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Research article

Vanishing viscosity limit to the planar rarefaction wave for the two-dimensional radiative hydrodynamics

Guangrong Ren*

School of Mathematics, Hefei University of Technology, Hefei 230009, China

* **Correspondence:** Email: 2022111444@mail.hfut.edu.cn.

Abstract: In this paper, we are concerned with the vanishing viscosity problem in two-dimensional radiative hydrodynamics. We prove that two-dimensional radiative hydrodynamics converge to the planar rarefaction wave solution for the corresponding two-dimensional compressible Euler equations. By introducing a different scaling and identifying cancellations within the flux terms, we establish a new convergence rate with the assistance of detailed energy estimates.

Keywords: Navier-Stokes equations; radiation term; planar rarefaction wave; vanishing viscosity
Mathematics Subject Classification: 35Q35, 76N10, 35M10

1. Introduction

The two-dimensional radiative hydrodynamics are formulated as follows:

$$\begin{cases} \rho_t + \mathrm{div}(\rho \boldsymbol{u}) = 0, \\ (\rho \boldsymbol{u})_t + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \mathrm{div}\mathcal{T}, \\ (\rho E)_t + \mathrm{div}(\rho E \boldsymbol{u} + p\boldsymbol{u}) + \mathrm{div}\boldsymbol{q} = \kappa_1 \Delta\theta + \mathrm{div}(\boldsymbol{u}\mathcal{T}), \\ a_1 \boldsymbol{q} + b_1 \nabla\theta^4 = \nabla \mathrm{div}\boldsymbol{q}, \end{cases} \tag{1.1}$$

where $\rho \geq 0, \boldsymbol{u} := (u_1, u_2)^{\mathrm{T}}$, p and θ represent the fluid's density, velocity, pressure, and absolute temperature, $\boldsymbol{q} := (q_1, q_2)^{\mathrm{T}}$ is the radiative heat flux. Moreover, $E := e + \frac{1}{2}|\boldsymbol{u}|^2$ is the specific total energy with the internal energy e. All variables mentioned above depend on t and $\boldsymbol{x}$, where t is time and $\boldsymbol{x} := (x_1, x_2) \in \Omega$ is the spatial variable. Here, we are concerned with the viscous fluid flow in an infinitely long flat nozzle domain $\Omega := \mathbb{R} \times \mathbb{T}$ with a real line $\mathbb{R}$ and a one-dimensional unit flat torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$.

The viscous stress tensor $\mathcal{T}$ is given by

$$\mathcal{T} = 2\mu_1 \mathbb{D}(\boldsymbol{u}) + \lambda_1 \mathrm{div}\boldsymbol{u}\mathbb{I},$$

where $\mathbb{D}(\boldsymbol{u}) = \frac{\nabla \boldsymbol{u}+(\nabla \boldsymbol{u})^{\top}}{2}$ stands for the deformation tensor, $\mathbb{I}$ represents the 2×2 identity matrix, parameters μ_1 and λ_1 represent the shear and bulk viscosity coefficients of the fluid, respectively, and they both are constants satisfying

$$\mu_1 > 0, \quad \mu_1 + \lambda_1 \geq 0. \tag{1.2}$$

Moreover, the constant $\kappa_1 > 0$ denotes the heat-conductivity coefficient, and $a_1 > 0, b_1 > 0$ depend only on the fluid itself.

Setting

$$\mu_1 = \mu\epsilon, \quad \lambda_1 = \lambda\epsilon, \quad \kappa_1 = \kappa\epsilon, \quad a_1 = a\epsilon^{-2}, \quad b_1 = b\epsilon^{-1}, \tag{1.3}$$

where $\epsilon > 0$ is the vanishing parameter, and constant $\mu, \lambda, \kappa, a, b$ are the prescribed uniformly in parameter ϵ. Such a setting for a_1 and b_1 is borrowed from [1]. Specifically, we investigate the ideal polytropic fluids such that the pressure p and the internal energy e are given by the following state equations:

$$p = R\rho\theta = A\rho^{\gamma}\exp\left(\frac{\gamma-1}{R}S\right), e = \frac{R}{\gamma-1}\theta,$$

where S is the entropy, $\gamma > 1$ is the adiabatic exponent, and both A and R are positive constants.

Under the Assumption (1.3), letting ϵ tend to zero, we formally derive that the solutions to the two-dimensional compressible Navier-Stokes equations with radiation term (1.1) converge to the corresponding two-dimensional compressible Euler equations

$$\begin{cases} \rho_t + \operatorname{div}(\rho\boldsymbol{u}) = 0, \\ (\rho\boldsymbol{u})_t + \operatorname{div}(\rho\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = 0, \\ (\rho E)_t + \operatorname{div}(\rho E\boldsymbol{u} + p\boldsymbol{u}) = 0. \end{cases} \tag{1.4}$$

Literature review. The vanishing viscosity limit for the compressible Navier-Stokes equations to the Euler equations with basic wave patterns has been extensively investigated. In [2], the zero-dissipation limit problem for the compressible and isentropic Navier-Stokes equations to the corresponding Euler equations with rarefaction wave solutions was considered. The solution to the compressible isentropic Navier-Stokes equations with shock data converges to the inviscid shock as the viscosity tends to zero and was derived in [3]. Concerning the Riemann solution to the Euler equations, which consists of the superposition of basic waves, such a vanishing viscosity limit problem was also investigated. This includes the superposition of a shock wave and a rarefaction wave in [4], and the superposition of rarefaction waves and contact discontinuity in [5]. For more results on the vanishing viscosity limit problem, further references can be found in [6–9] and the references therein. Up to now, all the results mentioned above are related to one-dimensional case. However, in thr high-dimensional case, the vanishing viscosity limit of the two-dimensional compressible and isentropic Navier-Stokes equations to the Euler equations with a planar rarefaction wave solution was studied in [10]; the three-dimensional compressible Navier-Stokes-Fourier equations were further considered in [11]; and the two dimensional full compressible Navier-Stokes equations were derived in [12]. The planar rarefaction wave to the two-dimensional compressible and isentropic Navier-Stokes equations was given in [13]; subsequently, the planar rarefaction wave for the three-dimensional full, compressible Navier–Stokes equations with the heat conductivities in an infinitely long flat nozzle domain was investigated in [14]. Very recently, the vanishing viscosity limit to the planar rarefaction wave with vacuum for the three-dimensional full compressible Navier-Stokes equations with temperature-dependent transport coefficients was presented in [15]. Nowadays, the nonlinear stability for the radiative hydrodynamics has been studied very thoroughly. The composite wave of two viscous shock waves for the one-dimensional radiative Euler equations was given in [16]; then the rarefaction wave case was addressed in [17] for the inflow problem and in [18] for the outflow problem.

We consider the two-dimensional radiative hydrodynamics (1.1) with the following initial data:

$$(\rho, \boldsymbol{u}, \theta)(0, x_1, x_2) = (\rho, u_1, u_2, \theta)(0, x_1, x_2) = (\rho_0, u_{10}, u_{20}, \theta_0)(x_1, x_2) \tag{1.5}$$

and the far field conditions of the solutions in the x_1-direction:

$$(\rho, u_1, u_2, \theta)(t, x_1, x_2) \to (\rho_\pm, u_{1\pm}, 0, \theta_\pm), \qquad \text{as} \quad x_1 \to \pm\infty \tag{1.6}$$

where $\rho_\pm > 0, u_{1\pm}, \theta_\pm$ are the prescribed constants. The periodic boundary condition is imposed on $x_2 \in \mathbb{T}$ for the solution $(\rho, u_1, u_2, \theta)(t, x_1, x_2)$ to (1.1), and the end states $(\rho_\pm, u_{1\pm}, \theta_\pm)$ are connected by the rarefaction wave solution to the Riemann problem of the corresponding one-dimensional compressible Euler system:

$$\begin{cases} \rho_t + (\rho u_1)_{x_1} = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_{x_1} = 0, \\ (\rho E)_t + (\rho E u_1 + p u_1)_{x_1} = 0, \end{cases} \tag{1.7}$$

with the Riemann initial data

$$(\rho_0^r, u_{10}^r, \theta_0^r)(x_1) = \begin{cases} (\rho_-, u_{1-}, \theta_-), & x_1 < 0, \\ (\rho_+, u_{1+}, \theta_+), & x_1 > 0. \end{cases} \tag{1.8}$$

With the above assumptions in hand, we could expect that as $\epsilon \to 0$, the solutions to the compressible Navier-Stokes equations (1.1), (1.5) ,and (1.6) converge to the corresponding planar rarefaction wave for the two-dimensional compressible Euler equations (1.4) with the following Riemann initial data:

$$(\rho, u, \theta)(0, x_1, x_2) = (\rho_0^r, u_{10}^r, u_{20}^r, \theta_0^r)(x_1) = \begin{cases} (\rho_-, u_{1-}, 0, \theta_-), & x_1 < 0, \\ (\rho_+, u_{1+}, 0, \theta_+), & x_1 > 0. \end{cases} \tag{1.9}$$

Direct calculations reveal that the solution (ρ, u_1, θ) to the Euler system (1.7) has three distinct eigenvalues

$$\lambda_i(\rho, u_1, S) = u_1 + (-1)^{\frac{i+1}{2}} \sqrt{p_\rho(\rho, S)}, i = 1, 3, \quad \lambda_2(\rho, u_1, S) = u_1.$$

What's more, three corresponding right eigenvectors are denoted by $\gamma_i(\rho, u_1, S)$, it has

$$\gamma_i(\rho, u_1, S) \cdot \nabla_{(\rho, u_1, S)} \lambda_i(\rho, u_1, S) \neq 0, \quad i = 1, 3,$$

and

$$\gamma_2(\rho, u_1, S) \cdot \nabla_{(\rho, u_1, S)} \lambda_2(\rho, u_1, S) \equiv 0.$$

Furthermore, the two i-Riemann invariants $\Sigma_i^j(\rho, u_1, S)$ $(i = 1, 3, j = 1, 2)$ to the Euler system (1.7) can be given by

$$\Sigma_i^{(1)} = u_1 + (-1)^{\frac{i-1}{2}} \int^\rho \frac{\sqrt{p_z(z, S)}}{z} \mathrm{d}z, \quad \Sigma_i^{(2)} = S, \tag{1.10}$$

satisfying $\nabla_{(\rho,u_1,S)}\Sigma_i^j(\rho,u_1,S)\cdot\gamma_i(\rho,u_1,S)\equiv 0\ \ (i=1,3,\ j=1,2)$ for all $\rho>0,u_1$ and S.

In this paper, we mainly consider the 3-rarefaction wave to the Euler systems (1.7) and (1.8). Actually, we also can manage the 1-rarefaction wave similarly without any substantial difference. For the given right state (ρ_+,u_{1+},θ_+) with $\rho_+>0,\theta_+>0$, we know that when $(\rho_-,u_{1-},\theta_-)\in R_3(\rho_+,u_{1+},\theta_+)$, where

$$R_3(\rho_+,u_{1+},\theta_+):=\left\{(\rho,u_1,\theta)|\lambda_{3x_1}(\rho,u_1,S)>0,\Sigma_3^j=\Sigma_3^{(j)}(\rho_+,u_{1+},S_+),j=1,2\right\},$$

the Euler systems (1.7) and (1.8) admit a 3-rarefaction wave $(\rho^r,u_1^r,\theta^r)(\frac{x_1}{t})$. For more details about basic waves, interested readers can refer to [19, 20]. Subsequently, the above analysis allows us to define the planar rarefaction wave solution to the two-dimensional compressible Euler equations (1.4) with initial data (1.5) and (1.6) as $(\rho^r,\boldsymbol{u}^r,\theta^r)(\frac{x_1}{t})=(\rho^r,u_1^r,0,\theta^r)(\frac{x_1}{t})$.

Our main result is given as follows:

Theorem 1.1. *Let $(\rho^r,\boldsymbol{u}^r,\theta^r)(\frac{x_1}{t})$ be the planar 3-rarefaction wave to the two-dimensional compressible Euler system* (1.4) *with Riemann initial data* (1.5)*. Then there exists a constant $\epsilon_0>0$ such that for any $\epsilon\in(0,\epsilon_0)$, we are able to construct smooth solutions $(\rho^\epsilon,\boldsymbol{u}^\epsilon,\theta^\epsilon)$ up to any arbitrarily large but fixed time T for the system* (1.1) *satisfying*

$$\begin{cases}(\rho^\epsilon-\rho^r,u_1^\epsilon-u_1^r,u_2^\epsilon,\theta^\epsilon-\theta^r)\in C^0(0,T;L^2(\Omega)),\\ (\nabla\rho^\epsilon,\nabla\boldsymbol{u}^\epsilon,\nabla\theta^\epsilon)\in C^0(0,T;H^1(\Omega)),\quad \boldsymbol{q}^\epsilon\in H^2(\Omega),\quad \mathrm{div}\boldsymbol{q}^\epsilon\in H^2(\Omega),\\ (\nabla^3\boldsymbol{u}^\epsilon,\nabla^3\theta^\epsilon)\in L^2(0,T;L^2(\Omega)).\end{cases}$$

In addition, for each small constant $\underline{T}>0$, there is an independent constant $C_{\underline{T},T}>0$ of ϵ, such that

$$\sup_{\underline{T}\le t\le T}\left\|(\rho^\epsilon,\boldsymbol{u}^\epsilon,\theta^\epsilon)(t,x_1,x_2)-(\rho^r,\boldsymbol{u}^r,\theta^r)\left(\frac{x_1}{t}\right)\right\|_{L^\infty(\Omega)}\le C_{\underline{T},T}\epsilon^\omega|\ln\epsilon|^2, \tag{1.11}$$

where $\omega=\frac{1}{4}$.

Moreover, letting $\epsilon\to0$, it holds that the smooth solution $(\rho^\epsilon,\boldsymbol{u}^\epsilon,\theta^\epsilon)$ converges to the planar rarefaction wave fan $(\rho^r,\boldsymbol{u}^r,\theta^r)(\frac{x_1}{t})$ point wisely, except for $(0,0)$*, and*

$$(\rho^\epsilon,\boldsymbol{u}^\epsilon,\theta^\epsilon)\to(\rho^r,\boldsymbol{u}^r,\theta^r)\left(\frac{x_1}{t}\right),\quad \text{a.e. in}\quad \mathbb{R}^+\times\Omega.$$

We make some comments on our analysis. Based on the observations of cancellations between the flux terms and viscosity terms for full compressible Navier–Stokes equations in [12], this method can also be used in the full system of hydrodynamic equations coupled with a nonlinear elliptic equation. Moreover, we will introduce the new cancellations considering the radiative term. Our concern lies in the convergence rate from the compressible radiative hydrodynamics to the planar rarefaction wave solution for the corresponding Euler equations. Instead of specifying the order of scaling, we provide a broad range of orders. Subsequently, under our chosen scaling setting, we derive a rate in (1.11) through elaborate energy estimates. This demonstrates that our result serves as a generalization of [11, 12] to radiative hydrodynamics.

The rest of the paper is organized as follows: in section 2, we present the smooth approximate rarefaction wave to the Euler equation, along with an introduction to the hyperbolic wave and the solution profile. Section 3 focuses on reformulating the system for the perturbation of the solution to the radiative hydrodynamics around the solution profile, which consists of the approximate rarefaction wave and the hyperbolic wave. Subsequently, we provide proof of the main result.

Notation. The notations listed below will be widely used in this paper. The standard Sobolev space with the norm $\|\cdot\|_k$ is denoted by $H^k(\mathbb{R}\times\mathbb{T})$ and $H^k(\mathbb{R}\times\mathbb{T}_\epsilon)(k\geq 0, k\in\mathbb{Z})$, where $\mathbb{T}_\epsilon := \mathbb{R}/\frac{1}{\epsilon^\alpha}\mathbb{Z}$ is the scaled torus. We set $H^0(\mathbb{R}\times\mathbb{T}_\epsilon) = L^2(\mathbb{R}\times\mathbb{T}_\epsilon)$, $H^0(\mathbb{R}\times\mathbb{T}) = L^2(\mathbb{R}\times\mathbb{T})$,and $\|\cdot\| = \|\cdot\|_0$. We additionally choose C as a general positive constant that is independent of T, ϵ and δ, and C_T as a positive constant that is independent of ϵ and δ but only relies on T.

2. Construction of profile

In this section, we construct the approximate rarefaction wave to the Euler system (1.7) through using the inviscid Burgers' equation,and then the hyperbolic wave will be constructed. We can refer to [2, 14, 21] for details. Last, we construct the solution profile that combines the rarefaction wave and the hyperbolic wave.

2.1. Smooth approximate rarefaction wave

The Riemann problem for the inviscid Burgers' equation is formulated as:

$$\begin{cases} w_t + ww_{x_1} = 0, \\ w(0,x_1) = w_0^r(x_1) = \begin{cases} w_-, & x_1 < 0, \\ w_+, & x_1 > 0. \end{cases} \end{cases} \tag{2.1}$$

If $w_- < w_+$, then (2.1) has a rarefaction wave $w^r(x_1,t) = w^r(x_1/t)$ given by

$$w^r(t,x_1) = w^r\left(\frac{x_1}{t}\right) = \begin{cases} w_-, & \frac{x_1}{t} < w_-, \\ \frac{x_1}{t}, & w_- < \frac{x_1}{t} < w_+, \\ w_+, & \frac{x_1}{t} > w_+. \end{cases} \tag{2.2}$$

Based on the fact that the rarefaction wave can only be Lipschitz continuous, we follow the method adopted in [1] to develop the approximation rarefaction wave with the smooth solution to the Burgers' equation,

$$\begin{cases} w_t + ww_{x_1} = 0 \\ w(0,x_1) = w_0(x_1) = \dfrac{w_+ + w_-}{2} + \dfrac{w_+ - w_-}{2}\tanh\dfrac{x_1}{\delta}, \end{cases} \tag{2.3}$$

where $\delta > 0$ is a small constant depending on the viscosity parameter ϵ. The following lemma is to show the properties of the solution $w(t,x_1)$ to (2.3), which will be frequently used in our analysis (see [2]).

Lemma 2.1. *Assume $w_+ > w_-$ and set $\tilde{w} = w_+ - w_-$, then the problem* (2.3) *has a unique smooth global solution $w(t,x_1)$ such that*
(1) $w_- < w(t,x_1) < w_+$, $w_{x_1} > 0$, for $x_1\in\mathbb{R}$ and $t\geq 0, \delta > 0$;

(2)The following estimates hold for all $t \geq 0, \delta > 0$, $p \in [1, +\infty]$, $k(\geq 2) \in \mathbb{N}^+$,

$$\left\| w_{x_1}(t, \cdot) \right\|_{L^p(\mathbb{R})} \leq C\tilde{w}^{1/p}(\delta + t)^{-1+1/p},$$
$$\left\| \frac{\partial^k}{\partial x_1^k} w(t, \cdot) \right\|_{L^p(\mathbb{R})} \leq C(\delta + t)^{-1}\delta^{-(k-1)+1/p};$$

(3)There exists a constant $\delta_0 \in (0, 1)$ *such that for* $\delta \in (0, \delta_0]$ *and* $t > 0$*,*

$$\left\| w(t, \cdot) - w^r\left(\frac{\cdot}{t}\right) \right\|_{L^\infty(\mathbb{R})} \leq C\delta t^{-1}[\ln(1 + t) + |\ln\delta|].$$

Denote $w_\pm = \lambda_3(\rho_\pm, u_{1\pm}, \theta_\pm)$, the 3-rarefaction wave $(\rho^r, u_1^r, \theta^r)(x_1, t) = (\rho^r, u_1^r, \theta^r)(x_1/t)$ to the Riemann problems (1.7) and (1.8) is given by

$$w\left(\frac{x_1}{t}\right) = \lambda_3(\rho^r, u_1^r, \theta^r)\left(\frac{x_1}{t}\right),$$
$$\Sigma_3^{(j)}(\rho^r, u_1^r, \theta^r)\left(\frac{x_1}{t}\right) = \Sigma_3^{(j)}(\rho_\pm, u_{1\pm}, \theta_\pm), \quad j = 1, 2,$$

where $\Sigma_3^{(j)}$ is the 3-Riemann invariant defined in (1.10). According to the smooth approximate rarefaction wave $(\bar{\rho}, \bar{u}_1, \bar{\theta})(t, x_1)$ of the 3-rarefaction wave fan $(\rho^r, u_1^r, \theta^r)(\frac{x_1}{t})$ can be formulated as

$$\begin{aligned} &w(t, x_1) = \lambda_3(\bar{\rho}, \bar{u}_1, \bar{\theta})(t, x_1), \\ &\Sigma_3^{(j)}(\bar{\rho}, \bar{u}_1, \bar{\theta})(t, x_1) = \Sigma_3^{(j)}(\rho_\pm, u_{1\pm}, \theta_\pm), \quad j = 1, 2, \end{aligned} \tag{2.4}$$

where $w(t, x_1)$ is the smooth solution to the Burgers' equation in (2.3). The $(\bar{\rho}, \bar{u}_1, \bar{\theta})$ satisfy the following equations:

$$\begin{cases} \bar{\rho}_t + (\bar{\rho}\bar{u}_1)_{x_1} = 0, \\ (\bar{\rho}\bar{u}_1)_t + (\bar{\rho}\bar{u}_1^2 + \bar{p})_{x_1} = 0, \\ \dfrac{R}{\gamma - 1}[(\bar{\rho}\bar{\theta})_t + (\bar{\rho}\bar{u}_1\bar{\theta})_{x_1}] + \bar{p}\bar{u}_{1x_1} = 0, \end{cases} \tag{2.5}$$

with the initial values $(\bar{\rho}, \bar{u}_1, \bar{\theta})(0, x_1) := (\bar{\rho}_0, \bar{u}_{10}, \bar{\theta}_0)(x_1)$. Furthermore, we can associate to the solution of (2.5) the following quantity (see [22])

$$\bar{q}_1 = \frac{b_1}{a_1}(\bar{\theta}^4)_{x_1}.$$

Based on the Lemma 2.1, we utilize the results of the properties on $w(t, x_1)$, which were obtained in [21].

