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*Research article*

# Applications of conformable double Sumudu-Elzaki transform

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**Abstract:** In this paper, we used an efficient new technique for solving fractional partial and integral equations that meet specific criteria. This technique is referred to as the conformable double Sumudu-Elzaki transform (CDSET) and combines two well-known transforms, namely, the Sumudu transform, and the Elzaki transform. We attempt to present the fundamental concepts and findings of the recently suggested transformation. Relying on this brand-new integral transform and its related features, users can transform the equations above into algebraic equations that are easier to understand. Consequently, exact solutions can efficiently and rapidly be acquired. We conclude that the suggested approach is dependable and effective, as demonstrated. Therefore, it facilitates accurate and feasible solutions for other fractional linear models such as conformable derivatives.

**Keywords:** conformable double Sumudu-Elzaki transform; conformable partial derivative; fractional partial and integral differential equations
**Mathematics Subject Classification:** 35A22, 35G10, 35G15, 44A30, 45A05

## 1. Introduction

The field of fractional calculus, as it is currently understood, encompasses a number of findings as an outcome of extensive study and practical application. These discoveries can be traced back to L'Hôpital's answer to a question asked by Leibnitz [1,2]. This context sets the core meanings of arbitrary order derivatives and integrals. The complicated nature of fractional derivatives has made it challenging to use them in mathematical, physical, and engineering contexts, which has caused a number of problems [3–6].

In [7,8], the authors propose a different concept known as the conformable fractional derivative (CFD), which is best for the fundamental principles of derivatives. The CFD is a useful concept that combines complex analysis and fractional calculus. It is especially useful for modelling complicated systems involving complicated shapes, like electromagnetism and fluid dynamics. In recent years, scientists have become increasingly interested in solving conformable fractional partial differential equations. The flexible feature of the CFD has sparked increased interest, opening new research prospects [9–12].

Researchers have developed a novel method for solving fractional and partial differential equations [13], which is called the double Sumudu-Elzaki transform (DSET). Unfortunately, this transformation, as well as other transformations, have problems dealing with nonlinear problems. Therefore, researchers have devised new techniques that combine transforms with other numerical methods to solve this issue. These include the variational iteration method [14,15], the decomposition method [16–19], the perturbation method [20,21], and the iterative method [22,23]. Our main objective of this study is to devise new ways to use the DSET, specifically appropriate for fixing various types of linear fractional partial and integral differential equations with conformable derivatives.

This study is divided into the following sections: In Section 2, we cover the fundamental characteristics of conformable fractional derivatives (CFDs). In Section 3, we introduce conformable double Sumudu-Elzaki transform (CDSET), a novel integral transformation that combines the Sumudu and Elzaki transforms, highlighting its key properties. In Section 4, we combine theoretical frameworks and the evidence supporting them to show how transformational it could be through four examples. This shows dependability, convergence, and efficiency. In Section 5, we present the statistical results of the study and conclude the paper in Section 6.

## 2. Preliminary

In this section, a definition of the CFD is given with demonstrations of essential features that are useful in obtaining significant results, as shown in the definition below [7].

**Definition 2.1.** [7] Let $\zeta:(0,\infty)\to\mathbb{R}$, then the CFD of order $0<\rho\le 1$ of $\zeta$ is defined by:

$$D^{\rho}\zeta(v)=\lim_{\delta\to 0}\frac{\zeta\left(v+\delta v^{1-\rho}\right)-\zeta(v)}{\delta},v>0,\rho\in(0,1]. \tag{1}$$

**Definition 2.2.** [24] Let $0<\rho\le 1,$ and $\zeta:(0,\infty)\to\mathbb{R}$, we define the conformable fractional partial derivatives (CFPDs) of orders $\sigma$ and $\rho$ of the function $\zeta(u,v)$ as follows:

$$D_{u}^{\sigma}\zeta(u,v)=\frac{\partial^{\sigma}}{\partial u^{\sigma}}\zeta(u,v)=\lim_{\varepsilon\to 0}\frac{\zeta\left(u+\varepsilon u^{1-\sigma},v\right)-\zeta(u,v)}{\varepsilon},u,v>0,\sigma\in(0,1], \tag{2}$$

$$D_{v}^{\rho}\zeta(u,v)=\frac{\partial^{\rho}}{\partial v^{\rho}}\zeta(u,v)=\lim_{\delta\to 0}\frac{\zeta\left(u,v+\delta v^{1-\rho}\right)-\zeta(u,v)}{\delta},u,v>0,\rho\in(0,1]. \tag{3}$$

The following theorem summarizes the relationships between $\frac{\partial^{\sigma}\zeta}{\partial u^{\sigma}}$, $\frac{\partial^{\rho}\zeta}{\partial v^{\rho}}$, $\frac{\partial\zeta}{\partial u}$ and $\frac{\partial\zeta}{\partial v}$.

**Theorem 2.1.** [25] Suppose $0<\sigma,\rho\le 1,$ and $\zeta(u,v)$ are $\sigma$ and $\rho$, differentiable at a point $u,v>0,$ then

$$\begin{aligned}\frac{\partial^{\sigma}\zeta}{\partial u^{\sigma}}&=u^{-\sigma+1}\frac{\partial\zeta}{\partial u},\\ \frac{\partial^{\rho}\zeta}{\partial v^{\rho}}&=v^{-\rho+1}\frac{\partial\zeta}{\partial v}.\end{aligned}\tag{4}$$

*Proof.* see [25].

The following proposition lists the CFPDs for several functions.

**Proposition 2.1.** [25] Suppose $0<\sigma,\rho\le 1, c_1, c_2, m_1, m_2, \gamma,$ and $\eta\in\mathbb{R};$ then

$$\frac{\partial^{\sigma}}{\partial u^{\sigma}}\left(c_1\zeta\left(\frac{u^{\sigma}}{\sigma},\frac{v^{\rho}}{\rho}\right)+c_2\psi\left(\frac{u^{\sigma}}{\sigma},\frac{v^{\rho}}{\rho}\right)\right)=c_1\left(\frac{\partial^{\sigma}}{\partial u^{\sigma}}\zeta\left(\frac{u^{\sigma}}{\sigma},\frac{v^{\rho}}{\rho}\right)\right)+c_2\left(\frac{\partial^{\sigma}}{\partial u^{\sigma}}\psi\left(\frac{u^{\sigma}}{\sigma},\frac{v^{\rho}}{\rho}\right)\right),$$

$$\frac{\partial^{\rho}}{\partial v^{\rho}}\left(e^{\gamma\frac{u^{\sigma}}{\sigma}+\eta\frac{v^{\rho}}{\rho}}\right)=\eta e^{\gamma\frac{u^{\sigma}}{\sigma}+\eta\frac{v^{\rho}}{\rho}},$$

$$\frac{\partial^{\sigma}}{\partial u^{\sigma}}\left(e^{\gamma\frac{u^{\sigma}}{\sigma}+\eta\frac{v^{\rho}}{\rho}}\right)=\gamma e^{\gamma\frac{u^{\sigma}}{\sigma}+\eta\frac{v^{\rho}}{\rho}},$$

$$\frac{\partial^{\sigma}}{\partial u^{\sigma}}\left(\sin\left(\frac{u^{\sigma}}{\sigma}\right)\sin\left(\frac{v^{\rho}}{\rho}\right)\right)=\cos\left(\frac{u^{\sigma}}{\sigma}\right)\sin\left(\frac{v^{\rho}}{\rho}\right),$$

$$\frac{\partial^{\rho}}{\partial v^{\rho}}\left(\sin\left(\frac{u^{\sigma}}{\sigma}\right)\sin\left(\frac{v^{\rho}}{\rho}\right)\right)=\sin\left(\frac{u^{\sigma}}{\sigma}\right)\cos\left(\frac{v^{\rho}}{\rho}\right).$$

## 3. Some results and theorem of the conformable double Sumudu-Elzaki transform

In this section, the initial results for formalizing a new version of the CDSET are presented, which are applicable to a wide range of FPDEs.

