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Research article

Dirichlet and Neumann boundary value problems for bi-polyanalytic functions on the bicylinder

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Abstract: Applying the Cauchy-Pompeiu formula and the properties of the singular integral operators on the unit disc, the specific representation of the solutions to the boundary value problems with the Dirichlet boundary conditions for bi-polyanalytic functions are obtained on the bicylinder. Also, the mixed-type boundary value problems of higher order for bi-polyanalytic functions were investigated. In addition, a system of complex partial differential equations with respect to polyanalytic functions with Neumann boundary conditions was discussed. On this foundation, the solutions to Neumann boundary value problems for bi-polyanalytic functions on the bicylinder were obtained. These results provide a favorable method for discussing other boundary value problems of bi-polyanalytic functions and the related systems of inhomogeneous complex partial differential equations of higher order in spaces of several complex variables.

Keywords: bi-polyanalytic functions; Dirichlet problems; Neumann problems; boundary value problems; complex partial differential equations
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1. Introduction

Bi-analytic functions arise from the research on systems of some partial differential equations and play an important role in studying elasticity problems. In 1961, Sander [1] studied a system of partial differential equations of first order:

$$\begin{cases} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \theta, & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \omega, \\ (k+1)\frac{\partial \theta}{\partial x} + \frac{\partial \omega}{\partial y} = 0, & (k+1)\frac{\partial \theta}{\partial y} - \frac{\partial \omega}{\partial x} = 0, \end{cases}$$

for $k \in \mathbb{R},\ k \neq -1$, and introduced the concept of bi-analytic functions of type k. He extended the elementary properties of analytic functions to bi-analytic functions and extended the related problems of the plane strain, the generalized plane stress, and the flow of viscous fluids to the theory of bi-analytic functions.

Later, Lin and Wu [2] introduced a more extensive class of functions, i.e., bi-analytic functions of type (λ, k), which are defined by the system of equations:

$$\begin{cases} \frac{1}{k}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \theta, & \frac{\partial u}{\partial y} + \frac{1}{k}\frac{\partial v}{\partial x} = \omega, \\ k\frac{\partial \theta}{\partial x} + \lambda\frac{\partial \omega}{\partial y} = 0, & k\frac{\partial \theta}{\partial y} - \lambda\frac{\partial \omega}{\partial x} = 0, \end{cases}$$

where $f(z) = u + iv$, λ, k are real constants with $\lambda \neq 0, 1, k^2$ and $0 < k < 1$. The complex form of the system of equations is

$$\frac{k+1}{2}\frac{\partial f}{\partial \bar{z}} - \frac{k-1}{2}\frac{\partial f}{\partial z} = \frac{\lambda - k}{4\lambda}\varphi(z) + \frac{\lambda + k}{4\lambda}\overline{\varphi(z)},$$

in which $\varphi(z) = k\vartheta - i\lambda\omega$ is analytic and is called the associated function of $f(z)$. Lin and Wu [2] obtained the general expression and some properties of bi-analytic functions of type (λ, k), including the Cauchy integral theorem and formula, the Morera theorem, the Weierstrass theorem, and the power series expansions.

Hua et al. [3,4] investigated the systems of partial differential equations of second order, which have close association with (λ, k) bi-analytic functions, and showed that (λ, k) bi-analytic functions provide a powerful tool to deal with problems of plane elasticity.

Mu [5] obtained the Cauchy integral representation, the removable singularity theorem, the Weierstrass theorem, and the mean value theorem of bi-analytic functions of type (λ, k) in the complex plane. Wen et al. [6,7] studied various types of boundary value problems for elliptic complex equations and systems, including boundary problems of (λ, k) bi-analytic functions. Begehr and Lin [8] studied a mix-contact problem by applying (λ, k) bi-analytic functions and singular integral operators. In addition, Lin and Zhao [9] reviewed the work on the applications of bi-analytic functions in elasticity, especially for solving the basic boundary value problems of plane elasticity (in isotropic or orthotropic cases). In [10], Lai investigated the properties of the vector-valued bi-analytic functions of type (λ, k) in a complex couple Banach space and obtained the solution of the corresponding Dirichlet problem.

The special case of bi-analytic functions of type (λ, k) for $k = 1$ is

$$\frac{\partial f}{\partial \bar{z}} = \frac{\lambda + 1}{4\lambda}\phi(z) + \frac{\lambda - 1}{4\lambda}\overline{\phi(z)}, \quad \frac{\partial \phi}{\partial \bar{z}} = 0,$$

which means

$$(1 - \lambda)\frac{\partial^2 f}{\partial \bar{z}^2} + (1 + \lambda)\overline{\frac{\partial^2 f}{\partial z \partial \bar{z}}} = 0.$$

Thus, bi-analytic functions are different from bianalytic functions (which only satisfy $\frac{\partial^2 f}{\partial \bar{z}^2} = 0$) in the sense of Bitsadze [11]. The generalization of a bianalytic function f is $\frac{\partial^n f}{\partial \bar{z}^n} = 0$ $(n \geq 1)$, which is called a polyanalytic function [12]. Obviously, bi-analytic functions can be similarly generalized.

In [13], Kumar extended bi-analytic functions on bounded simply connected domains. He considered the system of complex equations:

$$\frac{\partial f}{\partial \bar{z}} = \frac{\lambda + 1}{4\lambda}\phi(z) + \frac{\lambda - 1}{4\lambda}\overline{\phi(z)}, \quad \frac{\partial^n \phi}{\partial \bar{z}^n} = 0 \ \ (n \geq 1),$$

where f is called bi-polyanalytic functions, and obtained the expressions of solutions to the corresponding boundary value problems on the unit disc. In [13], the systems

$$\frac{\partial f}{\partial \bar{z}} = \frac{\lambda+1}{4\lambda}\phi(z) + \frac{\lambda-1}{4\lambda}\overline{\phi(z)}, \quad \frac{\partial^n \phi}{\partial \bar{z}^n} = h(z, \phi, f, f_z),$$
$$\frac{\partial f}{\partial \bar{z}} = \frac{\lambda+1}{4\lambda}\phi(z) + \frac{\lambda-1}{4\lambda}\overline{\phi(z)}, \quad \frac{\partial^n \phi}{\partial \bar{z}^n} = h(z, \phi, \phi_z, f),$$

and the associated boundary problems were also discussed.

Begehr et al. [14,15] investigated boundary value problems for bi-polyanalytic functions in the upper half plane and on the unit disc. Similar researches can also be found in the reference [16].

In 2022, Lin and Xu [17] successfully obtained the solutions of Riemann problems of (λ, k) bi-analytic functions. In 2023, Lin [18] discussed a type of inverse boundary value problems for $(\lambda, 1)$ bi-analytic functions. Nowadays, bi-analytic functions and polyanalytic functions have made great progress [19–22].

The above conclusions are all in the complex plane. If these results can be extended to spaces of several complex variables, there will be more applications. There have been some conclusions about bi-analytic functions or polyanalytic functions with several complex variables. For example, in [23], Begehr and Kumar studied complex bi-analytic functions of n variables. They successfully obtained the corresponding Cauchy integral formula, Taylor series and Poisson integral formula on the polydisc, and discussed the Dirichlet problem of the systems of partial differential equations on the polydisc. In [24], Kumar discussed a generalized Riemann boundary value problem by the Cauchy integral of bi-analytic functions with two complex variables. In 2023, Vasilevski [25] studied polyanalytic functions with several complex variables in detail.

However, there is relatively little research on bi-polyanalytic functions in spaces of several complex variables. Inspired by these, and on the basis of the previous works of the former researchers, we first investigate a kind of boundary value problem for polyanalytic functions with Schwarz conditions on the bicylinder, then we discuss boundary value problems with the Dirichlet, Neumann boundary conditions and mixed type boundary conditions for bi-polyanalytic functions on the bicylinder.

Throughout this paper, let the bicylinder $D^2 = D_1 \times D_2 = \{(z_1, z_2) : |z_1| < 1,\ |z_2| < 1\}$, whose characteristic boundary is denoted as $\partial_0 D^2$, where z_1 and z_2 are on the complex plane. Let $C^m(G)$ represent the set of functions whose partial derivatives of order m are all continuous within a bounded smooth domain G.

2. Some lemmas

Lemma 2.1. *[26] Let G be a bounded smooth domain in the complex plane, $f \in L_1(G; \mathbb{C})$ and*

$$Tf(z) = \frac{-1}{\pi}\int_G \frac{f(\zeta)}{\zeta - z} d\xi d\eta, \quad \zeta = \xi + i\eta.$$

Then, $\partial_{\bar{z}} Tf(z) = f(z)$.

Lemma 2.2. *[26] Let $w \in C^1(G; \mathbb{C}) \bigcap C(\overline{G}; \mathbb{C})$, where $m \geq 1$ and G is a bounded smooth domain in the complex plane, then*

$$w(z) = \frac{1}{2\pi i}\int_{\partial G} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_G w_{\bar{\zeta}}(\zeta)\frac{d\sigma_\zeta}{\zeta - z},$$

$$\int_G w_{\bar{z}}(z)d\sigma_z = \frac{1}{2i}\int_{\partial G} w(z)dz, \qquad \int_G w_z(z)d\sigma_z = -\frac{1}{2i}\int_{\partial G} w(z)d\bar{z},$$

where $d\sigma_\zeta = d\xi d\eta$ $(\zeta = \xi + i\eta)$ *and* $d\sigma_z = dxdy$ $(z = x + iy)$.

Lemma 2.3. *[27] For* $n \in \mathbb{N}$, $f \in L^p(D)$, $p \geq 1$ *with D being the unit disc, let*

$$T_n f(z) = \frac{(-1)^n}{2\pi(n-1)!}\int_D (\overline{\zeta - z} + \zeta - z)^{n-1}\Big[\frac{f(\zeta)}{\zeta}\frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}}\frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}}\Big]d\xi d\eta,$$

and let $T_0 f = f$. *Then*

$$\frac{\partial^l}{\partial \bar{z}^l}T_n f = T_{n-l}f, \ 1 \leq l \leq n,$$

$$\Re\frac{\partial^l}{\partial \bar{z}^l}T_n f = 0 \ (z \in \partial D), \quad \Im\frac{\partial^l}{\partial \bar{z}^l}T_n f(0) = 0, \ 0 \leq l \leq n - 1.$$

Lemma 2.4. *Let* $g_{\mu\nu} \in C(\partial_0 D^2; \mathbb{R})$ *for* $1 \leq \mu, \nu \leq m - 1$ $(m \geq 2)$*, and let*

$$\widetilde{\phi}(z) = \sum_{\widetilde{\nu}_1=\mu}^{m-1}\sum_{\widetilde{\nu}_2=\nu}^{m-1}\frac{\bar{z}_1^{\widetilde{\nu}_1}\bar{z}_2^{\widetilde{\nu}_2}}{\widetilde{\nu}_1!\widetilde{\nu}_2!}u_{\widetilde{\nu}_1\widetilde{\nu}_2}(z), \tag{2.1}$$

where

$$u_{\widetilde{\nu}_1\widetilde{\nu}_2} = \begin{cases} \dfrac{1}{(2\pi i)^2}\displaystyle\int_{\partial_0 D^2}\sum_{l_1=0}^{m-\mu-1}\sum_{l_2=0}^{m-\nu-1}\frac{g_{(\mu+l_1)(\nu+l_2)}(\zeta)}{l_1!l_2!}(A_1 - A_2 - A_3 + A_4)\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}, & \widetilde{\nu}_1 = \mu, \widetilde{\nu}_2 = \nu, \\ \dfrac{1}{(2\pi i)^2}\displaystyle\int_{\partial_0 D^2}\sum_{l_1=0}^{m-\mu-1}\sum_{l_2=0}^{m-1-\widetilde{\nu}_2}\frac{g_{(\mu+l_1)(\widetilde{\nu}_2+l_2)}(\zeta)}{l_1!l_2!}(B_1 - B_2)|_{\nu_2=\widetilde{\nu}_2-\nu}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}, & \widetilde{\nu}_1 = \mu, \nu < \widetilde{\nu}_2 \leq m-1, \\ \dfrac{1}{(2\pi i)^2}\displaystyle\int_{\partial_0 D^2}\sum_{l_1=0}^{m-1-\widetilde{\nu}_1}\sum_{l_2=0}^{m-\nu-1}\frac{g_{(\widetilde{\nu}_1+l_1)(\nu+l_2)}(\zeta)}{l_1!l_2!}(C_1 - C_2)|_{\nu_1=\widetilde{\nu}_1-\mu}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}, & \mu < \widetilde{\nu}_1 \leq m-1, \widetilde{\nu}_2 = \nu, \\ \dfrac{1}{(2\pi i)^2}\displaystyle\int_{\partial_0 D^2}\sum_{l_1=0}^{m-1-\widetilde{\nu}_1}\sum_{l_2=0}^{m-1-\widetilde{\nu}_2}\frac{g_{(\widetilde{\nu}_1+l_1)(\widetilde{\nu}_2+l_2)}(\zeta)}{l_1!l_2!}D|_{\substack{\nu_1=\widetilde{\nu}_1-\mu \\ \nu_2=\widetilde{\nu}_2-\nu}}\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}, & \mu < \widetilde{\nu}_1 \leq m-1, \nu < \widetilde{\nu}_2 \leq m-1, \end{cases}$$

