t)| &= |\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \beta \int_0^T h(s, x(s), \lambda_2 x(\gamma_2 s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \lambda_1 x(\gamma_1 s))ds| \\
&\leq \frac{t^{\alpha-1}}{\Gamma(\alpha)}|x_0| + \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T |h(s, x(s), \lambda_2 x(\gamma_2 s)))|ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s, x(s), \lambda_1 x(\gamma_1 s))|ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_0^T |Fx(t)|dt &\leq \int_0^T (\frac{t^{\alpha-1}}{\Gamma(\alpha)}|x_0| + \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T |h(s, x(s), \lambda_2 x(\gamma_2 s))|ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s, x(s), \lambda_1 x(\gamma_1 s))|ds)dt \\
&\leq \int_0^T \frac{t^{\alpha-1}}{\Gamma(\alpha)}|x_0|dt + \int_0^T \int_0^T \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)}dt\, |h(s, x(s), \lambda_2 x(\gamma_2 s))|ds \\
&+ \int_0^T \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s, x(s), \lambda_1 x(\gamma_1 s))|dsdt \\
&\leq \int_0^T \frac{t^{\alpha-1}}{\Gamma(\alpha)}|x_0|dt + \int_0^T \int_0^T \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)}dt\, |h(s, x(s), \lambda_2 x(\gamma_2 s))|ds \\
&+ \int_0^T |f(s, x(s), \lambda_1 x(\gamma_1 s))| \int_s^T \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}dtds \\
&\leq \frac{T^\alpha}{\Gamma(\alpha+1)}|x_0| + \frac{\beta\, T^\alpha}{\Gamma(\alpha+1)} \int_0^T |h(s, x(s), \lambda_2 x(\gamma_2 s))|ds \\
&+ \frac{T^\alpha}{\Gamma(\alpha+1)} \int_0^T |f(s, x(s), \lambda_1 x(\gamma_1 s))|ds \\
&\leq \frac{T^\alpha}{\Gamma(\alpha+1)}|x_0| + \frac{\beta\, T^\alpha}{\Gamma(\alpha+1)}(\int_0^T |a_2\ (s)|ds + K(\int_0^T |x(s)|ds + \lambda_2 \int_0^T |x(\gamma_2 s))|ds) \\
&+ \frac{T^\alpha}{\Gamma(\alpha+1)}(\int_0^T |a_1\ (s)|ds + L(\int_0^T |x(s)|ds + \lambda_1 \int_0^T |x(\gamma_1 s)|ds)) \\
&\leq \frac{T^\alpha}{\Gamma(\alpha+1)}|x_0| + \frac{\beta\, T^\alpha}{\Gamma(\alpha+1)}(\int_0^T |a_2\ (s)|ds + K(\int_0^T |x(s)|ds + \frac{\lambda_2}{\gamma_2} \int_0^T |x(\theta)|d\theta)) \\
&+ \frac{T^\alpha}{\Gamma(\alpha+1)}(\int_0^T |a_1\ (s)|ds + L(\int_0^T |x(s)|ds + \frac{\lambda_1}{\gamma_1} \int_0^T |x(\tau)|d\tau)) \\
&\leq \frac{T^\alpha}{\Gamma(\alpha+1)}\ |x_0| + \frac{\beta\, T^\alpha}{\Gamma(\alpha+1)}(||a_2||_1 + Kr(\ 1 + \frac{\lambda_2}{\gamma_2})) + \frac{T^\alpha}{\Gamma(\alpha+1)}(||a_1||_1 + Lr(\ 1 + \frac{\lambda_1}{\gamma_1}\ )).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
||Fx||_1 &\leq \frac{T^\alpha}{\Gamma(\alpha+1)}\ (|x_0| + \beta\, ||a_2|| + ||a_1||) + \frac{T^\alpha}{\Gamma(\alpha+1)}(\beta\, Kr(\ 1 + \frac{\lambda_2}{\gamma_2}) +\ Lr(\ 1 + \frac{\lambda_1}{\gamma_1}\ )) \\
&= r
\end{aligned}
$$

$$
\frac{T^\alpha}{\Gamma(\alpha+1)}\ (|x_0| + \beta\, ||a_2|| + ||a_1||) \quad = \quad r\ (1 - \frac{T^\alpha}{\Gamma(\alpha+1)}(\beta\, K(\ 1 + \frac{\lambda_2}{\gamma_2}) + L(1 + \frac{\lambda_1}{\gamma_1}\ ))).
$$

Hence the operator $F : L_1(I) \to L_1(I)$ and $\{Fx\}$ is uniformly bounded on Q_r. Now, let $x \in Q_r$, and then

$$\begin{aligned}
\|(Fx)_h - Fx\|_1 &= \int_0^T |(Fx)_h\,(t) - (Fx)(t)|dt \\
&= \int_0^T |\frac{1}{h}\int_t^{t+h} (Fx)(s)ds - (Fx)(t)|dt \\
&= \int_0^T \frac{1}{h}\int_t^{t+h} |(Fx)(s) - (Fx)(t)|ds\,dt.
\end{aligned}$$

Since $Fx \in L_1[0, T]$, then

$$\|(Fx)_h - Fx\|_1 \to 0 \ \ when \ \ h \to 0.$$

This means that $(Fx)_h \to (Fx)$ uniformly in $L_1(I)$. Thus $\{Fx\}$ is relatively compact [31]. Hence F is a compact operator. Now, let $\{x_n\} \subset Q_r$, and $x_n \to x$, and then

$$Fx_n(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \beta\int_0^T h(s, x_n(s), \lambda_2 x_n(\gamma_2 s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_n(s), \lambda_1 x_n(\gamma_1 s))ds,$$

$$\begin{aligned}
\lim_{n\to\infty} Fx_n(t) &= \lim_{n\to\infty} \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \beta\int_0^T h(s, x_n(s), \lambda_2 x_n(\gamma_2 s))ds) \\
&+ \lim_{n\to\infty}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_n(s), \lambda_1 x_n(\gamma_1 s))ds.
\end{aligned}$$

Applying the Lebesgue-dominated convergence theorem [22], then from assumption (i), we get

$$\begin{aligned}
\lim_{n\to\infty} Fx_n(t) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \beta\int_0^T h(s, \lim_{n\to\infty} x_n(s), \lambda_2 \lim_{n\to\infty} x_n(\gamma_2 s))ds) \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, \lim_{n\to\infty} x_n(s), \lambda_1 \lim_{n\to\infty} x_n(\gamma_1 s))ds \\
&= \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \beta\int_0^T h(s, x(s), \lambda_2 x(\gamma_2 s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \lambda_1 x(\gamma_1 s))ds \\
&= Fx(t).
\end{aligned}$$

This means that $Fx_n(t) \to Fx(t)$. Hence the operator F is continuous. Now by the Schauder fixed point theorem [31], there exists at least one solution $x \in L_1(I)$ of (2.1). Consequently, there exists at least one solution $x \in L_1(I)$ of the problems (1.1) and (1.2) or (1.1) and (1.3).