**Definition 3.1.** A function $\zeta(u,v)$ is considered to be of exponential order if there are positive constants $K,$ $a,$ and $b$ such that

$$\left|\zeta(u,v)\right|\le K\,e^{a\frac{u^{\sigma}}{\sigma}+b\frac{v^{\rho}}{\rho}}$$

for sufficiently large $u$ and $v.$ Now, under these conditions, we define the CDSET.

**Definition 3.2.** The conformable Sumudu transform (CST) of real-valued $\zeta:[0,\infty)\to\mathbb{R}$ order $\sigma$ has the following definition:

$$S_u^{\sigma}\left[\zeta(u,v):q\right]=\Omega_{\sigma}\left(q,v\right)=\frac{1}{q}\int_0^{\infty}e^{-\frac{u^{\sigma}}{q\sigma}}\zeta(u,v)d_{\sigma}u,u>0. \tag{5}$$

**Definition 3.3.** The conformable Elzaki transform (CET) of real-valued $\zeta:[0,\infty)\to\mathbb{R}$ order $\rho$ has the following definition:

$$E_v^{\rho}\left[\zeta(u,v):w\right]=\Omega_{\rho}\left(u,w\right)=w\int_0^{\infty}e^{-\frac{v^{\rho}}{w\rho}}\zeta(u,v)d_{\rho}v,v>0. \tag{6}$$

**Definition 3.4.** The DCSET of piecewise continuous $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ orders $\sigma$ , and $\rho$ has the following definition:

$$S_u^{\sigma}E_v^{\rho}\left[\zeta\left(u,v\right):\left(q,w\right)\right]=\Omega_{\sigma,\rho}\left(q,w\right)=\Omega\left(q,w\right)=\frac{w}{q}\int_0^{\infty}\int_0^{\infty}e^{-\frac{u^{\sigma}}{q\sigma}-\frac{v^{\rho}}{w\rho}}\zeta\left(u,v\right)d_{\sigma}u\,d_{\rho}v, \tag{7}$$

or

$$S_u^{\sigma}E_v^{\rho}\left[\zeta\left(u,v\right):\left(q,w\right)\right]=\Omega\left(q,w\right)=w^2\int_0^{\infty}\int_0^{\infty}e^{-\frac{u^{\sigma}}{\sigma}-\frac{v^{\rho}}{\rho}}\zeta\left(qu,wv\right)d_{\sigma}u\,d_{\rho}v, \tag{8}$$

where $q,w\in\mathbb{C}\,,0<\sigma,\rho\leq1,\ d_{\sigma}u=u^{\sigma-1}du,$ and $d_{\rho}v=v^{\rho-1}dv.$

**Remark 3.1.** According to CDSET definition, we can conclude that the CDSET is a linear integral transformation, as illustrated below:

$$\begin{aligned}
S_u^{\sigma}E_v^{\rho}\left[c_1\zeta\left(u,v\right)+c_2\psi\left(u,v\right)\right]&=\frac{w}{q}\int_0^{\infty}\int_0^{\infty}e^{-\frac{u^{\sigma}}{q\sigma}-\frac{v^{\rho}}{w\rho}}\left[c_1\zeta\left(u,v\right)+c_2\psi\left(u,v\right)\right]d_{\sigma}u\,d_{\rho}v\\
&=\frac{w}{q}\int_0^{\infty}\int_0^{\infty}e^{-\frac{u^{\sigma}}{q\sigma}-\frac{v^{\rho}}{w\rho}}c_1\zeta\left(u,v\right)d_{\sigma}u\,d_{\rho}v+\frac{w}{q}\int_0^{\infty}\int_0^{\infty}e^{-\frac{u^{\sigma}}{q\sigma}-\frac{v^{\rho}}{w\rho}}c_2\psi\left(u,v\right)d_{\sigma}u\,d_{\rho}v\\
&=\frac{wc_1}{q}\int_0^{\infty}\int_0^{\infty}e^{-\frac{u^{\sigma}}{q\sigma}-\frac{v^{\rho}}{w\rho}}\zeta\left(u,v\right)d_{\sigma}u\,d_{\rho}v+\frac{wc_2}{q}\int_0^{\infty}\int_0^{\infty}e^{-\frac{u^{\sigma}}{q\sigma}-\frac{v^{\rho}}{w\rho}}\psi\left(u,v\right)d_{\sigma}u\,d_{\rho}v\\
&=c_1\,S_u^{\sigma}E_v^{\rho}\left[\,\zeta\left(u,v\right)\right]+c_2\,S_u^{\sigma}E_v^{\rho}\left[\,\psi\left(u,v\right)\right],c_1,c_2\in R.
\end{aligned}$$

**Definition 3.5.** The inverse conformable double Sumudu-Elzaki transform,

$$\left(S_u^{\sigma}\right)^{-1}\left(E_v^{\rho}\right)^{-1}\left[\Omega\left(q,w\right)\right]=\zeta\left(u,v\right)$$

is defined by:

$$\left(S_u^{\sigma}\right)^{-1}\left(E_v^{\rho}\right)^{-1}\left[\Omega\left(q,w\right)\right]=\zeta\left(u,v\right)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{q}e^{\frac{u^{\sigma}}{q\sigma}}\left[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}we^{\frac{v^{\rho}}{w\rho}}\Omega\left(q,w\right)dw\right]dq. \tag{9}$$

If $\zeta\left(u,v\right)$ meets the required criteria [26], then

$$\frac{w}{q}\int_0^\infty\int_0^\infty e^{-\frac{u^\sigma}{q\sigma}-\frac{v^\rho}{w\rho}}\,\zeta(u,v)\,d_\sigma u\,d_\rho v=\frac{w}{q}\int_0^\infty\int_0^\infty e^{-\frac{v^\rho}{w\rho}-\frac{u^\sigma}{q\sigma}}\,\zeta(u,v)\,d_\rho v\,d_\sigma u, \tag{10}$$

and so,

$$S_u^\sigma E_v^\rho\left[\zeta(u,v):(q,w)\right]=E_v^\rho S_u^\sigma\left[\zeta(u,v):(q,w)\right]. \tag{11}$$

The following theorem sets the relationship between the normal DSET and the CDSET.