in which

$$\begin{cases} A_1 = [(z_1 - \zeta_1 - \bar{\zeta}_1)^{l_1}(z_2 - \zeta_2 - \bar{\zeta}_2)^{l_2} + (-z_1)^{l_1}(z_2 - \zeta_2 - \bar{\zeta}_2)^{l_2} + (z_1 - \zeta_1 - \bar{\zeta}_1)^{l_1}(-z_2)^{l_2}]\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} - 1\Big], \\ A_2 = (-\zeta_1 - \bar{\zeta}_1)^{l_1}\Big\{(z_2 - \zeta_2 - \bar{\zeta}_2)^{l_2}[\frac{2\zeta_1\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} - 1] + (-z_2)^{l_2}[\frac{2\zeta_2}{\zeta_2 - z_2} - 1]\Big\}, \\ A_3 = (-\zeta_2 - \bar{\zeta}_2)^{l_2}\Big\{(z_1 - \zeta_1 - \bar{\zeta}_1)^{l_2}[\frac{2\zeta_1\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} - 1] + (-z_1)^{l_1}[\frac{2\zeta_1}{\zeta_1 - z_1} - 1]\Big\}, \\ A_4 = (-\zeta_1 - \bar{\zeta}_1)^{l_1}(-\zeta_2 - \bar{\zeta}_2)^{l_2}[\frac{2\zeta_1}{\zeta_1 - z_1} + \frac{2\zeta_2}{\zeta_2 - z_2} - 2], \\ B_1 = [(z_1 - \zeta_1 - \bar{\zeta}_1)^{l_1}(z_2 - \zeta_2 - \bar{\zeta}_2)^{l_2} + (-z_1)^{l_1}(z_2 - \zeta_2 - \bar{\zeta}_2)^{l_2} + (z_1 - \zeta_1 - \bar{\zeta}_1)^{l_1}(-z_2)^{l_2}(-1)^{\nu_2}]\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} - 1\Big], \\ B_2 = (-\zeta_1 - \bar{\zeta}_1)^{l_1}\Big\{(z_2 - \zeta_2 - \bar{\zeta}_2)^{l_2}[\frac{2\zeta_1\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} - 1] + (-z_2)^{l_2}(-1)^{\nu_2}[\frac{2\zeta_2}{\zeta_2 - z_2} - 1]\Big\}, \\ C_1 = [(z_1 - \zeta_1 - \bar{\zeta}_1)^{l_1}(z_2 - \zeta_2 - \bar{\zeta}_2)^{l_2} + (-z_1)^{l_1}(-1)^{\nu_1}(z_2 - \zeta_2 - \bar{\zeta}_2)^{l_2} + (z_1 - \zeta_1 - \bar{\zeta}_1)^{l_1}(-z_2)^{l_2}]\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} - 1\Big], \\ C_2 = (-\zeta_2 - \bar{\zeta}_2)^{l_2}\Big\{(z_1 - \zeta_1 - \bar{\zeta}_1)^{l_2}[\frac{2\zeta_1\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} - 1] + (-z_1)^{l_1}(-1)^{\nu_1}[\frac{2\zeta_1}{\zeta_1 - z_1} - 1]\Big\}, \\ D = [(z_1 - \zeta_1 - \bar{\zeta}_1)^{l_1}(z_2 - \zeta_2 - \bar{\zeta}_2)^{l_2} + (-z_1)^{l_1}(-1)^{\nu_1}(z_2 - \zeta_2 - \bar{\zeta}_2)^{l_2} + (z_1 - \zeta_1 - \bar{\zeta}_1)^{l_1}(-z_2)^{l_2}(-1)^{\nu_2}]\Big[\frac{2\zeta_1\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} - 1\Big]. \end{cases}$$

Then, $\widetilde{\phi}(z)$ is a specific solution to the problem

$$\begin{cases} \partial_{\bar{z}_1}^m \partial_{\bar{z}_2}^m \widetilde{\phi}(z) = 0, \ z \in D^2, \\ \Re \partial_{\bar{z}_1}^\mu \partial_{\bar{z}_2}^\nu \widetilde{\phi}(z) = g_{\mu\nu}(z), \ z \in \partial_0 D^2, \\ \Im \partial_{\bar{z}_1}^\mu \partial_{\bar{z}_2}^\nu \widetilde{\phi}(0, z_2) = 0 = \Im \partial_{\bar{z}_1}^\mu \partial_{\bar{z}_2}^\nu \widetilde{\phi}(z_1, 0), \ z_1 \in D_1, z_2 \in D_2. \end{cases}$$

Proof. Obviously, $u_{\widetilde{\nu_1}\widetilde{\nu_2}}(z)$ is analytic on D^2, therefore, $\partial_{\bar{z}_1}^m \partial_{\bar{z}_2}^m \widetilde{\phi}(z) = 0$. In addition, from the proof of Theorem 2.1 in [28]: We obtain that $\Re \partial_{\bar{z}_1}^\mu \partial_{\bar{z}_2}^\nu \widetilde{\phi}(z) = g_{\mu\nu}(z)$ for $z \in \partial_0 D^2$, and

$$\Im \partial_{\bar{z}_1}^\mu \partial_{\bar{z}_2}^\nu \widetilde{\phi}(0, z_2) = 0 = \Im \partial_{\bar{z}_1}^\mu \partial_{\bar{z}_2}^\nu \widetilde{\phi}(z_1, 0)$$

for $z_1 \in D_1, z_2 \in D_2$. So, $\widetilde{\phi}(z)$ is a specific solution to the problem.

Lemma 2.5. *[15] Let D be the unit disc. For $1 \leq k \leq n-1$, let $g_k \in C(\partial D)$ and $c_k \in \mathbb{C}$ be given. Then, there exists an analytic function u_k on D satisfying*

$$\partial_\nu \sum_{\mu=0}^{k-1} \frac{1}{\mu!} u_{n+\mu-k}(z) \bar{z}^\mu = g_{n-k}(z) \ (z \in \partial D), \quad u_{n-k}(0) = c_{n-k}$$

if and only if for $\forall z \in D$

$$\begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\partial D} g_{n-1}(\zeta) \frac{d\zeta}{(1-\bar{z}\zeta)\zeta} = 0, \\ \displaystyle\sum_{\nu=0}^{k-1} \frac{(-1)^\nu}{\nu!} \frac{1}{2\pi i} \int_{\partial D} \bar{\zeta}^\nu g_{n+\nu-k}(\zeta) \frac{d\zeta}{1-\bar{z}\zeta} = c_{n-k+1}, \ 2 \leq k \leq n-1, \end{cases}$$

and u_k is uniquely represented as

$$\begin{aligned} u_{n-k}(z) = c_{n-k} &- \frac{1}{2\pi i} \int_{\partial D} \sum_{\nu=0}^{k-1} \frac{(-2)^\nu}{\nu!} \bar{\zeta}^\nu g_{n+\nu-k}(\zeta) \log(1-z\bar{\zeta}) \frac{d\zeta}{\zeta} \\ &- \sum_{\lambda=1}^{k-1} \frac{1}{2\pi i} \int_{\partial D} \sum_{\mu=0}^{k-1-\lambda} \frac{(-2)^\mu}{\mu!} \bar{\zeta}^\mu g_{n+\lambda+\mu-k}(\zeta) \frac{1}{\lambda! z^\lambda} \Big[\log(1-z\bar{\zeta}) + \sum_{\sigma=1}^{\lambda} \frac{z^\sigma \bar{\zeta}^\sigma}{\sigma} \Big] \frac{d\zeta}{\zeta}. \end{aligned}$$

Lemma 2.6. *[15] Let $f \in L_1(D)$, $\varphi \in C(\partial D)$ and $c \in \mathbb{C}$ be given; then the problem*

$$w_{\bar{z}} = f(z \in D), \ \ \partial_\nu w(z) = \varphi(z)(z \in \partial D), \ \ w(0) = c$$

has a unique solution

$$w(z) = c - \frac{1}{2\pi i} \int_{\partial D} \varphi(\zeta) \log(1-z\bar{\zeta}) \frac{d\zeta}{\zeta} - \frac{1}{2\pi i} \int_{\partial D} f(\zeta) \log(1-z\bar{\zeta}) d\bar{\zeta} - \frac{1}{\pi} \int_D \frac{z f(\zeta)}{\zeta(\zeta - z)} d\sigma_\zeta,$$

if and only if

$$\frac{1}{2\pi i} \int_{\partial D} \varphi(\zeta) \frac{d\zeta}{(1-\bar{z}\zeta)\zeta} + \frac{1}{2\pi i} \int_{\partial D} f(\zeta) \frac{d\bar{\zeta}}{1-\bar{z}\zeta} = -\frac{1}{\pi} \int_D f(\zeta) \frac{\bar{z}}{(1-\bar{z}\zeta)^2} d\sigma_\zeta.$$

Lemma 2.7. *Let u_0 be an analytic function on D^2. Then*

$$\begin{aligned}\frac{1}{\pi^2}\int_{D^2} u_0(\zeta)\frac{d\sigma_{\zeta_1}}{\zeta_1 - z_1}\frac{d\sigma_{\zeta_2}}{\zeta_2 - z_2} =&(\bar{z}_2 - \frac{1}{z_2})\Big[\bar{z}_1 u_0(z) - \frac{1}{z_1}(u_0(z) - u_0(0, z_2))\Big] \\ &+\frac{1}{z_2}\Big[\bar{z}_1 u_0(z_1, 0) - \frac{1}{z_1}(u_0(z_1, 0) - u_0(0, 0))\Big],\end{aligned} \tag{2.2}$$

and

$$\frac{1}{\pi^2}\int_{D^2} \overline{u_0(\zeta)}\frac{d\sigma_{\zeta_1}}{\zeta_1 - z_1}\frac{d\sigma_{\zeta_2}}{\zeta_2 - z_2} = \overline{u_1(z)} - \overline{u_1(0, z_2)} - (\overline{u_1(z_1, 0)} - \overline{u_1(0, 0)}), \tag{2.3}$$

where $\partial_{\zeta_1}\partial_{\zeta_2} u_1(\zeta) = u_0(\zeta)$.