2.1. Uniqueness of the solution

Consider the following assumptions:

$(i)^*$ $f, h{:}I \times R \times R \to R$ are measurable in $t \in I$, $\forall\ x,\ y \in R$, and satisfy the Lipschitz condition such that

$$\begin{aligned}
|f(t, x, y) - f(t, \bar{x}, \bar{y}| &\leq L\,(|x - \bar{x}| + |y - \bar{y}|) \qquad (2.2)\\
|h(t, x, y) - h(t, \bar{x}, \bar{y}| &\leq K\,(|x - \bar{x}| + |y - \bar{y}|),\ \forall\ t \in I,\ x,\ y \in R.
\end{aligned}$$

Theorem 2. *Let the assumptions* $(i)^*$ *and* (ii) *be satisfied, and then the solution of problems (1.1) and (1.2) or (1.1) and (1.3) is unique.*

Proof. Assumption (i) of Theorem (2) can be deduced from (i^*), and then the solution of problems (1.1) and (1.2) or (1.1) and (1.3) exists. Now let $x_1,\ x_2$ be two solutions of (2.1), and then

$$
\begin{aligned}
& |x_2(t) - x_1(t)| \\
= \ & |\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \beta \int_0^T h(s, x_2(s), \lambda_2 x_2(\gamma_2 s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_2(s), \lambda_1 x_2(\gamma_1 s))ds \\
- \ & \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \beta \int_0^T h(s, x_1(s), \lambda_2 x_1(\gamma_2 s))ds) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_1(s), \lambda_1 x_1(\gamma_1 s))ds| \\
\leq \ & \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T |h(s, x_2(s), \lambda_2 x_2(\gamma_2 s)) - h(s, x_1(s), \lambda_2 x_1(\gamma_2 s))|ds \\
+ \ & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x_2(s), \lambda_1 x_2(\gamma_1 s)) - f(s, x_1(s), \lambda_1 x_1(\gamma_1 s))|ds \\
\leq \ & \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T K(|x_2(s) - x_1(s)| + \lambda_2 |x_2(\gamma_2 s) - x_1(\gamma_2 s)|)ds \\
+ \ & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} L(|x_2(s) - x_1(s)| + \lambda_1 |x_2(\gamma_1 s) - x_1(\gamma_1 s)|)ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
& \int_0^T |x_2(t) - x_1(t)|dt \\
\leq \ & \int_0^T (\frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T K(|x_2(s) - x_1(s)| + \lambda_2 |x_2(\gamma_2 s) - x_1(\gamma_2 s)|)ds \\
+ \ & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} L(|x_2(s) - x_1(s)| + \lambda_1 |x_2(\gamma_1 s) - x_1(\gamma_1 s)|)ds)dt \\
\leq \ & \int_0^T \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T K(|x_2(s) - x_1(s)| + \lambda_2 |x_2(\gamma_2 s) - x_1(\gamma_2 s)|)dsdt \\
+ \ & \int_0^T \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} L(|x_2(s) - x_1(s)| + \lambda_1 |x_2(\gamma_1 s) - x_1(\gamma_1 s)|)dsdt \\
\leq \ & \int_0^T \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T K(|x_2(s) - x_1(s)| + \lambda_2 |x_2(\gamma_2 s) - x_1(\gamma_2 s)|)dsdt \\
+ \ & \int_0^T \int_s^T \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} L(|x_2(s) - x_1(s)| + \lambda_1 |x_2(\gamma_1 s) - x_1(\gamma_1 s)|)dtds \\
\leq \ & \frac{\beta\, T^{\alpha}}{\Gamma(\alpha+1)} \int_0^T K(|x_2(s) - x_1(s)| + \lambda_2 |x_2(\gamma_2 s) - x_1(\gamma_2 s)|)ds \\
+ \ & \int_0^T \frac{T^{\alpha}}{\Gamma(\alpha+1)} L(|x_2(s) - x_1(s)| + \lambda_1 |x_2(\gamma_1 s) - x_1(\gamma_1 s)|)ds, \\
\leq \ & \frac{\beta\, T^{\alpha}}{\Gamma(\alpha+1)} K(\int_0^T |x_2(s) - x_1(s)|ds + \lambda_2 \int_0^T |x_2(\gamma_2 s) - x_1(\gamma_2 s)|ds)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{T^{\alpha}}{\Gamma(\alpha+1)} L(\int_0^T |x_2(s)-x_1(s)|ds + \lambda_1 \int_0^T |x_2(\gamma_1 s)-x_1(\gamma_1 s)|ds) \\
&\leq \quad \frac{\beta\, T^{\alpha}}{\Gamma(\alpha+1)} K(\int_0^T |x_2(s)-x_1(s)|ds + \frac{\lambda_2}{\gamma_2} \int_0^T |x_2(\tau)-x_1(\tau)|d\tau) \\
&+ \quad \frac{T^{\alpha}}{\Gamma(\alpha+1)} L(\int_0^T |x_2(s)-x_1(s)|ds + \frac{\lambda_1}{\gamma_1} \int_0^T |x_2(\tau)-x_1(\tau)|d\tau) \\
&\leq \quad \frac{\beta\, T^{\alpha}}{\Gamma(\alpha+1)} K(\|x_2-x_1\|_1 + \frac{\lambda_2}{\gamma_2}\|x_2-x_1\|_1) + \frac{T^{\alpha}}{\Gamma(\alpha+1)} L(\|x_2-x_1\|_1 + \frac{\lambda_1}{\gamma_1}\|x_2-x_1\|_1) \\
&\leq \quad \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|x_2-x_1\|_1((L(1+\frac{\lambda_1}{\gamma_1}) + \beta\, K(1+\frac{\lambda_2}{\gamma_2})) \\
&\leq \quad \|x_2-x_1\|_1 \frac{T^{\alpha}}{\Gamma(\alpha+1)} (L(1+\frac{\lambda_1}{\gamma_1}) + \beta\, K(1+\frac{\lambda_2}{\gamma_2})).
\end{aligned}
$$

Hence

$$
\|x_2-x_1\|_1\ (1 - \frac{T^{\alpha}}{\Gamma(\alpha+1)}(L(1+\frac{\lambda_1}{\gamma_1}) + \beta\, K(1+\frac{\lambda_2}{\gamma_2})) \quad \leq \quad 0.
$$

Since

$$
\frac{T^{\alpha}}{\Gamma(\alpha+1)}(L(1+\frac{\lambda_1}{\gamma_1}) + \beta\, K(1+\frac{\lambda_2}{\gamma_2})) < 1,
$$

this implies that

$$
\|x_2-x_1\|_1 \leq 0
$$

and hence $\|x_2-x_1\|_1 = 0$. Then $x_1 = x_2$ and the solution of (2.1) is unique. Consequently the solution of problems (1.1) and (1.2) or (1.1) and (1.3) is unique.