**Theorem 3.1.** [9] Assume $\zeta:(0,\infty)\to\mathbb{R}$ such that $S_u^\sigma E_v^\rho\left[\zeta(u,v):(q,w)\right]=\Omega(q,w)$ exists, then

$$S_u^\sigma E_v^\rho\left[\zeta(u,v):(q,w)\right]=\Omega(q,w)=S_u E_v\left[\zeta(u,v):(q,w)\right], \tag{12}$$

where

$$S_u E_v\left[\zeta(u,v):(q,w)\right]=\Omega(q,w)=\frac{w}{q}\int_0^\infty\int_0^\infty e^{-\frac{u}{q}-\frac{v}{w}}\,\zeta(u,v)\,du\,dv. \tag{13}$$

Table 1 sets the CDSET for variables $u$ and $v$, as cited in [13,14,16].

**Table 1.** The CDSET for certain essential functions.

| $\zeta\left(\frac{u^\sigma}{\sigma},\frac{v^\rho}{\rho}\right)$ | $S_u^\sigma E_v^\rho\left[\zeta\left(\frac{u^\sigma}{\sigma},\frac{v^\rho}{\rho}\right)\right]=\Omega(q,w)$ |
|---|---|
| $c_1, c_1\in\mathbb{R}$ | $c_1 w^2$ |
| $\left(\frac{u^\sigma}{\sigma}\right)^m\left(\frac{v^\rho}{\rho}\right)^n, m,n\in\mathbb{Z}^+$ | $m!n!q^m w^{n+2}$ |
| $e^{c_1\frac{u^\sigma}{\sigma}+c_2\frac{v^\rho}{\rho}}$ | $\frac{w^2}{(1-c_1q)(1-c_2 w)},(1-c_1q)(1-c_2 w)>0$ |
| $\sin\left(c_1\frac{u^\sigma}{\sigma}+c_2\frac{v^\rho}{\rho}\right), c_1, c_2\in\mathbb{R}$ | $\frac{w^2(c_1q+c_2 w)}{(1+c_1^2q^2)(1+c_2^2 w^2)}$ |
| $\cos\left(c_1\frac{u^\sigma}{\sigma}+c_2\frac{v^\rho}{\rho}\right)$ | $\frac{w^2(1-c_1c_2 qw)}{(1+c_1^2q^2)(1+c_2^2 w^2)}$ |
| $\sinh\left(c_1\frac{u^\sigma}{\sigma}+c_2\frac{v^\rho}{\rho}\right)$ | $\frac{w^2(c_1q+c_2 w)}{(1-c_1^2q^2)(1-c_2^2 w^2)},(1-c_1^2q^2)(1-c_2^2 w^2)>0$ |
| $\cosh\left(c_1\frac{u^\sigma}{\sigma}+c_2\frac{v^\rho}{\rho}\right)$ | $\frac{w^2(1+c_1c_2 qw)}{(1-c_1^2q^2)(1-c_2^2 w^2)},(1-c_1^2q^2)(1-c_2^2 w^2)>0$ |
| $J_0\left(b\sqrt{\frac{u^\sigma}{\sigma}\frac{v^\rho}{\rho}}\right), b\in\mathbb{R}$ | $\frac{4w^2}{4+b^2qw}$ |

**Theorem 3.2.** [9] (CDL-CDSE duality) If the CDSET of $\zeta(u,v)$ exists, then

$$S_u^\sigma E_v^\rho\left[\zeta(u,v):(q,w)\right]=\frac{w}{q}L_u^\sigma L_v^\rho\left[\zeta(u,v):\left(\frac{1}{q},\frac{1}{w}\right)\right], \tag{14}$$

where

$$L_u^\sigma L_v^\rho\left[\zeta(u,v):(q,w)\right]=\Omega(q,w)=\int_0^\infty\int_0^\infty e^{-q\frac{u^\sigma}{\sigma}-w\frac{v^\rho}{\rho}}\,\zeta(u,v)\,d_\sigma u\,d_\rho v. \tag{15}$$

**Theorem 3.3.** (First shifting property) Assume that $S_u^\sigma E_v^\rho\left[\zeta(u,v)\right]=\Omega(q,w)$, then

$$S_u^\sigma E_v^\rho\left[e^{-c_1\frac{u^\sigma}{\sigma}-c_2\frac{v^\rho}{\rho}}\zeta(u,v):(q,w)\right]=\frac{(1+c_2w)}{(1+c_1q)}\Omega\left(\frac{q}{1+c_1q},\frac{w}{1+c_2w}\right),c_1,c_2\in\mathbb{R}. \tag{16}$$

*Proof.* Relevant to Definition 3.4,

$$\begin{aligned}\left[e^{-c_1\frac{u^\sigma}{\sigma}-c_2\frac{v^\rho}{\rho}}\zeta(u,v):(q,w)\right]&=\frac{w}{q}\int_0^\infty\int_0^\infty e^{-\frac{u^\sigma}{q\sigma}-\frac{v^\rho}{w\rho}}\,e^{-c_1\frac{u^\sigma}{\sigma}-c_2\frac{v^\rho}{\rho}}\zeta(u,v)\,d_\sigma u\,d_\rho v\\&=\frac{w}{q}\int_0^\infty\int_0^\infty e^{-\left(\frac{1}{q}+c_1\right)\frac{u^\sigma}{\sigma}-\left(\frac{1}{w}+c_2\right)\frac{v^\rho}{\rho}}\,\zeta(u,v)\,d_\sigma u\,d_\rho v.\end{aligned}$$

Put $p=\dfrac{q}{1+c_1q}$, $s=\dfrac{w}{1+c_2w}$, then

$$\begin{aligned}S_u^\sigma E_v^\rho\left[e^{-c_1\frac{u^\sigma}{\sigma}-c_2\frac{v^\rho}{\rho}}\zeta(u,v):(q,w)\right]&=\frac{1+c_2w}{1+c_1q}\left(\frac{s}{p}\int_0^\infty\int_0^\infty e^{-\frac{u^\sigma}{p\sigma}-\frac{v^\rho}{s\rho}}\,\zeta(u,v)\,d_\sigma u\,d_\rho v\right)\\&=\frac{1+c_2w}{1+c_1q}\Omega(p,s)=\frac{1+c_2w}{1+c_1q}\Omega\left(\frac{q}{1+c_1q},\frac{w}{1+c_2w}\right).\end{aligned}$$

**Theorem 3.4.** If $S_u^\sigma E_v^\rho\left[\zeta(u,v)\right]=\Omega(q,w)$, then the CDSET of the CFPDs $\dfrac{\partial^\sigma\zeta}{\partial u^\sigma}$ and $\dfrac{\partial^\rho\zeta}{\partial v^\rho}$ can be represented as follows:

$$S_u^\sigma E_v^\rho\left[\frac{\partial^\sigma\zeta}{\partial u^\sigma}\right]=\frac{1}{q}\Omega(q,w)-\frac{1}{q}\Omega(0,w). \tag{17}$$

$$S_u^\sigma E_v^\rho\left[\frac{\partial^\rho\zeta}{\partial v^\rho}\right]=\frac{1}{w}\Omega(q,w)-w\Omega(q,0). \tag{18}$$

*Proof.* Here, we proceed with the result's proof (17).