Proof. By Lemma 2.2,

$$\begin{aligned}&\frac{1}{\pi^2}\int_{D^2} u_0(\zeta)\frac{d\sigma_{\zeta_1}}{\zeta_1 - z_1}\frac{d\sigma_{\zeta_2}}{\zeta_2 - z_2} \\ =& \frac{-1}{\pi}\int_{D_1}\Big[\frac{-1}{\pi}\int_{D_2}\partial_{\bar{\zeta}_2}(\bar{\zeta}_2 u_0(\zeta))\frac{d\sigma_{\zeta_2}}{\zeta_2 - z_2}\Big]\frac{d\sigma_{\zeta_1}}{\zeta_1 - z_1} \\ =& \frac{-1}{\pi}\int_{D_1}\Big[\bar{z}_2 u_0(\zeta_1, z_2) - \frac{1}{2\pi i}\int_{\partial D_2}\bar{\zeta}_2 u_0(\zeta)\frac{d\zeta_2}{\zeta_2 - z_2}\Big]\frac{d\sigma_{\zeta_1}}{\zeta_1 - z_1} \\ =& \frac{-1}{\pi}\int_{D_1}\Big[\bar{z}_2 u_0(\zeta_1, z_2) - \frac{1}{z_2}(u_0(\zeta_1, z_2) - u_0(\zeta_1, 0))\Big]\frac{d\sigma_{\zeta_1}}{\zeta_1 - z_1} \\ =& (\bar{z}_2 - \frac{1}{z_2})\frac{-1}{\pi}\int_{D_1} u_0(\zeta_1, z_2)\frac{d\sigma_{\zeta_1}}{\zeta_1 - z_1} + \frac{1}{z_2}\frac{-1}{\pi}\int_{D_1} u_0(\zeta_1, 0)\frac{d\sigma_{\zeta_1}}{\zeta_1 - z_1} \\ =& (\bar{z}_2 - \frac{1}{z_2})\frac{-1}{\pi}\int_{D_1}\partial_{\bar{\zeta}_1}(\bar{\zeta}_1 u_0(\zeta_1, z_2))\frac{d\sigma_{\zeta_1}}{\zeta_1 - z_1} + \frac{1}{z_2}\frac{-1}{\pi}\int_{D_1}\partial_{\bar{\zeta}_1}(\bar{\zeta}_1 u_0(\zeta_1, 0))\frac{d\sigma_{\zeta_1}}{\zeta_1 - z_1} \\ =& (\bar{z}_2 - \frac{1}{z_2})\Big[\bar{z}_1 u_0(z_1, z_2) - \frac{1}{2\pi i}\int_{\partial D_1}\bar{\zeta}_1 u_0(\zeta_1, z_2)\frac{d\zeta_1}{\zeta_1 - z_1}\Big] \\ &+ \frac{1}{z_2}\Big[\bar{z}_1 u_0(z_1, 0) - \frac{1}{2\pi i}\int_{\partial D_1}\bar{\zeta}_1 u_0(\zeta_1, 0)\frac{d\zeta_1}{\zeta_1 - z_1}\Big] \\ =& (\bar{z}_2 - \frac{1}{z_2})\Big[\bar{z}_1 u_0(z) - \frac{1}{z_1}(u_0(z) - u_0(0, z_2))\Big] + \frac{1}{z_2}\Big[\bar{z}_1 u_0(z_1, 0) - \frac{1}{z_1}(u_0(z_1, 0) - u_0(0, 0))\Big].\end{aligned}$$

So we obtain (2.2). Let $\partial_{\zeta_1} u_1(\zeta) = u_2(\zeta)$. Since $\partial_{\zeta_1}\partial_{\zeta_2} u_1(\zeta) = u_0(\zeta)$, then we have $\partial_{\zeta_2} u_2(\zeta) = u_0(\zeta)$, which follows $\partial_{\bar{\zeta}_2}\overline{u_2(\zeta)} = \overline{u_0(\zeta)}$. Therefore, by Lemma 2.2,

$$\begin{aligned}\frac{-1}{\pi}\int_{D_2}\overline{u_0(\zeta)}\frac{d\sigma_{\zeta_2}}{\zeta_2 - z_2} &= \frac{-1}{\pi}\int_{D_2}\partial_{\bar{\zeta}_2}\overline{u_2(\zeta)}\frac{d\sigma_{\zeta_2}}{\zeta_2 - z_2} \\ &= \overline{u_2(\zeta_1, z_2)} - \frac{1}{2\pi i}\int_{\partial D_2}\overline{u_2(\zeta)}\frac{d\zeta_2}{\zeta_2 - z_2} \\ &= \overline{u_2(\zeta_1, z_2)} - \overline{\frac{1}{2\pi i}\int_{\partial D_2} u_2(\zeta)\frac{d\zeta_2}{\zeta_2(1 - \bar{z}_2\zeta_2)}} \\ &= \overline{u_2(\zeta_1, z_2)} - \overline{u_2(\zeta_1, 0)}.\end{aligned} \tag{2.4}$$

In addition, $\partial_{\zeta_1} u_1(\zeta) = u_2(\zeta)$ follows $\partial_{\bar{\zeta}_1} \overline{u_1(\zeta)} = \overline{u_2(\zeta)}$. Similarly, we have

$$\frac{-1}{\pi} \int_{D_1} \overline{u_2(\zeta_1, z_2)} \frac{d\sigma_{\zeta_1}}{\zeta_1 - z_1} = \overline{u_1(z)} - \overline{u_1(0, z_2)}. \tag{2.5}$$

Therefore, by (2.4) and (2.5), we obtain

$$\begin{aligned} \frac{1}{\pi^2} \int_{D^2} \overline{u_0(\zeta)} \frac{d\sigma_{\zeta_1}}{\zeta_1 - z_1} \frac{d\sigma_{\zeta_2}}{\zeta_2 - z_2} &= \frac{-1}{\pi} \int_{D_1} \Big[\frac{-1}{\pi} \int_{D_2} \frac{d\sigma_{\zeta_2}}{\zeta_2 - z_2} \Big] \frac{d\sigma_{\zeta_1}}{\zeta_1 - z_1} \\ &= \frac{-1}{\pi} \int_{D_1} \overline{u_2(\zeta_1, z_2)} \frac{d\sigma_{\zeta_1}}{\zeta_1 - z_1} - \frac{-1}{\pi} \int_{D_1} \overline{u_2(\zeta_1, 0)} \frac{d\sigma_{\zeta_1}}{\zeta_1 - z_1} \\ &= \overline{u_1(z)} - \overline{u_1(0, z_2)} - (\overline{u_1(z_1, 0)} - \overline{u_1(0, 0)}). \end{aligned}$$

Lemma 2.8. *Let $\varphi \in C(\partial_0 D^2; \mathbb{C})$ and $g_{\mu\nu} \in C(\partial_0 D^2; \mathbb{R})$ for $1 \leq \mu, \nu \leq m-1$ $(m \geq 2)$, and let $\lambda \in \mathbb{R} \setminus \{-1, 0, 1\}$. Let $W(z)$ and $u_0(z)$ be analytic functions on D^2 and*

$$\begin{aligned} W(\tau) = \varphi(\tau) &- \frac{1}{\pi^2} \int_{D^2} \Big[\frac{\lambda - 1}{4\lambda} \widetilde{\phi}(\zeta) + \frac{\lambda + 1}{4\lambda} \overline{\widetilde{\phi}(\zeta)} \Big] \frac{d\sigma_{\zeta_1}}{\zeta_1 - \tau_1} \frac{d\sigma_{\zeta_2}}{\zeta_2 - \tau_2} \\ &- \frac{\lambda - 1}{4\lambda} \Big\{ (\bar{\tau}_2 - \frac{1}{\tau_2}) \Big[\bar{\tau}_1 u_0(\tau) - \frac{u_0(\tau) - u_0(0, \tau_2)}{\tau_1} \Big] + \frac{1}{\tau_2} \Big[\bar{\tau}_1 u_0(\tau_1, 0) - \frac{u_0(\tau_1, 0) - u_0(0)}{\tau_1} \Big] \Big\} \\ &- \frac{\lambda + 1}{4\lambda} [\overline{u_1(\tau)} - \overline{u_1(0, \tau_2)} - \overline{u_1(\tau_1, 0)} + \overline{u_1(0, 0)}] \end{aligned} \tag{2.6}$$

for $\tau \in \partial D^2$, where $\widetilde{\phi}$ is determined in Lemma 2.4 and $\partial_{\zeta_1} \partial_{\zeta_2} u_1(\zeta) = u_0(\zeta)$. Then

$$W(z) = \frac{1}{(2\pi i)^2} \int_{\partial_0 D^2} \varphi(\zeta) \frac{d\zeta_1 d\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)}, \tag{2.7}$$

and

$$\begin{aligned} u_0(z) = \frac{4\lambda}{\lambda + 1} \Big\{ & \overline{\frac{1}{(2\pi i)^2} \int_{\partial_0 D^2} \frac{\varphi(\zeta) d\zeta_1}{(1 - \bar{z}_1 \zeta_1)^2} \frac{d\zeta_2}{(1 - \bar{z}_2 \zeta_2)^2}} \\ & - \overline{\frac{1}{\pi^2} \int_{D^2} \Big[\frac{\lambda - 1}{4\lambda} \widetilde{\phi}(\zeta) + \frac{\lambda + 1}{4\lambda} \overline{\widetilde{\phi}(\zeta)} \Big] \frac{d\sigma_{\zeta_1} d\sigma_{\zeta_2}}{(1 - \bar{z}_1 \zeta_1)^2 (1 - \bar{z}_2 \zeta_2)^2}} \Big\} \\ & - (\lambda - 1) \Big\{ \frac{1}{(2\pi i)^2} \int_{\partial_0 D^2} \varphi(\zeta) d\zeta - \frac{1}{\pi^2} \int_{D^2} \Big[\frac{\lambda - 1}{4\lambda} \widetilde{\phi}(\zeta) + \frac{\lambda + 1}{4\lambda} \overline{\widetilde{\phi}(\zeta)} \Big] d\sigma_{\zeta_1} d\sigma_{\zeta_2} \Big\} \\ & + \frac{(\lambda - 1)^2}{\lambda + 1} \Big\{ \overline{\frac{1}{(2\pi i)^2} \int_{\partial_0 D^2} \varphi(\zeta) d\zeta} - \overline{\frac{1}{\pi^2} \int_{D^2} \Big[\frac{\lambda - 1}{4\lambda} \widetilde{\phi}(\zeta) + \frac{\lambda + 1}{4\lambda} \overline{\widetilde{\phi}(\zeta)} \Big] d\sigma_{\zeta_1} d\sigma_{\zeta_2}} \Big\}. \end{aligned} \tag{2.8}$$

Proof. As $W(z)$ is analytic, applying the properties of the Poisson kernel on D^2, $W(z)$ can be expressed as

$$W(z) = \frac{1}{(2\pi i)^2} \int_{\partial_0 D^2} \frac{W(\zeta)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 d\zeta_2 \tag{2.9}$$

if and only if

$$\frac{1}{(2\pi i)^2} \int_{\partial_0 D^2} W(\zeta) \Big[\frac{\bar{z}_1}{1 - \bar{z}_1 \zeta_1} \frac{\bar{z}_2}{1 - \bar{z}_2 \zeta_2} + \frac{\bar{z}_2}{1 - \bar{z}_2 \zeta_2} \frac{1}{\zeta_1 - z_1} + \frac{\bar{z}_1}{1 - \bar{z}_1 \zeta_1} \frac{1}{\zeta_2 - z_2} \Big] d\zeta_1 d\zeta_2 = 0. \tag{2.10}$$

(2.6) and (2.9) derive the expression of W:

$$
\begin{aligned}
W(z) &= \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\frac{W(\zeta)}{(\zeta_1-z_1)(\zeta_2-z_2)}d\zeta_1 d\zeta_2\\
&= \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\Big\{\varphi(\zeta)-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\widetilde{\zeta})}\Big]\frac{d\sigma_{\widetilde{\zeta}_1}}{\widetilde{\zeta}_1-\zeta_1}\frac{d\sigma_{\widetilde{\zeta}_2}}{\widetilde{\zeta}_2-\zeta_2}\\
&\quad-\frac{\lambda-1}{4\lambda}\Big[(\bar{\zeta}_2-\frac{1}{\zeta_2})\Big(\bar{\zeta}_1u_0(\zeta)-\frac{u_0(\zeta)-u_0(0,\zeta_2)}{\zeta_1}\Big)+\frac{1}{\zeta_2}\Big(\bar{\zeta}_1u_0(\zeta_1,0)-\frac{u_0(\zeta_1,0)-u_0(0)}{\zeta_1}\Big)\Big]\\
&\quad-\frac{\lambda+1}{4\lambda}[\overline{u_1(\zeta)}-\overline{u_1(0,\zeta_2)}-\overline{u_1(\zeta_1,0)}+\overline{u_1(0,0)}]\Big\}\frac{d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}\\
&= \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\Big\{\varphi(\zeta)-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\widetilde{\zeta})}\Big]\frac{d\sigma_{\widetilde{\zeta}_1}}{\widetilde{\zeta}_1-\zeta_1}\frac{d\sigma_{\widetilde{\zeta}_2}}{\widetilde{\zeta}_2-\zeta_2}-\frac{\lambda-1}{4\lambda}\frac{u_0(0)}{\zeta_1\zeta_2}\\
&\quad-\frac{\lambda+1}{4\lambda}[\overline{u_1(\zeta)}-\overline{u_1(0,\zeta_2)}-\overline{u_1(\zeta_1,0)}+\overline{u_1(0,0)}]\Big\}\frac{d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}\\
&= \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\Big\{\varphi(\zeta)-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\widetilde{\zeta})}\Big]\frac{d\sigma_{\widetilde{\zeta}_1}}{\widetilde{\zeta}_1-\zeta_1}\frac{d\sigma_{\widetilde{\zeta}_2}}{\widetilde{\zeta}_2-\zeta_2}\Big\}\frac{d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}\\
&\quad-\frac{\lambda-1}{4\lambda}u_0(0)\frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{1}{2\pi i}\int_{\partial D_1}\frac{d\zeta_1}{\zeta_1(\zeta_1-z_1)}\Big]\frac{d\zeta_2}{\zeta_2(\zeta_2-z_2)}\\
&\quad-\frac{\lambda+1}{4\lambda}\overline{\frac{1}{2\pi i}\int_{\partial D_2}\Big\{\frac{1}{2\pi i}\int_{\partial D_1}[u_1(\zeta)-u_1(0,\zeta_2)-u_1(\zeta_1,0)+u_1(0)]\frac{d\zeta_1}{\zeta_1(1-\bar{z}_1\zeta_1)}\Big\}\frac{d\zeta_2}{\zeta_2(1-\bar{z}_2\zeta_2)}}\\
&= \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\Big\{\varphi(\zeta)-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\widetilde{\zeta})}\Big]\frac{d\sigma_{\widetilde{\zeta}_1}}{\widetilde{\zeta}_1-\zeta_1}\frac{d\sigma_{\widetilde{\zeta}_2}}{\widetilde{\zeta}_2-\zeta_2}\Big\}\frac{d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}\\
&= \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\frac{d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}\\
&\quad-\frac{1}{2\pi i}\int_{\partial D_1}\Big\{\frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{1}{\pi^2}\int_{D^2}\Big(\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\widetilde{\zeta})}\Big)\frac{d\sigma_{\widetilde{\zeta}_1}}{\widetilde{\zeta}_1-\zeta_1}\frac{d\sigma_{\widetilde{\zeta}_2}}{\widetilde{\zeta}_2-\zeta_2}\Big]\frac{d\zeta_2}{\zeta_2-z_2}\Big\}\frac{d\zeta_1}{\zeta_1-z_1}\\
&= \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\frac{d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}\\
&\quad-\frac{1}{2\pi i}\int_{\partial D_1}\Big\{\frac{1}{\pi^2}\int_{D^2}\Big(\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\widetilde{\zeta})}\Big)\Big[\frac{1}{2\pi i}\int_{\partial D_2}\frac{1}{\widetilde{\zeta}_2-\zeta_2}\frac{d\zeta_2}{\zeta_2-z_2}\Big]\frac{d\sigma_{\widetilde{\zeta}_1}}{\widetilde{\zeta}_1-\zeta_1}d\sigma_{\widetilde{\zeta}_2}\\
&= \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\frac{d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)},
\end{aligned}
$$