3. Stability analysis

We investigate the stability of the problem using two approaches: The continuous dependence of the solution on some parameters and functions, and the Hyers-Ulam stability.

3.1. Continuous dependence

Theorem 3. *Let the assumptions of Theorem 2 be satisfied, and then the unique solution $x \in L_1(I)$ of (1.1) and (1.2) or (1.1) and (1.3) depends continuously on x_0, f, h, λ_1, and λ_2 in the sense that $\forall \epsilon > 0\ \exists\ \delta(\epsilon)$ such that*

$$
\max\{|\lambda_1-\lambda_1^*|,\ |\lambda_2-\lambda_2^*|,\ |x_0-x_0^*|,\ |f(t,x,y)-f^*(t,x,y)|,\ |h(t,x,y)-h^*(t,x,y)|\} < \delta,
$$

and then

$$
\|x-x^*\|_1 < \epsilon,
$$

where x^* is the unique solution of

$$
x^*(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0^* + \beta \int_0^T h^*(s, x^*(s), \lambda_2^* x^*(\gamma_2 s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f^*(s, x^*(s), \lambda_1^* x^*(\gamma_1 s))ds.
$$

Proof.

$$
\begin{aligned}
& |x(t) - x^*(t)| \\
= & \ |\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \beta \int_0^T h(s, x(s), \lambda_2 x(\gamma_2 s))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), \lambda_1 x(\gamma_1 s))ds \\
- & \ \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0^* + \beta \int_0^T h^*(s, x^*(s), \lambda_2^* x^*(\gamma_2 s))ds) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f^*(s, x^*(s), \lambda_1^* x^*(\gamma_1 s))ds| \\
\leq & \ \frac{t^{\alpha-1}}{\Gamma(\alpha)}|x_0 - x_0^*| + \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T |h(s, x(s), \lambda_2 x(\gamma_2 s)) - h^*(s, x^*(s), \lambda_2^* x^*(\gamma_2 s))|ds \\
+ & \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s), \lambda_1 x(\gamma_1 s)) - f^*(s, x^*(s), \lambda_1^* x^*(\gamma_1 s))|ds \\
\leq & \ \frac{t^{\alpha-1}}{\Gamma(\alpha)}|x_0 - x_0^*| + \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (|h(s, x(s), \lambda_2 x(\gamma_2 s)) - h(s, x^*(s), \lambda_2^* x^*(\gamma_2 s)) \\
+ & \ h(s, x^*(s), \lambda_2^* x^*(\gamma_2 s)) - h^*(s, x^*(s), \lambda_2^* x^*(\gamma_2 s))|)ds \\
+ & \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (|f(s, x(s), \lambda_1 x(\gamma_1 s)) - f(s, x^*(s), \lambda_1^* x^*(\gamma_1 s)) \\
+ & \ f(s, x^*(s), \lambda_1^* x^*(\gamma_1 s)) - f^*(s, x^*(s), \lambda_1^* x^*(\gamma_1 s))|)ds \\
\leq & \ \frac{t^{\alpha-1}}{\Gamma(\alpha)}|x_0 - x_0^*| + \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (|h(s, x^*(s), \lambda_2^* x^*(\gamma_2 s)) - h^*(s, x^*(s), \lambda_2^* x^*(\gamma_2 s))| \\
+ & \ |h(s, x(s), \lambda_2 x(\gamma_2 s)) - h(s, x^*(s), \lambda_2^* x^*(\gamma_2 s))|)ds \\
+ & \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (|f(s, x^*(s), \lambda_1^* x^*(\gamma_1 s)) - f^*(s, x^*(s), \lambda_1^* x^*(\gamma_1 s))| \\
+ & \ |f(s, x(s), \lambda_1 x(\gamma_1 s)) - f(s, x^*(s), \lambda_1^* x^*(\gamma_1 s))|)ds \\
\leq & \ \frac{t^{\alpha-1}}{\Gamma(\alpha)}|x_0 - x_0^*| + \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (|h(s, x^*(s), \lambda_2^* x^*(\gamma_2 s)) - h^*(s, x^*(s), \lambda_2^* x^*(\gamma_2 s))| \\
+ & \ K(|x(s) - x^*(s)| + |\lambda_2 x(\gamma_2 s) - \lambda_2^* x^*(\gamma_2 s)|))ds \\
+ & \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (|f(s, x^*(s), \lambda_1^* x^*(\gamma_1 s)) - f^*(s, x^*(s), \lambda_1^* x^*(\gamma_1 s))| \\
+ & \ L(|x(s) - x^*(s)| + |\lambda_1 x(\gamma_1 s) - \lambda_1^* x^*(\gamma_1 s)|))ds \\
\leq & \ \frac{\delta\, t^{\alpha-1}}{\Gamma(\alpha)} + \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (\delta + K(|x(s) - x^*(s)| + |\lambda_2 x(\gamma_2 s) - \lambda_2 x^*(\gamma_2 s) + \lambda_2 x^*(\gamma_2 s) - \lambda_2^* x^*(\gamma_2 s)|))ds \\
+ & \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (\delta + L(|x(s) - x^*(s)| + |\lambda_1 x(\gamma_1 s) - \lambda_1^* x^*(\gamma_1 s) + \lambda_1 x^*(\gamma_1 s) - \lambda_1^* x^*(\gamma_1 s)|))ds \\
\leq & \ \frac{\delta\, t^{\alpha-1}}{\Gamma(\alpha)} + \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (\delta + K(|x(s) - x^*(s)| + |x^*(\gamma_2 s)||\lambda_2 - \lambda_2^*| + \lambda_2 |x(\gamma_2 s) - x^*(\gamma_2 s)|))ds \\
+ & \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [\delta + L(|x(s) - x^*(s)| + |x^*(\gamma_1 s)||\lambda_1 - \lambda_1^*| + \lambda_1 |x(\gamma_1 s) - x^*(\gamma_1 s)|)]ds \\
\leq & \ \frac{\delta\, t^{\alpha-1}}{\Gamma(\alpha)} + \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (\delta + K(|x(s) - x^*(s)| + |x^*(\gamma_2 s)|\delta + \lambda_2 |x(\gamma_2 s) - x^*(\gamma_2 s)|))ds \\
+ & \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (\delta + L(|x(s) - x^*(s)| + |x^*(\gamma_1 s)|\delta + \lambda_1 |x(\gamma_1 s) - x^*(\gamma_1 s)|))ds,
\end{aligned}
$$