$$S_u^\sigma E_v^\rho\left[\frac{\partial^\sigma\zeta}{\partial u^\sigma}\right]=\frac{w}{q}\int_0^\infty\int_0^\infty e^{-\frac{u^\sigma}{q\sigma}-\frac{v^\rho}{w\rho}}\ \frac{\partial^\sigma\zeta}{\partial u^\sigma}d_\sigma u\, d_\rho v=w\int_0^\infty e^{-\frac{v^\rho}{w\rho}}v^{\rho-1}\left\{\int_0^\infty\frac{1}{q}e^{-\frac{u^\sigma}{q\sigma}}\ \frac{\partial^\sigma\zeta}{\partial u^\sigma}u^{\sigma-1}du\right\}dv. \tag{19}$$

Since Theorem 2.1 states that $\frac{\partial^\sigma\zeta}{\partial u^\sigma}=u^{-\sigma+1}\frac{\partial\zeta}{\partial u}$, we put this result into Eq (19). So, Eq (10) becomes

$$S_u^\sigma E_v^\rho\left[\frac{\partial^\sigma\zeta}{\partial u^\sigma}\right]=w\int_0^\infty e^{-\frac{v^\rho}{w\rho}}\left\{\int_0^\infty\frac{1}{q}e^{-\frac{u^\sigma}{q\sigma}}\ \frac{\partial\zeta}{\partial u}du\right\}d_\rho v. \tag{20}$$

Let's put

$$\begin{gathered} w=e^{-\frac{u^\sigma}{q\sigma}}\Rightarrow dw=-\frac{1}{q}e^{-\frac{u^\sigma}{q\sigma}}\ du,\\ dt=\frac{\partial\zeta}{\partial u}du\Rightarrow t=\zeta\left(u,v\right). \end{gathered} \tag{21}$$

The integral contained within the bracket is provided by:

$$\int_0^\infty\frac{1}{q}e^{-\frac{u^\sigma}{q\sigma}}\ \frac{\partial\zeta}{\partial u}du=-\frac{1}{q}\zeta\left(0,v\right)+\frac{1}{q}\Omega\left(q,v\right). \tag{22}$$

By substituting Eq (22) into Eq (20), we obtain

$$S_u^\sigma E_v^\rho\left[\frac{\partial^\sigma\zeta}{\partial u^\sigma}\right]=w\int_0^\infty e^{-\frac{v^\rho}{w\rho}}\left\{-\frac{1}{q}\zeta\left(0,v\right)+\frac{1}{q}\Omega\left(q,v\right)\right\}d_\rho v=\frac{1}{q}\Omega\left(q,w\right)-\frac{1}{q}\Omega\left(0,w\right).$$

The same method can be used to illustrate the final result (18).

The following examples are a further generalization of the findings described above.

$$S_u^\sigma E_v^\rho\left[\frac{\partial^{n\sigma}\zeta}{\partial u^{n\sigma}}\right]=q^{-n}\Omega\left(q,w\right)-\sum_{k=0}^{n-1}q^{-n+k}E_v^\rho\left[\frac{\partial^{k\sigma}}{\partial u^{k\sigma}}\zeta\left(0,v\right)\right], \tag{23}$$

$$S_u^\sigma E_v^\rho\left[\frac{\partial^{m\rho}\zeta}{\partial v^{m\rho}}\right]=w^{-m}\,\Omega\left(q,w\right)-\sum_{j=0}^{m-1}w^{2-m+j}S_u^\sigma\left[\frac{\partial^{j\rho}}{\partial v^{j\rho}}\zeta\left(u,0\right)\right]. \tag{24}$$

**Theorem 3.5.** (Convolution theorem) Assume that $\zeta\left(u,v\right)$ and $\varphi\left(u,v\right)$ are two functions with the CDSET, then

$$S_u^\sigma E_v^\rho\left[\left(\zeta**\varphi\right)(u,v):\left(q,w\right)\right]=\frac{q}{w}\Omega\left(q,w\right)\Psi\left(q,w\right), \tag{25}$$

where

$$\left(\zeta**\varphi\right)(u,v)=\int_0^u\int_0^v\zeta\left(u-\eta,v-\gamma\right)\varphi\left(\eta,\gamma\right)d\eta\,d\gamma. \tag{26}$$

*Proof.* By using Theorems 2.2 and 3.2 from [12], we derive

$$\begin{aligned}S_u^\sigma E_v^\rho\left[(\zeta **\varphi)(u,v):(q,w)\right] &= \frac{w}{q}L_u^\sigma L_v^\rho\left[(\zeta **\varphi)(u,v):\left(\frac{1}{q},\frac{1}{w}\right)\right] \\ &= \frac{w}{q}L_u^\sigma L_v^\rho\left[\zeta(u,v):\left(\frac{1}{q},\frac{1}{w}\right)\right]L_u^\sigma L_v^\rho\left[\varphi(u,v):\left(\frac{1}{q},\frac{1}{w}\right)\right] \\ &= \frac{q}{w}S_u^\sigma E_v^\rho\left[\zeta(u,v):(q,w)\right]S_u^\sigma E_v^\rho\left[\varphi(u,v):(q,w)\right] = \frac{q}{w}\Omega(q,w)\Psi(q,w).\end{aligned}$$

## 4. Elucidative examples

In this part, four examples of linearly conformable fractional partial and integral differential equations are dealt with to show how well and feasible the CDSET method works. Each example includes its corresponding equation, initial and boundary conditions, solution steps, and the final solution.

**Example 4.1.** Consider the linear Euler-Bernoulli equation of CFPD:

$$\frac{\partial^{2\rho}\zeta}{\partial v^{2\rho}}+\frac{\partial^{4\sigma}\zeta}{\partial u^{4\sigma}}=\left(\frac{u^\sigma}{\sigma}\right)\left(\frac{v^\rho}{\rho}\right)+\left(\frac{v^\rho}{\rho}\right)^2, \tag{27}$$

with the IC,

$$\zeta\left(\frac{u^\sigma}{\sigma},0\right)=0,\zeta_v\left(\frac{u^\sigma}{\sigma},0\right)=\frac{1}{120}\left(\frac{u^\sigma}{\sigma}\right)^5, \tag{28}$$

and the BCs:

$$\zeta\left(0,\frac{v^\rho}{\rho}\right)=\frac{1}{12}\left(\frac{v^\rho}{\rho}\right)^4,\ \frac{\partial^{k\sigma}}{\partial u^{k\sigma}}\zeta\left(0,\frac{v^\rho}{\rho}\right)=0, k=1,2,3. \tag{29}$$

**Solution.** Applying the CDSET to Eq (27).

$$\begin{aligned}&w^{-2}\,\Omega(q,w)-\Omega(u,0)-w^{-1}\Omega_v(u,0)\\&+q^{-4}\,\Omega(q,w)-q^{-4}\,\Omega(0,v)-q^{-3}\,\Omega_u(0,v)-q^{-2}\,\Omega_{uu}(0,v)-q^{-1}\Omega_{uuu}(0,v)=qw^3+2w^4.\end{aligned} \tag{30}$$