Lemma 2.2. *The smooth approximate 3-rarefaction wave* $(\bar{\rho}, \bar{u}_1, \bar{\theta})$ *constructed in* (2.4) *satisfies the following properties:*
(1) $\bar{u}_{1x_1} = \frac{2}{\gamma+1}w_{x_1} > 0$ *for all* $x_1 \in \mathbb{R}$ *and* $t \geq 0$*,*

$$\bar{\rho}_{x_1} = \frac{1}{\sqrt{A\gamma \exp(\frac{\gamma-1}{R}S_+)}}\bar{\rho}^{\frac{3-\gamma}{2}}\bar{u}_{1x_1} > 0 \text{ and } \bar{\theta}_{x_1} = \frac{\gamma - 1}{\sqrt{R\gamma}}\bar{\theta}^{\frac{1}{2}}\bar{u}_{1x_1} > 0.$$

(2)The following estimates hold for all $t \geq 0, \delta > 0,\ p \in [1, +\infty],\ k(\geq 2) \in \mathbb{N}^+$,

$$\left\|(\bar{\rho}_{x_1}, \bar{u}_{1x_1}, \bar{\theta}_{x_1})\right\|_{L^p(\mathbb{R})} \leq C\tilde{w}^{1/p}(\delta+t)^{-1+1/p},$$
$$\left\|\frac{\partial}{\partial x_1^k}(\bar{\rho}, \bar{u}, \bar{\theta})\right\|_{L^p(\mathbb{R})} \leq C(\delta+t)^{-1}\delta^{-(k-1)+1/p}.$$

(3)There exists a constant $\delta_0 \in (0,1)$ such that for $\delta \in (0, \delta_0]$ and $t > 0$,

$$\left\|(\bar{\rho}, \bar{u}_1, \bar{\theta})(t,\cdot) - (\rho^r, u_1^r, \theta^r)\left(\frac{\cdot}{t}\right)\right\|_{L^\infty(\mathbb{R})} \leq C\delta t^{-1}[\ln(1+t) + |\ln\delta|].$$

2.2. *Hyperbolic wave*

To compensate for the error arising from the dissipation terms, the hyperbolic wave was introduced in [11] and [12]. We also develop hyperbolic wave $(d_1, d_2, d_3)(t, x_1)$ corresponding to the rarefaction wave $(\bar{\rho}, \bar{u}_1, \bar{\theta})$, at the same time we neglect the hyperbolic wave $d_4(=0)$ with respect to $\bar{q}_1$.

$$\begin{cases} d_{1t} + d_{2x_1} = 0, \\ d_{2t} + \left(\dfrac{2\bar{m}_1}{\bar{\rho}}d_2 - \dfrac{\bar{m}_1^2}{\bar{\rho}^2}d_1 + \bar{p}_{\bar{\rho}}d_1 + \bar{p}_{\bar{m}_1}d_2 + \bar{p}_{\bar{\mathcal{E}}}d_3\right)_{x_1} = (2\mu+\lambda)\epsilon\bar{u}_{1x_1x_1}, \\ d_{3t} + \left(\dfrac{\bar{\mathcal{E}}}{\bar{\rho}}d_2 + \dfrac{\bar{m}_1}{\bar{\rho}}d_3 - \dfrac{\bar{m}_1\bar{\mathcal{E}}}{\bar{\rho}^2}d_1 + \dfrac{\bar{m}_1}{\bar{\rho}}\bar{p}_{\bar{\rho}}d_1 + \dfrac{\bar{m}_1}{\bar{\rho}}\bar{p}_{\bar{m}_1}d_2 + \dfrac{\bar{m}_1}{\bar{\rho}}\bar{p}_{\bar{\mathcal{E}}}d_3 + \dfrac{\bar{p}}{\bar{\rho}}d_2 - \dfrac{\bar{m}_1\bar{p}}{\bar{\rho}^2}d_1\right)_{x_1} \\ = \kappa\epsilon\bar{\theta}_{x_1x_1} + (2\mu+\lambda)\epsilon(\bar{u}_1\bar{u}_{1x_1})_{x_1}, \\ (d_1, d_2, d_3)(0, x_1) = (0,0,0), \end{cases} \tag{2.6}$$

where $\bar{m}_1 := \bar{\rho}\bar{u}_1$, $\bar{\mathcal{E}} := \bar{\rho}\bar{E} = \bar{\rho}(\frac{R}{\gamma-1}\bar{\theta} + \frac{1}{2}\bar{u}_1^2)$ represent the momentum and the total energy of the approximate rarefaction wave, respectively. The first three lines of (2.6) are linear hyperbolic systems, which can be rewritten as

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}_t + \left(\mathbf{A}\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}\right)_{x_1} = \begin{bmatrix} 0 \\ (2\mu+\lambda)\epsilon\bar{u}_{1x_1x_1} \\ \kappa\epsilon\bar{\theta}_{x_1x_1} + (2\mu+\lambda)\epsilon(\bar{u}_1\bar{u}_{1x_1})_{x_1} \end{bmatrix},$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{\bar{m}_1^2}{\bar{\rho}^2} + \bar{p}_{\bar{\rho}} & \frac{2\bar{m}_1}{\bar{\rho}} + \bar{p}_{\bar{m}_1} & \bar{p}_{\bar{\mathcal{E}}} \\ -\frac{\bar{m}_1\bar{\mathcal{E}}}{\bar{\rho}^2} + \frac{\bar{m}_1}{\bar{\rho}}\bar{p}_{\bar{\rho}} - \frac{\bar{m}_1\bar{p}}{\bar{\rho}^2} & \frac{\bar{\mathcal{E}}}{\bar{\rho}} + \frac{\bar{m}_1}{\bar{\rho}}\bar{p}_{\bar{m}_1} + \frac{\bar{p}}{\bar{\rho}} & \frac{\bar{m}_1}{\bar{\rho}} + \frac{\bar{m}_1}{\bar{\rho}}\bar{p}_{\bar{\mathcal{E}}} \end{bmatrix}$$

with three distinct eigenvalues $\bar{\lambda}_i = \bar{\lambda}_i(\bar{\rho}, \bar{u}_1, S_\pm) = \bar{u}_1 + (-1)^{\frac{i+1}{2}}\sqrt{\bar{p}_{\bar{\rho}}(\bar{\rho}, S_\pm)}(i=1,3), \bar{\lambda}_2 = \bar{\lambda}_2(\bar{\rho}, \bar{u}_1, S_\pm) = \bar{u}_1$ and the corresponding left and right eigenvectors $\bar{l}_i = \bar{l}_i(\bar{\rho}, \bar{u}_1, S_\pm), \bar{r}_i = \bar{r}_i(\bar{\rho}, \bar{u}_1, S_\pm)(i = 1,2,3)$ satisfying

$$\bar{L}\mathbf{A}\bar{R} = \mathrm{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3) := \mathbf{\Lambda}, \quad \bar{L}\bar{R} = \mathbb{I},$$

where $\bar{L} = (\bar{l}_1, \bar{l}_2, \bar{l}_3)^\top, \bar{R} = (\bar{r}_1, \bar{r}_2, \bar{r}_3)$ and $\mathbb{I}$ is the 3×3 identity matrix.

Define

$$(D_1, D_2, D_3)^\top = \bar{L}(d_1, d_2, d_3)^\top,$$

then it has

$$(d_1, d_2, d_3)^\top = \bar{R}(D_1, D_2, D_3)^\top,$$

and we find that (D_1, D_2, D_3) satisfies the following system:

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}_t + \left(\mathbf{\Lambda} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \right)_{x_1} = \bar{L} \begin{bmatrix} 0 \\ (2\mu + \lambda)\epsilon \bar{u}_{1x_1x_1} \\ \kappa\epsilon\bar{\theta}_{x_1x_1} + (2\mu + \lambda)\epsilon(\bar{u}_1\bar{u}_{1x_1})_{x_1} \end{bmatrix} + (\bar{L}_t\bar{R} + \bar{L}_{x_1}\mathbf{A}\bar{R}) \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}. \tag{2.7}$$

Based on the fact that the 3-Riemann invariant is a constant on the 3-rarefaction wave curve, we obtain

$$\bar{L}_t = -\bar{\lambda}_3 \bar{L}_{x_1}. \tag{2.8}$$

We use (2.8) to rewrite (2.7) as

$$\begin{cases} (D_1)_t + (\bar{\lambda}_1 D_1)_{x_1} = (2\mu + \lambda)\epsilon \bar{l}_{12}\bar{u}_{1x_1x_1} + \bar{l}_{13}(\kappa\epsilon\bar{\theta}_{x_1x_1} + (2\mu + \lambda)\epsilon(\bar{u}_1\bar{u}_{1x_1})_{x_1} \\ \qquad + (\bar{\lambda}_1 - \bar{\lambda}_3)\bar{l}_{1x_1} \cdot \bar{r}_1 D_1 + (\bar{\lambda}_2 - \bar{\lambda}_3)\bar{l}_{1x_1} \cdot \bar{r}_2 D_2, \\ (D_2)_t + (\bar{\lambda}_2 D_2)_{x_1} = (2\mu + \lambda)\epsilon \bar{l}_{22}\bar{u}_{1x_1x_1} + \bar{l}_{23}(\kappa\epsilon\bar{\theta}_{x_1x_1} + (2\mu + \lambda)\epsilon(\bar{u}_1\bar{u}_{1x_1})_{x_1} \\ \qquad + (\bar{\lambda}_1 - \bar{\lambda}_3)\bar{l}_{2x_1} \cdot \bar{r}_1 D_1 + (\bar{\lambda}_2 - \bar{\lambda}_3)\bar{l}_{2x_1} \cdot \bar{r}_2 D_2, \\ (D_3)_t + (\bar{\lambda}_3 D_3)_{x_1} = (2\mu + \lambda)\epsilon \bar{l}_{32}\bar{u}_{1x_1x_1} + \bar{l}_{33}(\kappa\epsilon\bar{\theta}_{x_1x_1} + (2\mu + \lambda)\epsilon(\bar{u}_1\bar{u}_{1x_1})_{x_1} \\ \qquad + (\bar{\lambda}_1 - \bar{\lambda}_3)\bar{l}_{3x_1} \cdot \bar{r}_1 D_1 + (\bar{\lambda}_2 - \bar{\lambda}_3)\bar{l}_{3x_1} \cdot \bar{r}_2 D_2, \\ (D_1, D_2, D_3)(0, x_1) = (0, 0, 0). \end{cases} \tag{2.9}$$

We easily find that D_1 and D_2 are decoupled from D_3 in (2.8). This enables us to solve the linear hyperbolic system (2.9) on the finite time interval $[0, T]$. Furthermore, we borrow the result in [11] to get the following estimates for the hyperbolic wave (d_1, d_2, d_3).

Lemma 2.3. *There exists a positive constant C_T independent of δ and ϵ, such that*

$$\sup_{t \in [0,T]} \left\| \frac{\partial^k}{\partial x_1^k}(d_1, d_2, d_3)(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C_T \left(\frac{\epsilon}{\delta^{k+1}} \right)^2, \quad k = 0, 1, 2, 3.$$

In particular, it holds that

$$\sup_{t \in [0,T]} \left\| \frac{\partial^k}{\partial x_1^k}(d_1, d_2, d_3)(t, \cdot) \right\|_{L^\infty(\mathbb{R})} \leq O\left(\frac{\epsilon}{\delta^{\frac{3}{2}+k}} \right), \quad k = 0, 1, 2.$$

2.3. Approximate solution profile

We set the approximate solution profile $(\tilde{\rho}, \tilde{u}_1, \tilde{\theta})$ of the two-dimensional radiative hydrodynamics as

$$\tilde{\rho} = \bar{\rho} + d_1, \tilde{m}_1 = \bar{m}_1 + d_2 := \tilde{\rho}\tilde{u}_1, \mathcal{E} = \bar{\mathcal{E}} + d_3 := \tilde{\rho}\tilde{\mathcal{E}} = \tilde{\rho}\Big(\frac{R}{\gamma - 1}\tilde{\theta} + \frac{1}{2}\tilde{u}_1^2\Big), \tag{2.10}$$

then the approximate wave profile $(\tilde{\rho}, \tilde{u}_1, \tilde{\theta})$ satisfies

$$\begin{cases} \tilde{\rho}_t + (\tilde{\rho}\tilde{u}_1)_{x_1} = 0, \\ (\tilde{\rho}\tilde{u}_1)_t + (\tilde{\rho}\tilde{u}_1^2 + R\tilde{\rho}\tilde{\theta})_{x_1} = (2\mu+\lambda)\epsilon\bar{u}_{1x_1x_1} + \left(\frac{3-\gamma}{2\tilde{\rho}}(-\bar{u}_1 d_1 + d_2)^2\right)_{x_1}, \\ \frac{R}{\gamma-1}[(\tilde{\rho}\tilde{\theta})_t + (\tilde{\rho}\tilde{u}_1\tilde{\theta})_{x_1}] + R\tilde{\rho}\tilde{\theta}\tilde{u}_{1x_1} + \bar{q}_{1x_1} = \kappa\epsilon\bar{\theta}_{x_1x_1} + \frac{b}{a}\epsilon(\bar{\theta}^4)_{x_1x_1} \\ + \left[\frac{-\bar{u}_1 d_1 + d_2}{\tilde{\rho}}(\gamma d_3 - (\gamma-1)\bar{u}_1 d_2 - \frac{R\gamma}{\gamma-1}\bar{\theta}d_1 + \frac{\gamma-2}{2}\bar{u}_1^2 d_1) - (\gamma-1)\tilde{u}_1\frac{(-\bar{u}_1 d_1 + d_2)^2}{2\tilde{\rho}}\right]_{x_1} \\ +(2\mu+\lambda)\epsilon\bar{u}_{1x_1}^2 - (2\mu+\lambda)\epsilon\bar{u}_{1x_1x_1} - \tilde{u}_1\left(\frac{3-\gamma}{2\tilde{\rho}}(-\bar{u}_1 d_1 + d_2)^2\right)_{x_1}, \\ \frac{1}{b}\bar{q}_1 = \frac{1}{ab}\epsilon^2(\bar{q}_1)_{x_1x_1} - \frac{1}{a^2}\epsilon^3(\bar{\theta}^4)_{x_1x_1x_1} + \frac{1}{a}\epsilon(\bar{\theta}^4)_{x_1}, \end{cases} \tag{2.11}$$

with the initial data

$$(\tilde{\rho}, \tilde{u}_1, \tilde{\theta}, \tilde{q}_1)(0, x_1) = (\bar{\rho}_0, \bar{u}_{10}, \bar{\theta}_0, \bar{q}_0)(x_1). \tag{2.12}$$

3. Proof of Theorem 1.1

We denote the perturbation around the approximate wave profile

$$(\tilde{\rho}, \tilde{\boldsymbol{u}}, \tilde{\theta}, \tilde{\mathbf{q}}) := (\tilde{\rho}, \tilde{u}_1, 0, \tilde{\theta}, \bar{q}_1, 0)$$

by

$$\begin{aligned} &\phi(\tau, y) := \rho^\epsilon(\tau, y) - \tilde{\rho}(\tau, y), \\ &\Psi(\tau, y) := (\psi_1, \psi_2)^\top(\tau, y) := (u_1^\epsilon, u_2^\epsilon)^\top(\tau, y) - (\tilde{u}_1, 0)^\top(\tau, y), \\ &\zeta(\tau, y) := \theta^\epsilon(\tau, y) - \tilde{\theta}(\tau, y), \\ &Q(\tau, y) := (Q_1, Q_2)^\top(\tau, y) := (q_1^\epsilon, q_2^\epsilon)^\top(\tau, y) - (\bar{q}_1, 0)^\top(\tau, y), \end{aligned} \tag{3.1}$$

where

$$\tau = \frac{t}{\epsilon^\alpha}, \quad y_1 = \frac{x_1}{\epsilon^\alpha}, \quad y_2 = \frac{x_2}{\epsilon^\alpha}, \quad y = (y_1, y_2), \quad \frac{5}{8} < \alpha < 1, \tag{3.2}$$

and

$$(\rho^\epsilon, \boldsymbol{u}^\epsilon, \theta^\epsilon, \boldsymbol{q}^\epsilon) = (\rho^\epsilon, u_1^\epsilon, u_2^\epsilon, \theta^\epsilon, q_1^\epsilon, q_2^\epsilon)$$

is the solution to the problem (1.1) with the following initial data:

$$(\rho^\epsilon, u_1^\epsilon, u_2^\epsilon, \theta^\epsilon, q_1^\epsilon, q_2^\epsilon)(0, y) := \left(\bar{\rho}_0, \bar{u}_{10}, 0, \bar{\theta}_0, \bar{q}_{10}, 0\right)(y_1) + (\phi_0, \psi_{10}, \psi_{20}, \zeta_0, Q_{10}, Q_{20})(y). \tag{3.3}$$

For simplicity, the superscript ϵ in $(\rho^\epsilon, \boldsymbol{u}^\epsilon, \theta^\epsilon, \boldsymbol{q}^\epsilon)$ will be omitted. We use (1.1) and (2.11) to derive the equations for the perturbation (ϕ, Ψ, ζ, Q) :

$$\begin{cases} \phi_\tau + \boldsymbol{u}\cdot\nabla\phi + \rho\mathrm{div}\Psi + \tilde{\rho}_{y_1}\psi_1 + \tilde{u}_{1y_1}\phi = 0, \\ \rho\Psi_t + \rho\boldsymbol{u}\cdot\nabla\Psi + R\theta\nabla\phi + R\rho\nabla\zeta + \left(\rho\tilde{u}_{1y_1}\psi_1, 0\right)^\top + (R\tilde{\rho}_{y_1}(\theta - \frac{\rho}{\tilde{\rho}}\tilde{\theta}), 0)^\top \\ = \mu\epsilon^{1-\alpha}\Delta\Psi + (\mu+\lambda)\epsilon^{1-\alpha}\nabla\mathrm{div}\Psi + ((2\mu+\lambda))\epsilon^{1-\alpha}(\frac{-\bar{u}_1 d_1 + d_2}{\tilde{\rho}})_{y_1y_1}, 0)^\top \\ \quad -((2\mu+\lambda)\epsilon^{1-\alpha}\frac{\bar{u}_{1y_1y_1}}{\tilde{\rho}}\phi, 0)^\top - ((\frac{3-\gamma}{2\tilde{\rho}}(-\bar{u}_1 d_1 + d_2)^2)_{y_1}\frac{\rho}{\tilde{\rho}}, 0)^\top, \\ \frac{R}{\gamma-1}(\rho\zeta_\tau + \rho\boldsymbol{u}\cdot\nabla\zeta) + R\rho\theta\mathrm{div}\Psi + \mathrm{div}Q + \frac{R}{\gamma-1}\rho\tilde{\theta}_{y_1}\psi_1 + R\rho\tilde{u}_{1y_1}\zeta \\ = \kappa\epsilon^{1-\alpha}\Delta\zeta + \frac{\mu}{2}\epsilon^{1-\alpha}\left|\nabla\Psi + (\nabla\Psi)^\top\right|^2 + \lambda\epsilon^{1-\alpha}(\mathrm{div}\Psi)^2 + 2\tilde{u}_{1y_1}\epsilon^{1-\alpha}(2\mu\psi_{1y_1} + \lambda\mathrm{div}\Psi) \\ \quad +F_1 + F_2 + F_3, \\ \frac{1}{b}Q = \frac{1}{ab}\epsilon^{2-2\alpha}\nabla\mathrm{div}Q - \frac{1}{a}\epsilon^{1-\alpha}\nabla(\theta^4 - \bar{\theta}^4) + (\frac{1}{a^2}\epsilon^{3-3\alpha}(\bar{\theta}^4)_{y_1y_1y_1}, 0)^\top, \end{cases} \tag{3.4}$$

with initial data

$$(\phi,\Psi,\zeta,Q)(0,y)=(\phi_0,\Psi_0,\zeta_0,Q_0)(y)\\ =(\phi_0,\psi_{10},\psi_{20},\zeta_0,Q_{10},Q_{20})(y),\tag{3.5}$$

where

$$\begin{aligned}F_1=&\frac{\gamma-1}{R}\kappa\epsilon^{1-\alpha}\left[\frac{1}{\tilde\rho}((\frac{1}{2}\bar u_1^2-\frac{R}{\gamma-1}\bar\theta)d_1-\bar u_1d_2+d_3)\right]_{y_1y_1}-\frac{\gamma-1}{2R}\kappa\epsilon^{1-\alpha}\left(\left(\frac{-\bar u_1d_1+d_2}{\tilde\rho}\right)^2\right)_{y_1y_1}\\&+2(2\mu+\lambda)\epsilon^{1-\alpha}\bar u_{1y_1}\left(\frac{-\bar u_1d_1+d_2}{\tilde\rho}\right)_{y_1}+(2\mu+\lambda)\epsilon^{1-\alpha}\bar u_{1y_1y_1}\frac{\rho}{\tilde\rho^2}(-\bar u_1d_1+d_2)\\&+(2\mu+\lambda)\epsilon^{1-\alpha}\left(\left(\frac{-\bar u_1d_1+d_2}{\tilde\rho}\right)_{y_1}\right)^2-\frac{b}{a}\epsilon^{1-\alpha}\frac{\rho}{\tilde\rho}(\bar\theta^4)_{y_1y_1},\\F_2=&-\frac{\rho}{\tilde\rho}\Big[\frac{-\bar u_1d_1+d_2}{\tilde\rho}(\gamma d_3-(\gamma-1)\bar u_1d_2-\frac{R\gamma}{\gamma-1}\bar\theta d_1+\frac{\gamma-2}{2}\bar u_1^2d_1)\\&-(\gamma-1)\tilde u_1\frac{(-\bar u_1d_1+d_2)^2}{2\tilde\rho}\Big]_{y_1}+\frac{\rho\tilde u_1}{\tilde\rho}\left(\frac{3-\gamma}{2\tilde\rho}(-\bar u_1d_1+d_2)^2\right)_{y_1},\\F_3=&-\kappa\epsilon^{1-\alpha}\frac{\bar\theta_{y_1y_1}}{\tilde\rho}\phi-(2\mu+\lambda)\epsilon^{1-\alpha}\frac{\bar u_{1y_1}^2}{\tilde\rho}\phi+\frac{b}{a}\epsilon^{1-\alpha}\frac{(\bar\theta^4)_{y_1y_1}}{\tilde\rho}\phi,\end{aligned}$$

and the initial perturbation is chosen to satisfy

$$\begin{aligned}&\|(\phi_0,\psi_{10},\psi_{20},\zeta_0)\|_{H^2(\mathbb{R}\times\mathbb{T}_\epsilon)}=O(\epsilon^{1-\alpha}|\ln\epsilon|^{-1}),\\&\|(Q_{10},Q_{20})\|_{H^2(\mathbb{R}\times\mathbb{T}_\epsilon)}=O(\epsilon^{2-2\alpha}|\ln\epsilon|^{-1}).\end{aligned}\tag{3.6}$$

Our aim is to find a solution (ϕ,Ψ,ζ,Q) to (3.4)–(3.6) in the space $X(0,\frac{T}{\epsilon^\alpha})$, which is defined to be

$$X(0,\tau_1)=\left\{(\phi,\Psi,\zeta,Q)\left|\begin{array}{c}(\phi,\Psi,\zeta)\in C^0(0,\tau_1;H^2),\\\nabla\phi\in L^2(0,\tau_1;H^1),(\nabla\Psi,\nabla\zeta)\in L^2(0,\tau_1;H^2),\\Q\in L^\infty(0,\tau_1;H^2)\cap L^2(0,\tau_1;H^2),\\\mathrm{div}Q\in L^\infty(0,\tau_1;H^2)\cap L^2(0,\tau_1;H^2).\end{array}\right.\right\}$$

with $0\le\tau_1\le\frac{T}{\epsilon^\alpha}$.

Proposition 3.1. *There exists a positive constant $\epsilon_0<1$ such that if $0<\epsilon\le\epsilon_0$, then the perturbation problems* (3.4)–(3.6) *admits a unique solution $(\phi,\Psi,\zeta,Q)\in X(0,\frac{T}{\epsilon^\alpha})$ satisfying*

$$\begin{aligned}&\sup_{0\le\tau\le\frac{T}{\epsilon^\alpha}}\|(\phi,\Psi,\zeta)\|_2^2(\tau)+\int_0^{\frac{T}{\epsilon^\alpha}}\Big[\|\bar u_{1y_1}^{1/2}(\phi,\psi_1,\zeta)\|^2+\|\bar u_{1y_1}^{1/2}(\nabla\phi,\nabla^2\phi)\|^2\\&\quad+\epsilon^{1-\alpha}\|\nabla\phi\|_1^2+\epsilon^{\alpha-1}\|Q\|_2^2+\epsilon^{1-\alpha}\|(\nabla\Psi,\nabla\zeta,\mathrm{div}Q)\|_2^2\Big]\mathrm{d}s\\&\le C_T\Big(\frac{\epsilon^{3-2\alpha}}{\delta^4}+\frac{\epsilon^{4-2\alpha}}{\delta^7}\Big)+C_T\|(\phi_0,\Psi_0,\zeta_0)\|_2^2,\end{aligned}\tag{3.7}$$

and

$$\sup_{0\le\tau\le\frac{T}{\epsilon^\alpha}}(\|Q\|_2^2+\epsilon^{2-2\alpha}\|\mathrm{div}Q\|_2^2)(\tau)\le C_T\Big(\frac{\epsilon^{3-2\alpha}}{\delta^4}+\frac{\epsilon^{4-2\alpha}}{\delta^7}\Big)\epsilon^{2-2\alpha}+\frac{\epsilon^4}{\delta^5}+\frac{\epsilon^{4+2\alpha}}{\delta^9}+C_T\|(\phi_0,\Psi_0,\zeta_0)\|_2^2, \tag{3.8}$$

for some constant C_T is independent of ϵ,δ, but may depend on T.

According to the above estimate (3.7), we further derive that

$$\begin{aligned}
&\|(\rho,u_1,u_2,\theta)(t,x_1,x_2)-(\rho^r,u_1^r,0,\theta^r)\left(\frac{x_1}{t}\right)\|_{L^\infty(\Omega)}\\
&\le\|(\phi,\Psi,\zeta)(t,x_1,x_2)\|_{L^\infty(\Omega)}+C\|(d_1,d_2,d_3)(t,x_1)\|_{L^\infty(\mathbb{R})}\\
&\quad+\|(\bar\rho,\bar u_1,\bar\theta)(t,x_1)-(\rho^r,u_1^r,\theta^r)\left(\frac{x_1}{t}\right)\|_{L^\infty(\mathbb{R})}\\
&\le C\|(\phi,\Psi,\zeta)(\tau)\|_2+C_T\frac{\epsilon}{\delta^{3/2}}+C\delta t^{-1}[\ln(1+t)+|\ln\delta|]\\
&\le C_T\Big(\frac{\epsilon^{\frac{3-2\alpha}{2}}}{\delta^2}+\frac{\epsilon^{2-\alpha}}{\delta^{\frac72}}\Big)+C_T\frac{\epsilon}{\delta^{3/2}}+C\delta t^{-1}[\ln(1+t)+|\ln\delta|].
\end{aligned}\tag{3.9}$$

Under the setting $\delta=\epsilon^\omega|\ln\epsilon|$ and (3.41), we derive that the upper bound in (3.9) is $C_T\epsilon^\omega|\ln\epsilon|^2$; thus, the proof of Theorem 1.1 is completed.