and then

$$
\begin{aligned}
&\int_0^T |x(t)-x^*(t)|dt\\
\leq\ & \int_0^T (\frac{\delta\, t^{\alpha-1}}{\Gamma(\alpha)}+\frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)}\int_0^T (\delta + K(|x(s)-x^*(s)| + |x^*(\gamma_2 s)|\delta + \lambda_2|x(\gamma_2 s)-x^*(\gamma_2 s)|))ds\\
+\ & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(\delta + L(|x(s)-x^*(s)| + |x^*(\gamma_1 s)|\delta + \lambda_1|x(\gamma_1 s)-x^*(\gamma_1 s)|))ds)dt\\
\leq\ & \frac{\delta\, t^{\alpha-1}}{\Gamma(\alpha)}+\frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)}\int_0^T (\delta + K(|x(s)-x^*(s)| + |x^*(\gamma_2 s)|\delta + \lambda_2|x(\gamma_2 s)-x^*(\gamma_2 s)|))dsdt\\
+\ & \int_0^T\int_s^T \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(\delta + L(|x(s)-x^*(s)| + |x^*(\gamma_1 s)|\delta + \lambda_1|x(\gamma_1 s)-x^*(\gamma_1 s)|))dtds\\
\leq\ & \int_0^T (\frac{\delta\, t^{\alpha-1}}{\Gamma(\alpha)}+\frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)}\int_0^T (\delta + K(|x(s)-x^*(s)| + |x^*(\gamma_2 s)|\delta + \lambda_2|x(\gamma_2 s)-x^*(\gamma_2 s)|))dsdt\\
+\ & \int_0^T \frac{T^\alpha}{\Gamma(\alpha+1)}(\delta + L(|x(s)-x^*(s)| + |x^*(\gamma_1 s)|\delta + \lambda_1|x(\gamma_1 s)-x^*(\gamma_1 s)|))ds\\
\leq\ & \int_0^T \frac{\delta\, t^{\alpha-1}}{\Gamma(\alpha)}dt + \int_0^T \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)}dt(\int_0^T \delta\, ds + K(\int_0^T |x(s)-x^*(s)|ds\\
+\ & \delta\int_0^T |x^*(\gamma_2 s)|ds + \lambda_2\int_0^T |x^*(\gamma_2 s)-x^*(\gamma_2 s)|ds))\\
+\ & \frac{T^\alpha}{\Gamma(\alpha+1)}(\int_0^T \delta\, ds + L(\int_0^T |x(s)-x^*(s)|ds + \delta\int_0^T |x^*(\gamma_1 s)|ds + \lambda_1\int_0^T |x(\gamma_1 s)-x^*(\gamma_1 s)|ds))\\
\leq\ & \delta\int_0^T \frac{t^{\alpha-1}}{\Gamma(\alpha)}dt + \int_0^T \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)}dt(\int_0^T \delta\, ds + K(\int_0^T |x(s)-x^*(s)|ds\\
+\ & \frac{\delta}{\gamma_2}\int_0^T |x^*(\tau)|d\tau + \frac{\lambda_2}{\gamma_2}\int_0^T |x(\tau)-x^*(\tau)|d\tau))\\
+\ & \frac{T^\alpha}{\Gamma(\alpha+1)}(\int_0^T \delta\, ds + L(\int_0^T |x(s)-x^*(s)|ds + \frac{\delta}{\gamma_1}\int_0^T |x^*(\tau)|d\tau + \frac{\lambda_1}{\gamma_1}\int_0^T |x(\tau)-x^*(\tau)|d\tau)).
\end{aligned}
$$

Then

$$
\begin{aligned}
\|x(t)-x^*(t)\|_1 \leq\ & \frac{\delta\, T^\alpha}{\Gamma(\alpha+1)} + \frac{\beta\, T^\alpha}{\Gamma(\alpha+1)}(T\delta + K(\|x-x^*\|_1 + \frac{\delta}{\gamma_2}\|x^*\|_1 + \frac{\lambda_2}{\gamma_2}\|x-x^*\|_1))\\
+\ & \frac{T^\alpha}{\Gamma(\alpha+1)}(T\delta + L(\|x-x^*\|_1 + \frac{\delta}{\gamma_1}\|x^*\|_1 + \frac{\lambda_1}{\gamma_1}\|x-x^*\|_1))\\
\leq\ & \frac{\delta\, T^\alpha}{\Gamma(\alpha+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)}(T\beta\,\delta + \beta\, K(\|x-x^*\|_1 + \frac{\delta}{\gamma_2}r + \frac{\lambda_2}{\gamma_2}\|x-x^*\|_1))\\
+\ & \frac{T^\alpha}{\Gamma(\alpha+1)}(T\delta + L(\|x-x^*\|_1 + \frac{\delta}{\gamma_1}r + \frac{\lambda_1}{\gamma_1}\|x-x^*\|_1))\\
\leq\ & \frac{\delta\, T^\alpha}{\Gamma(\alpha+1)} + \frac{T^\alpha}{\Gamma(\alpha+1)}(T\beta\,\delta + T\delta + \frac{K\beta\,\delta}{\gamma_2}r + \frac{L\delta}{\gamma_1}r)\\
+\ & \frac{T^\alpha}{\Gamma(\alpha+1)}\|x-x^*\|_1(L + \beta\, K(1+\frac{\lambda_2}{\gamma_2}) + \frac{L\lambda_1}{\gamma_1}).
\end{aligned}
$$

Hence

$$\|x(t)-x^*(t)\|_1 \quad \leq \quad \frac{\frac{\delta\, T^\alpha}{\Gamma(\alpha+1)}(1+T\beta+T+\frac{K\beta}{\gamma_2}r+\frac{L}{\gamma_1}r)}{1-\frac{T^\alpha}{\Gamma(\alpha+1)}(\beta\, K(1\ +\frac{\lambda_2}{\gamma_2})+L(1+\frac{\lambda_1}{\gamma_1}))}.$$

Then

$$\|x(t)-x^*(t)\|_1 \quad \leq \quad \epsilon.$$

This finalizes the proof.