By applying the CSE for the ICs (28) and the CET for the BCs (29), we get

$$\begin{aligned}&\Omega(u,0)=0,\ \Omega_v(u,0)=q^5,\Omega(0,v)=2w^6,\\&\Omega_u(0,v)=\Omega_{uu}(0,v)=\Omega_{uuu}(0,v)=0.\end{aligned} \tag{31}$$

Using Eq (31) and conducting subsequent algebraic calculations for Eq (30), we obtain

$$\Omega(q,w)=2w^6+w^3q^5. \tag{32}$$

Taking $\left(S_u^\sigma\right)^{-1}\left(E_v^\rho\right)^{-1}$ of Eq (32), we get the solution $\zeta\left(\frac{u^\sigma}{\sigma},\frac{v^\rho}{\rho}\right)$ to Eq (27) as

$$\begin{aligned}\zeta\left(\frac{u^\sigma}{\sigma},\frac{v^\rho}{\rho}\right)&=\left(S_u^\sigma\right)^{-1}\left(E_v^\rho\right)^{-1}\left[2w^6+w^3q^5\right]\\&=\frac{1}{12}\left(\frac{v^\rho}{\rho}\right)^4+\frac{1}{120}\left(\frac{v^\rho}{\rho}\right)\left(\frac{u^\sigma}{\sigma}\right)^5.\end{aligned}\tag{33}$$

**Example 4.2.** Consider the telegraph equation of CFPD:

$$\frac{\partial^{2\sigma}\zeta}{\partial u^{2\sigma}}-\frac{\partial^{2\rho}\zeta}{\partial v^{2\rho}}-\frac{\partial^{\rho}\zeta}{\partial v^{\rho}}-\zeta=1-\left(\frac{u^\sigma}{\sigma}\right)^2-\left(\frac{v^\rho}{\rho}\right),\tag{34}$$

with the ICs,

$$\zeta\left(\frac{u^\sigma}{\sigma},0\right)=\left(\frac{u^\sigma}{\sigma}\right)^2,\zeta_v\left(\frac{u^\sigma}{\sigma},0\right)=1,\tag{35}$$

and the BCs:

$$\zeta\left(0,\frac{v^\rho}{\rho}\right)=\frac{v^\rho}{\rho},\ \zeta_u\left(0,\frac{v^\rho}{\rho}\right)=0.\tag{36}$$

**Solution.** Continuing the same steps explained in Example 4.1, we get

$$\Omega(q,w)=2q^2+w^3.\tag{37}$$

Hence, the solution to Eq (34) can be obtained as follows:

$$\zeta\left(\frac{u^\sigma}{\sigma},\frac{v^\rho}{\rho}\right)=\left(S_u^\sigma\right)^{-1}\left(E_v^\rho\right)^{-1}\left[2q^2+w^3\right]=\left(\frac{u^\sigma}{\sigma}\right)^2+\left(\frac{u^\sigma}{\sigma}\right).\tag{38}$$

**Example 4.3.** Consider the Volterra integro-partial differential equation of CFPD:

$$\zeta\left(\frac{u^\sigma}{\sigma},\frac{v^\rho}{\rho}\right)=a-\lambda\int_0^u\int_0^v\zeta\left(\frac{(u-\eta)^\sigma}{\sigma},\frac{(v-\gamma)^\rho}{\rho}\right)d\eta\,d\gamma,\sigma,\rho\in(0,1],\tag{39}$$

**Solution.** Operating the CDSET on both sides of Eq (39) and using Theorem 3.5, we get

$$\Omega(q,w)=a\,w^2-\lambda\,q\,w\Omega(q,w).\tag{40}$$

By conducting algebraic operations for Eq (40), we obtain the solution to Eq (39) as follows:

$$\begin{aligned}\zeta\left(\frac{u^\sigma}{\sigma},\frac{v^\rho}{\rho}\right)&=\left(S_u^\sigma\right)^{-1}\left(E_v^\rho\right)^{-1}\left[\frac{a\,w^2}{1+\lambda\,q\,w}\right]\\&=a\,J_0\left(2\sqrt{\lambda\frac{u^\sigma}{\sigma}\frac{v^\rho}{\rho}}\right).\end{aligned}\tag{41}$$

**Example 4.4.** Consider the integro-partial differential equation of CFPD:

$$\frac{\partial^{2\rho}\zeta}{\partial v^{2\rho}}-\frac{\partial^{2\sigma}\zeta}{\partial u^{2\sigma}}+\zeta\left(\frac{u^{\sigma}}{\sigma},\frac{v^{\rho}}{\rho}\right)+\lambda\int_0^u\int_0^v e^{\left(\frac{(u-\eta)^{\sigma}}{\sigma}+\frac{(v-\gamma)^{\rho}}{\rho}\right)}\zeta(\eta,\gamma)\,d\eta\,d\gamma$$
$$=e^{\frac{u^{\sigma}}{\sigma}+\frac{v^{\rho}}{\rho}}+\left(\frac{u^{\sigma}}{\sigma}\right)\left(\frac{v^{\rho}}{\rho}\right)e^{\frac{u^{\sigma}}{\sigma}+\frac{v^{\rho}}{\rho}}, \tag{42}$$

with the ICs,

$$\zeta\left(\frac{u^{\sigma}}{\sigma},0\right)=e^{\frac{u^{\sigma}}{\sigma}},\zeta_v\left(\frac{u^{\sigma}}{\sigma},0\right)=e^{\frac{u^{\sigma}}{\sigma}}, \tag{43}$$

and the BCs:

$$\zeta\left(0,\frac{v^{\rho}}{\rho}\right)=e^{\frac{v^{\rho}}{\rho}},\ \zeta_u\left(0,\frac{v^{\rho}}{\rho}\right)=e^{\frac{v^{\rho}}{\rho}}. \tag{44}$$

**Solution.** Operating the CDSET on Eq (42), the CST on Eq (43), and the CET on Eq (44), and making some algebraic calculations, we get

$$\Omega(q,w)=\frac{w^2}{(1-q)(1-w)}. \tag{45}$$

Taking $\left(S_u^{\sigma}\right)^{-1}\left(E_v^{\rho}\right)^{-1}$ of Eq (45), we get the solution $\zeta\left(\frac{u^{\sigma}}{\sigma},\frac{v^{\rho}}{\rho}\right)$ to Eq (42) as