In order to prove Proposition 3.1, we will perform the analysis under a priori assumption

$$\sup_{\tau\in[0,\tau_1(\epsilon)]}(\|(\phi,\Psi,\zeta)\|_2)\le\epsilon^{1-\alpha},\ \sup_{\tau\in[0,\tau_1(\epsilon)]}\|Q\|_2\le\epsilon^{2-2\alpha}. \tag{3.10}$$

Proposition 3.2. *(A priori estimate). Assume that the problems* (3.4)–(3.6) *admit a solution* $(\phi,\Psi,\zeta,Q)\in X(0,\tau_1(\epsilon))$ *for some* $\tau_1(\epsilon)(>0)$. *Then there exists a positive constant* ϵ_1 *which is independent of* ϵ,δ *and* $\tau_1(\epsilon)$, *such that if* $0<\epsilon\le\epsilon_1$ *and the a priori Assumptions* (3.10), *then we have the following uniform estimate:*

$$\begin{aligned}
&\sup_{0\le\tau\le\tau_1(\epsilon)}\|(\phi,\Psi,\zeta)(\tau)\|_2^2+\int_0^{\tau_1(\epsilon)}\Big[\|\bar u_{1y_1}^{1/2}(\phi,\psi_1,\zeta)\|^2+\|\bar u_{1y_1}^{1/2}(\nabla\phi,\nabla^2\phi)\|^2\\
&\quad+\epsilon^{1-\alpha}\|\nabla\phi\|_1^2+\epsilon^{\alpha-1}\|Q\|_2^2+\epsilon^{1-\alpha}\|(\nabla\Psi,\nabla\zeta,\mathrm{div}Q)\|_2^2\Big]d\tau\\
&\le C_T\Big(\frac{\epsilon^{3-2\alpha}}{\delta^4}+\frac{\epsilon^{4-2\alpha}}{\delta^7}\Big)+C_T\|(\phi_0,\Psi_0,\zeta_0)\|_2^2,
\end{aligned}\tag{3.11}$$

and

$$\sup_{0\le\tau\le\frac{T}{\epsilon^\alpha}}(\|Q\|_2^2+\epsilon^{2-2\alpha}\|\mathrm{div}Q\|_2^2)(\tau)\le C_T\Big(\frac{\epsilon^{3-2\alpha}}{\delta^4}+\frac{\epsilon^{4-2\alpha}}{\delta^7}\Big)\epsilon^{2-2\alpha}+\frac{\epsilon^4}{\delta^5}+\frac{\epsilon^{4+2\alpha}}{\delta^9}+C_T\|(\phi_0,\Psi_0,\zeta_0)\|_2^2, \tag{3.12}$$

where the constant C_T is independent of ϵ,δ, but may depend on T.

Before performing the energy estimates, we use the *a priori* assumption to derive that the density and temperature are uniformly bounded from below and above. Choosing $\frac{\epsilon}{\delta^{\frac32}}$ and ϵ small enough, then it holds

$$|d_i|\le C_T\frac{\epsilon}{\delta^{\frac32}}\le\frac14\rho_-,\quad i=1,2,3,$$

$$0<\frac{3}{4}\rho_-<\tilde\rho=\bar\rho+d_1\le\rho_++\frac{1}{4}\rho_-,$$
$$|\tilde u_1|=\left|\bar u_1+\frac{-\bar u_1d_1+d_2}{\tilde\rho}\right|\le C,$$
$$|\tilde q_1|=\left|\frac{b}{a}\epsilon(\bar\theta^4)_{x_1}\right|\le C_T\frac{\epsilon}{\delta}\le C,$$

and

$$\tilde\theta=\bar\theta+\frac{\gamma-1}{R\tilde\rho}\left(\left(\frac{1}{2}\bar u_1^2-\frac{R}{\gamma-1}\bar\theta\right)d_1-\bar u_1d_2+d_3\right)-\frac{\gamma-1}{2R\tilde\rho^2}(-\bar u_1d_1+d_2)^2,$$
$$0<\frac{3}{4}\theta_-\le\tilde\theta\le\theta_++\frac{1}{4}\theta_-.$$

Moreover, we use Sobolev's inequality and *a priori* assumption to get

$$0<\frac{1}{2}\rho_-\le\rho\le\rho_++\frac{1}{2}\rho_-,0<\frac{1}{2}\theta_-\le\theta\le\theta_++\frac{1}{2}\theta_-,|\boldsymbol{u}|\le C,|\boldsymbol{q}|\le C. \tag{3.13}$$

Zero-order energy estimate for the perturbation system is formulated as follows.

Lemma 3.1. *There exists a positive constant C_T such that*

$$\begin{aligned}&\sup_{0\le\tau\le\tau_1(\epsilon)}\|(\phi,\Psi,\zeta)(\tau)\|^2+\int_0^{\tau_1(\epsilon)}\left[\|\bar u_{1_{y_1}}^{1/2}(\phi,\psi_1,\zeta)\|^2+\epsilon^{1-\alpha}\|(\nabla\Psi,\nabla\zeta)\|^2\right]\mathrm{d}\tau\\&+\int_0^{\tau_1(\epsilon)}\left[\epsilon^{\alpha-1}\|Q\|^2+\epsilon^{1-\alpha}\|\mathrm{div}Q\|^2\right]\mathrm{d}\tau\\&\le C_T\left(\frac{\epsilon^{3-2\alpha}}{\delta^4}+\frac{\epsilon^{4-2\alpha}}{\delta^7}\right)+C\|(\phi_0,\Psi_0,\zeta_0)\|^2.\end{aligned} \tag{3.14}$$

Proof. Define $V(x)=x-\ln x-1$, then the entropy and entropy flux are given by:

$$\begin{cases}\eta=R\rho\tilde\theta V\left(\frac{\tilde\rho}{\rho}\right)+\frac{R}{\gamma-1}\rho\tilde\theta V\left(\frac{\theta}{\tilde\theta}\right)+\frac{1}{2}\rho|\boldsymbol{u}-\tilde{\boldsymbol{u}}|^2,\\ \mathfrak{q}=\boldsymbol{u}\eta+R(\boldsymbol{u}-\tilde{\boldsymbol{u}})(\rho\theta-\tilde\rho\tilde\theta).\end{cases} \tag{3.15}$$

Under the assumption (3.13), there is a positive C such that $\frac{1}{C}|\phi,\Psi,\zeta|^2\le\eta\le C|\phi,\Psi,\zeta|^2$. Straightforward calculations lead to

$$\begin{aligned}&\eta_\tau+\mathrm{div}\mathfrak{q}-\mathrm{div}\left[\Psi\epsilon^{1-\alpha}(2\mu D(\Psi)+\lambda\mathrm{div}\Psi\mathbb{I})+\kappa\epsilon^{1-\alpha}\frac{\zeta}{\theta}\nabla\zeta\right]+\frac{\tilde\theta}{\theta}\left(\frac{\mu\epsilon^{1-\alpha}}{2}|(\nabla\Psi)+(\nabla\Psi^\top)|^2\right.\\&\left.+\lambda\epsilon^{1-\alpha}(\mathrm{div}\Psi)^2\right)+\kappa\epsilon^{1-\alpha}\frac{\tilde\theta}{\theta^2}|\nabla\zeta|^2+\tilde u_{1y_1}\left[\rho\psi_1^2+R(\gamma-1)\rho\tilde\theta V(\frac{\tilde\rho}{\rho})+R\rho\tilde\theta V(\frac{\theta}{\tilde\theta})\right]\\&+\tilde\theta_{y_1}\rho\psi_1\left(R\ln\frac{\tilde\rho}{\rho}+\frac{R}{\gamma-1}\ln\frac{\theta}{\tilde\theta}\right)\\&=\frac{2\epsilon^{1-\alpha}}{\theta}\tilde u_{1y_1}\zeta(2\mu\psi_{1y_1}+\lambda\mathrm{div}\Psi)+\frac{\kappa\epsilon^{1-\alpha}}{\theta^2}\tilde\theta_{y_1}\zeta\zeta_{y_1}-\frac{(2\mu+\lambda)\epsilon^{1-\alpha}}{\tilde\rho}\bar u_{1y_1y_1}\phi\psi_1\end{aligned}$$

$$
\begin{aligned}
&+(2\mu+\lambda)\epsilon^{1-\alpha}\left(\frac{-\bar{u}_1 d_1+d_2}{\tilde{\rho}}\right)_{y_1y_1}\psi_1-\frac{\rho}{\tilde{\rho}}\psi_1\left(\frac{3-\gamma}{2\tilde{\rho}}(-\bar{u}_1d_1+d_2)^2\right)_{y_1} \qquad (3.16)\\
&+\frac{\rho}{\tilde{\rho}}\left((\gamma-1)V\left(\frac{\tilde{\rho}}{\rho}\right)-V\left(\frac{\tilde{\theta}}{\theta}\right)\right)\Big(\kappa\epsilon^{1-\alpha}\bar{\theta}_{y_1y_1}+(2\mu+\lambda)\epsilon^{1-\alpha}\bar{u}_{1y_1}^2\\
&-(2\mu+\lambda)\epsilon^{1-\alpha}\bar{u}_{1y_1y_1}\frac{-\bar{u}_1d_1+d_2}{\tilde{\rho}}\Big)-\mathrm{div}Q\frac{\zeta}{\theta}\\
&+F_3\frac{\zeta}{\theta}+F_1\frac{\zeta}{\theta}-F_2\left((\gamma-1)V(\frac{\tilde{\rho}}{\rho})-V(\frac{\tilde{\theta}}{\theta})-\frac{\zeta}{\theta}\right).
\end{aligned}
$$

Multiplying the fourth equation of (3.4) by $\frac{a\epsilon^{\alpha-1}}{4\theta\tilde{\theta}^3}Q$, and using the notation $\mathcal{P}:=(\tilde{\theta}-\bar{\theta})$, we have

$$
\begin{aligned}
&\frac{a\epsilon^{\alpha-1}}{4b\theta\tilde{\theta}^3}Q^2+\mathrm{div}\left(\frac{Q\zeta}{\theta}\right)-\frac{\epsilon^{1-\alpha}}{4b}\mathrm{div}\left(\frac{Q\mathrm{div}Q}{\theta\tilde{\theta}^3}\right)+\frac{\epsilon^{1-\alpha}}{4b}\frac{|\mathrm{div}Q|^2}{\theta\tilde{\theta}^3}\\
&=\frac{\zeta}{\theta}\mathrm{div}Q-\frac{Q}{4\theta\tilde{\theta}^3}\nabla\left(\zeta^4+4\zeta^3\tilde{\theta}+6\zeta^2\tilde{\theta}^2\right)-\frac{\zeta Q\cdot\nabla\zeta}{\theta^2}-\frac{\tilde{\theta}_{y_1}Q_1\zeta}{\theta^2} \qquad (3.17)\\
&+\frac{\epsilon^{1-\alpha}}{4b}\mathrm{div}QQ\cdot\nabla\left(\frac{1}{\theta\tilde{\theta}^3}\right)-\frac{\zeta Q_1}{\theta\tilde{\theta}^3}(\tilde{\theta}^3)_{y_1}-\frac{\epsilon^{2-2\alpha}}{4a\theta\tilde{\theta}^3}(\bar{\theta}^4)_{y_1y_1y_1}Q_1\\
&+\frac{Q_1}{4\theta\tilde{\theta}^3}\left(\mathcal{P}^4+4\mathcal{P}^3\bar{\theta}+6\mathcal{P}^2\bar{\theta}^2+4\mathcal{P}\bar{\theta}^3\right)_{y_1}.
\end{aligned}
$$

Actually, the term $\mathrm{div}Q\frac{\zeta}{\theta}$ in (3.16) cannot be estimated directly. Our strategy involves using cancellation, which combines (3.16) and (3.17) together.

Here, we present some preparatory work before performing detailed estimates. Specifically, we use the definition of the approximate solution profile $(\tilde{\rho},\tilde{u}_1,\tilde{\theta})$ to rewrite the last two terms on the left side of Eq (3.16). The estimation we conduct thereafter focuses on extracting the dissipation part.

$$
\begin{aligned}
&\tilde{u}_{1y_1}\left[\rho\psi_1^2+R(\gamma-1)\rho\tilde{\theta}V(\frac{\tilde{\rho}}{\rho})+R\rho\tilde{\theta}V(\frac{\theta}{\tilde{\theta}})\right]+\tilde{\theta}_{y_1}\rho\psi_1\left(R\ln\frac{\tilde{\rho}}{\rho}+\frac{R}{\gamma-1}\ln\frac{\theta}{\tilde{\theta}}\right)\\
&=\left(\frac{-\bar{u}_1d_1+d_2}{\tilde{\rho}}\right)_{y_1}\left[\rho\psi_1^2+R(\gamma-1)\rho\tilde{\theta}V(\frac{\tilde{\rho}}{\rho})+R\rho\tilde{\theta}V(\frac{\theta}{\tilde{\theta}})\right]\\
&+\bar{u}_{1y_1}\left[\rho\psi_1^2+R(\gamma-1)\rho\tilde{\theta}V(\frac{\tilde{\rho}}{\rho})+R\rho\tilde{\theta}V(\frac{\theta}{\tilde{\theta}})\right]\\
&+\left(\frac{(\frac{1}{2}\bar{u}_1^2-\frac{R}{\gamma-1}\bar{\theta})d_1-\bar{u}_1d_2+d_3)}{\tilde{\rho}}-\frac{(-\bar{u}_1d_1+d_2)^2}{2\tilde{\rho}^2}\right)_{y_1}\rho\psi_1\Big((\gamma-1)\ln\frac{\tilde{\rho}}{\rho}+\ln\frac{\theta}{\tilde{\theta}}\Big)\\
&+\frac{\gamma-1}{\sqrt{R\gamma}}\bar{\theta}^{\frac{1}{2}}\bar{u}_{1y_1}\rho\psi_1\Big(R\ln\frac{\tilde{\rho}}{\rho}+\frac{R}{\gamma-1}\ln\frac{\theta}{\tilde{\theta}}\Big)\\
&:=\sum_{i=1}^{4}D_i.
\end{aligned}
$$

Due to the positive quantity $\bar{u}_{1y_1}$, term D_2 serves as the dissipation. Terms D_1, D_3, and D_4 consist of approximated waves and their solutions. By appropriately choosing parameters ϵ and δ, we then utilize the smallness of the wave's length to bound them effectively.

We integrate D_1 over $[0,\tau']\times\mathbb{R}\times\mathbb{T}_\epsilon$ with $\tau'\le\tau_1(\epsilon)$ to get that

$$\begin{aligned}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon} D_1\mathrm{d}y\mathrm{d}\tau &\le C\sup_{0\le t\le T}\|(d_1,d_2,d_3)\|_{L^\infty(\mathbb{R})}\int_0^{\tau'}\|\bar{u}_{1y_1}^{\frac{1}{2}}(\phi,\psi_1,\zeta)\|^2\mathrm{d}\tau\\ &\quad + C\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}|(d_1,d_2,d_3)_{y_1}|(\phi^2+\psi_1^2+\zeta^2)\mathrm{d}y\mathrm{d}\tau\\ &\le C_T\frac{\epsilon}{\delta^{\frac{3}{2}}}\int_0^{\tau'}\|\bar{u}_{1y_1}^{\frac{1}{2}}(\phi,\psi_1,\zeta)\|^2\mathrm{d}\tau + C_T\frac{\epsilon}{\delta^{\frac{5}{2}}}\sup_{0\le\tau\le\tau_1(\epsilon)}\|(\phi,\psi_1,\zeta)\|^2.\end{aligned}$$

Then, integrating D_3 over $[0,\tau']\times\mathbb{R}\times\mathbb{T}_\epsilon$, we can obtain

$$\begin{aligned}&\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon} D_3\mathrm{d}y\mathrm{d}\tau\\ &\le \int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon} C\Big(\frac{(\frac{1}{2}\bar{u}_1^2-\frac{R}{\gamma-1}\bar{\theta})d_1-\bar{u}_1d_2+d_3)}{\tilde{\rho}}-\frac{(-\bar{u}_1d_1+d_2)^2}{2\tilde{\rho}^2}\Big)_{y_1}\rho\Big(\psi_1^2+V\Big(\frac{\tilde{\rho}}{\rho}\Big)+V\Big(\frac{\theta}{\tilde{\theta}}\Big)\Big)\mathrm{d}y\mathrm{d}\tau\\ &\le C\sup_{0\le t\le T}\|(d_1,d_2,d_3)\|_{L^\infty(\mathbb{R})}\int_0^{\tau'}\|\bar{u}_{1y_1}^{\frac{1}{2}}(\phi,\psi_1,\zeta)\|^2\mathrm{d}\tau\\ &\quad + C\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}|(d_1,d_2,d_3)_{y_1}|(\phi^2+\psi_1^2+\zeta^2)\mathrm{d}y\mathrm{d}\tau\\ &\le C_T\frac{\epsilon}{\delta^{\frac{3}{2}}}\int_0^{\tau'}\|\bar{u}_{1y_1}^{\frac{1}{2}}(\phi,\psi_1,\zeta)\|^2\mathrm{d}\tau + C_T\frac{\epsilon}{\delta^{\frac{5}{2}}}\sup_{0\le\tau\le\tau_1(\epsilon)}\|(\phi,\psi_1,\zeta)\|^2.\end{aligned}$$

For any $\frac{1}{2}<\tilde{\iota}<1$, and $\iota\in\Big(\frac{1}{2\tilde{\iota}},1\Big)$, $z_1,z_2\sim 1$, we consider the following function

$$f_{\tilde{\iota},\iota}(z_1,z_2)=\tilde{\iota}(z_1-\ln z_1-1)+\tilde{\iota}(\gamma-1)(z_2-\ln z_2-1)-\frac{1}{4\iota\gamma}\Big((\gamma-1)\ln z_2+\ln z_1\Big)^2.$$

It is easy to check that

$$f_{\tilde{\iota},\iota}(1,1)=\partial_{z_1}f_{\tilde{\iota},\iota}(1,1)=\partial_{z_2}f_{\tilde{\iota},\iota}(1,1)=0,\quad \det\nabla_z^2 f_{\tilde{\iota},\iota}(1,1)>0.$$

So we obtain

$$-\frac{R}{4\iota\gamma}\bar{\theta}\bar{u}_{1y_1}\rho\Big((\gamma-1)\ln\frac{\tilde{\rho}}{\rho}+\ln\frac{\theta}{\tilde{\theta}}\Big)^2\ge-\tilde{\iota}R\bar{u}_{1y_1}\rho\bar{\theta}V\left(\frac{\theta}{\tilde{\theta}}\right)-\tilde{\iota}R(\gamma-1)\bar{u}_{1y_1}\rho\bar{\theta}V\left(\frac{\tilde{\rho}}{\rho}\right).$$

Based on the above analysis, it holds that

$$D_4\ge-\iota\bar{u}_{1y_1}\rho\psi_1^2-\frac{R}{4\iota\gamma}\bar{\theta}\bar{u}_{1y_1}\rho\Big((\gamma-1)\ln\frac{\tilde{\rho}}{\rho}+\ln\frac{\theta}{\tilde{\theta}}\Big)^2,$$

and

$$D_2+D_4\ge(1-\iota)\bar{u}_{1y_1}\rho\psi_1^2-\frac{R}{4\iota\gamma}\bar{\theta}\bar{u}_{1y_1}\rho\Big((\gamma-1)\ln\frac{\tilde{\rho}}{\rho}+\ln\frac{\theta}{\tilde{\theta}}\Big)^2$$

$$
\begin{aligned}
&+ R\bar{u}_{1y_1}\rho\tilde{\theta}V\left(\frac{\theta}{\tilde{\theta}}\right) + R(\gamma-1)\bar{u}_{1y_1}\rho\tilde{\theta}V\left(\frac{\tilde{\rho}}{\rho}\right)\\
\geq\ & (1-\iota)\bar{u}_{1y_1}\rho\psi_1^2 + (1-\tilde{\iota})R\bar{u}_{1y_1}\rho\bar{\theta}V\left(\frac{\theta}{\bar{\theta}}\right) + (1-\tilde{\iota})R(\gamma-1)\bar{u}_{1y_1}\rho\bar{\theta}V\left(\frac{\tilde{\rho}}{\rho}\right)\\
&+ (\gamma-1)^2\bar{u}_{1y_1}\rho\Big[\frac{(\frac{1}{2}\bar{u}_1^2 - \frac{R}{\gamma-1}\bar{\theta})d_1 - \bar{u}_1 d_2 + d_3}{\tilde{\rho}} - \frac{(-\bar{u}_1 d_1 + d_2)^2}{2\tilde{\rho}^2}\Big]V\left(\frac{\tilde{\rho}}{\rho}\right)\\
&+ (\gamma-1)\bar{u}_{1y_1}\rho\Big[\frac{(\frac{1}{2}\bar{u}_1^2 - \frac{R}{\gamma-1}\bar{\theta})d_1 - \bar{u}_1 d_2 + d_3)}{\tilde{\rho}} - \frac{(-\bar{u}_1 d_1 + d_2)^2}{2\tilde{\rho}^2}\Big]V\left(\frac{\theta}{\bar{\theta}}\right).
\end{aligned}
$$

Moreover, by integrating the above inequality over the interval $[0,\tau']\times\mathbb{R}\times\mathbb{T}_\epsilon$, we easily find that the last two terms on the right-hand side can be bounded by $C_T\frac{\epsilon}{\delta^{\frac{5}{2}}}\sup_{0\leq\tau\leq\tau_1(\epsilon)}\|(\phi,\zeta)\|^2$.