3.2. Hyres-Ulam stability

Many authors have studied and further developed the definition of Hyers-Ulam stability across various types of problems, see [1, 5, 16, 30]. In light of these definitions and based on the equivalence between the problems (1.1) and (1.2) or (1.1) and (1.3) and the integral Eq (2.1), we present the next definition of the Hyers-Ulam stability of the problems (1.1) and (1.2) or (1.1) and (1.3) as follows:

Definition 1. *Let the solution $x \in L_1(I)$ of (1.1) and (1.2) or (1.1) and (1.3) exist, and then the constrained problems (1.1) and (1.2) or (1.1) and (1.3) are Hyers-Ulam stable if $\forall\epsilon > 0\ \exists\ \delta(\epsilon)$ such that for any $\delta-$approximate solution x_s satisfies*

$$|\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0+\beta\int_0^T h(\theta,x(\theta),\lambda_2 x(\gamma_2\theta))d\theta)+\int_0^t\frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)}\,f(\theta,x(\theta),\lambda_1 x(\gamma_1\theta))d\theta - x_s(t)| < \delta, \tag{3.1}$$

and then $\|x-x_s\|_1 < \epsilon$.

Theorem 4. *Let the assumptions of Theorem 2 be satisfied, and then the constrained problems (1.1) and (1.2) or (1.1) and (1.3) are Hyers-Ulam stable.*

Proof. From (3.1), we have

$$-\delta < \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0+\beta\int_0^T h(\theta,x_s(\theta),\lambda_2 x_s(\gamma_2\theta))d\theta)+\int_0^t\frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)}\,f(\theta,x_s(\theta),\lambda_1 x_s(\gamma_1\theta))d\theta) - x_s(t) < \delta.$$

Now we have

$$\begin{aligned}
& |x(t)-x_s(t)| \\
= \ & |\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0+\beta\int_0^T h(\theta,x(\theta),\lambda_2 x(\gamma_2\theta))d\theta)+\int_0^t\frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)}f(\theta,x(\theta),\lambda_1 x(\gamma_1\theta))d\theta - x_s(t)| \\
\leq \ & |\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0+\beta\int_0^T h(\theta,x(\theta),\lambda_2 x(\gamma_2\theta))d\theta)+\int_0^t\frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)}\,f(\theta,x(\theta),\lambda_1 x(\gamma_1\theta))d\theta \\
- \ & \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0+\beta\int_0^T h(\theta,x_s(\theta),\lambda_2 x_s(\gamma_2\theta))d\theta)-\int_0^t\frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)}\,f(\theta,x_s(\theta),\lambda_1 x_s(\gamma_1\theta))d\theta \\
+ \ & \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0+\beta\int_0^T h(\theta,x_s(\theta),\lambda_2 x_s(\gamma_2\theta))d\theta)+\int_0^t\frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)}\,f(\theta,x_s(\theta),\lambda_1 x_s(\gamma_1\theta))d\theta - x_s(t)| \\
\leq \ & \delta + |\frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)}\int_0^T h(\theta,x(\theta),\lambda_2 x(\gamma_2\theta))d\theta+\int_0^t\frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)}\,f(\theta,x(\theta),\lambda_1 x(\gamma_1\theta))d\theta
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T h(\theta, x_s(\theta), \lambda_2 x_s(\gamma_2\theta))d\theta - \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} f(\theta, x_s(\theta), \lambda_1 x_s(\gamma_1\theta))d\theta| \\
&\leq \delta + \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T |h(\theta, x(\theta), \lambda_2 x(\gamma_2\theta)) - h(\theta, x_s(\theta), \lambda_2 x_s(\gamma_2\theta))|d\theta \\
&+ \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} |f(\theta, x(\theta), \lambda_1 x(\gamma_1\theta)) - f(\theta, x_s(\theta), \lambda_1 x_s(\gamma_1\theta))|d\theta \\
&\leq \delta + \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (K(|x(\theta) - x_s(\theta)| + \lambda_2|x(\gamma_2\theta) - x_s(\gamma_2\theta)|))d\theta \\
&+ \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} (L(|x(\theta) - x_s(\theta)| + \lambda_1|x(\gamma_1\theta) - x_s(\gamma_1\theta)|))d\theta,
\end{aligned}
$$

and then

$$
\begin{aligned}
&\int_0^T |x(t) - x_s(t)|dt \\
&\leq \int_0^T [\delta + \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} \int_0^T (K(|x(\theta) - x_s(\theta)| + \lambda_2|x(\gamma_2\theta) - x_s(\gamma_2\theta)|))d\theta \\
&+ \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} (L(|x(\theta) - x_s(\theta)| + \lambda_1|x(\gamma_1\theta) - x_s(\gamma_1\theta)|))d\theta]dt \\
&\leq \delta\, T + \int_0^T \frac{\beta\, t^{\alpha-1}}{\Gamma(\alpha)} dt \int_0^T (K(|x(\theta) - x_s(\theta)| + \lambda_2|x(\gamma_2\theta) - x_s(\gamma_2\theta)|))d\theta \\
&+ \int_0^T \int_\theta^T \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} (L(|x(\theta) - x_s(\theta)| + \lambda_1|x(\gamma_1\theta) - x_s(\gamma_1\theta)|))dt d\theta \\
&\leq \delta\, T + \frac{\beta\, T^{\alpha}}{\Gamma(\alpha+1)} \int_0^T (K(|x(\theta) - x_s(\theta)| + \lambda_2|x(\gamma_2\theta) - x_s(\gamma_2\theta)|))d\theta \\
&+ \int_0^T \frac{T^{\alpha}}{\Gamma(\alpha+1)} L(|x(\theta) - x_s(\theta)| + \lambda_1|x(\gamma_1\theta) - x_s(\gamma_1\theta)|)d\theta \\
&\leq \delta\, T + \frac{\beta\, T^{\alpha}}{\Gamma(\alpha+1)} K(\int_0^T |x(\theta) - x_s(\theta)|d\theta + \lambda_2 \int_0^T |x(\gamma_2\theta) - x_s(\gamma_2\theta)|d\theta) \\
&+ \frac{T^{\alpha}}{\Gamma(\alpha+1)} L(\int_0^T |x(\theta) - x_s(\theta)|d\theta + \lambda_1 \int_0^T |x(\gamma_1\theta) - x_s(\gamma_1\theta)|d\theta) \\
&\leq \delta\, T + \frac{\beta\, T^{\alpha}}{\Gamma(\alpha+1)} K(\int_0^T |x(\theta) - x_s(\theta)|d\theta + \frac{\lambda_2}{\gamma_2} \int_0^T |x(\tau) - x_s(\tau)|d\tau) \\
&+ \frac{T^{\alpha}}{\Gamma(\alpha+1)} L(\int_0^T |x(\theta) - x_s(\theta)|d\theta + \frac{\lambda_1}{\gamma_1} \int_0^T |x(\tau) - x_s(\tau)|d\tau) \\
&\leq \delta\, T + \frac{\beta\, T^{\alpha}}{\Gamma(\alpha+1)} K(\|x - x_s\|_1 + \frac{\lambda_2}{\gamma_2}\|x - x_s\|_1) \\
&+ \frac{T^{\alpha}}{\Gamma(\alpha+1)} L(\|x - x_s\|_1 + \frac{\lambda_1}{\gamma_1}\, \|x - x_s\|_1) \\
&\leq \quad T + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|x - x_s\|_1 (\beta\, K + \frac{\beta\lambda_2 K}{\gamma_2} + L + \frac{L\lambda_1}{\gamma_1}),
\end{aligned}
$$