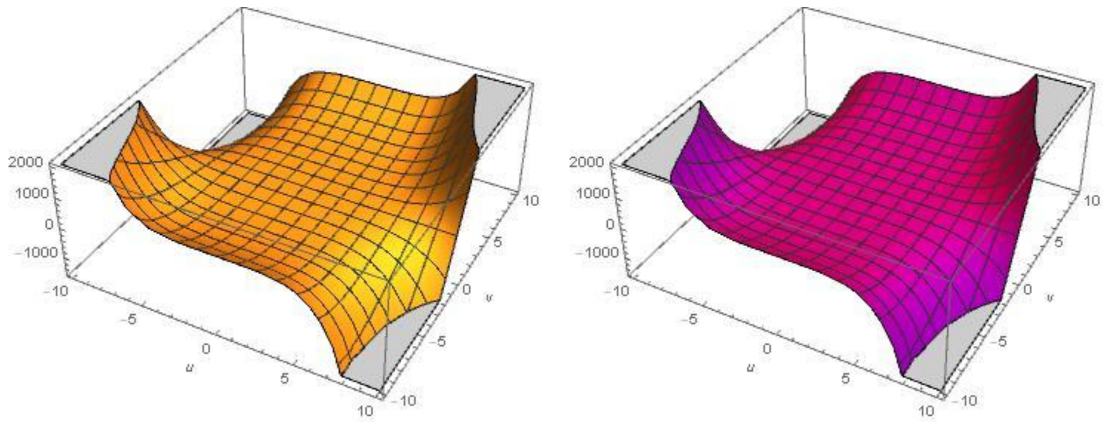
$$\zeta\left(\frac{u^{\sigma}}{\sigma},\frac{v^{\rho}}{\rho}\right)=\left(S_u^{\sigma}\right)^{-1}\left(E_v^{\rho}\right)^{-1}\left[\frac{w^2}{(1-q)(1-w)}\right]=e^{\frac{u^{\sigma}}{\sigma}+\frac{v^{\rho}}{\rho}}. \tag{46}$$

## 5. Numerical results and discussion

In this part, an evaluation of the proposed method's accuracy and usefulness by comparing approximate and exact results is presented using graphs and tables.

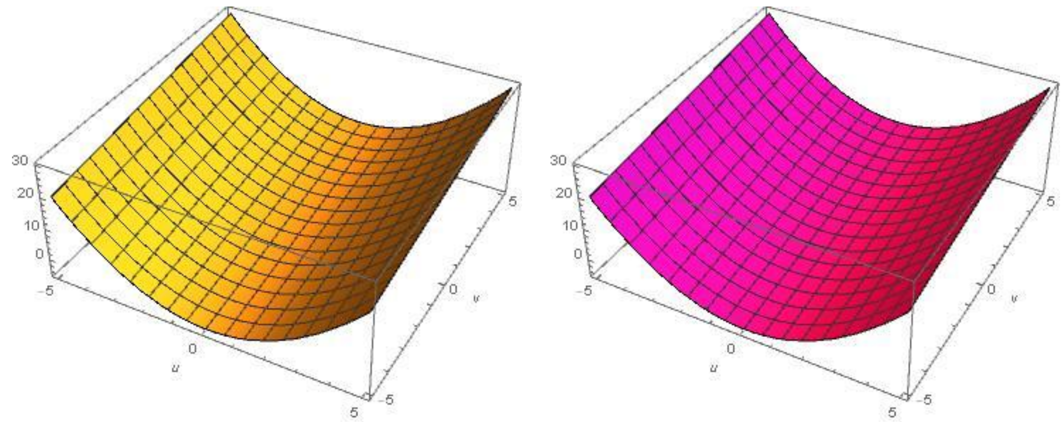
### *5.1. Graphical analysis*

Graphs of the 3D answers to Examples 4.1–4.4 in Figures 1–4, respectively, are given blow. The graphs (a) on the left show what happens with the current method, and the graphs (b) on the right show the exact answers, both at $\sigma=\rho=1$. These graphs show that they are in close contact.



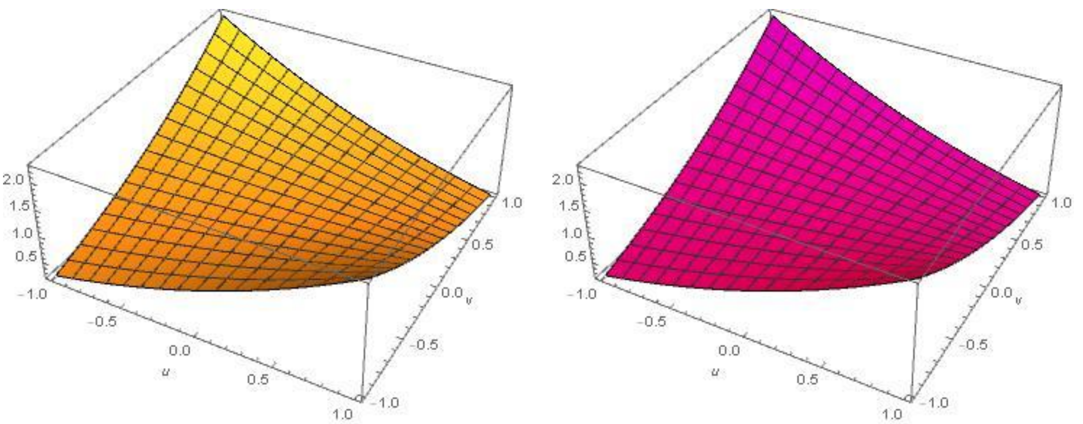

(a) (b)

**Figure 1.** Example 1 actual and approximate solutions of $\zeta\left(\frac{u^{\sigma}}{\sigma},\frac{v^{\rho}}{\rho}\right)$ for Eq (33) at $\sigma=\rho=1$.



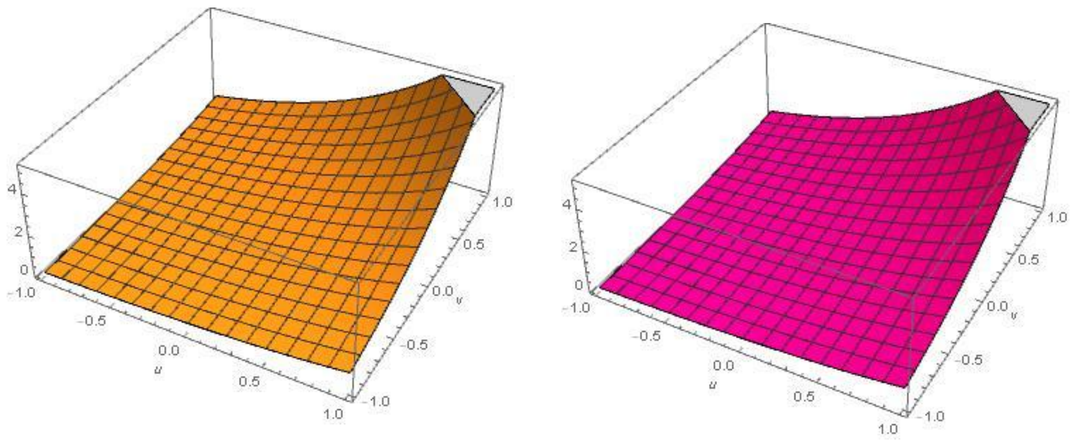

(a) (b)

**Figure 2.** Example 2 actual and approximate solutions of $\zeta\left(\frac{u^{\sigma}}{\sigma},\frac{v^{\rho}}{\rho}\right)$ for Eq (38) at $\sigma=\rho=1$.




(a) (b)

**Figure 3.** Example 3 actual and approximate solutions of $\zeta\left(\frac{u^{\sigma}}{\sigma},\frac{v^{\rho}}{\rho}\right)$ for Eq (41) at $\sigma=\rho=1$.




(a) (b)

**Figure 4.** Example 4 actual and approximate solutions of $\zeta\left(\frac{u^{\sigma}}{\sigma}, \frac{v^{\rho}}{\rho}\right)$ for Eq (46) at $\sigma = \rho = 1$.