Collecting the above estimates together, we choose $\frac{\epsilon}{\delta^{\frac{3}{2}}}\ll 1$ to deduce that

$$
\begin{aligned}
&\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\Big\{\tilde{u}_{1y_1}\Big[\rho\psi_1^2 + R(\gamma-1)\rho\tilde{\theta}V(\frac{\tilde{\rho}}{\rho}) + R\rho\tilde{\theta}V(\frac{\theta}{\tilde{\theta}})\Big] + \tilde{\theta}_{y_1}\rho\psi_1\left(R\ln\frac{\tilde{\rho}}{\rho} + \frac{R}{\gamma-1}\ln\frac{\theta}{\tilde{\theta}}\right)\Big\}\mathrm{d}y\mathrm{d}\tau\\
\geq\ & C^{-1}\int_0^{\tau'}\|\bar{u}_{1y_1}^{\frac{1}{2}}(\phi,\psi,\zeta)\|^2\mathrm{d}\tau + \mathcal{R},
\end{aligned}
$$

where

$$
|\mathcal{R}| \leq C_T\frac{\epsilon}{\delta^{\frac{5}{2}}}\sup_{0\leq\tau\leq\tau_1(\epsilon)}\|(\phi,\psi_1,\zeta)\|^2.
$$

Based on the above analysis, we combine (3.16) and (3.17); and then integrate the resulting equation over $[0,\tau']\times\mathbb{R}\times\mathbb{T}_\epsilon$ to deduce that

$$
\begin{aligned}
&\|(\phi,\Psi,\zeta)(\tau')\|^2 + \int_0^{\tau'}\Big[\|\bar{u}_{1y_1}^{1/2}(\phi,\psi_1,\zeta)\|^2 + \epsilon^{1-\alpha}\|\nabla(\Psi,\zeta)\|^2\Big]\mathrm{d}\tau + \int_0^{\tau'}\Big[\epsilon^{\alpha-1}\|Q\|^2 + \epsilon^{1-\alpha}\|\mathrm{div}Q\|^2\Big]\mathrm{d}\tau\\
\leq\ & C\|(\phi_0,\Psi_0,\zeta_0)\|^2 + C\Big|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\frac{(2\mu+\lambda)\epsilon^{1-\alpha}}{\tilde{\rho}}\bar{u}_{1y_1y_1}\phi\psi_1\mathrm{d}y\mathrm{d}\tau\Big|\\
&+ C\Big|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\Big[\frac{2\epsilon^{1-\alpha}}{\theta}\tilde{u}_{1y_1}\zeta(2\mu\psi_{1y_1} + \lambda\mathrm{div}\Psi) + \frac{\kappa\epsilon^{1-\alpha}}{\theta^2}\tilde{\theta}_{y_1}\zeta\zeta_{y_1}\Big]\mathrm{d}y\mathrm{d}\tau\Big|\\
&+ C\Big|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\Big[(2\mu+\lambda)\epsilon^{1-\alpha}\left(\frac{-\bar{u}_1 d_1 + d_2}{\tilde{\rho}}\right)_{y_1y_1}\psi_1 - \frac{\rho}{\tilde{\rho}}\psi_1\left(\frac{3-\gamma}{2\tilde{\rho}}(-\bar{u}_1 d_1 + d_2)^2\right)_{y_1}\Big]\mathrm{d}y\mathrm{d}\tau\Big|\\
&+ C\Big|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\frac{\rho}{\tilde{\rho}}\left((\gamma-1)V\left(\frac{\tilde{\rho}}{\rho}\right) - V\left(\frac{\tilde{\theta}}{\theta}\right)\right)\Big(\kappa\epsilon^{1-\alpha}\bar{\theta}_{y_1y_1} + (2\mu+\lambda)\epsilon^{1-\alpha}\bar{u}_{1y_1}^2\\
&- (2\mu+\lambda)\epsilon^{1-\alpha}\bar{u}_{1y_1y_1}\frac{-\bar{u}_1 d_1 + d_2}{\tilde{\rho}}\Big)\mathrm{d}y\mathrm{d}\tau\Big|
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
&+C\Big|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\left[F_3\frac{\zeta}{\theta}+F_1\frac{\zeta}{\theta}-F_2\left((\gamma-1)V\left(\frac{\tilde\rho}{\rho}\right)-V\left(\frac{\tilde\theta}{\theta}\right)-\frac{\zeta}{\theta}\right)\right]\mathrm{d}y\mathrm{d}\tau\Big|\\
&+C\Big|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\frac{Q}{4\theta\tilde\theta^3}\nabla\left(\zeta^4+4\zeta^3\tilde\theta+6\zeta^2\tilde\theta^2\right)\mathrm{d}y\mathrm{d}\tau\Big|+C\Big|\int_0^{\tau}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\frac{\zeta Q\cdot\nabla\zeta}{\theta^2}\mathrm{d}y\mathrm{d}\tau\Big|\\
&+C\Big|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\frac{\tilde\theta_{y_1}Q_1\zeta}{\theta^2}\mathrm{d}y\mathrm{d}\tau\Big|+C\Big|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\frac{\epsilon^{1-\alpha}}{4b}\mathrm{div}QQ\cdot\nabla\left(\frac{1}{\theta\tilde\theta^3}\right)\mathrm{d}y\mathrm{d}\tau\Big|\\
&+C\Big|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\frac{\zeta Q_1}{\theta\tilde\theta^3}(\tilde\theta^3)_{y_1}\mathrm{d}y\mathrm{d}\tau\Big|+C\Big|\int_0^{\tau}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\frac{\epsilon^{2-2\alpha}}{4a\theta\tilde\theta^3}(\bar\theta^4)_{y_1y_1y_1}Q_1\mathrm{d}y\mathrm{d}\tau\Big|\\
&+C\Big|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\frac{Q_1}{4\theta\tilde\theta^3}\left(\mathcal{P}^4+4\mathcal{P}^3\bar\theta+6\mathcal{P}^2\bar\theta^2+4\mathcal{P}\bar\theta^3\right)_{y_1}\mathrm{d}y\mathrm{d}\tau\Big|+C\mathcal{R}\\
&:=C\|(\phi_0,\Psi_0,\zeta_0)\|^2+\sum_{i=1}^{12}\mathcal{I}_i+C\mathcal{R}.
\end{aligned}
$$

According to the scaling argument in (3.2), we use Lemmas 2.2 and 2.3 to deduce that

$$
\mathcal{I}_1\le C_T\epsilon\sup_{0\le t\le T}\|\bar u_{1x_1x_1}\|_{L^\infty(\mathbb{R})}\sup_{0\le\tau\le\tau_1(\epsilon)}\|(\phi,\psi_1)\|^2\le C_T\frac{\epsilon}{\delta^2}\sup_{0\le\tau\le\tau_1(\epsilon)}\|(\phi,\psi_1)\|^2,
$$

and

$$
\begin{aligned}
\mathcal{I}_2&\le\frac{\epsilon^{1-\alpha}}{13}\int_0^{\tau'}\|(\nabla\Psi,\nabla\zeta)\|^2\mathrm{d}\tau+C\epsilon^{1-\alpha}\int_0^{\tau'}\|(\tilde u_{1y_1},\tilde\theta_{y_1})\zeta\|^2\mathrm{d}\tau\\
&\le\frac{\epsilon^{1-\alpha}}{13}\int_0^{\tau'}\|(\nabla\Psi,\nabla\zeta)\|^2\mathrm{d}\tau+C\epsilon^{1-\alpha}\sup_{0\le\tau\le\tau_1(\epsilon)}\|\bar u_{1y_1}\|_{L^\infty(\mathbb{R})}\int_0^{\tau'}\|\bar u_{1y_1}^{1/2}\zeta\|^2\mathrm{d}\tau\\
&\quad+C_T\epsilon^{1-2\alpha}\sup_{0\le\tau\le\tau(\epsilon)}\left\|\left(\frac{-\bar u_1d_1+d_2}{\tilde\rho}\right)_{y_1}\right\|^2_{L^\infty(\mathbb{R})}\sup_{0\le\tau\le\tau(\epsilon)}\|\zeta\|^2\\
&\le\frac{\epsilon^{1-\alpha}}{13}\int_0^{\tau'}\|(\nabla\Psi,\nabla\zeta)\|^2\mathrm{d}\tau+C_T\frac{\epsilon}{\delta}\int_0^{\tau'}\|\bar u_{1y_1}^{1/2}\zeta\|^2\mathrm{d}\tau+C_T\frac{\epsilon^3}{\delta^5}\sup_{0\le\tau\le\tau_1(\epsilon)}\|\zeta\|^2.
\end{aligned}
$$

For any fixed time T, it follows from the Holder's inequality and Lemmas 2.2 and 2.3 that

$$
\mathcal{I}_3\le\frac{\epsilon^\alpha}{13T}\int_0^{\tau'}\|\psi_1\|^2\mathrm{d}\tau+C_T\epsilon^{2-3\alpha}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\left|\left(\frac{-\bar u_1d_1+d_2}{\tilde\rho}\right)_{y_1y_1}\right|^2\mathrm{d}y\mathrm{d}\tau
$$

$$+ C_T\epsilon^{-\alpha}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\left|\left(\frac{(\bar{u}_1 d_1 - d_2)^2}{\tilde{\rho}}\right)_{y_1}\right|^2 \mathrm{d}y\mathrm{d}\tau$$

$$\leq \frac{1}{3}\sup_{0\leq\tau\leq\tau_1(\epsilon)}\|\psi_1\|^2 + C_T\epsilon^{2-2\alpha}\int_0^t\int_{\mathbb{R}}\left|\left(\frac{-\bar{u}_1 d_1 + d_2}{\tilde{\rho}}\right)_{x_1x_1}\right|^2 \mathrm{d}x_1\mathrm{d}s$$

$$+C_T\epsilon^{-2\alpha}\int_0^t\int_{\mathbb{R}}\left|\left(\frac{(\bar{u}_1 d_1 - d_2)^2}{\tilde{\rho}}\right)_{x_1}\right|^2 \mathrm{d}x_1\mathrm{d}s$$

$$\leq \frac{1}{3}\sup_{0\leq\tau\leq\tau_1(\epsilon)}\|\psi_1\|^2 + C_T\frac{\epsilon^{4-2\alpha}}{\delta^6} + C_T\frac{\epsilon^{4-2\alpha}}{\delta^7}.$$

Moreover, we use Lemmas 2.2 and 2.3 to obtain that

$$\mathcal{I}_4 \leq C\int_0^{\tau'}\left(\epsilon^{1-\alpha}\|(\bar{\theta}_{y_1y_1}, \bar{u}^2_{1y_1})\|_{L^\infty(\mathbb{R})}+\epsilon^{1-\alpha}\left\|\bar{u}_{1y_1y_1}\left(\frac{-\bar{u}_1 d_1 + d_2}{\tilde{\rho}}\right)\right\|_{L^\infty(\mathbb{R})}\right)\|(\phi,\zeta)\|^2\mathrm{d}\tau$$

$$\leq C_T\left(\frac{\epsilon}{\delta^2} + \frac{\epsilon^2}{\delta^{7/2}}+\frac{\epsilon^3}{\delta^4}\right)\sup_{0\leq\tau\leq\tau_1(\epsilon)}\|(\phi,\zeta)\|^2.$$

As for the term $\mathcal{I}_5$, it contains similar terms as $\mathcal{I}_i$ with $1 \leq i \leq 4$. Therefore, it satisfies similar estimates, and for the sake of simplicity, we omit the details.

By using the definition of $\tilde{\theta}$, *a priori* Assumption (3.10) ,and (3.13), we have

$$\mathcal{I}_6 + \mathcal{I}_7 \leq C\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\Big(|Q||\nabla\zeta|(|\zeta| + |\zeta|^3 + |\zeta|^2) + |Q||\zeta^2||\tilde{\theta}_{y_1}|\Big)\mathrm{d}y\mathrm{d}\tau$$

$$\leq C\epsilon^{1-\alpha}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}(|Q||\nabla\zeta| + |Q||\zeta||\tilde{\theta}_{y_1}|)\mathrm{d}y\mathrm{d}\tau$$

$$\leq \frac{\epsilon^{1-\alpha}}{13}\int_0^{\tau'}\|\nabla\zeta\|^2\mathrm{d}s + C\epsilon^{1-\alpha}\int_0^{\tau'}\|Q\|^2\mathrm{d}s + C\epsilon^{1-\alpha}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}|\zeta|^2|\tilde{\theta}_{y_1}|^2\mathrm{d}y\mathrm{d}\tau$$

$$\leq \frac{\epsilon^{1-\alpha}}{13}\int_0^{\tau'}\|\nabla\zeta\|^2\mathrm{d}s + C\epsilon^{1-\alpha}\int_0^{\tau'}\|Q\|^2\mathrm{d}s + C\epsilon^{1-\alpha}\sup_{0\leq\tau\leq\tau_1(\epsilon)}\|\bar{u}_{1y_1}\|_{L^\infty(\mathbb{R})}\int_0^{\tau'}\|\bar{u}^{1/2}_{1y_1}\zeta\|^2\mathrm{d}\tau$$

$$+ C\epsilon^{1-\alpha}\sup_{0\leq\tau\leq\tau_1(\epsilon)}\left\|\mathcal{P}_{y_1}\right\|^2_{L^\infty(\mathbb{R})}\int_0^{\tau'}\|\zeta\|^2\mathrm{d}\tau$$

$$\leq \frac{\epsilon^{1-\alpha}}{13}\int_0^{\tau'}\|\nabla\zeta\|^2\mathrm{d}s + C\epsilon^{1-\alpha}\int_0^{\tau'}\|Q\|^2\mathrm{d}\tau + C_T\frac{\epsilon}{\delta}\int_0^{\tau'}\|\bar{u}^{1/2}_{1y_1}\zeta\|^2\mathrm{d}\tau + C_T\frac{\epsilon^{3-\alpha}}{\delta^5}\sup_{0\leq\tau\leq\tau_1(\epsilon)}\|\zeta\|^2,$$

where we used Sobolev's inequality and *a priori* assumption $\|\zeta\|_{L^\infty} \leq C\|\zeta\|_2 \leq C\epsilon^{1-\alpha}$ in the second line. Moreover, in the last line of the preceding estimate, we employed the expression

$$\mathcal{P} = \frac{\gamma-1}{R\tilde{\rho}}\left(\left(\frac{1}{2}\bar{u}_1^2 - \frac{R}{\gamma-1}\bar{\theta}\right)d_1 - \bar{u}_1 d_2 + d_3\right) - \frac{\gamma-1}{2R\tilde{\rho}^2}(-\bar{u}_1 d_1 + d_2)^2$$

which consist of different waves.

Similarly, it holds that

$$\begin{aligned}
\mathcal{I}_8 + \mathcal{I}_{10} &\leq C\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon} |\tilde{\theta}_{y_1}||Q||\zeta| \mathrm{d}y\mathrm{d}\tau \\
&\leq \frac{\epsilon^{\alpha-1}}{3}\int_0^{\tau'} \|Q\|^2 \mathrm{d}y\mathrm{d}\tau + C\epsilon^{1-\alpha}\sup_{0\leq\tau\leq\tau_1(\epsilon)}\left(\|\bar{\theta}_{y_1}\|^2_{L^\infty(\mathbb{R})} + \|\mathcal{P}_{y_1}\|^2_{L^\infty(\mathbb{R})}\right)\int_0^{\tau'}\|\zeta\|^2\mathrm{d}\tau \\
&\leq \frac{\epsilon^{\alpha-1}}{3}\int_0^{\tau'} \|Q\|^2 \mathrm{d}y\mathrm{d}\tau + C_T\left(\frac{\epsilon}{\delta^2} + \frac{\epsilon^3}{\delta^5}\right)\sup_{0\leq\tau\leq\tau_1(\epsilon)}\|\zeta\|^2.
\end{aligned}$$

By using Holder's inequality, Sobolev's inequality, Young's inequality and *a priori* Assumption (3.10), we have

$$\begin{aligned}
\mathcal{I}_9 &\leq \left|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon} \epsilon^{1-\alpha}\mathrm{div}Q Q\cdot\nabla\left(\frac{1}{4b\theta\tilde{\theta}^3}\right)\mathrm{d}y\mathrm{d}\tau\right| \\
&\leq C\epsilon^{1-\alpha}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}(|Q||\nabla\zeta||\mathrm{div}Q| + |Q_1||\tilde{\theta}_{y_1}||\mathrm{div}Q|)\mathrm{d}y\mathrm{d}\tau \\
&\leq C\epsilon^{1-\alpha}\int_0^{\tau'}\|\mathrm{div}Q\|\|\nabla\zeta\|\|Q\|_{L^\infty(\mathbb{R}\times\mathbb{T}_\epsilon)}\mathrm{d}s + \frac{\epsilon^{\alpha-1}}{3}\int_0^{\tau'}\|Q\|^2\mathrm{d}s + C\epsilon^{2-2\alpha}\int_0^{\tau'}\|\mathrm{div}Q\|^2\|\tilde{\theta}_{y_1}\|^2\mathrm{d}\tau \\
&\leq C\epsilon^{1-\alpha}\int_0^{\tau'}\|\mathrm{div}Q\|\|\nabla\zeta\|\|Q\|_2\mathrm{d}\tau + \frac{\epsilon^{\alpha-1}}{3}\int_0^{\tau'}\|Q\|^2\mathrm{d}\tau \\
&\quad + C\epsilon^{2-2\alpha}\sup_{0\leq\tau\leq\tau_1(\epsilon)}\|\bar{\theta}_{y_1}\|^2_{L^\infty(\mathbb{R})}\int_0^{\tau'}\|\mathrm{div}Q\|^2\mathrm{d}\tau + C\epsilon^{2-2\alpha}\sup_{0\leq\tau\leq\tau_1(\epsilon)}\|\mathcal{P}_{y_1}\|^2_{L^\infty(\mathbb{R})}\int_0^{\tau'}\|\mathrm{div}Q\|^2\mathrm{d}\tau \\
&\leq C\epsilon^{3-3\alpha}\int_0^{\tau'}\|\mathrm{div}Q\|^2\mathrm{d}\tau + C\epsilon^{3-3\alpha}\int_0^{\tau'}\|\nabla\zeta\|^2\mathrm{d}\tau + \frac{\epsilon^{\alpha-1}}{3}\int_0^{\tau'}\|Q\|^2\mathrm{d}\tau + C_T\left(\frac{\epsilon^2}{\delta^2} + \frac{\epsilon^4}{\delta^5}\right)\int_0^{\tau'}\|\mathrm{div}Q\|^2\mathrm{d}\tau,
\end{aligned}$$

and

$$\mathcal{I}_{11} + \mathcal{I}_{12} \leq C\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\left(\epsilon^{2-2\alpha}|Q||(\bar{\theta}^4)_{y_1y_1y_1}| + |Q||\mathcal{P}_{y_1}| + |Q||\mathcal{P}||\bar{\theta}_{y_1}|\right)\mathrm{d}y\mathrm{d}\tau$$

$$\leq \frac{\epsilon^{\alpha-1}}{3}\int_0^{\tau'}\|Q\|^2 d\tau + C\epsilon^{5(1-\alpha)}\int_0^{\tau'}\|(\bar{\theta}^4)_{y_1y_1y_1}\|^2 d\tau$$
$$+ C\epsilon^{1-\alpha}\int_0^{\tau'}\|\mathcal{P}_{y_1}\|^2 d\tau + C\epsilon^{1-\alpha}\sup_{0\leq t\leq T}\|\mathcal{P}\|^2_{L^\infty(\mathbb{R})}\int_0^{\tau'}\|\bar{\theta}_{y_1}\|^2 d\tau$$
$$\leq \frac{\epsilon^{\alpha-1}}{3}\int_0^{\tau'}\|Q\|^2 d\tau + C_T\frac{\epsilon^{5-2\alpha}}{\delta^5} + C_T\frac{\epsilon^{3-2\alpha}}{\delta^4}.$$

Substituting the above estimates into (3.18) and taking ϵ and $\frac{\epsilon}{\delta^{5/2}}$ $(\ll 1)$ suitably small, we can complete the proof of the (3.14) in Lemma 3.1. □

Lemma 3.2. *There exists a positive constant C_T such that*

$$\sup_{0\leq\tau\leq\tau_1(\epsilon)}\|(\nabla\phi, \nabla\Psi, \nabla\zeta)(\tau)\|^2 + \int_0^{\tau_1(\epsilon)}\Big(\|\bar{u}_{1y_1}^{1/2}\nabla\phi\|^2 + \epsilon^{1-\alpha}\|(\nabla^2\Psi, \nabla^2\zeta)\|^2\Big)d\tau$$
$$+\int_0^{\tau_1(\epsilon)}\Big[\epsilon^{\alpha-1}\|\nabla Q\|^2 + \epsilon^{1-\alpha}\|\nabla \mathrm{div} Q\|^2\Big]d\tau \leq C\|(\phi_0, \Psi_0, \zeta_0)\|_1^2 + C_T\Big(\frac{\epsilon^{3-2\alpha}}{\delta^4} + \frac{\epsilon^{4-2\alpha}}{\delta^7}\Big). \tag{3.19}$$

Proof. Applying the operator ∇ to the first equation of (3.4) and then multiplying the resulting equation by $\frac{R\theta}{\rho}\nabla\phi$ yields

$$\begin{aligned}
&\Big(R\frac{\theta}{\rho}\frac{|\nabla\phi|^2}{2}\Big)_\tau + \mathrm{div}\Big(R\frac{\theta}{\rho}\boldsymbol{u}\frac{|\nabla\phi|^2}{2} - R\theta\phi_{y_i}\nabla\psi_i\Big) + (R\theta\nabla\phi\cdot\nabla\psi_i)_{y_i}\\
&+\frac{R(\gamma-1)\theta}{\rho}\bar{u}_{1y_1}\frac{|\nabla\phi|^2}{2} + R\frac{\theta}{\rho}\bar{u}_{1y_1}\phi_{y_1}^2 + R\theta\nabla\phi\cdot\Delta\Psi\\
&= -\frac{R(\gamma-1)\theta}{\rho}\mathrm{div}\Psi\frac{|\nabla\phi|^2}{2} - \frac{R\theta}{\rho}\phi_{y_i}\nabla\phi\cdot\nabla\psi_i + R\zeta_{y_i}\phi\cdot\nabla\psi_i - R\phi_{y_i}\nabla\zeta\cdot\nabla\psi_i\\
&+\frac{\gamma-1}{\rho^2}\frac{|\nabla\phi|^2}{2}\epsilon^{1-\alpha}\Big[\kappa\Delta\zeta + \kappa\tilde{\theta}_{y_1y_1} + \frac{\mu}{2}|\nabla\Psi + (\nabla\Psi)^\top|^2 + 2\tilde{u}_{1y_1}(2\mu\psi_{1y_1} + \lambda\mathrm{div}\Psi)\\
&+\lambda(\mathrm{div}\Psi)^2 + (2\mu+\lambda)(\tilde{u}_1\tilde{u}_{1y_1})_{y_1}\Big] - \frac{R(\gamma-1)\theta}{\rho}\Big(\frac{-\bar{u}_1d_1 + d_2}{\tilde{\rho}}\Big)_{y_1}\frac{|\nabla\phi|^2}{2}\\
&-\frac{R\theta}{\rho}\Big(\frac{-\bar{u}_1d_1+d_2}{\tilde{\rho}}\Big)_{y_1}\phi_{y_1}^2 + R\tilde{\theta}_{y_1}\nabla\phi\cdot\nabla\psi_1 - R\tilde{\theta}_{y_1}\nabla\phi\cdot\Psi_{y_1} - \frac{R\theta}{\rho}\tilde{\rho}_{y_1}\phi_{y_1}\mathrm{div}\Psi\\
&-\frac{R\theta}{\rho}\tilde{\rho}_{y_1y_1}\psi_1\phi_{y_1} - \frac{R\theta}{\rho}\tilde{\rho}_{y_1}\nabla\phi\cdot\nabla\psi_1 - \frac{R\theta}{\rho}\tilde{u}_{1y_1y_1}\phi\phi_{y_1}\\
&-\frac{\gamma-1}{\rho^2}\frac{|\nabla\phi|^2}{2}\mathrm{div}\mathrm{Q} - \frac{\gamma-1}{\rho^2}\frac{|\nabla\phi|^2}{2}\bar{q}_{y_1}\\
&:= \mathcal{J}_1(\tau, y).
\end{aligned}\tag{3.20}$$

Multiplying the second equation of (3.4) by $-\Delta\Psi$ results in

$$
\begin{aligned}
&\Big(\rho\frac{|\nabla\Psi|^2}{2}\Big)_\tau - \mathrm{div}\Big(\rho\psi_{i\tau}\nabla\psi_i + \rho u_i\psi_{jy_i}\nabla\psi_j - \rho\boldsymbol{u}\frac{|\nabla\Psi|^2}{2}\\
&+(\mu+\lambda)\epsilon^{1-\alpha}\mathrm{div}\Psi\nabla\mathrm{div}\Psi - (\mu+\lambda)\epsilon^{1-\alpha}\mathrm{div}\Psi\Delta\Psi\Big)\\
&+\mu\epsilon^{1-\alpha}|\Delta\Psi|^2 + (\mu+\lambda)\epsilon^{1-\alpha}|\nabla\mathrm{div}\Psi|^2 - R\theta\nabla\phi\cdot\Delta\Psi - R\rho\nabla\zeta\cdot\Delta\Psi\\
&= -\phi_{y_i}\Psi_{y_i}\cdot\Psi_\tau - \tilde\rho_{y_1}\Psi_{y_1}\cdot\Psi_\tau - u_i\psi_{jy_i}\nabla\rho\cdot\nabla\psi_j - \rho\psi_{jy_i}\nabla\psi_i\cdot\nabla\psi_j - \rho\tilde u_{1y_1}|\Psi_{y_1}|^2 \qquad (3.21)\\
&+\rho\tilde u_{1y_1}\psi_1\Delta\psi_1 + R\tilde\rho_{y_1}\Big(\theta - \frac{\rho}{\tilde\rho}\tilde\theta\Big)\Delta\psi_1 - (2\mu+\lambda)\epsilon^{1-\alpha}\Big(\frac{-\bar u_1 d_1 + d_2}{\tilde\rho}\Big)_{y_1y_1}\Delta\psi_1\\
&+(2\mu+\lambda)\epsilon^{1-\alpha}\frac{\bar u_{1y_1y_1}}{\tilde\rho}\phi\Delta\psi_1 + \Big(\frac{3-\gamma}{2\tilde\rho}(-\bar u_1 d_1 + d_2)^2\Big)_{y_1}\frac{\rho}{\tilde\rho}\Delta\psi_1 \qquad (3.22)\\
&:= \mathcal{J}_2(\tau, y).
\end{aligned}
$$