$$\|x - x_s\|_1 \leq \frac{\delta\, T}{1 - \frac{T^\alpha}{\Gamma(\alpha+1)}(L(1 + \frac{\lambda_1}{\gamma_1}) + \beta\, K(1 + \frac{\lambda_2}{\gamma_2}))}.$$

Thus

$$\|x - x_s\|_1 \leq \epsilon.$$

4. Examples

Example 1. Consider the following fractional-order-pantograph differential equation:

$$^R D^{\frac{1}{5}} x(t) = \frac{e^{-t}}{t+1} + \frac{1}{8}(x(t) + \frac{1}{2}x(\frac{1}{4}t)), \quad a.e. \;\; t \in (0, 3] \tag{4.1}$$

subject to the nonlocal and weighted pantograph integral constraints

$$I^{\frac{4}{5}} x(t)|_{t=0} = \frac{1}{7} + \frac{1}{6}\int_0^3 (\frac{s}{s^2+1} sins + \frac{1}{4}(x(s) + \frac{1}{3}x(\frac{1}{5}s)))ds \tag{4.2}$$

or

$$t^{\frac{4}{5}} x(t)|_{t=0} = \frac{1}{\Gamma(\frac{1}{5})}(\frac{1}{7} + \frac{1}{6}\int_0^3 (\frac{s}{s^2+1} sins + \frac{1}{4}(x(s) + \frac{1}{3}x(\frac{1}{5}s)))ds). \tag{4.3}$$

This problem can be expressed by the fractional-order integral equation

$$x(t) = \frac{t^{\frac{-4}{5}}}{\Gamma(\frac{1}{5})}(\frac{1}{7} + \frac{1}{6}\int_0^3 (\frac{s}{s^2+1}\; sins + \frac{1}{4}(x(s) + \frac{1}{3}x(\frac{1}{5}s)))ds) + I^{\frac{1}{5}}\; \frac{e^{-t}}{t+1} + \frac{1}{8}(x(t) + \frac{1}{2}x(\frac{1}{4}t)). \tag{4.4}$$

Set

$$f(t, x(t), \lambda_1 x(\gamma_1 t)) = \frac{e^{-t}}{t+1} + \frac{1}{8}(x(t) + \frac{1}{2}x(\frac{1}{4}t)),$$

$$|f(t, x, y)| \leq \frac{1}{t+1} + \frac{1}{8}(|x| + |y|).$$

Similarly,

$$h(t, x(t), \lambda_2 x(\gamma_2 t)) = \frac{t}{t^2+1} sint + \frac{1}{4}(x(t) + \frac{1}{3}x(\frac{1}{5}t)),$$

$$|h(t, x, y)| \leq \frac{t}{t^2+1} + \frac{1}{4}(|x| + |y|),$$

where

$$a_1(t) = \frac{1}{t+1} \quad and \quad \|a_1\| = \int_0^3 \frac{1}{s+1} ds = \ln(4),$$

$$a_2(t) = \frac{t}{t^2+1} \quad and \quad \|a_2\| = \int_0^3 \frac{s}{s^2+1} ds = \frac{\ln(10)}{2}.$$

Now, we have $T = 3,\ \alpha = \frac{1}{5},\ x_0 = \frac{1}{7},\ \lambda_1 = \frac{1}{2},\ \gamma_1 = \frac{1}{4},\ \lambda_2 = \frac{1}{3},\ \gamma_2 = \frac{1}{5},\ \beta = \frac{1}{6},\ L = \frac{1}{8},\ K = \frac{1}{4},\ and\ r =$ 6.85831695596841.

Then

$$\frac{T^{\alpha}}{\Gamma(\alpha+1)}(L(1+\frac{\lambda_1}{\gamma_1})+\beta\, K(1+\frac{\lambda_2}{\gamma_2})) = 0.6595341610293 < 1.$$

Now all the conditions of Theorem 1 are satisfied, and then the problems (4.1) and (4.2) or (4.1) and (4.3) have at least one solution $x \in L_1[0,3]$. Moreover,

$$\begin{aligned} |f(t,x,y)-f(t,\bar{x},\bar{y}| &\leq \frac{1}{8}(|x-\bar{x}|+|y-\bar{y}|), \\ |h(t,x,y)-h(t,\bar{x},\bar{y}| &\leq \frac{1}{4}(|x-\bar{x}|+|y-\bar{y}|),\ \forall\, t\in I,\ x,\ y\in R. \end{aligned} \tag{4.5}$$

Then the solution of the problems (4.1) and (4.2) or (4.1) and (4.3) is unique.