which is due to

$$
\frac{1}{2\pi i}\int_{\partial D_1}\frac{d\zeta_1}{\zeta_1(\zeta_1-z_1)}=0,\quad \frac{1}{2\pi i}\int_{\partial D_2}\frac{1}{\widetilde{\zeta}_2-\zeta_2}\frac{d\zeta_2}{\zeta_2-z_2}=0
$$

and

$$
\frac{1}{2\pi i}\int_{\partial D_1}[u_1(\zeta_1,\zeta_2)-u_1(0,\zeta_2)-u_1(\zeta_1,0)+u_1(0,0)]\frac{d\zeta_1}{\zeta_1(1-\bar{z}_1\zeta_1)}=0
$$

for $z\in D^2$.

Similarly, by (2.6) and (2.10), we obtain

$$\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2} W(\zeta)\Big[\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{\bar z_2}{1-\bar z_2\zeta_2}+\frac{\bar z_2}{1-\bar z_2\zeta_2}\frac{1}{\zeta_1-z_1}+\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{1}{\zeta_2-z_2}\Big]d\zeta_1 d\zeta_2\\
=&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\Big\{\varphi(\zeta)-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi(\widetilde{\zeta})}+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi(\widetilde{\zeta})}}\Big]\frac{d\sigma_{\widetilde{\zeta_1}}}{\widetilde{\zeta_1}-\zeta_1}\frac{d\sigma_{\widetilde{\zeta_2}}}{\widetilde{\zeta_2}-\zeta_2}\\
&-\frac{\lambda-1}{4\lambda}\Big[(\bar\zeta_2-\frac{1}{\zeta_2})\Big(\bar\zeta_1 u_0(\zeta)-\frac{u_0(\zeta)-u_0(0,\zeta_2)}{\zeta_1}\Big)+\frac{1}{\zeta_2}\Big(\bar\zeta_1 u_0(\zeta_1,0)-\frac{u_0(\zeta_1,0)-u_0(0)}{\zeta_1}\Big)\Big]\\
&-\frac{\lambda+1}{4\lambda}[\overline{u_1(\zeta)}-\overline{u_1(0,\zeta_2)}-\overline{u_1(\zeta_1,0)}+\overline{u_1(0,0)}]\Big\}\\
&\cdot\Big[\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{\bar z_2}{1-\bar z_2\zeta_2}+\frac{\bar z_2}{1-\bar z_2\zeta_2}\frac{1}{\zeta_1-z_1}+\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{1}{\zeta_2-z_2}\Big]d\zeta_1 d\zeta_2\\
=&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\Big\{\varphi(\zeta)-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi(\widetilde{\zeta})}+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi(\widetilde{\zeta})}}\Big]\frac{d\sigma_{\widetilde{\zeta_1}}}{\widetilde{\zeta_1}-\zeta_1}\frac{d\sigma_{\widetilde{\zeta_2}}}{\widetilde{\zeta_2}-\zeta_2}\\
&-\frac{\lambda-1}{4\lambda}\frac{1}{\zeta_1\zeta_2}u_0(0)-\frac{\lambda+1}{4\lambda}[\overline{u_1(\zeta)}-\overline{u_1(0,\zeta_2)}-\overline{u_1(\zeta_1,0)}+\overline{u_1(0,0)}]\Big\}\\
&\cdot\Big[\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{\bar z_2}{1-\bar z_2\zeta_2}+\frac{\bar z_2}{1-\bar z_2\zeta_2}\frac{1}{\zeta_1-z_1}+\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{1}{\zeta_2-z_2}\Big]d\zeta_1 d\zeta_2\\
=&\,0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\Big\{\varphi(\zeta)-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi(\widetilde{\zeta})}+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi(\widetilde{\zeta})}}\Big]\frac{d\sigma_{\widetilde{\zeta_1}}}{\widetilde{\zeta_1}-\zeta_1}\frac{d\sigma_{\widetilde{\zeta_2}}}{\widetilde{\zeta_2}-\zeta_2}\Big\}\\
&\cdot\Big[\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{\bar z_2}{1-\bar z_2\zeta_2}+\frac{\bar z_2}{1-\bar z_2\zeta_2}\frac{1}{\zeta_1-z_1}+\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{1}{\zeta_2-z_2}\Big]d\zeta_1 d\zeta_2\\
=&\frac{\lambda-1}{4\lambda}u_0(0)\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\Big[\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{\bar z_2}{1-\bar z_2\zeta_2}+\frac{\bar z_2}{1-\bar z_2\zeta_2}\frac{1}{\zeta_1-z_1}+\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{1}{\zeta_2-z_2}\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}\\
&+\frac{\lambda+1}{4\lambda}\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}[\overline{u_1(\zeta)}-\overline{u_1(0,\zeta_2)}-\overline{u_1(\zeta_1,0)}+\overline{u_1(0,0)}]\Big\}\\
&\cdot\Big[\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{\bar z_2}{1-\bar z_2\zeta_2}+\frac{\bar z_2}{1-\bar z_2\zeta_2}\frac{1}{\zeta_1-z_1}+\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{1}{\zeta_2-z_2}\Big]d\zeta_1 d\zeta_2\\
=&\frac{\lambda-1}{4\lambda}u_0(0)\Big\{\frac{1}{2\pi i}\int_{\partial D_1}\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{d\zeta_1}{\zeta_1}\Big[\frac{1}{2\pi i}\int_{\partial D_2}\frac{\bar z_2}{1-\bar z_2\zeta_2}\frac{d\zeta_2}{\zeta_2}+\frac{1}{2\pi i}\int_{\partial D_2}\frac{d\zeta_2}{(\zeta_2-z_2)\zeta_2}\Big]\\
&+\frac{1}{2\pi i}\int_{\partial D_1}\frac{d\zeta_1}{(\zeta_1-z_1)\zeta_1}\cdot\frac{1}{2\pi i}\int_{\partial D_2}\frac{\bar z_2}{1-\bar z_2\zeta_2}\frac{d\zeta_2}{\zeta_2}\Big\}\\
&+\frac{\lambda+1}{4\lambda}\Big\{\overline{\frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{1}{2\pi i}\int_{\partial D_1}[u_1(\zeta)-u_1(0,\zeta_2)-u_1(\zeta_1,0)+u_1(0)]\frac{z_1 d\zeta_1}{\zeta_1(\zeta_1-z_1)}\Big]\frac{z_2 d\zeta_2}{\zeta_2(\zeta_2-z_2)}}\\
&+\overline{\frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{1}{2\pi i}\int_{\partial D_1}[u_1(\zeta)-u_1(0,\zeta_2)-u_1(\zeta_1,0)+u_1(0)]\frac{d\zeta_1}{\zeta_1(1-\bar z_1\zeta_1)}\Big]\frac{z_2 d\zeta_2}{\zeta_2(\zeta_2-z_2)}}\\
&+\overline{\frac{1}{2\pi i}\int_{\partial D_1}\Big[\frac{1}{2\pi i}\int_{\partial D_2}[u_1(\zeta)-u_1(0,\zeta_2)-u_1(\zeta_1,0)+u_1(0)]\frac{d\zeta_2}{\zeta_2(1-\bar z_2\zeta_2)}\Big]\frac{z_1 d\zeta_1}{\zeta_1(\zeta_1-z_1)}}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda-1}{4\lambda}u_0(0)\bar{z}_1\bar{z}_2 + \frac{\lambda+1}{4\lambda}\Big\{\overline{\frac{1}{2\pi i}\int_{\partial D_2}[u_1(z_1,\zeta_2)-u_1(0,\zeta_2)-u_1(z_1,0)+u_1(0)]\frac{z_2 d\zeta_2}{\zeta_2(\zeta_2-z_2)}}\\
&+\overline{\frac{1}{2\pi i}\int_{\partial D_2}\frac{u_1(\zeta)-u_1(0,\zeta_2)-u_1(\zeta_1,0)+u_1(0)}{1-\bar{z}_1\zeta_1}\Big|_{\zeta_1=0}\frac{z_2 d\zeta_2}{\zeta_2(\zeta_2-z_2)}}\\
&+\overline{\frac{1}{2\pi i}\int_{\partial D_1}\frac{u_1(\zeta)-u_1(0,\zeta_2)-u_1(\zeta_1,0)+u_1(0)}{1-\bar{z}_2\zeta_2}\Big|_{\zeta_2=0}\frac{z_1 d\zeta_1}{\zeta_1(\zeta_1-z_1)}}\Big\}\\
&= \frac{\lambda-1}{4\lambda}u_0(0)\bar{z}_1\bar{z}_2 + \frac{\lambda+1}{4\lambda}\Big\{\overline{[u_1(z_1,\zeta_2)-u_1(0,\zeta_2)-u_1(z_1,0)+u_1(0)]_{\zeta_2=z_2}}\\
&-\overline{[u_1(z_1,\zeta_2)-u_1(0,\zeta_2)-u_1(z_1,0)+u_1(0)]_{\zeta_2=0}}\Big\}\\
&= \frac{\lambda-1}{4\lambda}u_0(0)\bar{z}_1\bar{z}_2 + \frac{\lambda+1}{4\lambda}\overline{[u_1(z)-u_1(0,z_2)-u_1(z_1,0)+u_1(0)]},
\end{aligned} \tag{2.11}
$$

which is in virtue of

$$
\frac{1}{2\pi i}\int_{\partial D_1}\frac{\bar{z}_1}{1-\bar{z}_1\zeta_1}\frac{d\zeta_1}{\zeta_1}=\bar{z}_1,\quad \frac{1}{2\pi i}\int_{\partial D_2}\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}\frac{d\zeta_2}{\zeta_2}=\bar{z}_2
$$

and

$$
\frac{1}{2\pi i}\int_{\partial D_1}\frac{d\zeta_1}{\zeta_1(\zeta_1-z_1)}=0=\frac{1}{2\pi i}\int_{\partial D_2}\frac{d\zeta_2}{\zeta_2(\zeta_2-z_2)}.
$$