We multiply the third equation of (3.4) by $-\frac{1}{\theta}\Delta\zeta$ to get

$$
\begin{aligned}
&\Big(\frac{R}{\gamma-1}\frac{\rho}{\theta}\frac{|\nabla\zeta|^2}{2}\Big)_\tau - \mathrm{div}\Big(\frac{R}{\gamma-1}\frac{\rho}{\theta}\zeta_\tau\nabla\zeta + \frac{R}{\gamma-1}\frac{\rho}{\theta}u_i\zeta_{y_i}\nabla\zeta - \frac{R}{\gamma-1}\frac{\rho}{\theta}\boldsymbol{u}\frac{|\nabla\zeta|^2}{2}\\
&+R\rho\mathrm{div}\Psi\nabla\zeta + R\rho\nabla\psi_i\zeta_{y_i} + \frac{1}{\theta}\mathrm{div}Q\nabla\zeta\Big) + (R\rho\nabla\psi_i\cdot\nabla\zeta)_{y_i} + \frac{\kappa\epsilon^{1-\alpha}}{\theta}|\Delta\zeta|^2 + R\rho\nabla\zeta\cdot\Delta\Psi \qquad (3.23)\\
&= \sum_{i=1}^{4}\mathcal{J}_{3,i}(\tau, y),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{J}_{3,1}(\tau, y) :=& -\frac{R}{\gamma-1}\Big(\frac{1}{\theta}\nabla\phi\cdot\nabla\zeta\zeta_\tau + \frac{1}{\theta}\tilde\rho_{y_1}\nabla\zeta_{y_1}\zeta_\tau - \frac{\rho}{\theta^2}|\nabla\zeta|^2\zeta_\tau - \frac{\rho}{\theta^2}\tilde\theta_{y_1}\zeta_{y_1}\zeta_\tau\\
&+\frac{1}{\theta}u_i\zeta_{y_i}\nabla\rho\cdot\nabla\zeta - \frac{\rho}{\theta^2}u_i\zeta_{y_i}\nabla\theta\cdot\nabla\zeta + \frac{\rho}{\theta}\zeta_{y_i}\nabla\psi_i\cdot\nabla\zeta + \frac{\rho}{\theta}\tilde u_{1y_1}|\zeta_{y_1}|^2\Big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_{3,2}(\tau, y) :=& R\frac{\rho}{\theta}\mathrm{div}\Psi\frac{|\nabla\zeta|^2}{2} + R\frac{\rho}{\theta}\tilde u_{1y_1}\frac{|\nabla\zeta|^2}{2} - R\mathrm{div}\Psi\nabla\phi\cdot\nabla\zeta - R\tilde\rho_{y_1}\zeta_{y_1}\mathrm{div}\Psi\\
&+R\phi_{y_i}\nabla\psi_i\cdot\nabla\zeta + R\tilde\rho_{y_1}\nabla\psi_1\cdot\nabla\zeta - R\nabla\phi\cdot\nabla\psi_i\zeta_{y_i} - R\tilde\rho_{y_1}\psi_{iy_1}\zeta_{y_i}\\
&+\frac{R}{\gamma-1}\frac{\rho}{\theta}\tilde\theta_{y_1}\psi_1\Delta\zeta + \frac{R\rho}{\theta}\tilde u_{1y_1}\zeta\Delta\zeta + \frac{3}{2}\frac{|\nabla\zeta|^2}{\theta^2}\mathrm{div}Q + \frac{\tilde\theta_{y_1}}{\theta^2}\mathrm{div}Q\nabla\zeta\\
&+\frac{|\nabla\zeta|^2}{2\theta^2}\bar q_{y_1} - \frac{1}{\theta}\nabla\mathrm{div}Q\cdot\nabla\zeta,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_{3,3}(\tau, y) :=& -\epsilon^{1-\alpha}\Big[\frac{1}{\theta^2}\frac{|\nabla\zeta|^2}{2}\Big(\kappa\Delta\zeta + \kappa\tilde\theta_{y_1y_1} + \frac{\mu}{2}|\nabla\Psi + (\nabla\Psi)^\top|^2 + \lambda(\mathrm{div}\Psi)^2\\
&+2\tilde u_{1y_1}(2\mu\psi_{1y_1} + \lambda\mathrm{div}\Psi) + (2\mu+\lambda)(\tilde u_1\tilde u_{1y_1})_{y_1}\Big) + \frac{\mu}{2\theta}|\nabla\Psi + (\nabla\Psi)^\top|^2\Delta\zeta\\
&+\frac{\lambda}{\theta}(\mathrm{div}\Psi)^2\Delta\zeta + \frac{2\tilde u_{1y_1}}{\theta}(2\mu\psi_{1y_1} + \lambda\mathrm{div}\Psi)\Delta\zeta\Big],
\end{aligned}
$$

$$\mathcal{J}_{3,4}(\tau,y) := -(F_1+F_2+F_3)\frac{\Delta\zeta}{\theta}.$$

Applying the operator ∇ to the fourth equation of (3.4) and then multiplying the resulting equation by $\frac{a\epsilon^{1-\alpha}}{4\theta\tilde{\theta}^3}\nabla Q$ we obtain

$$\begin{aligned}
&\frac{a\epsilon^{\alpha-1}}{4b\theta\tilde{\theta}^3}|\nabla Q|^2 + \mathrm{div}\left(\frac{\nabla Q\cdot\nabla\zeta}{\theta}\right) - \frac{\epsilon^{1-\alpha}}{4b}\mathrm{div}\left(\frac{\nabla Q\cdot\nabla\mathrm{div}Q}{\theta\tilde{\theta}^3}\right) + \frac{\epsilon^{1-\alpha}}{4b}\frac{|\nabla\mathrm{div}Q|^2}{\theta\tilde{\theta}^3}\\
&= \frac{1}{\theta}\nabla\mathrm{div}Q\cdot\nabla\zeta - \frac{\nabla Q_i}{4\theta\tilde{\theta}^3}\cdot\nabla\left(\zeta^4+4\zeta^3\tilde{\theta}+6\zeta^2\tilde{\theta}^2\right)_{y_i} - \frac{\nabla Q_i\cdot\nabla\zeta\zeta_{yi}}{\theta^2}\\
&\quad - \frac{Q_{1y_1}\zeta_{y_1}}{\theta^2}\tilde{\theta}_{y_1} + \epsilon^{1-\alpha}\mathrm{div}Q_{y_i}\nabla Q\cdot\nabla\left(\frac{1}{4b\theta\tilde{\theta}^3}\right) - \frac{6\zeta_{y_1}Q_{1y_1}}{\theta\tilde{\theta}}\tilde{\theta}_{y_1}\\
&\quad - \frac{3\zeta Q_{1y_1}}{\theta\tilde{\theta}}\tilde{\theta}_{y_1y_1} - \frac{6\zeta Q_{1y_1}}{\theta\tilde{\theta}^2}\tilde{\theta}^2_{y_1} + \frac{\epsilon^{2-2\alpha}}{4a\theta\tilde{\theta}^3}(\bar{\theta}^4)_{y_1y_1y_1y_1}Q_{1y_1}\\
&\quad - \frac{Q_{1y_1}}{4\theta\tilde{\theta}^3}\left(\mathcal{P}^4+4\mathcal{P}^3\bar{\theta}+6\mathcal{P}^2\bar{\theta}^2+4\mathcal{P}\bar{\theta}^3\right)_{y_1y_1}\\
&:= \mathcal{J}_4(\tau,y).
\end{aligned} \tag{3.24}$$

Now, we add Eqs (3.20)–(3.24) together. Then, we integrate the resulting equation over the domain $[0,\tau']\times\mathbb{R}\times\mathbb{T}_\epsilon$ to obtain

$$\begin{aligned}
&\|(\nabla\phi,\nabla\Psi,\nabla\zeta)(\tau')\|^2 + \int_0^{\tau'}\Big[\|\bar{u}_{1y_1}^{1/2}(\nabla\phi)\|^2\\
&+\epsilon^{1-\alpha}\|(\nabla^2\Psi,\nabla^2\zeta)\|^2\Big]d\tau + \int_0^{\tau'}\Big[\epsilon^{\alpha-1}\|\nabla Q\|^2 + \epsilon^{1-\alpha}\|\nabla\mathrm{div}Q\|^2\Big]d\tau\\
&\le C\|(\nabla\phi_0,\nabla\Psi_0,\nabla\zeta_0)\|^2 + C\Big|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\Big[\mathcal{J}_1(\tau,y)+\mathcal{J}_2(\tau,y)+\sum_{i=1}^4\mathcal{J}_{3,i}(\tau,y)+\mathcal{J}_4(\tau,y)\Big]dyd\tau\Big|,
\end{aligned} \tag{3.25}$$

where we used the cancellations in the flux terms.

Now we are in a position to estimate the right-hand terms in Eq (3.25). For simplicity, we will focus on estimating some typical terms, while others can be treated similarly.

Straightforward calculations give that

$$\begin{aligned}
&C\Big|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\frac{(\gamma-1)\lambda\epsilon^{1-\alpha}}{\rho^2}\frac{|\nabla\phi|^2}{2}(\mathrm{div}\Psi)^2dyd\tau\Big|\\
&\le C\epsilon^{1+\alpha}\int_0^{\tau'}\|\nabla\phi\|_{L^2(\mathbb{R}\times\mathbb{T})}\|\nabla\phi\|_{L^4(\mathbb{R}\times\mathbb{T})}\|\nabla\Psi\|^2_{L^8(\mathbb{R}\times\mathbb{T})}d\tau\\
&\le C\epsilon^{1+\alpha}\int_0^{\tau'}\|\nabla\phi\|^{\frac{3}{2}}_{L^2(\mathbb{R}\times\mathbb{T})}\|\nabla\phi\|^{\frac{1}{2}}_{H^1(\mathbb{R}\times\mathbb{T})}\|\nabla\Psi\|^2_{H^1(\mathbb{R}\times\mathbb{T})}d\tau
\end{aligned}$$

$$\leq C\epsilon^{1-\alpha}\int_0^{\tau'}\|\nabla\phi\|_{L^2(\mathbb{R}\times\mathbb{T}_\epsilon)}^{\frac{3}{2}}\|\nabla\phi\|_{H^1(\mathbb{R}\times\mathbb{T}_\epsilon)}^{\frac{1}{2}}\|\nabla\Psi\|_{H^1(\mathbb{R}\times\mathbb{T}_\epsilon)}^2\,\mathrm{d}\tau$$
$$\leq C\epsilon^{3-3\alpha}\int_0^{\tau'}\|\nabla\Psi\|_1^2\,\mathrm{d}\tau.$$

Using the definition of $\bar{q}$, and Lemmas 2.2 and 2.3, it holds that

$$C\Big|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\Big(\frac{\gamma-1}{\rho^2}\frac{|\nabla\phi|^2}{2}\mathrm{divQ}-\frac{\gamma-1}{\rho^2}\frac{|\nabla\phi|^2}{2}\bar{q}_{y_1}\Big)\mathrm{d}y\mathrm{d}\tau\Big|$$
$$\leq\frac{\epsilon^{\alpha-1}}{13}\int_0^{\tau'}\|\nabla Q\|^2\mathrm{d}\tau+C\epsilon^{1-\alpha}\int_0^{\tau'}\|\nabla\phi\|_{L^4(\mathbb{R}\times\mathbb{T}_\epsilon)}^4\mathrm{d}\tau$$
$$+C\sup_{0\leq\tau\leq\tau_1(\epsilon)}\|(\bar{\theta}^4)_{y_1y_1}\|_{L^\infty(\mathbb{R})}\int_0^{\tau'}\|\nabla\phi\|^2\mathrm{d}\tau$$
$$\leq\frac{\epsilon^{\alpha-1}}{13}\int_0^{\tau'}\|\nabla Q\|^2\mathrm{d}\tau+C_T\Big(\frac{\epsilon^{2\alpha}}{\delta^2}+\epsilon^{3-3\alpha}\Big)\int_0^{\tau'}\|\nabla\phi\|^2\mathrm{d}\tau,$$

where we used Sobolev inequalities

$$\|\nabla\phi\|_{L^4(\mathbb{R}\times\mathbb{T}_\epsilon)}=\epsilon^{\frac{\alpha}{2}}\|\nabla\phi\|_{L^4(\mathbb{R}\times\mathbb{T})}\leq C\epsilon^{\frac{\alpha}{2}}\|\nabla\phi\|_{L^2(\mathbb{R}\times\mathbb{T})}^{1/2}\|\nabla\phi\|_{H^1(\mathbb{R}\times\mathbb{T})}^{1/2}=C\|\nabla\phi\|_{L^2(\mathbb{R}\times\mathbb{T}_\epsilon)}^{1/2}\|\nabla\phi\|_{H^1(\mathbb{R}\times\mathbb{T}_\epsilon)}^{1/2},$$

and

$$\|\nabla\Psi\|_{L^8(\mathbb{R}\times\mathbb{T}_\epsilon)}=\epsilon^{\frac{3}{4}\alpha}\|\nabla\Psi\|_{L^8(\mathbb{R}\times\mathbb{T})}\leq C\epsilon^{\frac{3}{4}\alpha}\|\nabla\Psi\|_{H^1(\mathbb{R}\times\mathbb{T})}^{3/4}\|\nabla\Psi\|_{L^2(\mathbb{R}\times\mathbb{T})}^{1/4}=C\|\nabla\Psi\|_{H^1(\mathbb{R}\times\mathbb{T}\epsilon)}^{3/4}\|\nabla\Psi\|_{L^2(\mathbb{R}\times\mathbb{T}_\epsilon)}^{1/4}.$$

We use the definition of $\tilde{u}_1$ and Lemmas 2.2 and 2.3 to get that

$$C\Big|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\rho\tilde{u}_{1y_1}\psi_1\Delta\psi_1\mathrm{d}y\mathrm{d}\tau\Big|\leq\frac{\epsilon^{1-\alpha}}{150}\int_0^{\tau'}\|\nabla^2\psi_1\|^2\mathrm{d}\tau+C\epsilon^{\alpha-1}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\Big[|\bar{u}_{1y_1}\psi_1|^2+\Big|\Big(\frac{-\bar{u}_1d_1+d_2}{\tilde{\rho}}\Big)_{y_1}\psi_1\Big|^2\Big]\mathrm{d}y\mathrm{d}\tau$$
$$\leq\frac{\epsilon^{1-\alpha}}{150}\int_0^{\tau'}\|\nabla^2\psi_1\|^2\mathrm{d}\tau+C\epsilon^{\alpha-1}\sup_{0\leq\tau\leq\tau_1(\epsilon)}\|\psi_1\|^2\int_0^{\tau'}\|\bar{u}_{1y_1}\|_{L^\infty(\mathbb{R})}^2\mathrm{d}\tau$$
$$+C\epsilon^{3\alpha-1}\sup_{0\leq t\leq T}\Big\|\Big(\frac{-\bar{u}_1d_1+d_2}{\tilde{\rho}}\Big)_{x_1}\Big\|_{L^\infty(\mathbb{R})}^2\int_0^{\tau'}\|\psi_1\|^2\mathrm{d}\tau$$
$$\leq\frac{\epsilon^{1-\alpha}}{150}\int_0^{\tau'}\|\nabla^2\psi_1\|^2\mathrm{d}\tau+C_T\left(\frac{\epsilon^{2\alpha-1}}{\delta}+\frac{\epsilon^{1+2\alpha}}{\delta^5}\right)\sup_{0\leq\tau\leq\tau_1(\epsilon)}\|\psi_1\|^2.$$

Similarly, it holds that

$$
\begin{aligned}
&C\Big|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\Big(\frac{3-\gamma}{2\tilde{\rho}}(-\bar{u}_1d_1+d_2)^2\Big)_{y_1}\frac{\rho}{\tilde{\rho}}\Delta\psi_1\mathrm{d}y\mathrm{d}\tau\Big|\\
&\leq\frac{\epsilon^{1-\alpha}}{160}\int_0^{\tau'}\|\nabla^2\psi_1\|^2\mathrm{d}\tau+C\epsilon^{-1}\int_0^{\tau'}\int_{\mathbb{R}}\Big|\Big(\frac{(-\bar{u}_1d_1+d_2)^2}{\tilde{\rho}}\Big)_{x_1}\Big|^2\mathrm{d}x_1\mathrm{d}s\\
&\leq\frac{\epsilon^{1-\alpha}}{150}\int_0^{\tau'}\|\nabla^2\psi_1\|^2\mathrm{d}\tau+C_T\frac{\epsilon^3}{\delta^7}.
\end{aligned}
$$

Using the definition of $\tilde{\theta}$, it holds that

$$
\begin{aligned}
&C\Big|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}R\tilde{\theta}_{1y_1}\nabla\phi\cdot\nabla\psi_1\mathrm{d}y\mathrm{d}\tau\Big|\leq C\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\Big(|\bar{\theta}_{y_1}+\mathcal{P}_{y_1}||\nabla\phi||\nabla\psi_1|\Big)\mathrm{d}y\mathrm{d}\tau\\
&\leq\frac{1}{150}\int_0^{\tau'}\|\bar{u}_{1y_1}^{1/2}\nabla\phi\|^2\mathrm{d}\tau+C\int_0^{\tau'}\|\bar{\theta}_{y_1}\|_{L^\infty(\mathbb{R})}\|\nabla\psi_1\|^2\mathrm{d}\tau+C_T\frac{\epsilon^{1+\alpha}}{\delta^{5/2}}\int_0^{\tau'}(\|\nabla\phi\|^2+\|\nabla\psi_1\|^2)\mathrm{d}\tau\\
&\leq\frac{1}{150}\int_0^{\tau'}\|\bar{u}_{1y_1}^{1/2}\nabla\phi\|^2\mathrm{d}\tau+C_T\Big(\frac{\epsilon^\alpha}{\delta}+\frac{\epsilon^{1+\alpha}}{\delta^{5/2}}\Big)\int_0^{\tau'}\|\nabla\psi_1\|^2\mathrm{d}\tau+C_T\frac{\epsilon^{1+\alpha}}{\delta^{5/2}}\int_0^{\tau'}\|\nabla\phi_1\|^2\mathrm{d}\tau.
\end{aligned}
$$

When it comes to the estimate for $J_4(\tau,y)$, we just focus on the following two terms, and others can be well controlled. Specifically, we have

$$
\begin{aligned}
&C\left|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\frac{\nabla Q_i}{4\theta\tilde{\theta}^3}\cdot\nabla\left(\zeta^4+4\zeta^3\tilde{\theta}+6\zeta^2\tilde{\theta}^2\right)_{y_i}-\frac{\nabla Q_i\cdot\nabla\zeta\zeta_{yi}}{\theta^2}\mathrm{d}y\mathrm{d}\tau\right|\\
&\leq C\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}|\nabla Q||\zeta|\left(|\nabla\zeta|^2+|\nabla^2\zeta|+|\nabla\zeta||\tilde{\theta}_{y_1}|+|\zeta||\tilde{\theta}_{y_1y_1}|+|\zeta||\tilde{\theta}^2_{y_1}|\right)\mathrm{d}y\mathrm{d}\tau\\
&\leq C\epsilon^{1-\alpha}\int_0^{\tau'}\|\nabla\zeta\|^2_{L^4}\|\nabla Q\|\mathrm{d}\tau+\epsilon^{1-\alpha}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}|\nabla Q||\nabla^2\zeta|\mathrm{d}y\mathrm{d}\tau\\
&\quad+\epsilon^{2-2\alpha}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}|Q_{1y_1}\tilde{\theta}_{y_1y_1}|\mathrm{d}y\mathrm{d}\tau+C\epsilon^{1-\alpha}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}|Q_{1y_1}\zeta_{y_1}\bar{\theta}_{y_1}|\mathrm{d}y\mathrm{d}\tau\\
&\quad+C\epsilon^{1-\alpha}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}|Q_{1y_1}\zeta_{y_1}\mathcal{P}_{y_1}|\mathrm{d}y\mathrm{d}\tau+C\epsilon^{2-2\alpha}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}|Q_{1y_1}\bar{\theta}^2_{y_1}|\mathrm{d}y\mathrm{d}\tau+C\epsilon^{2-2\alpha}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}|Q_{1y_1}\mathcal{P}^2_{y_1}|\mathrm{d}y\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le C\epsilon^{2-2\alpha}\int_0^{\tau'}\|\nabla\zeta\|^2\|\nabla\zeta\|_1^2\mathrm{d}\tau + C\epsilon^{2-2\alpha}\int_0^{\tau'}\|\nabla^2\zeta\|^2\mathrm{d}\tau + \frac{\epsilon^{\alpha-1}}{16}\int_0^{\tau'}\|\nabla Q\|^2\mathrm{d}\tau \\
&\quad + C\epsilon^{1-\alpha}\sup_{0\le\tau\le\tau_1(\epsilon)}\|\bar{\theta}_{y_1}\|_{L^\infty(\mathbb{R})}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}|Q_{1y_1}\zeta_{y_1}|\mathrm{d}y\mathrm{d}\tau + C\epsilon^{1-\alpha}\sup_{0\le\tau\le\tau_1(\epsilon)}\|\mathcal{P}_{y_1}\|_{L^\infty(\mathbb{R})}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}|Q_{1y_1}\zeta_{y_1}|\mathrm{d}y\mathrm{d}\tau \\
&\quad + C\epsilon^{3-3\alpha}\int_0^{\tau'}\Big(\|\mathcal{P}_{y_1y_1}\|^2 + \|\bar{\theta}_{y_1y_1}\|^2 + \|\bar{\theta}_{y_1}\|^4 + \|\mathcal{P}_{y_1}\|^4\Big)\mathrm{d}\tau \\
&\le C\epsilon^{4-4\alpha}\int_0^{\tau'}\|\nabla\zeta\|^2\mathrm{d}\tau + C\epsilon^{2-2\alpha}\int_0^{\tau'}\|\nabla^2\zeta\|^2\mathrm{d}\tau + \frac{\epsilon^{\alpha-1}}{13}\int_0^{\tau'}\|\nabla Q\|^2\mathrm{d}\tau \\
&\quad + C_T\Big(\frac{\epsilon^{3-\alpha}}{\delta^2} + \frac{\epsilon^{5-\alpha}}{\delta^5}\Big)\int_0^{\tau'}\|\nabla\zeta\|^2\mathrm{d}\tau + C_T\Big(\frac{\epsilon^{5-2\alpha}}{\delta^6} + \frac{\epsilon^{3-2\alpha}}{\delta^3} + \frac{\epsilon^{3-2\alpha}}{\delta^2} + \frac{\epsilon^{7-2\alpha}}{\delta^8}\Big),
\end{aligned}
$$

where we have used the facts $\|\nabla\zeta\| \le \|\nabla\zeta\|^{1/2}\|\nabla\zeta\|_1^{1/2}$ and the *a priori* Assumption (3.10).

Moreover, the standard calculations give that

$$
\begin{aligned}
&C\left|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\epsilon^{1-\alpha}\mathrm{div}Q_{y_i}\nabla Q\cdot\nabla\left(\frac{1}{4b\theta\tilde{\theta}^3}\right)\mathrm{d}y\mathrm{d}\tau\right| \\
&\le C\epsilon^{1-\alpha}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}(|\nabla Q||\nabla\zeta||\nabla\mathrm{div}Q| + |Q_{1y_1}||\tilde{\theta}_{y_1}||\nabla\mathrm{div}Q|)\mathrm{d}y\mathrm{d}\tau \\
&\le C\epsilon^{1-\alpha}\int_0^{\tau'}\|\nabla\mathrm{div}Q\|\|\nabla\zeta\|_{L^4(\mathbb{R}\times\mathbb{T}_\epsilon)}\|\nabla Q\|_{L^4(\mathbb{R}\times\mathbb{T}_\epsilon)}\mathrm{d}\tau + \frac{\epsilon^{\alpha-1}}{13}\int_0^{\tau'}\|\nabla Q\|^2\mathrm{d}\tau + C\epsilon^{2-2\alpha}\int_0^{\tau'}\|\nabla\mathrm{div}Q\|^2\|\tilde{\theta}_{y_1}\|^2\mathrm{d}\tau \\
&\le C\epsilon^{1-\alpha}\int_0^{\tau'}\|\nabla\mathrm{div}Q\|\|\nabla\zeta\|_1\|\nabla Q\|_1\mathrm{d}\tau + \frac{\epsilon^{\alpha-1}}{13}\int_0^{\tau'}\|\nabla Q\|^2\mathrm{d}\tau \\
&\quad + C\epsilon^{2-2\alpha}\sup_{0\le\tau\le\tau_1(\epsilon)}\|\bar{\theta}_{y_1}\|^2_{L^\infty(\mathbb{R})}\int_0^{\tau'}\|\nabla\mathrm{div}Q\|^2\mathrm{d}\tau + C\epsilon^{2-2\alpha}\sup_{0\le\tau\le\tau_1(\epsilon)}\|\mathcal{P}_{y_1}\|^2_{L^\infty(\mathbb{R})}\int_0^{\tau'}\|\nabla\mathrm{div}Q\|^2\mathrm{d}\tau \\
&\le \frac{\epsilon^{1-\alpha}}{16}\int_0^{\tau'}\|\nabla\mathrm{div}Q\|^2\mathrm{d}\tau + \frac{\epsilon^{\alpha-1}}{13}\int_0^{\tau'}\|\nabla Q\|^2\mathrm{d}\tau + C\epsilon^{5-5\alpha}\int_0^{\tau'}\|\nabla\zeta\|_1^2\mathrm{d}\tau + C_T\Big(\frac{\epsilon^2}{\delta^2} + \frac{\epsilon^4}{\delta^5}\Big)\int_0^{\tau'}\|\nabla\mathrm{div}Q\|^2\mathrm{d}\tau.
\end{aligned}
$$

Furthermore, the remaining terms in Eq (3.25) can be estimated in a similar manner, and we will omit the details for the sake of simplicity.