The proposed method's approximate solutions in Figures 5–8 are compared with the exact answers. Thus, these comparisons give us valuable information about how the solution works in different fractional situations for $\sigma$ and $\rho$, which enables us to fully examine its performance.

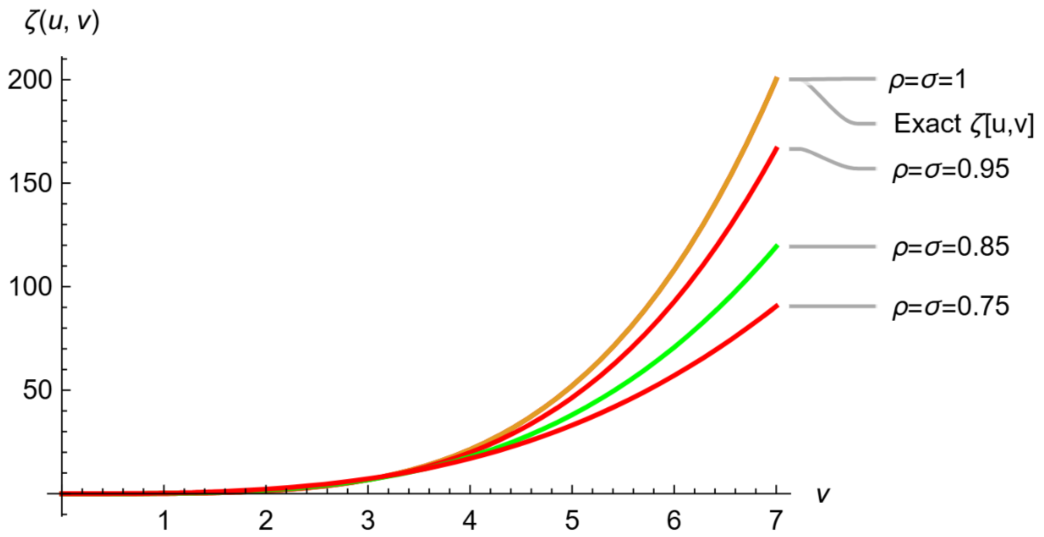



**Figure 5.** Example 1 approximate solution graph of $\zeta\left(\frac{u^{\sigma}}{\sigma}, \frac{v^{\rho}}{\rho}\right)$ for Eq (33) at varying values of $\sigma$ and $\rho$.

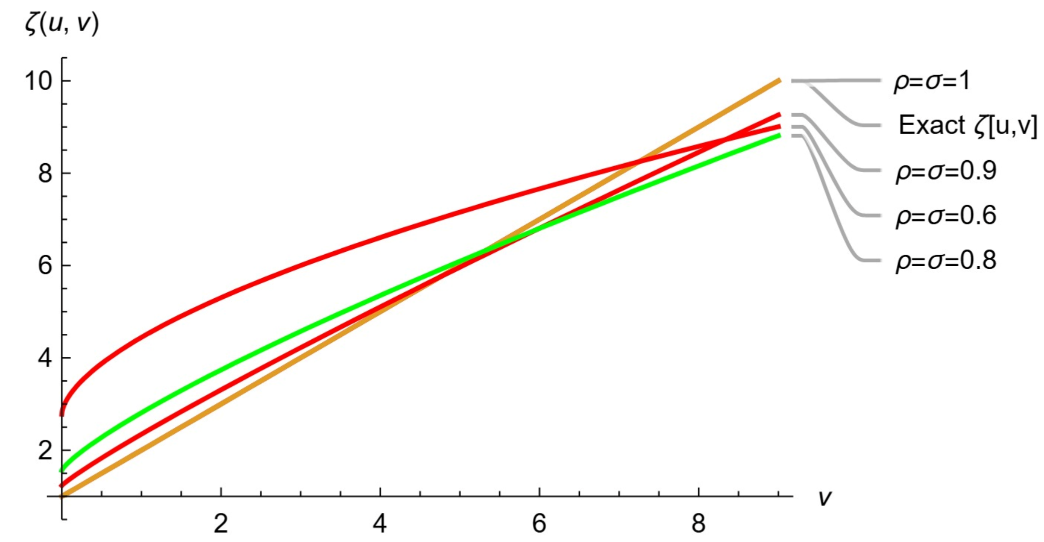



**Figure 6.** Example 2 approximate solution graph of $\zeta\left(\frac{u^{\sigma}}{\sigma},\frac{v^{\rho}}{\rho}\right)$ for Eq (38) at varying values of $\sigma$ and $\rho$.

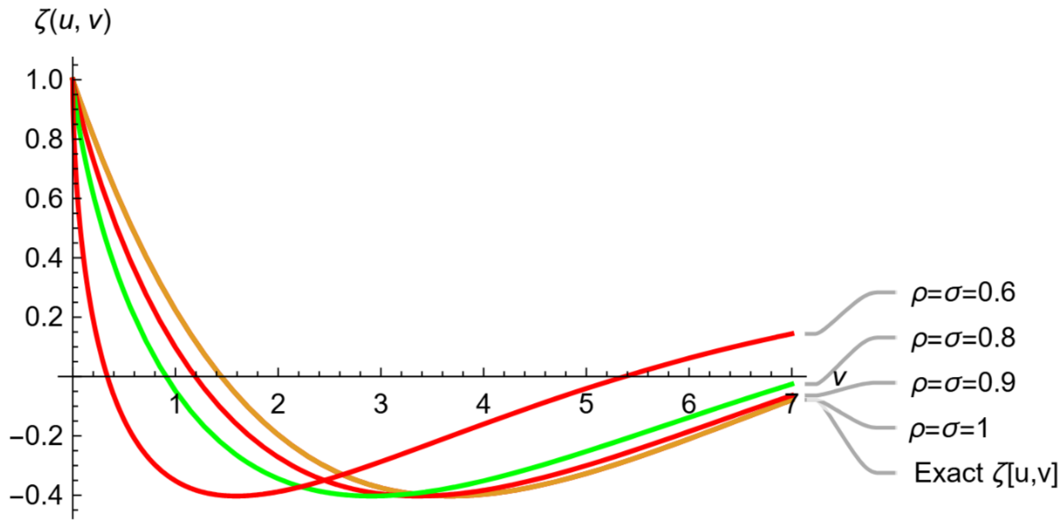



**Figure 7.** Example 3 approximate solution graph of $\zeta\left(\frac{u^{\sigma}}{\sigma},\frac{v^{\rho}}{\rho}\right)$ for Eq (41) at varying values of $\sigma$ and $\rho$.