Example 2. Consider the following fractional order-pantograph differential equation:

$$^{R}D^{\frac{1}{2}}x(t) = \frac{e^{-t}sint}{1+10t}+\frac{1}{12}(x(t)+\frac{1}{12}x(\frac{1}{8}t)), \quad a.e.\ \ t\in(0,\frac{1}{2}] \tag{4.6}$$

subject to the nonlocal and weighted pantograph integral constraints

$$I^{\frac{1}{2}}x(t)|_{t=0} = \frac{1}{8}+\frac{1}{16}\int_0^{\frac{1}{2}}(\frac{sins}{7}+\frac{1}{9}(x(s)+\frac{1}{9}x(\frac{1}{3}s)))ds, \tag{4.7}$$

or

$$t^{\frac{1}{2}}x(t)|_{t=0} = \frac{1}{\Gamma(\frac{1}{2})}(\frac{1}{8}+\frac{1}{16}\int_0^{\frac{1}{2}}(\frac{sins}{7}+\frac{1}{9}(x(s)+\frac{1}{9}x(\frac{1}{3}s)))ds). \tag{4.8}$$

This problem can be expressed by the fractional-order integral equation

$$x(t) = \frac{t^{\frac{-1}{2}}}{\Gamma(\frac{1}{2})}(\frac{1}{8}+\frac{1}{16}\int_0^{\frac{1}{2}}(\frac{sins}{7}+\frac{1}{9}(x(s)+\frac{1}{9}x(\frac{1}{3}s))ds)+I^{\frac{1}{2}}\,\frac{e^{-t}sint}{1+10t}+\frac{1}{12}(x(t)+\frac{1}{12}x(\frac{1}{8}t)). \tag{4.9}$$

Set

$$f(t,x(t),\lambda_1 x(\gamma_1 t)) = \frac{e^{-t}sint}{1+10t}+\frac{1}{12}(x(t)+\frac{1}{12}x(\frac{1}{8}t)),$$

$$|f(t,x,y)| \leq \frac{1}{1+10t}+\frac{1}{12}(|x|+|y|).$$

Also

$$h(t,x(t),\lambda_2 x(\gamma_2 t)) = \frac{sint}{7}+\frac{1}{9}(x(t)+\frac{1}{9}x(\frac{1}{3}t)),$$

$$|h(t,x,y)| \leq \frac{1}{7}+\frac{1}{9}(|x|+|y|),$$

where

$$a_1(t) = \frac{1}{1+10t} \quad and \quad ||a_1|| = \int_0^{\frac{1}{2}}\frac{1}{1+10s}ds = \frac{\ln(6)}{10},$$

$$a_2(t) = \frac{1}{7} \quad and \quad ||a_2|| = \int_0^{\frac{1}{2}}\frac{1}{7}\,ds = \frac{1}{14}.$$

Now, we have $T = \frac{1}{2}$, $\alpha = \frac{1}{2}$, $x_0 = \frac{1}{8}$, $\lambda_1 = \frac{1}{12}$, $\gamma_1 = \frac{1}{8}$, $\lambda_2 = \frac{1}{9}$, $\gamma_2 = \frac{1}{3}$, $\beta = \frac{1}{16}$, $L = \frac{1}{12}$, $K = \frac{1}{9}$, *and* $r = 0.39494809573195$.

Then

$$\frac{T^{\alpha}}{\Gamma(\alpha+1)}(L(1+\frac{\lambda_1}{\gamma_1})+\beta K(1+\frac{\lambda_2}{\gamma_2})) = 0.118205120118943 < 1.$$

Now all the conditions of Theorem 1 are satisfied, then and the problems (4.6) and (4.7) or (4.6) and (4.8) have at least one solution $x \in L_1[0, \frac{1}{2}]$. Moreover,

$$\begin{aligned} |f(t,x,y) - f(t,\bar{x},\bar{y}| &\leq \frac{1}{12}(|x-\bar{x}|+|y-\bar{y}|), \qquad (4.10)\\ |h(t,x,y) - h(t,\bar{x},\bar{y}| &\leq \frac{1}{9}(|x-\bar{x}|+|y-\bar{y}|),\ \forall\, t\in I,\ x,\ y\in R. \end{aligned}$$

Then the solution of problems (4.6) and (4.7) or (4.6) and (4.8) is unique.

Example 3. Consider the following fractional-order pantograph differential equation:

$$^{R}D^{\frac{1}{4}}x(t) = \frac{e^{-t}}{8-t} + \frac{1}{14}(x(t)+\frac{1}{2}x(\frac{1}{5}t)), \quad a.e.\ t\in (0,1] \qquad (4.11)$$

subject to the nonlocal and weighted pantograph integral constraints

$$I^{\frac{3}{4}}x(t)|_{t=0} = \frac{1}{9}+\frac{1}{6}\int_0^1 (s^2e^{-s}cos^2s+\frac{1}{6}(x(s)+\frac{1}{7}x(\frac{1}{3}s)))ds, \qquad (4.12)$$

or

$$t^{\frac{3}{4}}x(t)|_{t=0} = \frac{1}{\Gamma(\frac{1}{4})}(\frac{1}{9}+\frac{1}{6}\int_0^1 (s^2e^{-s}cos^2s+\frac{1}{6}(x(s)+\frac{1}{7}x(\frac{1}{3}s)))ds). \qquad (4.13)$$

This problem can be expressed by the fractional-order integral equation

$$x(t) = \frac{t^{\frac{-3}{4}}}{\Gamma(\frac{1}{4})}(\frac{1}{9}+\frac{1}{6}\int_0^1 (s^2e^{-s}cos^2s+\frac{1}{6}(x(s)+\frac{1}{7}x(\frac{1}{3}s)))ds) + I^{\frac{1}{4}}\ \frac{e^{-t}}{8-t}+\frac{1}{14}(x(t)+\frac{1}{2}x(\frac{1}{5}t)). \qquad (4.14)$$

Set

$$f(t,x(t),\lambda_1 x(\gamma_1 t)) = \frac{e^{-t}}{8-t}+\frac{1}{14}(x(t)+\frac{1}{2}x(\frac{1}{5}t)),$$

$$|f(t,x,y)| \leq \frac{1}{8-t}+\frac{1}{14}(|x|+|y|),$$

and

$$h(t,x(t),\lambda_2 x(\gamma_2 t)) = t^2e^{-t}cos^2t+\frac{1}{6}(x(t)+\frac{1}{7}x(\frac{1}{3})),$$

$$|h(t,x,y)| \leq t^2+\frac{1}{6}(|x|+|y|),$$

where

$$a_1(t) = \frac{1}{8-t} \quad and \quad \|a_1\| = \int_0^1 \frac{1}{8-s}ds = \ln(8)-\ln(7),$$

$$a_2(t) = t^2 \quad and \quad \|a_2\| = \int_0^1 s^2 ds = \frac{1}{3}.$$

Now, we have $T = 1,\ \alpha = \frac{1}{4},\ x_0 = \frac{1}{9},\ \lambda_1 = \frac{1}{2},\ \gamma_1 = \frac{1}{5},\ \lambda_2 = \frac{1}{7},\ \gamma_2 = \frac{1}{3},\ \beta = \frac{1}{6},\ L = \frac{1}{14},\ K = \frac{1}{6},\ and\ r =$ 0.486765614502368.
Then

$$\frac{T^\alpha}{\Gamma(\alpha+1)}(L(1+\frac{\lambda_1}{\gamma_1}) + \beta\, K(1+\frac{\lambda_2}{\gamma_2})) = 0.331197306814979 < \ 1.$$