To close the estimate of (3.25), we need to have the estimate of $\int_0^{\tau'}\|\nabla\phi\|^2\mathrm{d}\tau$. Following similar procedures in Lemma 3.2 [12], and keeping in mind the different scalings, we have

$$\int_0^{\tau_1(\epsilon)} \|\nabla\phi\|^2 d\tau \le C_T\Big(\frac{\epsilon^{3-2\alpha}}{\delta^4}+\frac{\epsilon^{4-2\alpha}}{\delta^7}\Big)\cdot \epsilon^{\alpha-1} + \epsilon^{\alpha}\int_0^{\tau_1(\epsilon)} \|\nabla^2\Psi\|^2 d\tau + C\sup_{0\le\tau\le\tau_1(\epsilon)}\|\nabla\phi\|^2 + C_T\epsilon^{\alpha-1}\|(\phi_0,\Psi_0,\zeta_0)\|^2 + C\|\nabla\phi_0\|^2. \tag{3.26}$$

Thus, plugging the above estimates into (3.25) and taking $\frac{\epsilon^{2\alpha-1}}{\delta}$, $\frac{\epsilon^{\alpha}}{\delta^2}$ and ϵ suitably small, we can complete the proof of (3.19). □

Lemma 3.3. *There exists a positive constant C_T such that*

$$\sup_{0\le\tau\le\tau_1(\epsilon)}\|(\nabla^2\phi,\nabla^2\Psi,\nabla^2\zeta)(\tau)\|^2 + \int_0^{\tau_1(\epsilon)}\Big[\|\bar u_{1y_1}^{1/2}(\nabla^2\phi)\|^2 + \epsilon^{1-\alpha}\|(\nabla^3\Psi,\nabla^3\zeta)\|^2\Big]d\tau + \int_0^{\tau_1(\epsilon)}\Big[\epsilon^{\alpha-1}\|\nabla^2 Q\|^2 + \epsilon^{1-\alpha}\|\nabla^2 \mathrm{div} Q\|^2\Big]d\tau \le C_T\Big(\frac{\epsilon^{3-2\alpha}}{\delta^4}+\frac{\epsilon^{4-2\alpha}}{\delta^7}\Big) + C\|(\phi_0,\Psi_0,\zeta_0)\|_2^2. \tag{3.27}$$

Proof. We apply the operator ∇^2 to the first equation of (3.4), then multiply it by $R\nabla^2\psi$ to obtain

$$\begin{aligned}
&\Big(R\frac{|\nabla^2\phi|^2}{2}\Big)_\tau + \mathrm{div}\Big(R\boldsymbol{u}\frac{|\nabla^2\phi|^2}{2} - R\rho\phi_{y_iy_j}\nabla\psi_{iy_j}\Big) + \big(R\rho\nabla^2\phi\cdot\nabla^2\psi_i\big)_{y_i} + R\bar u_{1y_1}\frac{|\nabla^2\phi|^2}{2}\\
&+ 2R\bar u_{1y_1}\left|\nabla\phi_{y_1}\right|^2 + R\rho\nabla^2\phi\cdot\nabla\Delta\Psi\\
&= -R\mathrm{div}\Psi\frac{|\nabla^2\phi|^2}{2} - R\Big(\frac{-\bar u_1 d_1 + d_2}{\tilde\rho}\Big)_{y_1}\frac{|\nabla^2\phi|^2}{2} - 2R\psi_{iy_j}\nabla\phi_{y_i}\cdot\nabla\phi_{y_j}\\
&- 2R\Big(\frac{-\bar u_1 d_1 + d_2}{\tilde\rho}\Big)_{y_1}\left|\nabla\phi_{y_1}\right|^2 - R\tilde u_{1y_1y_1}\phi_{y_1}\phi_{y_1y_1} - R\tilde\rho_{y_1y_1}\mathrm{div}\Psi\phi_{y_1y_1}\\
&- 2R\phi_{y_i}\nabla\phi_{y_i}\cdot\nabla\mathrm{div}\Psi - 2R\tilde\rho_{y_1}\nabla\phi_{y_1}\cdot\nabla\mathrm{div}\Psi - R\nabla\phi\cdot\nabla\psi_{iy_j}\phi_{y_iy_j} - R\tilde\rho_{y_1}\nabla^2\phi\cdot\nabla\Psi_{y_1}\\
&- R\tilde\rho_{y_1y_1y_1}\psi_1\phi_{y_1y_1} - 2R\tilde\rho_{y_1y_1}\nabla\psi_1\cdot\nabla\phi_{y_1} - R\tilde u_{1y_1y_1y_1}\phi\phi_{y_1y_1} - 2R\tilde u_{1y_1y_1}\nabla\phi\cdot\nabla\phi_{y_1}\\
&:= \mathcal{H}(\tau,y).
\end{aligned} \tag{3.28}$$

Next, we are dividing the second equation of (3.4) by ρ, applying the operator ∇ and then multiplying the resulting equation by $-\frac{\rho^2}{\theta}\nabla\Delta\Psi$ to obtain

$$\begin{aligned}
&\Big(\frac{\rho^2}{\theta}\frac{|\nabla^2\Psi|^2}{2}\Big)_\tau - \Big(\frac{\rho^2}{\theta}\nabla\Psi_\tau\cdot\nabla\Psi_{y_i}\Big)_{y_i} - \Big(\frac{\rho^2}{\theta}u_i\nabla\Psi_{y_i}\cdot\nabla\Psi_{y_j}\Big)_{y_j} + \mathrm{div}\Big(\frac{\rho^2}{\theta}u\frac{|\nabla^2\Psi|^2}{2}\Big)\\
&- (\mu+\lambda)\epsilon^{1-\alpha}\mathrm{div}\Big(\frac{\rho}{\theta}\mathrm{div}\Psi_{y_j}\nabla\mathrm{div}\Psi_{y_j}\Big) + (\mu+\lambda)\epsilon^{1-\alpha}\Big(\frac{\rho}{\theta}\mathrm{div}\Psi_{y_j}\Delta\psi_{iy_j}\Big)_{y_i} + \mu\epsilon^{1-\alpha}\frac{\rho}{\theta}|\nabla\Delta\Psi|^2\\
&+ (\mu+\lambda)\epsilon^{1-\alpha}\frac{\rho}{\theta}|\nabla^2\mathrm{div}\Psi|^2 - R\rho\nabla^2\phi\cdot\nabla\Delta\Psi - R\frac{\rho^2}{\theta}\nabla^2\zeta\cdot\nabla\Delta\Psi := \sum_{i=1}^{4}\mathcal{L}_i(\tau,y),
\end{aligned} \tag{3.29}$$

where

$$\begin{aligned}\mathcal{L}_1(\tau,y)=&-\frac{2\rho}{\theta}\phi_{y_i}\nabla\Psi_{y_i}\cdot\nabla\Psi_\tau-\frac{2\rho}{\theta}\tilde{\rho}_{y_1}\nabla\Psi_{y_1}\cdot\nabla\Psi_\tau+\frac{\rho^2}{\theta^2}\zeta_{y_i}\nabla\Psi_{y_i}\cdot\nabla\Psi_\tau+\frac{\rho^2}{\theta^2}\tilde{\theta}_{y_1}\nabla\Psi_{y_1}\cdot\nabla\Psi_\tau\\&-\frac{2\rho}{\theta}\rho_{y_j}u_i\nabla\Psi_{y_i}\cdot\nabla\Psi_{y_j}+\frac{\rho^2}{\theta^2}\theta_{y_j}u_i\nabla\Psi_{y_i}\cdot\nabla\Psi_{y_j}-\frac{\rho^2}{\theta}\psi_{iy_j}\nabla\Psi_{y_i}\cdot\nabla\Psi_{y_j}\\&-\frac{\rho^2}{\theta}\tilde{u}_{1y_1}\left|\nabla\Psi_{y_1}\right|^2+(\gamma-2)\frac{\rho^2}{\theta}\tilde{u}_{1y_1}\frac{|\nabla^2\Psi|^2}{2}+(\gamma-2)\frac{\rho^2}{\theta}\mathrm{div}\Psi\frac{|\nabla^2\Psi|^2}{2},\end{aligned}$$

$$\begin{aligned}\mathcal{L}_2(\tau,y)=&-\frac{(\gamma-1)}{R}\frac{\rho}{\theta^2}\frac{|\nabla^2\Psi|^2}{2}\epsilon^{1-\alpha}[\kappa\Delta\zeta+\kappa\tilde{\theta}_{y_1y_1}+\frac{\mu}{2}|\nabla\Psi+(\nabla\Psi)^\top|^2+(2\mu+\lambda)\left(\tilde{u}_1\tilde{u}_{1y_1}\right)_{y_1}\\&+\lambda(\mathrm{div}\Psi)^2+2\tilde{u}_{1y_1}(2\mu\psi_{1y_1}+\lambda\mathrm{div}\Psi)]+\frac{\rho^2}{\theta}\psi_{jy_i}\nabla\psi_i\cdot\nabla\Delta\psi_j+\frac{\rho^2}{\theta}\tilde{u}_{1y_1}\Psi_{y_1}\cdot\Delta\Psi_{y_1}\\&+\frac{R\rho}{\theta}\phi_{y_i}\nabla\zeta\cdot\nabla\Delta\psi_i+\frac{R\rho}{\theta}\tilde{\theta}_{y_1}\nabla\phi\cdot\Delta\Psi_{y_1}-R\phi_{y_i}\nabla\phi\cdot\nabla\Delta\psi_i-R\tilde{\rho}_{y_1}\nabla\phi\cdot\Delta\Psi_{y_1}\\&+\frac{\rho^2}{\theta}\tilde{u}_{1y_1y_1}\psi_1\Delta\psi_{1y_1}+\frac{\rho^2}{\theta}\tilde{u}_{1y_1}\nabla\psi_1\cdot\nabla\Delta\psi_1+\frac{R\rho^2}{\theta}\tilde{\rho}_{y_1y_1}(\frac{\theta}{\rho}-\frac{\tilde{\theta}}{\tilde{\rho}})\Delta\psi_{1y_1}\\&+\frac{R\rho}{\theta}\tilde{\rho}_{y_1}\nabla\zeta\cdot\nabla\Delta\psi_1-\frac{R\rho}{\theta\tilde{\rho}}\tilde{\rho}_{y_1}\tilde{\theta}_{y_1}\phi\Delta\psi_{1y_1}-R\tilde{\rho}_{y_1}\nabla\phi\cdot\nabla\Delta\psi_1-\frac{R\rho^2}{\theta}\tilde{\rho}_{y_1}^2(\frac{\theta}{\rho^2}-\frac{\tilde{\theta}}{\tilde{\rho}^2})\Delta\psi_{1y_1}\\&-\frac{(\gamma-1)}{R}\frac{\rho}{\theta^2}|\nabla^2\Psi|^2\mathrm{div}Q-\frac{(\gamma-1)}{R}\frac{\rho}{\theta^2}|\nabla^2\Psi|^2\bar{q}_{y_1},\end{aligned}$$

$$\begin{aligned}\mathcal{L}_3(\tau,y)=&\frac{\mu\epsilon^{1-\alpha}}{\theta}(\Delta\psi_i\nabla\phi\cdot\nabla\Delta\psi_i+\tilde{\rho}_{y_1}\Delta\Psi\cdot\Delta\Psi_{y_1})+\frac{(\mu+\lambda)\epsilon^{1-\alpha}}{\theta}\phi_{y_i}\mathrm{div}\Psi_{y_j}\Delta\psi_{iy_j}\\&+\frac{(\mu+\lambda)\epsilon^{1-\alpha}}{\theta}\tilde{\rho}_{y_1}\mathrm{div}\Psi_{y_j}\Delta\psi_{1y_j}-(\mu+\lambda)\epsilon^{1-\alpha}\frac{\rho}{\theta^2}\zeta_{y_i}\mathrm{div}\Psi_{y_j}\Delta\psi_{iy_j}\\&-(\mu+\lambda)\epsilon^{1-\alpha}\frac{\rho}{\theta^2}\tilde{\theta}_{y_1}\mathrm{div}\Psi_{y_j}\Delta\psi_{1y_j}-\frac{(\mu+\lambda)\epsilon^{\alpha}}{\theta}\mathrm{div}\Psi_{y_j}\nabla\phi\cdot\nabla\mathrm{div}\Psi_{y_j}\\&-\frac{(\mu+\lambda)\epsilon^{1-\alpha}}{\theta}\tilde{\rho}_{y_1}\mathrm{div}\Psi_{y_j}\mathrm{div}\Psi_{y_1y_j}+(\mu+\lambda)\epsilon^{\alpha}\frac{\rho}{\theta^2}\mathrm{div}\Psi_{y_j}\nabla\zeta\cdot\nabla\mathrm{div}\Psi_{y_j}\\&+(\mu+\lambda)\epsilon^{1-\alpha}\frac{\rho}{\theta^2}\tilde{\theta}_{y_1}\mathrm{div}\Psi_{y_j}\mathrm{div}\Psi_{y_1y_j}+\frac{(\mu+\lambda)\epsilon^{\alpha}}{\theta}\mathrm{div}\Psi_{y_i}\nabla\phi\cdot\nabla\Delta\psi_i\\&+\frac{(\mu+\lambda)\epsilon^{1-\alpha}}{\theta}\tilde{\rho}_{y_1}\nabla\mathrm{div}\Psi\cdot\Delta\Psi_{y_1},\end{aligned}$$

$$\begin{aligned}\mathcal{L}_4(\tau,y)=&-(2\mu+\lambda)\epsilon^{1-\alpha}\frac{\rho}{\theta}\Big(\frac{-\bar{u}_1d_1+d_2}{\tilde{\rho}}\Big)_{y_1y_1y_1}\Delta\psi_{1y_1}+\frac{(2\mu+\lambda)\epsilon^{1-\alpha}}{\theta}\Big(\frac{-\bar{u}_1d_1+d_2}{\tilde{\rho}}\Big)_{y_1y_1}\nabla\phi\cdot\nabla\Delta\psi_1\\&+\frac{(2\mu+\lambda)\epsilon^{1-\alpha}}{\theta}\Big(\frac{-\bar{u}_1d_1+d_2}{\tilde{\rho}}\Big)_{y_1y_1}\tilde{\rho}_{y_1}\Delta\psi_{1y_1}+\frac{(2\mu+\lambda)\epsilon^{1-\alpha}\rho}{\tilde{\rho}\theta}\bar{u}_{1y_1y_1y_1}\phi\Delta\psi_{1y_1}\\&+\frac{(2\mu+\lambda)\epsilon^{\alpha}}{\theta}\bar{u}_{1y_1y_1}\nabla\phi\cdot\nabla\Delta\psi_1-\frac{(2\mu+\lambda)\epsilon^{1-\alpha}}{\tilde{\rho}^2\theta}(\tilde{\rho}+\rho)\tilde{\rho}_{y_1}\bar{u}_{1y_1y_1}\phi\Delta\psi_{1y_1}\\&+\frac{\rho^2}{\theta}\Big[\frac{1}{\tilde{\rho}}\Big(\frac{3-\gamma}{2\tilde{\rho}}(-\bar{u}_1d_1+d_2)^2\Big)_{y_1}\Big]_{y_1}\Delta\psi_{1y_1}.\end{aligned}$$

We divide the third equation of (3.4) by ρ, and apply the operator ∇ to the resulting equation, then multiply it by $-\frac{\rho^2}{\theta^2}\nabla\Delta\zeta$ to deduce that

$$\begin{aligned}
&\left(\frac{R}{\gamma-1}\frac{\rho^2}{\theta^2}\frac{\left|\nabla^2\zeta\right|^2}{2}\right)_\tau-\left(\frac{R}{\gamma-1}\frac{\rho^2}{\theta^2}\nabla\zeta_\tau\cdot\nabla\zeta_{y_i}\right)_{y_i}-\left(\frac{R}{\gamma-1}\frac{\rho^2}{\theta^2}u_i\nabla\zeta_{y_i}\cdot\nabla\zeta_{y_j}\right)_{y_j}\\
&+\operatorname{div}\left(\frac{R}{\gamma-1}\frac{\rho^2}{\theta^2}\mathbf{u}\frac{\left|\nabla^2\zeta\right|^2}{2}-R\frac{\rho^2}{\theta}\nabla\psi_{jy_i}\zeta_{y_iy_j}\right)-\left(R\frac{\rho^2}{\theta}\nabla\operatorname{div}\Psi\cdot\nabla\zeta_{y_i}\right)_{y_i}\\
&+\left(R\frac{\rho^2}{\theta}\nabla\psi_{jy_i}\cdot\nabla\zeta_{y_i}\right)_{y_j}-\left(\frac{\rho}{\theta^2}\nabla\operatorname{div}Q\cdot\nabla\zeta_{y_i}\right)_{y_i}+\frac{\kappa\epsilon^{1-\alpha}\rho}{\theta^2}|\nabla\Delta\zeta|^2+R\frac{\rho^2}{\theta}\nabla^2\zeta\cdot\nabla\Delta\Psi\\
&:=\sum_{i=1}^4\mathcal{N}_i(\tau,y),
\end{aligned}\tag{3.30}$$

where

$$\begin{aligned}
\mathcal{N}_1(\tau,y)=&-\frac{R}{\gamma-1}\frac{2\rho}{\theta^2}\phi_{y_i}\nabla\zeta_{y_i}\cdot\nabla\zeta_\tau-\frac{R}{\gamma-1}\frac{2\rho}{\theta^2}\tilde{\rho}_{y_1}\nabla\zeta_{y_1}\cdot\nabla\zeta_\tau+\frac{R}{\gamma-1}\frac{2\rho^2}{\theta^3}\zeta_{y_i}\nabla\zeta_{y_i}\cdot\nabla\zeta_\tau\\
&+\frac{R}{\gamma-1}\frac{2\rho^2}{\theta^3}\tilde{\theta}_{y_1}\nabla\zeta_{y_1}\cdot\nabla\zeta_\tau-\frac{R}{\gamma-1}\frac{2\rho}{\theta^2}\rho_{y_j}u_i\nabla\zeta_{y_i}\cdot\nabla\zeta_{y_j}+\frac{R}{\gamma-1}\frac{2\rho^2}{\theta^3}\theta_{y_j}u_i\nabla\zeta_{y_i}\cdot\nabla\zeta_{y_j}\\
&-R\frac{2\rho}{\theta}\rho_{y_i}\nabla\operatorname{div}\Psi\cdot\nabla\zeta_{y_i}+R\frac{\rho^2}{\theta^2}\theta_{y_i}\nabla\operatorname{div}\Psi\cdot\nabla\zeta_{y_i}-\frac{1}{\theta^2}\rho_{y_i}\nabla\operatorname{div}Q\cdot\nabla\zeta_{y_i}\\
&+\frac{2\rho}{\theta^3}\theta_{y_i}\nabla\operatorname{div}Q\cdot\nabla\zeta_{y_i}-\frac{\rho}{\theta^2}\nabla\operatorname{div}Q_{y_i}\cdot\nabla\zeta_{y_i}-\frac{R}{\gamma-1}\frac{\rho^2}{\theta^2}\psi_{iy_j}\nabla\zeta_{y_i}\cdot\nabla\zeta_{y_j}\\
&-\frac{R}{\gamma-1}\frac{\rho^2}{\theta^2}\tilde{u}_{1y_1}|\nabla\zeta_{y_1}|^2+\frac{(2\gamma-3)R}{\gamma-1}\frac{\rho^2}{\theta^2}\operatorname{div}\Psi\frac{\left|\nabla^2\zeta\right|^2}{2}+\frac{(2\gamma-3)R}{\gamma-1}\frac{\rho^2}{\theta^2}\tilde{u}_{1y_1}\frac{\left|\nabla^2\zeta\right|^2}{2},
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}_2(\tau,y)=&-\frac{\rho}{\theta^3}\frac{\left|\nabla^2\zeta\right|^2}{2}\epsilon^{1-\alpha}[\kappa\Delta\zeta+\kappa\tilde{\theta}_{y_1y_1}+\frac{\mu}{2}|\nabla\Psi+(\nabla\Psi)^\top|^2+\lambda(\operatorname{div}\Psi)^2\\
&+2\tilde{u}_{1y_1}(2\mu\psi_{1y_1}+\lambda\operatorname{div}\Psi)+(2\mu+\lambda)\left(\tilde{u}_1\tilde{u}_{1y_1}\right)_{y_1}]+R\frac{2\rho}{\theta}\phi_{y_j}\nabla\psi_{jy_i}\cdot\nabla\zeta_{y_i}\\
&+R\frac{2\rho}{\theta}\tilde{\rho}_{y_1}\nabla\psi_{1y_i}\cdot\nabla\zeta_{y_i}-R\frac{\rho^2}{\theta^2}\zeta_{y_j}\nabla\psi_{jy_i}\cdot\nabla\zeta_{y_i}-R\frac{\rho^2}{\theta^2}\tilde{\theta}_{y_1}\nabla\psi_{1y_i}\cdot\nabla\zeta_{y_i}\\
&-R\frac{2\rho}{\theta}\nabla\phi\cdot\nabla\psi_{jy_i}\zeta_{y_iy_j}-R\frac{2\rho}{\theta}\tilde{\rho}_{y_1}\nabla\Psi_{y_1}\cdot\nabla^2\zeta+R\frac{\rho^2}{\theta^2}\nabla\zeta\cdot\nabla\psi_{jy_i}\zeta_{y_iy_j}\\
&+R\frac{\rho^2}{\theta^2}\tilde{\theta}_{y_1}\nabla\Psi_{y_1}\cdot\nabla^2\zeta+\frac{R}{\gamma-1}\frac{\rho^2}{\theta^2}\zeta_{y_i}\nabla\psi_i\cdot\nabla\Delta\zeta+\frac{R}{\gamma-1}\frac{\rho^2}{\theta^2}\tilde{u}_{1y_1}\zeta_{y_1}\Delta\zeta_{y_1}\\
&+\frac{R\rho^2}{\theta^2}\operatorname{div}\Psi\nabla\zeta\cdot\nabla\Delta\zeta+\frac{R\rho^2}{\theta^2}\tilde{\theta}_{y_1}\operatorname{div}\Psi\Delta\zeta_{y_1}+\frac{R}{\gamma-1}\frac{\rho^2}{\theta^2}\tilde{\theta}_{y_1y_1}\psi_1\Delta\zeta_{y_1}\\
&+\frac{R}{\gamma-1}\frac{\rho^2}{\theta^2}\tilde{\theta}_{y_1}\nabla\psi_1\cdot\nabla\Delta\zeta+\frac{R\rho^2}{\theta^2}\tilde{u}_{1y_1y_1}\zeta\Delta\zeta_{y_1}+\frac{R\rho^2}{\theta^2}\tilde{u}_{1y_1}\nabla\zeta\cdot\nabla\Delta\zeta\\
&+\frac{\rho}{\theta^3}|\nabla^2\zeta|^2\operatorname{div}Q+\frac{\rho}{\theta^3}|\nabla^2\zeta|^2\bar{q}_{y_1}-\frac{1}{\theta^2}\operatorname{div}Q\nabla\phi\cdot\nabla\Delta\zeta-\frac{1}{\theta^2}\tilde{\rho}_{y_1}\operatorname{div}Q\Delta\zeta_{y_1},
\end{aligned}$$

$$
\begin{aligned}
\mathcal{N}_3(\tau, y) = &- \frac{\kappa\epsilon^{1-\alpha}}{\theta^2}\Delta\zeta\nabla\phi\cdot\nabla\Delta\zeta + \frac{\kappa\epsilon^{1-\alpha}}{\theta^2}\tilde{\rho}_{y_1}\Delta\zeta\cdot\Delta\zeta_{y_1} - \frac{\mu\epsilon^{1-\alpha}\rho}{2\theta^2}\nabla(|\nabla\Psi + (\nabla\Psi)^\top|^2)\cdot\nabla\Delta\zeta \\
&+ \frac{\mu\epsilon^{1-\alpha}}{2\theta^2}|\nabla\Psi + (\nabla\Psi)^\top|^2\nabla\phi\cdot\nabla\Delta\zeta + \frac{\mu\epsilon^{1-\alpha}}{2\theta^2}\tilde{\rho}_{y_1}|\nabla\Psi + (\nabla\Psi)^\top|^2\Delta\zeta_{y_1} \\
&- \frac{\lambda\epsilon^{1-\alpha}\rho}{\theta^2}\nabla(\mathrm{div}\Psi)^2\cdot\nabla\Delta\zeta + \frac{\lambda\epsilon^{1-\alpha}}{\theta^2}(\mathrm{div}\Psi)^2\nabla\phi\cdot\nabla\Delta\zeta + \frac{\lambda\epsilon^{1-\alpha}}{\theta^2}\tilde{\rho}_{y_1}(\mathrm{div}\Psi)^2\Delta\zeta_{y_1} \\
&- \frac{2\rho}{\theta^2}\tilde{u}_{1y_1y_1}\epsilon^{1-\alpha}(2\mu\psi_{1y_1} + \lambda\mathrm{div}\Psi)\Delta\zeta_{y_1} - \frac{2\rho}{\theta^2}\tilde{u}_{1y_1}\epsilon^{\alpha-1}(2\mu\nabla\psi_{1y_1} + \lambda\nabla\mathrm{div}\Psi)\cdot\nabla\Delta\zeta \\
&+ \frac{2\tilde{u}_{1y_1}}{\theta^2}\epsilon^{1-\alpha}(2\mu\psi_{1y_1} + \lambda\mathrm{div}\Psi)\nabla\phi\cdot\nabla\Delta\zeta + \frac{2\tilde{u}_{1y_1}\tilde{\rho}_{y_1}}{\theta^2}\epsilon^{1-\alpha}\left(2\mu\psi_{1y_1} + \lambda\mathrm{div}\Psi\right)\Delta\zeta_{y_1},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{N}_4(\tau, y) = &- \frac{\rho}{\theta^2}\nabla F_1\cdot\nabla\Delta\zeta + \frac{F_1}{\theta^2}\nabla\phi\cdot\nabla\Delta\zeta + \frac{F_1}{\theta^2}\tilde{\rho}_{y_1}\Delta\zeta_{y_1} - \frac{\rho}{\theta^2}\nabla F_2\cdot\nabla\Delta\zeta \\
&+ \frac{F_2}{\theta^2}\nabla\phi\cdot\nabla\Delta\zeta + \frac{F_2}{\theta^2}\tilde{\rho}_{y_1}\Delta\zeta_{y_1} - \frac{\rho}{\theta^2}\nabla F_3\cdot\nabla\Delta\zeta + \frac{F_3}{\theta^2}\nabla\phi\cdot\nabla\Delta\zeta + \frac{F_3}{\theta^2}\tilde{\rho}_{y_1}\Delta\zeta_{y_1}.
\end{aligned}
$$