Taking the partial derivative on both sides of Eq (2.11) with respect to $\bar{z}_1\bar{z}_2$ gives

$$
\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\Big\{\varphi(\zeta)-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\widetilde{\zeta})}\Big]\frac{d\sigma_{\widetilde{\zeta}_1}}{\widetilde{\zeta}_1-\zeta_1}\frac{d\sigma_{\widetilde{\zeta}_2}}{\widetilde{\zeta}_2-\zeta_2}\Big\}\frac{d\zeta_1}{(1-\bar{z}_1\zeta_1)^2}\frac{d\zeta_2}{(1-\bar{z}_2\zeta_2)^2}\\
&=\frac{\lambda-1}{4\lambda}u_0(0)+\frac{\lambda+1}{4\lambda}\overline{u_0(z)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\frac{\lambda-1}{4\lambda}u_0(0)+\frac{\lambda+1}{4\lambda}\overline{u_0(z)}\\
&=\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\frac{d\zeta_1}{(1-\bar{z}_1\zeta_1)^2}\frac{d\zeta_2}{(1-\bar{z}_2\zeta_2)^2}-\frac{1}{2\pi i}\int_{\partial D_2}\Big\{\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\widetilde{\zeta})}\Big]\\
&\cdot\Big[\frac{1}{2\pi i}\int_{\partial D_1}\frac{1}{\widetilde{\zeta}_1-\zeta_1}\frac{d\zeta_1}{(1-\bar{z}_1\zeta_1)^2}\Big]d\sigma_{\widetilde{\zeta}_1}\frac{d\sigma_{\widetilde{\zeta}_2}}{\widetilde{\zeta}_2-\zeta_2}\Big\}\frac{d\zeta_2}{(1-\bar{z}_2\zeta_2)^2}\\
&=\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\frac{d\zeta_1}{(1-\bar{z}_1\zeta_1)^2}\frac{d\zeta_2}{(1-\bar{z}_2\zeta_2)^2}\\
&-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\widetilde{\zeta})}\Big]\frac{-d\sigma_{\widetilde{\zeta}_1}}{(1-\bar{z}_1\widetilde{\zeta}_1)^2}\Big[\frac{1}{2\pi i}\int_{\partial D_2}\frac{1}{\widetilde{\zeta}_2-\zeta_2}\frac{d\zeta_2}{(1-\bar{z}_2\zeta_2)^2}\Big]d\sigma_{\widetilde{\zeta}_2}\\
&=\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\frac{\varphi(\zeta)d\zeta_1}{(1-\bar{z}_1\zeta_1)^2}\frac{d\zeta_2}{(1-\bar{z}_2\zeta_2)^2}-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]\frac{d\sigma_{\zeta_1}d\sigma_{\zeta_2}}{(1-\bar{z}_1\zeta_1)^2(1-\bar{z}_2\zeta_2)^2}.
\end{aligned} \tag{2.12}
$$

Particularly,

$$
\frac{\lambda-1}{4\lambda}u_0(0)+\frac{\lambda+1}{4\lambda}\overline{u_0(0)}=\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)d\zeta-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2}, \tag{2.13}
$$

which follows

$$\frac{\lambda-1}{4\lambda}\overline{u_0(0)}+\frac{\lambda+1}{4\lambda}u_0(0)=\overline{\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)d\zeta}-\overline{\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2}},\tag{2.14}$$

(2.13) and (2.14) derive

$$\begin{aligned}u_0(0)=(\lambda+1)\Big\{&\overline{\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)d\zeta}-\overline{\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2}}\Big\}\\&-(\lambda-1)\Big\{\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)d\zeta-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2}\Big\}.\end{aligned}\tag{2.15}$$

Plugging (2.15) into (2.12), we obtain the expression of $u_0(z)$, i.e., (2.8).

3. Dirichlet boundary value problems

Theorem 3.1. *Let* $\varphi\in C(\partial_0 D^2;\mathbb{C})$ *and* $g_{\mu\nu}\in C(\partial_0 D^2;\mathbb{R})$ *for* $1\le\mu,\nu\le m-1$ *(*$m\ge 2$*), and let* $\lambda\in\mathbb{R}\setminus\{-1,0,1\}$. *Then the problem*

$$\partial_{\bar z_1}\partial_{\bar z_2}f(z)=\frac{\lambda-1}{4\lambda}\phi(z)+\frac{\lambda+1}{4\lambda}\bar\phi(z),\ \partial_{\bar z_1}^m\partial_{\bar z_2}^m\phi(z)=0,\ z\in D^2,$$

with the conditions

$$f(z)=\varphi(z),\ \Re\partial_{\bar z_1}^{\mu}\partial_{\bar z_2}^{\nu}\phi(z)=g_{\mu\nu}(z)\ (z\in\partial_0 D^2),\ \Im\partial_{\bar z_1}^{\mu}\partial_{\bar z_2}^{\nu}\phi(0,z_2)=0=\Im\partial_{\bar z_1}^{\mu}\partial_{\bar z_2}^{\nu}\phi(z_1,0)$$

for $1\le\mu,\nu\le m-1$ *has a unique solution*

$$\begin{aligned}f(z)=&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\Big(\frac{\zeta_1}{\zeta_1-z_1}+\frac{\bar\zeta_1}{\overline{\zeta_1-z_1}}-1\Big)\Big(\frac{\zeta_2}{\zeta_2-z_2}+\frac{\bar\zeta_2}{\overline{\zeta_2-z_2}}-1\Big)\frac{d\zeta}{\zeta}\\&+\frac{\lambda-1}{4\lambda}\Big[\frac{(|z_1|^2-1)(|z_2|^2-1)}{z_1z_2}u_0(z)+\frac{|z_2|^2-1}{z_1z_2}u_0(0,z_2)+\frac{|z_1|^2-1}{z_1z_2}u_0(z_1,0)+\frac{1-|z_1|^2|z_2|^2}{z_1z_2}u_0(0)\Big]\\&+\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]\Big[\frac{1}{(\zeta_1-z_1)(\zeta_2-z_2)}-\frac{\bar z_1\bar z_2}{(\bar z_1\zeta_1-1)(\bar z_2\zeta_2-1)}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2},\end{aligned}\tag{3.1}$$

where $\widetilde{\phi}$ *is determined by (2.1) and*

$$\begin{aligned}u_0(z)=\frac{4\lambda}{\lambda+1}\Big\{&\overline{\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\frac{\varphi(\zeta)d\zeta_1}{(1-\bar z_1\zeta_1)^2}\frac{d\zeta_2}{(1-\bar z_2\zeta_2)^2}}\\&-\overline{\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]\frac{d\sigma_{\zeta_1}d\sigma_{\zeta_2}}{(1-\bar z_1\zeta_1)^2(1-\bar z_2\zeta_2)^2}}\Big\}\\&-(\lambda-1)\Big\{\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)d\zeta-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2}\Big\}\\&+\frac{(\lambda-1)^2}{\lambda+1}\Big\{\overline{\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)d\zeta}-\overline{\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2}}\Big\}.\end{aligned}\tag{3.2}$$

Proof. (1) Applying Lemma 2.1,

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}\Big\{\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\phi(\zeta)+\frac{\lambda+1}{4\lambda}\bar{\phi}(\zeta)\Big]\frac{d\sigma_{\zeta_1}}{\zeta_1-z_1}\frac{d\sigma_{\zeta_2}}{\zeta_2-z_2}\Big\}=\frac{\lambda-1}{4\lambda}\phi(z)+\frac{\lambda+1}{4\lambda}\bar{\phi}(z),$$

which means

$$\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\phi(\zeta)+\frac{\lambda+1}{4\lambda}\bar{\phi}(\zeta)\Big]\frac{d\sigma_{\zeta_1}}{\zeta_1-z_1}\frac{d\sigma_{\zeta_2}}{\zeta_2-z_2}$$

is a special solution to

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}f(z)=\frac{\lambda-1}{4\lambda}\phi(z)+\frac{\lambda+1}{4\lambda}\bar{\phi}(z).$$

Therefore, by Lemma 2.4, the solution of the problem is

$$f(z)=W(z)+\frac{1}{\pi^2}\int_{D^2}\Big\{\frac{\lambda-1}{4\lambda}[u_0(\zeta)+\widetilde{\phi}(\zeta)]+\frac{\lambda+1}{4\lambda}\overline{[u_0(\zeta)+\widetilde{\phi}(\zeta)]}\Big\}\frac{d\sigma_{\zeta_1}}{\zeta_1-z_1}\frac{d\sigma_{\zeta_2}}{\zeta_2-z_2},\tag{3.3}$$

where $\widetilde{\phi}$ is determined in Lemma 2.4, W and u_0 are analytic functions on D^2 to be determined, and $f=\varphi$.

Applying Lemma 2.7 and plugging (2.2) and (2.3) into (3.3), we obtain

$$\begin{aligned}f(z)=&\,W(z)+\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]\frac{d\sigma_{\zeta_1}}{\zeta_1-z_1}\frac{d\sigma_{\zeta_2}}{\zeta_2-z_2}\\&+\frac{\lambda-1}{4\lambda}\Big\{(\bar{z}_2-\frac{1}{z_2})\Big[\bar{z}_1u_0(z)-\frac{u_0(z)-u_0(0,z_2)}{z_1}\Big]+\frac{1}{z_2}\Big[\bar{z}_1u_0(z_1,0)-\frac{u_0(z_1,0)-u_0(0)}{z_1}\Big]\Big\}\\&+\frac{\lambda+1}{4\lambda}[\overline{u_1(z)}-\overline{u_1(0,z_2)}-\overline{u_1(z_1,0)}+\overline{u_1(0,0)}],\end{aligned}\tag{3.4}$$

which leads to

$$\begin{aligned}W(z)=&\,f(z)-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]\frac{d\sigma_{\zeta_1}}{\zeta_1-z_1}\frac{d\sigma_{\zeta_2}}{\zeta_2-z_2}\\&-\frac{\lambda-1}{4\lambda}\Big\{(\bar{z}_2-\frac{1}{z_2})\Big[\bar{z}_1u_0(z)-\frac{u_0(z)-u_0(0,z_2)}{z_1}\Big]+\frac{1}{z_2}\Big[\bar{z}_1u_0(z_1,0)-\frac{u_0(z_1,0)-u_0(0)}{z_1}\Big]\Big\}\\&-\frac{\lambda+1}{4\lambda}[\overline{u_1(z)}-\overline{u_1(0,z_2)}-\overline{u_1(z_1,0)}+\overline{u_1(0,0)}].\end{aligned}\tag{3.5}$$

Considering the boundary condition $f=\varphi$ and applying Lemma 2.8, we obtain that $W(z)$ and $u_0(z)$ are determined by (2.7) and (2.8), respectively. Therefore, we have

$$\begin{aligned}u_0(0,z_2)=&\,\frac{4\lambda}{\lambda+1}\Big\{\overline{\frac{1}{(2\pi i)^2}\int_{\partial_0D^2}\frac{\varphi(\zeta)d\zeta}{(1-\bar{z}_2\zeta_2)^2}}-\overline{\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]\frac{d\sigma_{\zeta_1}d\sigma_{\zeta_2}}{(1-\bar{z}_2\zeta_2)^2}}\Big\}\\&-(\lambda-1)\Big\{\frac{1}{(2\pi i)^2}\int_{\partial_0D^2}\varphi(\zeta)d\zeta-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2}\Big\}\\&+\frac{(\lambda-1)^2}{\lambda+1}\Big\{\overline{\frac{1}{(2\pi i)^2}\int_{\partial_0D^2}\varphi(\zeta)d\zeta}-\overline{\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2}}\Big\},\end{aligned}\tag{3.6}$$

and

$$
\begin{aligned}
u_0(z_1,0)=&\ \frac{4\lambda}{\lambda+1}\Big\{\overline{\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\frac{\varphi(\zeta)d\zeta}{(1-\bar{z}_1\zeta_1)^2}}-\overline{\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]\frac{d\sigma_{\zeta_1}d\sigma_{\zeta_2}}{(1-\bar{z}_1\zeta_1)^2}}\Big\}\\
&-(\lambda-1)\Big\{\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)d\zeta-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2}\Big\}\\
&+\frac{(\lambda-1)^2}{\lambda+1}\Big\{\overline{\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)d\zeta}-\overline{\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2}}\Big\}.
\end{aligned}
\tag{3.7}
$$