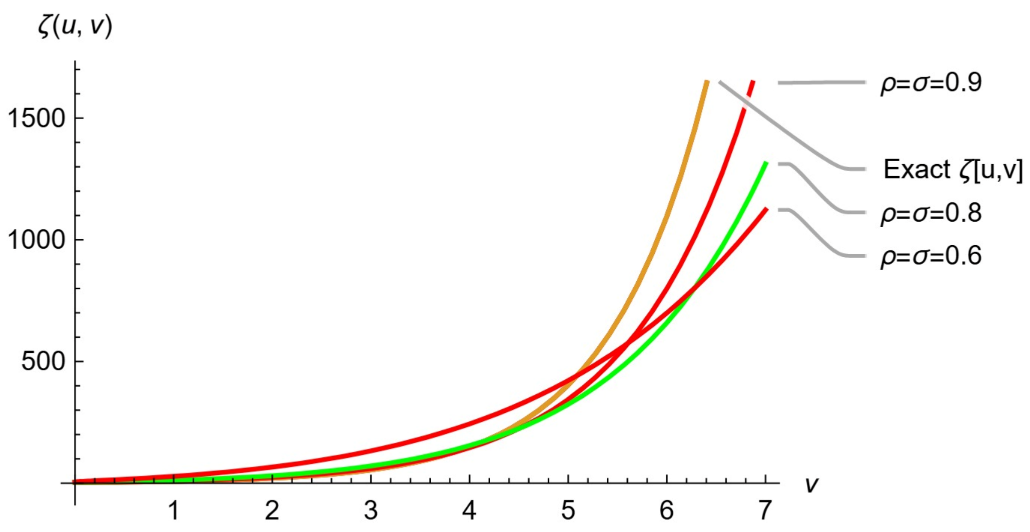



**Figure 8.** Example 4 approximate solution graph of $\zeta\left(\frac{u^{\sigma}}{\sigma},\frac{v^{\rho}}{\rho}\right)$ for Eq (46) at varying values of $\sigma$ and $\rho$.

Based on these results, the CDSET method is more accurate and comes closer to exact solutions when $\sigma,\rho\to1$.

### *5.2. Tabular results*

Tables 2–5 show how the approximate answers to the proposed method would be compared to the actual answers for each example, where $\sigma$ and $\rho$ are different fractional orders, $u=1$, and $v$ takes different values. The tables clearly show a significant consistency between the actual results and those of the proposed method.

**Table 2.** The actual and CDSET solutions of $\zeta\left(\frac{u^{\sigma}}{\sigma},\frac{v^{\rho}}{\rho}\right)$ for Example 1 at the values of $\sigma$ and $\rho$ are different; $u=1$, and $v$ takes various values.

| $u$ | $v$ | CDSET at $\sigma=\rho=0.8$ | CDSET at $\sigma=\rho=0.9$ | CDSET at $\sigma=\rho=1$ | actual at $\sigma=\rho=1$ |
|---|---|---|---|---|---|
| | 0 | 0 | 0 | 0 | 0 |
| | 3 | 6.9196 | 6.672 | 6.775 | 6.775 |
| 1 | 5 | 35.2035 | 41.767 | 52.125 | 52.125 |
| | 7 | 103.135 | 140.114 | 200.142 | 200.142 |
| | 9 | 230.347 | 346.15 | 546.825 | 546.825 |

**Table 3.** The actual and CDSET solutions of $\zeta\left(\frac{u^\sigma}{\sigma},\frac{v^\rho}{\rho}\right)$ for Example 2 at the values of $\sigma$ and $\rho$ are different; $u=1$, and $v$ takes various values.

| $u$ | $v$ | CDSET at $\sigma=\rho=0.8$ | CDSET at $\sigma=\rho=0.9$ | CDSET at $\sigma=\rho=1$ | actual at $\sigma=\rho=1$ |
|---|---|---|---|---|---|
| | 0 | 1.5625 | 1.23457 | 1 | 1 |
| | 3 | 4.57278 | 4.2211 | 4 | 4 |
| 1 | 5 | 6.09237 | 5.96423 | 6 | 6 |
| | 7 | 7.4916 | 7.63701 | 8 | 8 |
| | 9 | 8.81193 | 9.26198 | 10 | 10 |

**Table 4.** The actual and CDSET solutions of $\zeta\left(\frac{u^\sigma}{\sigma},\frac{v^\rho}{\rho}\right)$ for Example 3 at the values of $\sigma$ and $\rho$ are different; $u=1$, and $v$ takes various values.

| $u$ | $v$ | CDSET at $\sigma=\rho=0.8$ | CDSET at $\sigma=\rho=0.9$ | CDSET at $\sigma=\rho=1$ | actual at $\sigma=\rho=1$ |
|---|---|---|---|---|---|
| | 0 | 1 | 1 | 1 | 1 |
| | 3 | −0.402299 | −0.395509 | −0.374926 | −0.374926 |
| 1 | 5 | −0.252471 | −0.3 | −0.326875 | −0.326875 |
| | 7 | −0.0257781 | −0.063915 | −0.0787425 | −0.0787425 |
| | 9 | 0.156292 | 0.14312 | 0.150645 | 0.150645 |

**Table 5.** The actual and CDSET solutions of $\zeta\left(\frac{u^\sigma}{\sigma},\frac{v^\rho}{\rho}\right)$ for Example 4 at the values of $\sigma$ and $\rho$ are different; $u=1$, and $v$ takes various values.

| $u$ | $v$ | CDSET at $\sigma=\rho=0.8$ | CDSET at $\sigma=\rho=0.9$ | CDSET at $\sigma=\rho=1$ | actual at $\sigma=\rho=1$ |
|---|---|---|---|---|---|
| | 0 | 3.49034 | 3.0377 | 2.71828 | 2.71828 |
| | 3 | 70.8299 | 60.198 | 54.5982 | 54.5982 |
| 1 | 5 | 323.718 | 344.047 | 403.429 | 403.429 |
| | 7 | 1311.72 | 1832.72 | 2980.96 | 2980.96 |
| | 9 | 4911.98 | 9307.04 | 22026.5 | 22026.5 |

The figures and tables show that there is a significant consistency between the actual results and those of the proposed method. The test case shows that the suggested method is effective, reliable, and feasible enough to solve the linear models and provide more accurate results.

## 6. Conclusions

We discuss the major features of CDSET. Our focus is to show how to use CDSET to solve different linear partial and integral differential equations involving CFDs. Moreover, graphs and tables illustrate the proposed method's accuracy and usefulness, comparing it with exact solutions for varying values of $\sigma$ and $\rho$. The results show that the suggested method is feasible, appropriate, dependable, and convenient to resolve linear partial differential and integral equations with starting and ending conditions. As a result, this method can be applied to numerous linear FPDE systems. The CDSET method could be better than other methods because it turns differential equations into algebraic equations that are easier to understand. This means that it is time and effort saving when we have to get the exact solutions. This method works without making any assumptions about the answer (see [4,15,23,27]). Unfortunately, we cannot apply this transformation and similar ones to solve integral equations unless they are of the convolution type. In the future, we hope we could use the CDSET method to solve nonlinear differential equations with CFDs. We could combine it with other numerical techniques to enhance its capabilities, which potentially could lead to more specialized solutions and broader applications across fields.

## Author contributions

Shams A. Ahmed: drafting and designing the article; formal analysis; data interpretation; critical revision; methodology, final approval of the published version. Abdelgabar Adam Hassan and Husam E Dargail: original draft writing, study design, article revision through critical analysis, and final approval of the published version. Adam Zakria, Ibrahim-Elkhalil Ahmed and Ahmed Yahya: Study design; data collection and acquisition; writing; initial draft; critical revision; final approval of the published version. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

All authors declare no conflicts of interest in this paper.

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