Now all the conditions of Theorem 1 are satisfied, then and the problems (4.11) and (4.12) or (4.11) and (4.13) have at least one solution $x \in \ L_1[0,1]$. Moreover,

$$\begin{aligned} |f(t,x,y) - f(t,\bar{x},\bar{y}| &\leq \frac{1}{14}\,(|x-\bar{x}| + |y-\bar{y}|), \quad (4.15)\\ |h(t,x,y) - h(t,\bar{x},\bar{y}| &\leq \frac{1}{6}\,(|x-\bar{x}| + |y-\bar{y}|),\ \forall\, t \in I,\ x,\ y \in R. \end{aligned}$$

Then the solution of the problems (4.11) and (4.12) or (4.11) and (4.13) is unique.
Example 4. Consider the following fractional-order pantograph differential equation:

$$^{R}D^{\frac{1}{10}}x(t) = \frac{1+2t}{15} + \frac{1}{9}(\frac{x^2(t)}{1+|x(t)|} + \frac{e^{-t}sinx(\frac{1}{4}t)}{3}), \quad a.e.\ \ t \in \ (0,1] \tag{4.16}$$

subject to the nonlocal and weighted pantograph integral constraints

$$I^{\frac{9}{10}}x(t)|_{t=0} = \frac{1}{6} + \frac{1}{4}\int_0^1 (\frac{e^{-5s}sin(5(s+1))}{5-s} + \frac{1}{16}(\ln(1+|x(s)|) + \frac{1}{2}\ \frac{x(\frac{1}{2}s)}{1+|x(\frac{1}{2}s)|}))ds, \tag{4.17}$$

or

$$t^{\frac{9}{10}}x(t)|_{t=0} = \frac{1}{\Gamma(\frac{1}{10})}(\frac{1}{6} + \frac{1}{4}\int_0^1 (\frac{e^{-5s}sin(5(s+1))}{5-s} + \frac{1}{16}(\ln(1+|x(s)|) + \frac{1}{2}\ \frac{x(\frac{1}{2}s)}{1+|x(\frac{1}{2}s)|}))ds). \tag{4.18}$$

This problem can be expressed by the fractional-order integral equation

$$\begin{aligned} x(t) = &\frac{t^{\frac{-9}{10}}}{\Gamma(\frac{1}{10})}(\frac{1}{6} + \frac{1}{4}\int_0^1 (\frac{e^{-5s}sin(5(s+1))}{5-s} + \frac{1}{16}(\ln(1+|x(s)|) + \frac{1}{2}\ \frac{x(\frac{1}{2}s)}{1+|x(\frac{1}{2}s)|}))ds)\\ &+ \ I^{\frac{1}{10}}\ \frac{1+2t}{15} + \frac{1}{9}(\frac{x^2(t)}{1+|x(t)|} + \frac{e^{-t}sinx(\frac{1}{4}t)}{3}). \end{aligned} \tag{4.19}$$

Set

$$f(t,x(t),\lambda_1 x(\gamma_1 t)) = \frac{1+2t}{15} + \frac{1}{9}(\frac{x^2(t)}{1+|x(t)|} + \frac{e^{-t}sinx(\frac{1}{4}t)}{3}),$$

$$|f(t,x,y)| \leq 1 + 2t + \frac{1}{9}(|x| + |y|).$$

Also

$$h(t,x(t),\lambda_2 x(\gamma_2 t)) = \frac{e^{-5t}sin(5(t+1))}{5-t} + \frac{1}{16}(\ln(1+|x(t)|) + \frac{1}{2}\ \frac{x(\frac{1}{2}t)}{1+|x(\frac{1}{2}t)|}),$$

$$|h(t,x,y)| \leq \frac{1}{5-t} + \frac{1}{16}(|x|+|y|),$$

where

$$a_1(t) = 1+2t \quad and \quad \|a_1\| = \int_0^1 (1+2s)ds = 2,$$

$$a_2(t) = \frac{1}{5-t} \quad and \quad \|a_2\| = \int_0^1 \frac{1}{5-s}ds = \ln(5) - \ln(4).$$

Now, we have $T = 1$, $\alpha = \frac{1}{10}$, $x_0 = \frac{1}{6}$, $\lambda_1 = \frac{1}{3}$, $\gamma_1 = \frac{1}{4}$, $\lambda_2 = \frac{1}{2}$, $\gamma_2 = \frac{1}{2}$, $\beta = \frac{1}{4}$, $L = \frac{1}{9}$, $K = \frac{1}{16}$, *and* $r = 3.36281387755$.
Then

$$\frac{T^\alpha}{\Gamma(\alpha+1)}(L(1+\frac{\lambda_1}{\gamma_1}) + \beta\, K(1+\frac{\lambda_2}{\gamma_2})) = 0.3053650330255 < 1.$$

Now all the conditions of Theorem 1 are satisfied, then and the problems (4.16) and (4.17) or (4.16) and (4.18) have at least one solution $x \in L_1[0,1]$.

5. Conclusions

Fractional-order derivatives, which extend the concept of classical derivatives to non-integer orders, can raise a variety of theoretical and practical problems. Several theoretical frameworks and methodologies are used to establish the existence and uniqueness of fractional differential equation solutions. Stability analysis is a broad and diverse field with strong theoretical foundations and various applications in engineering, economics, biology, physics, and other disciplines. Hyers-Ulam stability assesses a problem's resilience to interruptions, while continuous dependency analyzes how modest parameter changes impact the problem's unique solution. In this investigation, the Riemann-Liouville fractional-order pantograph differential equation is constrained by nonlocal and weighted pantograph integral equations. We discussed the existence of an integrable solution of the Riemann-Liouville fractional-order differential equation (1.1) subject to each one of the nonlocal and weighted pantograph integral constraints (1.2) or (1.3) by applying the technique of Schauder's fixed point theorem and Kolmogorov's compactness criterion. Moreover, we established sufficient conditions to guarantee the uniqueness of the solution. We also studied the continuous dependence on the functions f, h and the parameters λ_i, $i = 1, 2$. Moreover, we thoroughly investigated the Hyers-Ulam stability of the constrained problems (1.1) and (1.2) or (1.1) and (1.3). Finally, some examples were provided to illustrate our results.

Author contributions

The authors contributed equally to this paper.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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