In addition, (2.11) gives

$$
\begin{aligned}
&\frac{\lambda+1}{4\lambda}\overline{[u_1(z)-u_1(0,z_2)-u_1(z_1,0)+u_1(0)]}\\
=&\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\Big\{\varphi(\zeta)-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\widetilde{\zeta})}\Big]\frac{d\sigma_{\widetilde{\zeta}_1}}{\widetilde{\zeta}_1-\zeta_1}\frac{d\sigma_{\widetilde{\zeta}_2}}{\widetilde{\zeta}_2-\zeta_2}\Big\}\\
&\cdot\Big[\frac{\bar{z}_1}{1-\bar{z}_1\zeta_1}\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}+\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}\frac{1}{\zeta_1-z_1}+\frac{\bar{z}_1}{1-\bar{z}_1\zeta_1}\frac{1}{\zeta_2-z_2}\Big]d\zeta_1d\zeta_2-\frac{\lambda-1}{4\lambda}u_0(0)\bar{z}_1\bar{z}_2\\
=&\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\Big[\frac{\bar{z}_1}{1-\bar{z}_1\zeta_1}\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}+\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}\frac{1}{\zeta_1-z_1}+\frac{\bar{z}_1}{1-\bar{z}_1\zeta_1}\frac{1}{\zeta_2-z_2}\Big]d\zeta_1d\zeta_2\\
&-\frac{1}{2\pi i}\int_{\partial D_2}\Big\{\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\widetilde{\zeta})}\Big]\Big[\frac{1}{2\pi i}\int_{\partial D_1}\frac{1}{\widetilde{\zeta}_1-\zeta_1}\frac{\bar{z}_1d\zeta_1}{1-\bar{z}_1\zeta_1}\Big]\frac{d\sigma_{\widetilde{\zeta}_1}d\sigma_{\widetilde{\zeta}_2}}{\widetilde{\zeta}_2-\zeta_2}\Big\}\frac{\bar{z}_2d\zeta_2}{1-\bar{z}_2\zeta_2}\\
&-\frac{1}{2\pi i}\int_{\partial D_2}\Big\{\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\widetilde{\zeta})}\Big]\Big[\frac{1}{2\pi i}\int_{\partial D_1}\frac{1}{\widetilde{\zeta}_1-\zeta_1}\frac{d\zeta_1}{\zeta_1-z_1}\Big]\frac{d\sigma_{\widetilde{\zeta}_1}d\sigma_{\widetilde{\zeta}_2}}{\widetilde{\zeta}_2-\zeta_2}\Big\}\frac{\bar{z}_2d\zeta_2}{1-\bar{z}_2\zeta_2}\\
&-\frac{1}{2\pi i}\int_{\partial D_1}\Big\{\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\widetilde{\zeta})}\Big]\Big[\frac{1}{2\pi i}\int_{\partial D_2}\frac{1}{\widetilde{\zeta}_2-\zeta_2}\frac{d\zeta_2}{\zeta_2-z_2}\Big]\frac{d\sigma_{\widetilde{\zeta}_1}d\sigma_{\widetilde{\zeta}_2}}{\widetilde{\zeta}_1-\zeta_1}\Big\}\frac{\bar{z}_1d\zeta_1}{1-\bar{z}_1\zeta_1}\\
&-\frac{\lambda-1}{4\lambda}u_0(0)\bar{z}_1\bar{z}_2\\
=&\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\Big[\frac{\bar{z}_1}{1-\bar{z}_1\zeta_1}\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}+\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}\frac{1}{\zeta_1-z_1}+\frac{\bar{z}_1}{1-\bar{z}_1\zeta_1}\frac{1}{\zeta_2-z_2}\Big]d\zeta_1d\zeta_2\\
&-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\widetilde{\zeta})}\Big]\frac{\bar{z}_1d\sigma_{\widetilde{\zeta}_1}}{\bar{z}_1\widetilde{\zeta}_1-1}\Big[\frac{1}{2\pi i}\int_{\partial D_2}\frac{1}{\widetilde{\zeta}_2-\zeta_2}\frac{\bar{z}_2d\zeta_2}{1-\bar{z}_2\zeta_2}\Big]d\sigma_{\widetilde{\zeta}_2}-\frac{\lambda-1}{4\lambda}u_0(0)\bar{z}_1\bar{z}_2\\
=&\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\Big[\frac{\bar{z}_1}{1-\bar{z}_1\zeta_1}\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}+\frac{\bar{z}_2}{1-\bar{z}_2\zeta_2}\frac{1}{\zeta_1-z_1}+\frac{\bar{z}_1}{1-\bar{z}_1\zeta_1}\frac{1}{\zeta_2-z_2}\Big]d\zeta_1d\zeta_2\\
&-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\widetilde{\zeta})}\Big]\frac{\bar{z}_1d\sigma_{\widetilde{\zeta}_1}}{\bar{z}_1\widetilde{\zeta}_1-1}\frac{\bar{z}_2d\sigma_{\widetilde{\zeta}_2}}{\bar{z}_2\widetilde{\zeta}_2-1}-\frac{\lambda-1}{4\lambda}u_0(0)\bar{z}_1\bar{z}_2,
\end{aligned}
\tag{3.8}
$$

in which

$$
\frac{1}{2\pi i}\int_{\partial D_1}\frac{1}{\widetilde{\zeta}_1-\zeta_1}\frac{\bar{z}_1d\zeta_1}{1-\bar{z}_1\zeta_1}=\frac{\bar{z}_1}{\bar{z}_1\widetilde{\zeta}_1-1},\quad \frac{1}{2\pi i}\int_{\partial D_1}\frac{1}{\widetilde{\zeta}_1-\zeta_1}\frac{d\zeta_1}{\zeta_1-z_1}=0=\frac{1}{2\pi i}\int_{\partial D_2}\frac{1}{\widetilde{\zeta}_2-\zeta_2}\frac{d\zeta_2}{\zeta_2-z_2}
$$

are used.

Plugging (2.7) and (3.8) into (3.4), $f(z)$ is determined as

$$
\begin{aligned}
f(z)=&\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\frac{d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}+\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]\frac{d\sigma_{\zeta_1}}{\zeta_1-z_1}\frac{d\sigma_{\zeta_2}}{\zeta_2-z_2}\\
&+\frac{\lambda-1}{4\lambda}\Big[\frac{(|z_1|^2-1)(|z_2|^2-1)}{z_1z_2}u_0(z)+\frac{|z_2|^2-1}{z_1z_2}u_0(0,z_2)+\frac{|z_1|^2-1}{z_1z_2}u_0(z_1,0)+\frac{u_0(0)}{z_1z_2}\Big]\\
&+\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\Big[\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{\bar z_2}{1-\bar z_2\zeta_2}+\frac{\bar z_2}{1-\bar z_2\zeta_2}\frac{1}{\zeta_1-z_1}+\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{1}{\zeta_2-z_2}\Big]d\zeta_1 d\zeta_2\\
&-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\widetilde{\zeta})}\Big]\frac{\bar z_1 d\sigma_{\widetilde{\zeta}_1}}{\bar z_1\widetilde{\zeta}_1-1}\frac{\bar z_2 d\sigma_{\widetilde{\zeta}_2}}{\bar z_2\widetilde{\zeta}_2-1}-\frac{\lambda-1}{4\lambda}u_0(0)\bar z_1\bar z_2\\
=&\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\frac{d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}+\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]\frac{d\sigma_{\zeta_1}}{\zeta_1-z_1}\frac{d\sigma_{\zeta_2}}{\zeta_2-z_2}\\
&+\frac{\lambda-1}{4\lambda}\Big[\frac{(|z_1|^2-1)(|z_2|^2-1)}{z_1z_2}u_0(z)+\frac{|z_2|^2-1}{z_1z_2}u_0(0,z_2)+\frac{|z_1|^2-1}{z_1z_2}u_0(z_1,0)+\frac{1-|z_1|^2|z_2|^2}{z_1z_2}u_0(0)\Big]\\
&+\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\Big[\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{\bar z_2}{1-\bar z_2\zeta_2}+\frac{\bar z_2}{1-\bar z_2\zeta_2}\frac{1}{\zeta_1-z_1}+\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{1}{\zeta_2-z_2}\Big]d\zeta_1 d\zeta_2\\
&-\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\widetilde{\zeta})+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\widetilde{\zeta})}\Big]\frac{\bar z_1 d\sigma_{\widetilde{\zeta}_1}}{\bar z_1\widetilde{\zeta}_1-1}\frac{\bar z_2 d\sigma_{\widetilde{\zeta}_2}}{\bar z_2\widetilde{\zeta}_2-1}\\
=&\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\Big(\frac{\zeta_1}{\zeta_1-z_1}+\frac{\bar\zeta_1}{\overline{\zeta_1-z_1}}-1\Big)\Big(\frac{\zeta_2}{\zeta_2-z_2}+\frac{\bar\zeta_2}{\overline{\zeta_2-z_2}}-1\Big)\frac{d\zeta}{\zeta}\\
&+\frac{\lambda-1}{4\lambda}\Big[\frac{(|z_1|^2-1)(|z_2|^2-1)}{z_1z_2}u_0(z)+\frac{|z_2|^2-1}{z_1z_2}u_0(0,z_2)+\frac{|z_1|^2-1}{z_1z_2}u_0(z_1,0)+\frac{1-|z_1|^2|z_2|^2}{z_1z_2}u_0(0)\Big]\\
&+\frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]\Big[\frac{1}{(\zeta_1-z_1)(\zeta_2-z_2)}-\frac{\bar z_1\bar z_2}{(\bar z_1\zeta_1-1)(\bar z_2\zeta_2-1)}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2},
\end{aligned}
\tag{3.9}
$$

where $u_0(z), u_0(0,z_2), u_0(z_1,0), u_0(0)$ are defined in (2.8), (2.15), (3.6), and (3.7), and the last equation is due to

$$
\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\Big[\frac{1}{(\zeta_1-z_1)(\zeta_2-z_2)}+\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{\bar z_2}{1-\bar z_2\zeta_2}+\frac{\bar z_2}{1-\bar z_2\zeta_2}\frac{1}{\zeta_1-z_1}+\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{1}{\zeta_2-z_2}\Big]d\zeta\\
=&\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\Big[\frac{1}{(\zeta_1-z_1)(\zeta_2-z_2)}+\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{\bar z_2}{1-\bar z_2\zeta_2}+\Big(\frac{z_1}{\zeta_1(\zeta_1-z_1)}\frac{\bar z_2}{1-\bar z_2\zeta_2}+\frac{\bar\zeta_1\bar z_2}{1-\bar z_2\zeta_2}\Big)\\
&+\Big(\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{z_2}{\zeta_2(\zeta_2-z_2)}+\frac{\bar z_1\bar\zeta_2}{1-\bar z_1\zeta_1}\Big)\Big]d\zeta\\
=&\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\Big[\frac{1}{(\zeta_1-z_1)(\zeta_2-z_2)}+\frac{\bar\zeta_1\bar z_2+\bar z_1\bar\zeta_2-\bar z_1\bar z_2}{(1-\bar z_1\zeta_1)(1-\bar z_2\zeta_2)}+\frac{\bar\zeta_1 z_1}{\zeta_1-z_1}\frac{\bar z_2}{1-\bar z_2\zeta_2}+\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{\bar\zeta_2 z_2}{\zeta_2-z_2}\Big]d\zeta\\
=&\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\Big[\frac{1}{(\zeta_1-z_1)(\zeta_2-z_2)}+\frac{\bar\zeta_1\bar\zeta_2}{(1-\bar z_1\zeta_1)(1-\bar z_2\zeta_2)}-\frac{1}{\zeta_1\zeta_2}+\frac{\bar\zeta_1 z_1}{\zeta_1-z_1}\frac{\bar z_2}{1-\bar z_2\zeta_2}+\frac{\bar z_1}{1-\bar z_1\zeta_1}\frac{\bar\zeta_2 z_2}{\zeta_2-z_2}\Big]d\zeta\\
=&\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\Big[\frac{\zeta_1\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}+\frac{\bar\zeta_1\bar\zeta_2}{(\overline{\zeta_1-z_1})(\overline{\zeta_2-z_2})}-1+\frac{z_1}{\zeta_1-z_1}\frac{\bar z_2}{\overline{\zeta_2-z_2}}+\frac{\bar z_1}{\overline{\zeta_1-z_1}}\frac{z_2}{\zeta_2-z_2}\Big]\frac{d\zeta}{\zeta}\\
=&\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\varphi(\zeta)\Big(\frac{\zeta_1}{\zeta_1-z_1}+\frac{\bar\zeta_1}{\overline{\zeta_1-z_1}}-1\Big)\Big(\frac{\zeta_2}{\zeta_2-z_2}+\frac{\bar\zeta_2}{\overline{\zeta_2-z_2}}-1\Big)\frac{d\zeta}{\zeta}.
\end{aligned}
$$

(2) In the following, we verify that (3.1) is the solution to the problem.