Applying the operator ∇^2 to the fourth equation of (3.4) and then multiplying the resulting equation by $\frac{a\rho\epsilon^{\alpha-1}}{4\theta^2\tilde{\theta}^3}\nabla Q$, we have

$$
\begin{aligned}
&\frac{\rho a\epsilon^{\alpha-1}}{4b\theta^2\tilde{\theta}^3}|\nabla^2 Q|^2 + \mathrm{div}\left(\frac{\rho\nabla Q_{iy_j}\cdot\nabla\zeta_{y_j}}{\theta}\right) + \epsilon^{1-\alpha}\mathrm{div}\left(\frac{\rho\nabla^2 Q\cdot\nabla^2\mathrm{div}Q}{4b\theta^2\tilde{\theta}^3}\right) - \epsilon^{1-\alpha}\frac{\rho|\nabla^2\mathrm{div}Q|^2}{4b\theta^2\tilde{\theta}^3} \\
&= \frac{\rho}{\theta^2}\nabla\mathrm{div}Q_{y_i}\cdot\nabla\zeta_{y_i} + \frac{\zeta_{y_iy_j}}{\theta^2}\nabla\phi\cdot\nabla Q_{iy_j} + \frac{\tilde{\rho}_{y_1}}{\theta^2}\nabla^2\zeta\cdot\nabla Q_{y_1} - \frac{2\rho\zeta_{y_iy_j}}{\theta^4}\nabla\zeta\cdot\nabla Q_{iy_j} \\
&- \frac{2\rho\tilde{\theta}_{y_1}}{\theta^4}\nabla^2\zeta\cdot\nabla Q_{y_1} + \epsilon^{1-\alpha}\mathrm{div}Q_{y_iy_j}\nabla Q_{iy_j}\cdot\nabla\left(\frac{\rho}{4b\theta^2\tilde{\theta}^3}\right) - \frac{3\rho}{\theta^2\tilde{\theta}^3}\nabla Q_{iy_j}\cdot\nabla(\zeta\tilde{\theta}^2\tilde{\theta}_{y_1})_{y_j} \\
&- \frac{3\rho}{\theta^2\tilde{\theta}^3}\nabla Q_{y_i}\cdot\nabla(\nabla\zeta\tilde{\theta}^2\tilde{\theta}_{y_1}) - \frac{3\rho\tilde{\theta}_{y_1}}{\theta^2\tilde{\theta}}\nabla Q_{y_1}\cdot\nabla^2\zeta + \frac{\rho\epsilon^{2-2\alpha}}{4a\theta^2\tilde{\theta}^3}(\bar{\theta}^4)_{y_1y_1y_1y_1y_1}Q_{1y_1y_1} \\
&- \frac{\rho Q_{1y_1y_1}}{4\theta^2\tilde{\theta}^3}\left(\mathcal{P}^4 + 4\mathcal{P}^3\bar{\theta} + 6\mathcal{P}^2\bar{\theta}^2 + 4\mathcal{P}\bar{\theta}^3\right)_{y_1y_1y_1} \\
&- \frac{\rho}{4\theta^2\tilde{\theta}^3}\nabla Q_{iy_j}\cdot\nabla\left(\zeta^4 + 4\zeta^3\tilde{\theta} + 6\zeta^2\tilde{\theta}^2\right)_{y_iy_j} \\
&:= \mathcal{M}(\tau, y).
\end{aligned} \tag{3.31}
$$

Now we add (3.28)–(3.31) together and integrate the resulting equation over $[0,\tau']\times\mathbb{R}\times\mathbb{T}_\epsilon$ to obtain

$$
\begin{aligned}
&\left\|\left(\nabla^2\phi, \nabla^2\Psi, \nabla^2\zeta\right)(\tau')\right\|^2 + \int_0^{\tau'}\left[\left\|\bar{u}_{1y_1}^{1/2}\nabla^2\phi\right\|^2 + \epsilon^{1-\alpha}\left\|\left(\nabla^3\Psi, \nabla^3\zeta\right)\right\|^2\right]\mathrm{d}\tau \\
&+ \int_0^{\tau'}\left[\epsilon^{\alpha-1}\|\nabla^2 Q\|^2 + \epsilon^{1-\alpha}\|\nabla^2\mathrm{div}Q\|^2\right]\mathrm{d}\tau \\
&\le C\left\|\left(\nabla^2\phi_0, \nabla^2\Psi_0, \nabla^2\zeta_0\right)\right\|^2 + C\left|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}[\mathcal{H}(\tau, y) + \sum_{i=1}^4\mathcal{L}_i(\tau, y) + \sum_{i=1}^4\mathcal{N}_i(\tau, y) + \mathcal{M}(\tau, y)]\mathrm{d}y\mathrm{d}\tau\right|.
\end{aligned} \tag{3.32}
$$

Now, we focus on estimating the terms on the right-hand side of Eq (3.32). We will begin by estimating some typical terms, and we anticipate that the remaining terms can be treated similarly. We use the *a priori* Assumption (3.10) and Sobolev's inequality to obtain

$$C\left|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon} R\mathrm{div}\Psi\frac{\left|\nabla^2\phi\right|^2}{2}\mathrm{d}y\mathrm{d}\tau\right| \le C\int_0^{\tau'}\|\nabla\Psi\|_{L^\infty(\mathbb{R}\times\mathbb{T}_\epsilon)}\left\|\nabla^2\phi\right\|^2\mathrm{d}\tau$$
$$\le C\int_0^{\tau'}\|\nabla\Psi\|_2\left\|\nabla^2\phi\right\|^2\mathrm{d}\tau \le \frac{\epsilon^{1-\alpha}}{150}\int_0^{\tau'}\|\nabla\Psi\|_2^2\mathrm{d}\tau + C\epsilon^{\alpha-1}\int_0^{\tau'}\left\|\nabla^2\phi\right\|^4\mathrm{d}\tau$$
$$\le \frac{\epsilon^{1-\alpha}}{150}\int_0^{\tau'}\|\nabla\Psi\|_2^2\mathrm{d}\tau + C\epsilon^{1-\alpha}\int_0^{\tau'}\left\|\nabla^2\phi\right\|^2\mathrm{d}\tau.$$

Similarly, using Hölder's inequality, Sobolev's inequality, and Young's inequality, we obtain

$$C\left|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon} 2R\phi_{y_i}\nabla\phi_{y_i}\cdot\nabla\mathrm{div}\Psi\mathrm{d}y\mathrm{d}\tau\right| \le C\int_0^{\tau'}\left\|\nabla^2\phi\right\|\|\nabla\phi\|_{L^4(\mathbb{R}\times\mathbb{T}_\epsilon)}\left\|\nabla^2\Psi\right\|_{L^4(\mathbb{R}\times\mathbb{T}_\epsilon)}\mathrm{d}\tau$$
$$\le C\int_0^{\tau'}\left\|\nabla^2\phi\right\|\|\nabla\phi\|^{1/2}\|\nabla\phi\|_1^{1/2}\left\|\nabla^2\Psi\right\|^{1/2}\left\|\nabla^2\Psi\right\|_1^{1/2}\mathrm{d}\tau$$
$$\le \frac{\epsilon^{1-\alpha}}{150}\int_0^{\tau'}\left\|\nabla^2\Psi\right\|_1^2\mathrm{d}\tau + C\epsilon^{(\alpha-1)/3}\int_0^{\tau'}\|\nabla\phi\|_1^2\|\nabla\phi\|^{2/3}\left\|\nabla^2\Psi\right\|^{2/3}\mathrm{d}\tau$$
$$\le \frac{\epsilon^{1-\alpha}}{150}\int_0^{\tau'}\left\|\nabla^2\Psi\right\|_1^2\mathrm{d}\tau + C\epsilon^{1-\alpha}\int_0^{\tau'}\|\nabla\phi\|_1^2\mathrm{d}\tau.$$

It follows from Lemmas 2.2 and 2.3 that

$$C\left|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon} R\tilde{u}_{1y_1y_1y_1}\phi\phi_{y_1y_1}\mathrm{d}y\mathrm{d}\tau\right| \le C\epsilon^{1-\alpha}\int_0^{\tau'}\left\|\nabla^2\phi\right\|^2\mathrm{d}\tau + C\epsilon^{\alpha-1}\int_0^{\tau'}\left\|\tilde{u}_{1y_1y_1y_1}\right\|_{L^\infty(\mathbb{R})}^2\|\phi\|^2\mathrm{d}\tau$$
$$\le C\epsilon^{1-\alpha}\int_0^{\tau'}\left\|\nabla^2\phi\right\|^2\mathrm{d}\tau + C\epsilon^{6\alpha-1}\sup_{0\le t\le T}\left\|\tilde{u}_{1x_1x_1x_1}\right\|_{L^\infty(\mathbb{R})}^2\sup_{0\le\tau\le\tau_1(\epsilon)}\|\phi\|^2$$
$$\le C\epsilon^{1-\alpha}\int_0^{\tau'}\left\|\nabla^2\phi\right\|^2\mathrm{d}\tau + C_T\epsilon^{6\alpha-1}\left(\frac{1}{\delta^6}+\frac{\epsilon^2}{\delta^9}\right)\sup_{0\le\tau\le\tau_1(\epsilon)}\|\phi\|^2.$$

Using *a priori* Assumption (3.10) and Sobolev's inequality, we have

$$C\left|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}(\gamma-2)\frac{\rho^2}{\theta}\mathrm{div}\Psi\frac{\left|\nabla^2\Psi\right|^2}{2}\mathrm{d}y\mathrm{d}\tau\right| \le C\int_0^{\tau'}\|\nabla\Psi\|\left\|\nabla^2\Psi\right\|_{L^4(\mathbb{R}\times\mathbb{T}_\epsilon)}^2\mathrm{d}\tau$$

$$\leq C\int_0^{\tau'} \|\nabla\Psi\| \left\|\nabla^2\Psi\right\| \left\|\nabla^2\Psi\right\|_1 \mathrm{d}\tau \leq C\epsilon^{1-\alpha}\int_0^{\tau'} \|\nabla\Psi\| \left\|\nabla^2\Psi\right\|_1 \mathrm{d}\tau$$

$$\leq \frac{\epsilon^{1-\alpha}}{150}\int_0^{\tau'} \left\|\nabla^2\Psi\right\|_1^2 \mathrm{d}\tau + C\epsilon^{1-\alpha}\int_0^{\tau'} \|\nabla\Psi\|^2\mathrm{d}\tau,$$

and

$$C\left|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon} \frac{\gamma-1}{R}\kappa\epsilon^{1-\alpha}\frac{\rho}{\theta^2}\frac{\left|\nabla^2\Psi\right|^2}{2}\Delta\zeta\mathrm{d}y\mathrm{d}\tau\right| \leq C\epsilon^{1-\alpha}\int_0^{\tau'} \left\|\nabla^2\zeta\right\| \left\|\nabla^2\Psi\right\|_{L^4(\mathbb{R}\times\mathbb{T}_\epsilon)}^2 \mathrm{d}\tau$$

$$\leq C\epsilon^{1-\alpha}\int_0^{\tau'} \left\|\nabla^2\zeta\right\| \left\|\nabla^2\Psi\right\|_1^2 \mathrm{d}\tau \leq C\epsilon^{2-2\alpha}\int_0^{\tau'} \left\|\nabla^2\Psi\right\|_1^2 \mathrm{d}\tau.$$

Based on Young's inequality, Lemma 2.2, Lemma 2.3, and *a priori* Assumption (3.10), it holds that

$$C\left|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon} (2\mu+\lambda)\epsilon^{1-\alpha}\left(\frac{-\bar{u}_1 d_1 + d_2}{\tilde{\rho}}\right)_{y_1y_1y_1} \Delta\psi_{1y_1}\mathrm{d}y\mathrm{d}\tau\right|$$

$$\leq \frac{\epsilon^{1-\alpha}}{150}\int_0^{\tau'} \left\|\nabla^2\psi_{1y_1}\right\|^2 d\tau + C\epsilon^{2\alpha+1}\int_0^{\tau'}\int_{\mathbb{R}} \left|\left(\frac{-\bar{u}_1 d_1 + d_2}{\tilde{\rho}}\right)_{x_1x_1x_1}\right|^2 \mathrm{d}x_1\mathrm{d}\tau$$

$$\leq \frac{\epsilon^{1-\alpha}}{150}\int_0^{\tau'} \left\|\nabla^2\psi_{1y_1}\right\|^2 \mathrm{d}\tau + C_T\frac{\epsilon^{3+2\alpha}}{\delta^8},$$

and

$$C\left|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon} \frac{\lambda\epsilon^{1-\alpha}}{\theta^2}(\operatorname{div}\Psi)^2\nabla\phi\cdot\nabla\Delta\zeta\mathrm{d}y\mathrm{d}\tau\right| \leq C\epsilon^{1-\alpha}\int_0^{\tau'} \|\nabla\Psi\|_{L^8(\mathbb{R}\times\mathbb{T}_\epsilon)}^2\|\nabla\phi\|_{L^4(\mathbb{R}\times\mathbb{T}_\epsilon)} \left\|\nabla^3\zeta\right\| \mathrm{d}\tau$$

$$\leq C\epsilon^{1-\alpha}\int_0^{\tau'} \|\nabla\Psi\|_1^2\|\nabla\phi\|_1 \left\|\nabla^3\zeta\right\| \mathrm{d}\tau \leq \frac{\epsilon^{1-\alpha}}{150}\int_0^{\tau'} \left\|\nabla^3\zeta\right\|^2 \mathrm{d}\tau + C\epsilon^{1-\alpha}\int_0^{\tau'} \|\nabla\Psi\|_1^4\|\nabla\phi\|_1^2\mathrm{d}\tau$$

$$\leq \frac{\epsilon^{1-\alpha}}{150}\int_0^{\tau'} \left\|\nabla^3\zeta\right\|^2 \mathrm{d}\tau + C\epsilon^{5-5\alpha}\int_0^{\tau'} \|\nabla\phi\|_1^2\mathrm{d}\tau.$$

For $\mathcal{M}(\tau, y)$, we will focus on several terms that may cause difficulties.
Integration by parts gives that

$$\epsilon^{1-\alpha}\int_{\mathbb{R}\times\mathbb{T}_\epsilon} \operatorname{div}Q_{y_iy_j}\nabla Q_{iy_j}\cdot\nabla\left(\frac{\rho}{4b\theta^2\tilde{\theta}^3}\right)\mathrm{d}y = -\epsilon^{1-\alpha}\int_{\mathbb{R}\times\mathbb{T}_\epsilon} \operatorname{div}Q_{y_i}\nabla(\operatorname{div}Q)_{y_j}\cdot\nabla\left(\frac{\rho}{4b\theta^2\tilde{\theta}^3}\right)\mathrm{d}y$$

$$- \epsilon^{1-\alpha} \int\limits_{\mathbb{R}\times\mathbb{T}_\epsilon} \mathrm{div} Q_{y_i} \nabla Q_{iy_j} \cdot \nabla \left(\frac{\rho}{4b\theta^2\tilde{\theta}^3} \right)_{y_i} \mathrm{d}y,$$

then, we obtain that

$$\begin{aligned}
&C \left| \int\limits_0^{\tau'} \int\limits_{\mathbb{R}\times\mathbb{T}_\epsilon} \epsilon^{1-\alpha} \mathrm{div} Q_{y_i} \nabla(\mathrm{div} Q)_{y_j} \cdot \nabla \left(\frac{\rho}{4b\theta^2\tilde{\theta}^3} \right) \mathrm{d}y \mathrm{d}\tau \right| \\
&\le C\epsilon^{1-\alpha} \int\limits_0^{\tau'} \int\limits_{\mathbb{R}\times\mathbb{T}_\epsilon} (|\nabla \mathrm{div} Q||\nabla\zeta||\nabla^2 \mathrm{div} Q| + |\nabla \mathrm{div} Q||\tilde{\theta}_{y_1}||\nabla^2 \mathrm{div} Q|) \mathrm{d}y \mathrm{d}\tau \\
&\quad + C\epsilon^{1-\alpha} \int\limits_0^{\tau'} \int\limits_{\mathbb{R}\times\mathbb{T}_\epsilon} (|\nabla \mathrm{div} Q||\nabla\phi||\nabla^2 \mathrm{div} Q| + |\nabla \mathrm{div} Q||\tilde{\rho}_{y_1}||\nabla^2 \mathrm{div} Q|) \mathrm{d}y \mathrm{d}\tau \\
&\le C\epsilon^{1-\alpha} \int\limits_0^{\tau'} \|\nabla^2 \mathrm{div} Q\| \|\nabla\zeta\|_{L^4(\mathbb{R}\times\mathbb{T}_\epsilon)} \|\nabla \mathrm{div} Q\|_{L^4(\mathbb{R}\times\mathbb{T}_\epsilon)} \mathrm{d}\tau + \frac{\epsilon^{\alpha-1}}{13} \int\limits_0^{\tau'} \|\nabla^2 Q\|^2 \mathrm{d}\tau \\
&\quad + C\epsilon^{1-\alpha} \int\limits_0^{\tau'} \|\nabla^2 \mathrm{div} Q\| \|\nabla\phi\|_{L^4(\mathbb{R}\times\mathbb{T}_\epsilon)} \|\nabla \mathrm{div} Q\|_{L^4(\mathbb{R}\times\mathbb{T}_\epsilon)} \mathrm{d}\tau \\
&\quad + C\epsilon^{2-2\alpha} \int\limits_0^{\tau'} \|\nabla^2 \mathrm{div} Q\|^2 \|\tilde{\theta}_{y_1}\|^2 \mathrm{d}\tau + C\epsilon^{2-2\alpha} \int\limits_0^{\tau'} \|\nabla^2 \mathrm{div} Q\|^2 \|\tilde{\rho}_{y_1}\|^2 \mathrm{d}\tau \\
&\le C\epsilon^{1-\alpha} \int\limits_0^{\tau'} \|\nabla^2 \mathrm{div} Q\| \|\nabla\zeta\|_1 \|\nabla \mathrm{div} Q\|_1 \mathrm{d}\tau + \frac{\epsilon^{\alpha-1}}{13} \int\limits_0^{\tau'} \|\nabla^2 Q\|^2 \mathrm{d}\tau \\
&\quad + C\epsilon^{1-\alpha} \int\limits_0^{\tau'} \|\nabla^2 \mathrm{div} Q\| \|\nabla\phi\|_1 \|\nabla \mathrm{div} Q\|_1 \mathrm{d}\tau \\
&\quad + C\epsilon^{2-2\alpha} \sup_{0\le\tau\le\tau_1(\epsilon)} \left(\|\bar{\theta}_{y_1}\|^2_{L^\infty(\mathbb{R})} + \|\bar{\rho}_{y_1}\|^2_{L^\infty(\mathbb{R})} \right) \int\limits_0^{\tau'} \|\nabla^2 \mathrm{div} Q\|^2 \mathrm{d}\tau \\
&\quad + C\epsilon^{2-2\alpha} \sup_{0\le\tau\le\tau_1(\epsilon)} \left(\|\mathcal{P}_{y_1}\|^2_{L^\infty(\mathbb{R})} + \|d_{1y_1}\|^2_{L^\infty(\mathbb{R})} \right) \int\limits_0^{\tau'} \|\nabla \mathrm{div} Q\|^2 \mathrm{d}\tau \\
&\le \frac{\epsilon^{1-\alpha}}{16} \int\limits_0^{\tau'} \|\nabla^2 \mathrm{div} Q\|^2 \mathrm{d}\tau + \frac{\epsilon^{\alpha-1}}{13} \int\limits_0^{\tau'} \|\nabla^2 Q\|^2 \mathrm{d}\tau + C\epsilon^{3-3\alpha} \int\limits_0^{\tau'} \|\nabla \mathrm{div} Q\|_1^2 \mathrm{d}\tau \\
&\quad + C_T \left(\frac{\epsilon^2}{\delta^2} + \frac{\epsilon^4}{\delta^5} \right) \int\limits_0^{\tau'} \|\nabla^2 \mathrm{div} Q\|^2 \mathrm{d}\tau.
\end{aligned}$$

Direct calculations give that

$$
\begin{aligned}
&C\left|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\epsilon^{1-\alpha}\mathrm{div}Q_{y_i}\nabla Q_{iy_j}\cdot\nabla\left(\frac{\rho}{4b\theta^2\tilde{\theta}^3}\right)_{y_i}\mathrm{d}y\mathrm{d}\tau\right|\\
&\leq C\epsilon^{1-\alpha}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}(|\nabla\mathrm{div}Q||\nabla^2Q|\left(|\nabla\zeta|^2+|\nabla^2\zeta|+|\nabla\phi|^2+|\nabla^2\phi|+|\nabla\zeta||\nabla\phi|\right)\mathrm{d}y\mathrm{d}\tau\\
&\quad+C\epsilon^{1-\alpha}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}(|\nabla\mathrm{div}Q||\nabla^2Q|\left(|\tilde{\theta}_{y_1}|^2+|\tilde{\theta}_{y_1y_1}|+|\tilde{\rho}_{y_1}|^2+|\tilde{\rho}_{y_1y_1}|+|\tilde{\rho}_{y_1}\tilde{\theta}_{y_1}|\right)\mathrm{d}y\mathrm{d}\tau,
\end{aligned}
$$

Then, we will only present estimates for the first two terms, while the remaining terms are treated in the same fashion.