(i) For $z \in \partial_0 D^2$ (i.e., $|z_1| = |z_2| = 1$), applying the properties of the Poisson kernel on D^2, (3.1) satisfies the boundary condition $f = \varphi$ obviously. In addition, applying Lemma 2.1, by (2.12) and (3.1), we obtain

$$\begin{aligned}
\partial_{\bar z_1}\partial_{\bar z_2} f(z) &= \frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\frac{\varphi(\zeta)d\zeta_1}{(1-\bar z_1\zeta_1)^2}\frac{d\zeta_2}{(1-\bar z_2\zeta_2)^2} + \frac{\lambda-1}{4\lambda}[u_0(z)-u_0(0)]\\
&\quad + \frac{\lambda-1}{4\lambda}\widetilde{\phi}(z) + \frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(z)} - \frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta) + \frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]\frac{d\sigma_{\zeta_1}d\sigma_{\zeta_2}}{(1-\bar z_1\zeta_1)^2(1-\bar z_2\zeta_2)^2}\\
&= \frac{\lambda-1}{4\lambda}\widetilde{\phi}(z) + \frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(z)} + \frac{\lambda-1}{4\lambda}u_0(z) + \frac{\lambda+1}{4\lambda}\Big\{-\frac{\lambda-1}{\lambda+1}u_0(0) + \frac{4\lambda}{\lambda+1}\\
&\quad \cdot\Big[\frac{1}{(2\pi i)^2}\int_{\partial_0 D^2}\frac{\varphi(\zeta)d\zeta_1}{(1-\bar z_1\zeta_1)^2}\frac{d\zeta_2}{(1-\bar z_2\zeta_2)^2} - \frac{1}{\pi^2}\int_{D^2}\Big[\frac{\lambda-1}{4\lambda}\widetilde{\phi}(\zeta) + \frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(\zeta)}\Big]\frac{d\sigma_{\zeta_1}d\sigma_{\zeta_2}}{(1-\bar z_1\zeta_1)^2(1-\bar z_2\zeta_2)^2}\Big]\\
&= \frac{\lambda-1}{4\lambda}\widetilde{\phi}(z) + \frac{\lambda+1}{4\lambda}\overline{\widetilde{\phi}(z)} + \frac{\lambda-1}{4\lambda}u_0(z) + \frac{\lambda+1}{4\lambda}\overline{u_0(z)}\\
&= \frac{\lambda-1}{4\lambda}[\widetilde{\phi}(z) + u_0(z)] + \frac{\lambda+1}{4\lambda}[\overline{\widetilde{\phi}(z)} + \overline{u_0(z)}]\\
&= \frac{\lambda-1}{4\lambda}\phi(z) + \frac{\lambda+1}{4\lambda}\overline{\phi(z)}.
\end{aligned}$$

(ii) By Lemma 2.4, $\phi(z) = \widetilde{\phi}(z) + u_0(z)$ satisfies $\partial^m_{\bar z_1}\partial^m_{\bar z_2}\phi(z) = 0$ ($z \in D^2$) and

$$\Re\partial^\mu_{\bar z_1}\partial^\nu_{\bar z_2}\phi(z) = g_{\mu\nu}(z)\ (z\in\partial_0 D^2),\ \Im\partial^\mu_{\bar z_1}\partial^\nu_{\bar z_2}\phi(0,z_2) = 0 = \Im\partial^\mu_{\bar z_1}\partial^\nu_{\bar z_2}\phi(z_1,0)$$

for $1 \le \mu,\nu \le m-1$.

From the above analysis in (1) and (2), and by the expression of f in (3.1) and the uniqueness of $u_0(z)$ in (3.2), it is shown that (3.1) is the unique solution of the problem.

Theorem 3.2. *Let $\varphi \in C(\partial_0 D^2;\mathbb{C})$ and $h_{\mu_1\nu_1}, g_{\mu_2\nu_2} \in C(\partial_0 D^2;\mathbb{R})$ for $0 \le \mu_1,\nu_1 \le n-2$, $1 \le \mu_2,\nu_2 \le m-1$ ($m,n \ge 2$), and let $\lambda \in \mathbb{R}\setminus\{-1,0,1\}$. Then*

$$f(z) = T_{n-1,D_1}(T_{n-1,D_2}F)(z) + \widetilde{\phi}(z)$$

is the solution to the problem

$$\partial^n_{\bar z_1}\partial^n_{\bar z_2} f(z) = \frac{\lambda-1}{4\lambda}\phi(z) + \frac{\lambda+1}{4\lambda}\bar\phi(z),\ \partial^m_{\bar z_1}\partial^m_{\bar z_2}\phi(z) = 0,\ z\in D^2,$$

with the conditions

$$\begin{cases}\partial^{n-1}_{\bar z_1}\partial^{n-1}_{\bar z_2} f = \varphi,\ \Re\partial^{\mu_1}_{\bar z_1}\partial^{\nu_1}_{\bar z_2} f = h_{\mu_1\nu_1}\ (z\in\partial_0 D^2),\ \Im\partial^{\mu_1}_{\bar z_1}\partial^{\nu_1}_{\bar z_2} f(0,z_2) = 0 = \Im\partial^{\mu_1}_{\bar z_1}\partial^{\nu_1}_{\bar z_2} f(z_1,0),\\ \Re\partial^{\mu_2}_{\bar z_1}\partial^{\nu_2}_{\bar z_2}\phi(z) = g_{\mu_2\nu_2}(z)\ (z\in\partial_0 D^2),\ \Im\partial^{\mu_2}_{\bar z_1}\partial^{\nu_2}_{\bar z_2}\phi(0,z_2) = 0 = \Im\partial^{\mu_2}_{\bar z_1}\partial^{\nu_2}_{\bar z_2}\phi(z_1,0),\end{cases}$$

where $F(z)$ is determined by (3.1) (in which $g_{\mu\nu}$ is replaced by $g_{\mu_2\nu_2}$), $\widetilde{\phi}(z)$ is determined by (2.1) (in which $g_{\mu\nu}$ is replaced by $h_{\mu_1\nu_1}$ and m is replaced by $n-1$), and

$$T_{n,D_i}F(z_i) = \frac{(-1)^n}{2\pi(n-1)!}\int_{D_i}(\overline{\zeta_i - z_i} + \zeta_i - z_i)^{n-1}\Big[\frac{F(\zeta_i)}{\zeta_i}\frac{\zeta_i + z_i}{\zeta_i - z_i} + \frac{\overline{F(\zeta_i)}}{\bar\zeta_i}\frac{1 + z_i\bar\zeta_i}{1 - z_i\bar\zeta_i}\Big]d\xi_i d\eta_i, \tag{3.10}$$

with $T_{0,D_i}F = F$ and $n \in \mathbb{N}$, $i = 1,2$.

Proof. Let $F(z) = \partial_{\bar{z}_1}^{n-1}\partial_{\bar{z}_2}^{n-1} f(z)$. Then, the problem is transformed to be

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}F(z) = \frac{\lambda-1}{4\lambda}\phi(z) + \frac{\lambda+1}{4\lambda}\bar{\phi}(z),\ \partial_{\bar{z}_1}^{m}\partial_{\bar{z}_2}^{m}\phi(z) = 0,\ z \in D^2,$$

with the conditions

$$\begin{cases} \partial_{\bar{z}_1}^{n-1}\partial_{\bar{z}_2}^{n-1} f = F,\ \Re\partial_{\bar{z}_1}^{\mu_1}\partial_{\bar{z}_2}^{\nu_1} f = h_{\mu_1\nu_1}\ (z \in \partial_0 D^2),\ \Im\partial_{\bar{z}_1}^{\mu_1}\partial_{\bar{z}_2}^{\nu_1} f(0, z_2) = 0 = \Im\partial_{\bar{z}_1}^{\mu_1}\partial_{\bar{z}_2}^{\nu_1} f(z_1, 0), \\ F = \varphi,\ \Re\partial_{\bar{z}_1}^{\mu_2}\partial_{\bar{z}_2}^{\nu_2}\phi(z) = g_{\mu_2\nu_2}(z)\ (z \in \partial_0 D^2),\ \Im\partial_{\bar{z}_1}^{\mu_2}\partial_{\bar{z}_2}^{\nu_2}\phi(0, z_2) = 0 = \Im\partial_{\bar{z}_1}^{\mu_2}\partial_{\bar{z}_2}^{\nu_2}\phi(z_1, 0). \end{cases}$$

By Theorem 3.1, $F(z)$ determined by (3.1) (where $g_{\mu\nu}$ is replaced by $g_{\mu_2\nu_2}$) is the unique solution to

$$\begin{cases} \partial_{\bar{z}_1}\partial_{\bar{z}_2}F(z) = \frac{\lambda-1}{4\lambda}\phi(z) + \frac{\lambda+1}{4\lambda}\bar{\phi}(z),\ \partial_{\bar{z}_1}^{m}\partial_{\bar{z}_2}^{m}\phi(z) = 0\ (z \in D^2), \\ F = \varphi,\ \Re\partial_{\bar{z}_1}^{\mu_2}\partial_{\bar{z}_2}^{\nu_2}\phi(z) = g_{\mu_2\nu_2}(z)\ (z \in \partial_0 D^2),\ \Im\partial_{\bar{z}_1}^{\mu_2}\partial_{\bar{z}_2}^{\nu_2}\phi(0, z_2) = 0 = \Im\partial_{\bar{z}_1}^{\mu_2}\partial_{\bar{z}_2}^{\nu_2}\phi(z_1, 0). \end{cases}$$

For $T_{n,D_i}F$ defined by (3.10); applying Lemma 2.3,

$$\partial_{\bar{z}_1}^{n-1}\partial_{\bar{z}_2}^{n-1} T_{n-1,D_1}(T_{n-1,D_2}F) = \partial_{\bar{z}_1}^{n-1}(T_{n-1,D_1}F) = F,$$

$$\Re\partial_{\bar{z}_1}^{\mu_1}\partial_{\bar{z}_2}^{\nu_1} T_{n-1,D_1}(T_{n-1,D_2}F) = \Re\partial_{\bar{z}_1}^{\mu_1} T_{n-1,D_1}(T_{n-1-\nu_1,D_2}F) = 0\ (z \in \partial D),$$

$$\Im\partial_{\bar{z}_1}^{\mu_1}\partial_{\bar{z}_2}^{\nu_1} T_{n-1,D_1}(T_{n-1,D_2}F)(z_1, 0) = \Im\partial_{\bar{z}_1}^{\mu_1} T_{n-1,D_1}(T_{n-1-\nu_1,D_2}F)(z_1, 0) = 0,$$

$$\Im\partial_{\bar{z}_1}^{\mu_1}\partial_{\bar{z}_2}^{\nu_1} T_{n-1,D_1}(T_{n-1,D_2}F)(0, z_2) = \Im\partial_{\bar{z}_1}^{\mu_1} T_{n-1,D_1}(T_{n-1-\nu_1,D_2}F)(0, z_2) = 0.$$

Therefore, $T_{n-1,D_1}(T_{n-1,D_2}F)$ is a special solution to $\partial_{\bar{z}_1}^{n-1}\partial_{\bar{z}_2}^{n-1} f = F$, and the solution to

$$\partial_{\bar{z}_1}^{n-1}\partial_{\bar{z}_2}^{n-1} f = F,\ \Re\partial_{\bar{z}_1}^{\mu_1}\partial_{\bar{z}_2}^{\nu_1} f = h_{\mu_1\nu_1}\ (z \in \partial_0 D^2),\ \Im\partial_{\bar{z}_1}^{\mu_1}\partial_{\bar{z}_2}^{\nu_1} f(0, z_2) = 0 = \Im\partial_{\bar{z}_1}^{\mu_1}\partial_{\bar{z}_2}^{\nu_1} f(z_1, 0)$$

is

$$f(z) = T_{n-1,D_1}(T_{n-1,D_2}F)(z) + \widetilde{\phi}(z),$$

where $\widetilde{\phi}(z)$ is a $n-1$-holomorphic function on D^2 satisfying

$$\Re\partial_{\bar{z}_1}^{\mu_1}\partial_{\bar{z}_2}^{\nu_1}\tilde{\phi} = h_{\mu_1\nu_1}\ (z \in \partial_0 D^2),\ \Im\partial_{\bar{z}_1}^{\mu_1}\partial_{\bar{z}_2}^{\nu_1}\tilde{\phi}(0, z_2) = 0 = \Im\partial_{\bar{z}_1}^{\mu_1}\partial_{\bar{z}_2}^{\nu_1}\tilde{\phi}(z_1, 0).$$

By Lemma 2.4, $\widetilde{\phi}(z)$ is determined by (2.1), in which $g_{\mu\nu}$ and m are replaced by $h_{\mu_1\nu_1}$ and $n-1$, respectively.

4. Neumann boundary value problems

In this section, we discuss systems of complex partial differential equations with Neumann boundary conditions on the bicylinder. Let $\partial\nu_i f(z) = z_i\partial z_i f(z) + \bar{z}_i\partial\bar{z}_i f(z)$ denote the directional derivative of $f(z)$ in relation to the outer normal vector, where $i = 1, 2$.