By Sobolev's inequality and the a priori Assumption (3.10), we obtain

$$
\begin{aligned}
&C\epsilon^{1-\alpha}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}(|\nabla\mathrm{div}Q||\nabla^2Q||\nabla\zeta|^2)\mathrm{d}\tau\leq C\epsilon^{1-\alpha}\int_0^{\tau'}\|\nabla^2Q\|\|\nabla\zeta\|^2_{L^8(\mathbb{R}\times\mathbb{T}_\epsilon)}\|\nabla\mathrm{div}Q\|_{L^4(\mathbb{R}\times\mathbb{T}_\epsilon)}\mathrm{d}\tau\\
&\leq C\epsilon^{1-\alpha}\int_0^{\tau'}\|\nabla^2Q\|\|\nabla\zeta\|_1^2\|\nabla\mathrm{div}Q\|_1\mathrm{d}\tau\leq\frac{\epsilon^{\alpha-1}}{16}\int_0^{\tau'}\|\nabla^2Q\|\mathrm{d}\tau+C\epsilon^{3-3\alpha}\int_0^{\tau'}\|\nabla\zeta\|_1^4\|\nabla\mathrm{div}Q\|_1^2\mathrm{d}\tau\\
&\leq\frac{\epsilon^{\alpha-1}}{16}\int_0^{\tau'}\|\nabla^2Q\|\mathrm{d}\tau+C\epsilon^{7-7\alpha}\int_0^{\tau'}\|\nabla\mathrm{div}Q\|_1^2\mathrm{d}\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
&C\epsilon^{1-\alpha}\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}(|\nabla\mathrm{div}Q||\nabla^2Q||\nabla^2\zeta|)\mathrm{d}\tau\leq C\epsilon^{1-\alpha}\int_0^{\tau'}\|\nabla^2Q\|\|\nabla\zeta\|_{L^4(\mathbb{R}\times\mathbb{T}_\epsilon)}\|\nabla\mathrm{div}Q\|_{L^4(\mathbb{R}\times\mathbb{T}_\epsilon)}\mathrm{d}\tau\\
&\leq C\epsilon^{1-\alpha}\int_0^{\tau'}\|\nabla^2Q\|\|\nabla\zeta\|_1\|\nabla\mathrm{div}Q\|_1\mathrm{d}\tau\leq\frac{\epsilon^{\alpha-1}}{16}\int_0^{\tau'}\|\nabla^2Q\|\mathrm{d}\tau+C\epsilon^{3-3\alpha}\int_0^{\tau'}\|\nabla\zeta\|_1^2\|\nabla\mathrm{div}Q\|_1^2\mathrm{d}\tau\\
&\leq\frac{\epsilon^{\alpha-1}}{16}\int_0^{\tau'}\|\nabla^2Q\|\mathrm{d}\tau+C_T\epsilon^{5-5\alpha}\int_0^{\tau'}\|\nabla\mathrm{div}Q\|_1^2\mathrm{d}\tau.
\end{aligned}
$$

By Lemmas 2.2 and 2.3, it holds that

$$
C\left|\int_0^{\tau'}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\frac{1}{\theta^2}\tilde{\rho}_{y_1}\Delta\zeta_{y_1}\mathrm{div}Q_{y_1}\mathrm{d}y\mathrm{d}\tau\right|\leq\frac{\epsilon^{1-\alpha}}{150}\int_0^{\tau'}\|\nabla^3\zeta\|^2\mathrm{d}\tau+C\epsilon^{\alpha-1}\int_0^{\tau'}(\|\bar{\rho}_{y_1}\nabla\mathrm{div}Q\|^2+\|d_{1y_1}\nabla\mathrm{div}Q\|^2)\mathrm{d}\tau
$$

$$\leq \frac{\epsilon^{1-\alpha}}{150}\int_0^{\tau'}\|\nabla^3\zeta\|^2\mathrm{d}\tau + C_T\Big(\frac{\epsilon^{3\alpha-1}}{\delta^2}+\frac{\epsilon^{3\alpha+1}}{\delta^5}\Big)\int_0^{\tau'}\|\nabla\mathrm{div}Q\|^2\mathrm{d}\tau.$$

In order to close the estimate of (3.32), we also introduce the estimate of $\int_0^{\tau'}\|\nabla^2\phi\|^2\mathrm{d}\tau$. Following the same procedures as in Lemma 3.4 of [12], and considering the new scaling argument introduced here, we finally derive that

$$\begin{aligned}\int_0^{\tau_1(\epsilon)}\left\|\nabla^2\phi\right\|^2\mathrm{d}\tau \leq & C_T\Big(\frac{\epsilon^{3-2\alpha}}{\delta^4}+\frac{\epsilon^{4-2\alpha}}{\delta^7}\Big)\cdot\epsilon^{\alpha-1} + C\sup_{0\leq\tau\leq\tau_1(\epsilon)}\left\|\nabla^2\phi\right\|^2 + C\epsilon^{\alpha}\int_0^{\tau_1(\epsilon)}\left\|\nabla^3\Psi\right\|^2\mathrm{d}\tau\\ &+ C_T\epsilon^{\alpha-1}\|(\phi_0,\Psi_0,\zeta_0)\|_1^2 + C\left\|\nabla^2\phi_0\right\|^2. \end{aligned}\tag{3.33}$$

By incorporating the above estimates into (3.32); and combining them with (3.33),complete the proof of (3.27). □

Lemma 3.4. *There exists a positive constant C_T such that*

$$\sup_{0\leq\tau\leq\tau_1(\epsilon)}\Big(\|Q\|_2^2+\epsilon^{2-2\alpha}\|\mathrm{div}Q\|_2^2\Big)(\tau)\leq C_T\Big(\frac{\epsilon^{3-2\alpha}}{\delta^4}+\frac{\epsilon^{4-2\alpha}}{\delta^7}\Big)\epsilon^{2-2\alpha}+\frac{\epsilon^4}{\delta^5}+\frac{\epsilon^{4+2\alpha}}{\delta^9}+C\|(\phi_0,\Psi_0,\zeta_0)\|_2^2.\tag{3.34}$$

Proof. It remains to show the energy estimate of Q.

• **Zero-order energy estimate of Q**:

We multiply the fourth equation of (3.4) with Q and integrate the resulting equation over $\mathbb{R}\times\mathbb{T}_\epsilon$ to get

$$\begin{aligned}\frac{1}{b}\|Q\|^2+\frac{\epsilon^{2-2\alpha}}{ab}\|\mathrm{div}Q\|^2 &= \frac{\epsilon^{1-\alpha}}{a}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\nabla(\bar\theta^4-\theta^4)Q\mathrm{d}y+\frac{\epsilon^{3-3\alpha}}{a^2}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}(\bar\theta^4)_{y_1y_1y_1}Q_1\mathrm{d}y\\ &:= \mathcal{W}_{1,1}+\mathcal{W}_{1,2}.\end{aligned}\tag{3.35}$$

Using Lemmas 2.2 and 2.3, we have

$$\begin{aligned}\mathcal{W}_{1,1}\leq & C\epsilon^{1-\alpha}\left|\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\nabla\left(\zeta^4+4\zeta^3\tilde\theta+6\zeta^2\tilde\theta^2+4\zeta\tilde\theta^3\right)Q\mathrm{d}y\right|\\ &+C\epsilon^{1-\alpha}\left|\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\left(\mathcal{P}^4+4\mathcal{P}^3\bar\theta+6\mathcal{P}^2\bar\theta^2+4\mathcal{P}\bar\theta^3\right)_{y_1}Q\mathrm{d}y\right|\\ \leq & C\epsilon^{1-\alpha}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\Big(|\nabla\zeta||Q|+|\zeta||Q||\tilde\theta_{y_1}|\Big)\mathrm{d}y+C\epsilon^{1-\alpha}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\Big(|\mathcal{P}||\bar\theta_{y_1}||Q|+|\mathcal{P}_{y_1}||Q|\Big)\mathrm{d}y\\ \leq & \frac{1}{5b}\|Q\|^2+C\epsilon^{2-2\alpha}\|\nabla\zeta\|^2+C_T\frac{\epsilon^2}{\delta^2}\|\zeta\|^2+C_T\frac{\epsilon^4}{\delta^5},\end{aligned}$$

and

$$\mathcal{W}_{1,2} \leq \epsilon^{3-3\alpha} \int\limits_{\mathbb{R}\times\mathbb{T}_\epsilon} |(\bar{\theta}^4)_{y_1y_1y_1}||Q_1|\mathrm{d}y \leq \frac{1}{5b}\|Q\|^2 + C\frac{\epsilon^{6-2\alpha}}{\delta^5}.$$

Thus, for any $\tau \in [0, \tau_1(\epsilon)]$, we plug the above estimates into (3.35), then use (3.14) and (3.19) to obtain

$$\begin{aligned}\Big(\|Q\|^2 + \epsilon^{2-2\alpha}\|\mathrm{div}Q\|^2\Big)(\tau) &\leq C\epsilon^{2-2\alpha}\|\nabla\zeta\|^2 + C_T\frac{\epsilon^2}{\delta^2}\|\zeta\|^2 + C_T\frac{\epsilon^4}{\delta^5} \\ &\leq C_T\Big(\epsilon^{2-2\alpha} + \frac{\epsilon^2}{\delta^2}\Big)\Big(\frac{\epsilon^{3-2\alpha}}{\delta^4} + \frac{\epsilon^{4-2\alpha}}{\delta^7}\Big) + C_T\frac{\epsilon^4}{\delta^5} + C\|(\phi_0, \Psi_0, \zeta_0)\|_1^2.\end{aligned} \tag{3.36}$$

• **First-order energy estimate of Q**:

Applying the operator ∇ to the fourth equation of (3.4), then multiplying with ∇Q and integrating the resulting equation over $\mathbb{R}\times\mathbb{T}_\epsilon$, we have

$$\begin{aligned}\frac{1}{b}\|\nabla Q\|^2 + \frac{\epsilon^{2-2\alpha}}{ab}\|\nabla\mathrm{div}Q\|^2 &= \frac{\epsilon^{1-\alpha}}{a}\int\limits_{\mathbb{R}\times\mathbb{T}_\epsilon} \nabla^2(\bar{\theta}^4 - \theta^4)\cdot\nabla Q\mathrm{d}y + \frac{\epsilon^{3-3\alpha}}{a^2}\int\limits_{\mathbb{R}\times\mathbb{T}_\epsilon} (\bar{\theta}^4)_{y_1y_1y_1y_1}Q_{1y_1}\mathrm{d}y \\ &:= \mathcal{W}_{2,1} + \mathcal{W}_{2,2}.\end{aligned} \tag{3.37}$$

Direct calculations give that

$$\begin{aligned}\mathcal{W}_{2,1} \leq\ & C\epsilon^{1-\alpha}\left|\int\limits_{\mathbb{R}\times\mathbb{T}_\epsilon} \nabla^2\left(\zeta^4 + 4\zeta^3\tilde{\theta} + 6\zeta^2\tilde{\theta}^2 + 4\zeta\tilde{\theta}^3\right)\cdot\nabla Q\mathrm{d}y\right| \\ &+ C\epsilon^{1-\alpha}\left|\int\limits_{\mathbb{R}\times\mathbb{T}_\epsilon} \left(\mathcal{P}^4 + 4\mathcal{P}^3\bar{\theta} + 6\mathcal{P}^2\bar{\theta}^2 + 4\mathcal{P}\bar{\theta}^3\right)_{y_1y_1} Q_{1y_1}\mathrm{d}y\right|.\end{aligned}$$

Furthermore, we use Lemmas 2.2 and 2.3 to derive that

$$\begin{aligned}& C\epsilon^{1-\alpha}\left|\int\limits_{\mathbb{R}\times\mathbb{T}_\epsilon} \nabla^2\left(\zeta^4 + 4\zeta^3\tilde{\theta} + 6\zeta^2\tilde{\theta}^2 + 4\zeta\tilde{\theta}^3\right)\cdot\nabla Q\mathrm{d}y\right| \\ &\leq C\epsilon^{1-\alpha}\int\limits_{\mathbb{R}\times\mathbb{T}_\epsilon} |\nabla Q||\zeta|\left(|\nabla\zeta|^2 + |\nabla^2\zeta| + |\nabla\zeta||\tilde{\theta}_{y_1}| + |\zeta||\tilde{\theta}_{y_1y_1}| + |\zeta||\tilde{\theta}^2_{y_1}|\right)\mathrm{d}y \\ &\leq C\epsilon^{4-4\alpha}\|\nabla\zeta\|^2\|\nabla\zeta\|_1^2 + C\epsilon^{4-4\alpha}\|\nabla^2\zeta\|^2 + \frac{1}{4b}\|\nabla Q\|^2 \\ &\quad + C\epsilon^{2-2\alpha}\|\bar{\theta}_{y_1}\|_{L^\infty(\mathbb{R})}\int\limits_{\mathbb{R}\times\mathbb{T}_\epsilon} |Q_{1y_1}\zeta_{y_1}|\mathrm{d}y + C\epsilon^{2-2\alpha}\|\mathcal{P}_{y_1}\|_{L^\infty(\mathbb{R})}\int\limits_{\mathbb{R}\times\mathbb{T}_\epsilon} |Q_{1y_1}\zeta_{y_1}|\mathrm{d}y \\ &\quad + C\epsilon^{6-6\alpha}\Big(\|\mathcal{P}_{y_1y_1}\|^2 + \|\bar{\theta}_{y_1y_1}\|^2 + \|\bar{\theta}_{y_1}\|^4 + \|\mathcal{P}_{y_1}\|^4\Big) \\ &\leq C\epsilon^{6-6\alpha}\|\nabla\zeta\|^2 + C\epsilon^{4-4\alpha}\|\nabla^2\zeta\|^2 + \frac{1}{3b}\|\nabla Q\|^2\end{aligned}$$

$$+ C_T\Big(\frac{\epsilon^{4-2\alpha}}{\delta^2} + \frac{\epsilon^{6-2\alpha}}{\delta^5}\Big)\|\nabla\zeta\|^2 + C_T\Big(\frac{\epsilon^{8-4\alpha}}{\delta^6} + \frac{\epsilon^{6-4\alpha}}{\delta^3} + \frac{\epsilon^{6-4\alpha}}{\delta^2} + \frac{\epsilon^{10-4\alpha}}{\delta^8}\Big),$$

and

$$\begin{aligned} &C\epsilon^{1-\alpha}\left|\int_{\mathbb{R}\times\mathbb{T}_\epsilon} \left(\mathcal{P}^4 + 4\mathcal{P}^3\bar\theta + 6\mathcal{P}^2\bar\theta^2 + 4\mathcal{P}\bar\theta^3\right)_{y_1y_1} Q_{1y_1}\mathrm{d}y\right| \\ &\le C\epsilon^{1-\alpha}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\Big(|\mathcal{P}_{y_1}|^2|Q_{1y_1}| + |\mathcal{P}_{y_1y_1}||Q_{1y_1}| + |\bar\theta_{y_1}|^2|\mathcal{P}||Q_{1y_1}| + |\bar\theta_{y_1y_1}||\mathcal{P}||Q_1y_1|\Big)\mathrm{d}y \\ &\le \frac{1}{3b}\|\nabla Q\|^2 + C_T\frac{\epsilon^{6+2\alpha}}{\delta^9} + C_T\frac{\epsilon^{4+2\alpha}}{\delta^6}, \end{aligned}$$

so we have

$$\mathcal{W}_{2,1} \le \frac{1}{3b}\|\nabla Q\|^2 + C\epsilon^{6-6\alpha}\|\nabla\zeta\|^2 + C\epsilon^{4-4\alpha}\|\nabla^2\zeta\|^2 + C_T\frac{\epsilon^{4+2\alpha}}{\delta^6}.$$

Obviously, it holds that

$$\mathcal{W}_{2,2} \le \epsilon^{3-3\alpha}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}|(\bar\theta^4)_{y_1y_1y_1y_1}||Q_{1y_1}|\mathrm{d}y \le \frac{1}{3b}\|\nabla Q\|^2 + C\frac{\epsilon^6}{\delta^7}.$$

Thus, for any $\tau \in [0, \tau_1(\epsilon)]$, inserting the above estimates into (3.37), then using (3.19) and (3.27), we obtain

$$\begin{aligned} \Big(\|\nabla Q\|^2 + \epsilon^{2-2\alpha}\|\nabla\mathrm{div}Q\|^2\Big)(\tau) &\le C\epsilon^{6-6\alpha}\|\nabla\zeta\|^2 + C\epsilon^{4-4\alpha}\|\nabla^2\zeta\|^2 + C_T\frac{\epsilon^{4+2\alpha}}{\delta^6} + C\frac{\epsilon^6}{\delta^7} \\ &\le C_T\epsilon^{4-4\alpha}\Big(\frac{\epsilon^{3-2\alpha}}{\delta^4} + \frac{\epsilon^{4-2\alpha}}{\delta^7}\Big) + C_T\frac{\epsilon^{4+2\alpha}}{\delta^6} + C\frac{\epsilon^6}{\delta^7} + C\|(\phi_0, \Psi_0, \zeta_0)\|_2^2. \end{aligned} \tag{3.38}$$

• **Second-order energy estimate of Q**:

Last, we apply the operator ∇^2 to the fourth equation of (3.4), multiply with $\nabla^2 Q$, and integrate the resulting equation over $\mathbb{R}\times\mathbb{T}_\epsilon$ to obtain

$$\begin{aligned} \frac{1}{b}\|\nabla^2 Q\|^2 + \frac{\epsilon^{2-2\alpha}}{ab}\|\nabla^2\mathrm{div}Q\|^2 &= \frac{\epsilon^{1-\alpha}}{a}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\nabla^2(\theta^4 - \bar\theta^4)\cdot\nabla^2\mathrm{div}Q\mathrm{d}y + \frac{\epsilon^{3-3\alpha}}{a^2}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}(\bar\theta^4)_{y_1y_1y_1y_1y_1}Q_{1y_1y_1}\mathrm{d}y \\ &:= \mathcal{W}_{3,1} + \mathcal{W}_{3,2}. \end{aligned} \tag{3.39}$$

We divide the $\mathcal{W}_{3,1}$ into two terms.

$$\begin{aligned} \mathcal{W}_{3,1} \le& C\epsilon^{1-\alpha}\left|\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\nabla^2\left(\zeta^4 + 4\zeta^3\tilde\theta + 6\zeta^2\tilde\theta^2 + 4\zeta\tilde\theta^3\right)\cdot\nabla^2\mathrm{div}Q\mathrm{d}y\right| \\ &+ C\epsilon^{1-\alpha}\left|\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\left(\mathcal{P}^4 + 4\mathcal{P}^3\bar\theta + 6\mathcal{P}^2\bar\theta^2 + 4\mathcal{P}\bar\theta^3\right)_{y_1y_1y_1} Q_{1y_1y_1}\mathrm{d}y\right|. \end{aligned}$$

We use Lemmas 2.2 and 2.3 to obtain

$$
\begin{aligned}
&C\epsilon^{1-\alpha}\left|\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\nabla^2\left(\zeta^4+4\zeta^3\tilde{\theta}+6\zeta^2\tilde{\theta}^2+4\zeta\tilde{\theta}^3\right)\cdot\nabla^2\mathrm{div}Q\mathrm{d}y\right|\\
&\le C\epsilon^{1-\alpha}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}|\nabla^2\mathrm{div}Q||\zeta|\left(|\nabla\zeta|^2+|\nabla^2\zeta|+|\nabla\zeta||\tilde{\theta}_{y_1}|+|\zeta||\tilde{\theta}_{y_1y_1}|+|\zeta||\tilde{\theta}^2_{y_1}|\right)\mathrm{d}y\\
&\le C\epsilon^{2-2\alpha}\|\nabla\zeta\|^2\|\nabla\zeta\|_1^2+C\epsilon^{2-2\alpha}\|\nabla^2\zeta\|^2+\frac{\epsilon^{2-2\alpha}}{3ab}\|\nabla^2\mathrm{div}Q\|^2+C\epsilon^{2-2\alpha}\|\bar{\theta}_{y_1}\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}|Q_{1y_1}\zeta_{y_1}|\mathrm{d}y\\
&\quad+C\epsilon^{2-2\alpha}\|\mathcal{P}_{y_1}\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}|Q_{1y_1}\zeta_{y_1}|\mathrm{d}y+C\epsilon^{4-4\alpha}\Big(\|\mathcal{P}_{y_1y_1}\|^2+\|\bar{\theta}_{y_1y_1}\|^2+\|\bar{\theta}_{y_1}\|^4+\|\mathcal{P}_{y_1}\|^4\Big)\\
&\le C\epsilon^{4-4\alpha}\|\nabla\zeta\|^2+C\epsilon^{2-2\alpha}\|\nabla^2\zeta\|^2+\frac{\epsilon^{2-2\alpha}}{3ab}\|\nabla^2\mathrm{div}Q\|^2\\
&\quad+C_T\Big(\frac{\epsilon^2}{\delta^2}+\frac{\epsilon^4}{\delta^5}\Big)\|\nabla\zeta\|^2+C_T\Big(\frac{\epsilon^{6-2\alpha}}{\delta^6}+\frac{\epsilon^{4-2\alpha}}{\delta^3}+\frac{\epsilon^{4-2\alpha}}{\delta^2}+\frac{\epsilon^{8-2\alpha}}{\delta^8}\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
&C\epsilon^{1-\alpha}\left|\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\left(\mathcal{P}^4+4\mathcal{P}^3\bar{\theta}+6\mathcal{P}^2\bar{\theta}^2+4\mathcal{P}\bar{\theta}^3\right)_{y_1y_1y_1}Q_{1y_1y_1}\mathrm{d}y\right|\\
&\le C\epsilon^{1-\alpha}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}\Big(|\mathcal{P}_{y_1}|^3|Q_{1y_1y_1}|+|\mathcal{P}_{y_1y_1y_1}||Q_{1y_1y_1}|+|\bar{\theta}_{y_1}|^3|\mathcal{P}||Q_{1y_1y_1}|+|\bar{\theta}_{y_1y_1y_1}||\mathcal{P}||Q_{1y_1y_1}|\\
&\quad+|\mathcal{P}_{y_1y_1}||\mathcal{P}_{y_1}||Q_{1y_1y_1}|+|\mathcal{P}_{y_1y_1}||\bar{\theta}_{y_1}||Q_{1y_1y_1}|+|\bar{\theta}_{y_1y_1}||\mathcal{P}_{y_1}||Q_{1y_1y_1}|+|\bar{\theta}_{y_1y_1}||\bar{\theta}_{y_1}||\mathcal{P}||Q_{1y_1y_1}|\Big)\mathrm{d}y\\
&\le\frac{1}{3b}\|\nabla^2Q\|^2+C_T\Big(\frac{\epsilon^{8+2\alpha}}{\delta^{14}}+\frac{\epsilon^{4+4\alpha}}{\delta^8}+\frac{\epsilon^{4+2\alpha}}{\delta^9}+\frac{\epsilon^{6+2\alpha}}{\delta^{12}}\Big).
\end{aligned}
$$

Then plugging the above estimates into $\mathcal{W}_{3,1}$, we have

$$
\mathcal{W}_{3,1}\le C\epsilon^{4-4\alpha}\|\nabla\zeta\|^2+C\epsilon^{2-2\alpha}\|\nabla^2\zeta\|^2+\frac{1}{3b}\|\nabla^2Q\|^2+\frac{\epsilon^{2-2\alpha}}{3ab}\|\nabla^2\mathrm{div}Q\|^2+C_T\frac{\epsilon^{4+2\alpha}}{\delta^9}.
$$

Also, it holds

$$
\mathcal{W}_{3,2}\le\epsilon^{3-3\alpha}\int_{\mathbb{R}\times\mathbb{T}_\epsilon}|(\bar{\theta}^4)_{y_1y_1y_1y_1y_1}||Q_{1y_1y_1}|\mathrm{d}y\le\frac{1}{3b}\|\nabla^2Q\|^2+C\frac{\epsilon^{6+2\alpha}}{\delta^9}.
$$

Thus, for any $\tau\in[0,\tau_1(\epsilon)]$, inserting the above estimates into (3.39), then using (3.19) and (3.27), we obtain

$$
\begin{aligned}
\Big(\|\nabla^2Q\|^2+\epsilon^{2-2\alpha}\|\nabla^2\mathrm{div}Q\|^2\Big)(\tau)&\le C\epsilon^{4-4\alpha}\|\nabla\zeta\|^2+C\epsilon^{2-2\alpha}\|\nabla^2\zeta\|^2+C\frac{\epsilon^{4+2\alpha}}{\delta^9}\\
&\le C_T\epsilon^{2-2\alpha}\Big(\frac{\epsilon^{3-2\alpha}}{\delta^4}+\frac{\epsilon^{4-2\alpha}}{\delta^7}\Big)+C_T\frac{\epsilon^{4+2\alpha}}{\delta^9}+C\|(\phi_0,\Psi_0,\zeta_0)\|_2^2.
\end{aligned}\tag{3.40}
$$

Finally, we combine (3.36), (3.38), and (3.40) together to end the proof of (3.34). □

Collecting Lemmas 3.1–3.4 together, choosing $\delta = \epsilon^{\omega}|\ln\epsilon|$ with

$$\omega \in \left(0, \frac{1}{4}\right] \quad \text{and} \quad \alpha \in \left(\frac{5}{8}, 1\right), \tag{3.41}$$

such that

$$\left.\begin{aligned} \frac{\epsilon^{3-2\alpha}}{\delta^4} + \frac{\epsilon^{4-2\alpha}}{\delta^7} \ll \epsilon^{2-2\alpha}, \\ \frac{\epsilon^4}{\delta^5} + \frac{\epsilon^{4+2\alpha}}{\delta^9} \ll \epsilon^{4-4\alpha}, \\ \left(\frac{\epsilon}{\delta^{\frac{5}{2}}}, \frac{\epsilon^{2\alpha-1}}{\delta}, \frac{\epsilon^{\alpha}}{\delta^2}\right) \ll 1, \end{aligned}\right\} \text{when} \quad \epsilon \ll 1, \tag{3.42}$$

we finally close the *a priori* Assumption (3.10) and obtain *a priori* estimates (3.11) and (3.12). Eventually, Proposition 3.1 is proved.

4. Conclusions

In this paper,we establish the vanishing viscosity limit for the two-dimensional radiative hydrodynamics system, demonstrating that its solutions converge to the planar rarefaction wave solution of the corresponding compressible Euler equations as the viscosity parameter $\epsilon \to 0$. Further, we develop new cancelation mechanisms to address the radiative term in the system. Instead of specifying the order of scaling, we provide a broad range of orders. Using carefully designed scaling relationships and performing meticulous energy estimates, we establish the explicit convergence rate presented in (1.11) . This shows that our result serves as a generalization of [11, 12] to radiative hydrodynamics.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declare no conflict of interest.

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