Theorem 4.1. *Let $g_{k_1k_2} \in C(\partial_0 D^2)$ and $c_{k_1k_2} \in \mathbb{C}$. Let $b_{k_1k_2}(z_1)$ and $d_{k_1k_2}(z_2)$ be analytic functions about z_1 and z_2, respectively, with $b_{k_1k_2}(0) = d_{k_1k_2}(0) = c_{k_1k_2}$ ($1 \le k_1, k_2 \le m-1$, $m \ge 2$). Then, there exists an* *analytic function $u_{(m-k_1)(m-k_2)}(z)$ on D^2:*

$$
\begin{aligned}
u_{(m-k_1)(m-k_2)}(z) = & d_{(m-k_1)(m-k_2)}(z_2) + b_{(m-k_1)(m-k_2)}(z_1) - c_{(m-k_1)(m-k_2)} + \frac{1}{(2\pi i)^2} \\
& \cdot \int_{\partial D_1}\int_{\partial D_2} \sum_{t_1=0}^{k_1-1}\sum_{t_2=0}^{k_2-1} \frac{(-2)^{t_1+t_2}}{t_1!t_2!}\overline{\zeta_1}^{t_1}\overline{\zeta_2}^{t_2} g_{(t_1+m-k_1)(t_2+m-k_2)}(\zeta)\log(1-z_1\overline{\zeta_1})\log(1-z_2\overline{\zeta_2})\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2} \\
& + \sum_{\lambda_1=1}^{k_1-1}\frac{1}{2\pi i}\int_{\partial D_1}\sum_{s_1=0}^{k_1-1-\lambda_1}\frac{(-2)^{s_1}}{s_1!}\overline{\zeta_1}^{s_1}\Big[\frac{1}{2\pi i}\int_{\partial D_2}\sum_{t_2=0}^{k_2-1}\frac{(-2)^{t_2}}{t_2!}\overline{\zeta_2}^{t_2} g_{(\lambda_1+s_1+m-k_1)(t_2+m-k_2)}(\zeta) \\
& \cdot \log(1-z_2\overline{\zeta_2})\frac{d\zeta_2}{\zeta_2}\Big]\frac{1}{\lambda_1! z_1^{\lambda_1}}\Big[\log(1-z_1\overline{\zeta_1}) + \sum_{\sigma_1=1}^{\lambda_1}\frac{z_1^{\sigma_1}\overline{\zeta_1}^{\sigma_1}}{\sigma_1}\Big]\frac{d\zeta_1}{\zeta_1} \\
& + \sum_{\lambda_2=1}^{k_2-1}\frac{1}{2\pi i}\int_{\partial D_2}\sum_{s_2=0}^{k_2-1-\lambda_2}\frac{(-2)^{s_2}}{s_2!}\overline{\zeta_2}^{s_2}\Big[\frac{1}{2\pi i}\int_{\partial D_1}\sum_{t_1=0}^{k_1-1}\frac{(-2)^{t_1}}{t_1!}\overline{\zeta_1}^{t_1} g_{(t_1+m-k_1)(\lambda_2+s_2+m-k_2)}(\zeta) \\
& \cdot \log(1-z_1\overline{\zeta_1})\frac{d\zeta_1}{\zeta_1}\Big]\frac{1}{\lambda_2! z_2^{\lambda_2}}\Big[\log(1-z_2\overline{\zeta_2}) + \sum_{\sigma_2=1}^{\lambda_2}\frac{z_2^{\sigma_2}\overline{\zeta_2}^{\sigma_2}}{\sigma_2}\Big]\frac{d\zeta_2}{\zeta_2} \\
& + \sum_{\lambda_1=1}^{k_1-1}\sum_{\lambda_2=1}^{k_2-1}\frac{1}{(2\pi i)^2}\int_{\partial D_1}\int_{\partial D_2}\sum_{s_1=0}^{k_1-1-\lambda_1}\sum_{s_2=0}^{k_2-1-\lambda_2}\frac{(-2)^{s_1+s_2}}{s_1!s_2!}\overline{\zeta_1}^{s_1}\overline{\zeta_2}^{s_2} g_{(\lambda_1+s_1+m-k_1)(\lambda_2+s_2+m-k_2)}(\zeta) \\
& \cdot \frac{1}{\lambda_1! z_1^{\lambda_1}}\frac{1}{\lambda_2! z_2^{\lambda_2}}\Big[\log(1-z_1\overline{\zeta_1}) + \sum_{\sigma_1=1}^{\lambda_1}\frac{z_1^{\sigma_1}\overline{\zeta_1}^{\sigma_1}}{\sigma_1}\Big]\Big[\log(1-z_2\overline{\zeta_2}) + \sum_{\sigma_2=1}^{\lambda_2}\frac{z_2^{\sigma_2}\overline{\zeta_2}^{\sigma_2}}{\sigma_2}\Big]\frac{d\zeta_1 d\zeta_2}{\zeta_1\zeta_2}
\end{aligned}
\tag{4.1}
$$

that satisfies

$$
\begin{cases}
\partial_{\nu_1}\partial_{\nu_2}\sum_{l_1=0}^{k_1-1}\sum_{l_2=0}^{k_2-1}\frac{\overline{z}_1^{l_1}\overline{z}_2^{l_2}}{l_1!l_2!}u_{(l_1+m-k_1)(l_2+m-k_2)}(z) = g_{(m-k_1)(m-k_2)}(z),\ z \in \partial_0 D^2, \\
u_{(m-k_1)(m-k_2)}(z_1, 0) = b_{(m-k_1)(m-k_2)}(z_1),\ z_1 \in D_1, \\
u_{(m-k_1)(m-k_2)}(0, z_2) = d_{(m-k_1)(m-k_2)}(z_2),\ z_2 \in D_2, \\
u_{(m-k_1)(m-k_2)}(0) = c_{(m-k_1)(m-k_2)}
\end{cases}
\tag{4.2}
$$

if and only if

$$
\begin{cases}
\frac{1}{2\pi i}\int_{\partial D_1} g_{(m-1)(m-k_2)}(\zeta_1, z_2)\frac{d\zeta_1}{(1-\overline{z_1}\zeta_1)\zeta_1} = 0, \\
\sum_{\tau_1=0}^{k_1-1}\frac{(-1)^{\tau_1}}{\tau_1!}\frac{1}{2\pi i}\int_{\partial D_1}\overline{\zeta_1}^{\tau_1} g_{(\tau_1+m-k_1)(m-k_2)}(\zeta_1, z_2)\frac{d\zeta_1}{1-\overline{z_1}\zeta_1} \\
= \partial_{\nu_2}\Big[\sum_{l_2=0}^{k_2-1}\frac{1}{l_2!}d_{(m-k_1+1)(l_2+m-k_2)}(z_2)\overline{z_2}^{l_2}\Big],\ 2 \le k_1 \le m-1,\ 1 \le k_2 \le m-1
\end{cases}
\tag{4.3}
$$

for $\forall z_1 \in D_1$ $(z_2 \in \partial D_2)$*, and*

$$
\begin{cases}
\int_{\partial D_2}\Big\{\int_{\partial D_1}\sum_{t_1=0}^{k_1-1}\frac{(-2)^{t_1}}{t_1!}\overline{\zeta_1}^{t_1}g_{(t_1+m-k_1)(m-1)}(\zeta)\log(1-z_1\overline{\zeta_1})\frac{d\zeta_1}{\zeta_1}+\sum_{\lambda_1=1}^{k_1-1}\int_{\partial D_1}\sum_{s_1=0}^{k_1-1-\lambda_1}\frac{(-2)^{s_1}}{s_1!}\overline{\zeta_1}^{s_1} \\
\cdot g_{(\lambda_1+s_1+m-k_1)(m-1)}(\zeta)\frac{1}{\lambda_1!z_1^{\lambda_1}}\Big[\log(1-z_1\overline{\zeta_1})+\sum_{\sigma_1=1}^{\lambda_1}\frac{z_1^{\sigma_1}\overline{\zeta_1}^{\sigma_1}}{\sigma_1}\Big]\frac{d\zeta_1}{\zeta_1}\Big\}\frac{d\zeta_2}{(1-\overline{z_2}\zeta_2)\zeta_2}=0, \\
\sum_{\tau_2=0}^{k_2-1}\frac{(-1)^{\tau_2}}{\tau_2!}\frac{1}{2\pi i}\int_{\partial D_2}\overline{\zeta_2}^{\tau_2}\Big\{\frac{-1}{2\pi i}\int_{\partial D_1}\sum_{t_1=0}^{k_1-1}\frac{(-2)^{t_1}}{t_1!}\overline{\zeta_1}^{t_1}g_{(t_1+m-k_1)(\tau_2+m-k_2)}(\zeta)\log(1-z_1\overline{\zeta_1})\frac{d\zeta_1}{\zeta_1} \\
-\sum_{\lambda_1=1}^{k_1-1}\frac{1}{2\pi i}\int_{\partial D_1}\sum_{s_1=0}^{k_1-1-\lambda_1}\frac{(-2)^{s_1}}{s_1!}\overline{\zeta_1}^{s_1}\frac{g_{(\lambda_1+s_1+m-k_1)(\tau_2+m-k_2)}(\zeta)}{\lambda_1!z_1^{\lambda_1}}\Big[\log(1-z_1\overline{\zeta_1})+\sum_{\sigma_1=1}^{\lambda_1}\frac{z_1^{\sigma_1}\overline{\zeta_1}^{\sigma_1}}{\sigma_1}\Big]\frac{d\zeta_1}{\zeta_1}\Big\}\frac{d\zeta_2}{1-\overline{z_2}\zeta_2} \\
=b_{(m-k_1)(m-k_2+1)}(z_1)-c_{(m-k_1)(m-k_2+1)},\quad 1\le k_1\le m-1,\ 2\le k_2\le m-1.
\end{cases} \tag{4.4}
$$

for $\forall z_2\in D_2$, $z_1\in D_1$.

Proof. From (4.2), we obtain

$$
\begin{cases}
\partial_{\nu_1}\sum_{l_1=0}^{k_1-1}\frac{1}{l_1!}\Big\{\partial_{\nu_2}\Big[\sum_{l_2=0}^{k_2-1}\frac{1}{l_2!}u_{(l_1+m-k_1)(l_2+m-k_2)}(z)\bar{z}_2^{l_2}\Big]\Big\}\bar{z}_1^{l_1}=g_{(m-k_1)(m-k_2)}(z), \\
\partial_{\nu_2}\Big[\sum_{l_2=0}^{k_2-1}\frac{1}{l_2!}u_{(m-k_1)(l_2+m-k_2)}(0,z_2)\bar{z}_2^{l_2}\Big]=\partial_{\nu_2}\Big[\sum_{l_2=0}^{k_2-1}\frac{1}{l_2!}d_{(m-k_1)(l_2+m-k_2)}(z_2)\bar{z}_2^{l_2}\Big]
\end{cases} \tag{4.5}
$$

for $z_1\in\partial D_1$ (at the same time $z_2\in\partial D_2$). Applying Lemma 2.5 on (4.5) for $z_1\in\partial D_1$, there exists an analytic function about z_1, i.e., $\partial_{\nu_2}\Big[\sum_{l_2=0}^{k_2-1}\frac{\bar{z}_2^{l_2}}{l_2!}u_{(m-k_1)(l_2+m-k_2)}(z)\Big]$, satisfying (4.5) if and only if (4.3) is satisfied for $\forall z_1\in D_1$ ($z_2\in\partial D_2$). Besides that, $\partial_{\nu_2}\Big[\sum_{l_2=0}^{k_2-1}\frac{\bar{z}_2^{l_2}}{l_2!}u_{(m-k_1)(l_2+m-k_2)}(z)\Big]$ is determined by

$$
\begin{aligned}
\partial_{\nu_2}\Big[\sum_{l_2=0}^{k_2-1}\frac{1}{l_2!}u_{(m-k_1)(l_2+m-k_2)}(z)\bar{z}_2^{l_2}\Big]=&\ \partial_{\nu_2}\Big[\sum_{l_2=0}^{k_2-1}\frac{1}{l_2!}d_{(m-k_1)(l_2+m-k_2)}(z_2)\bar{z}_2^{l_2}\Big] \\
&-\frac{1}{2\pi i}\int_{\partial D_1}\sum_{t_1=0}^{k_1-1}\frac{(-2)^{t_1}}{t_1!}\overline{\zeta_1}^{t_1}g_{(t_1+m-k_1)(m-k_2)}(\zeta_1,z_2)\log(1-z_1\overline{\zeta_1})\frac{d\zeta_1}{\zeta_1} \\
&-\sum_{\lambda_1=1}^{k_1-1}\frac{1}{2\pi i}\int_{\partial D_1}\sum_{s_1=0}^{k_1-1-\lambda_1}\frac{(-2)^{s_1}}{s_1!}\overline{\zeta_1}^{s_1}g_{(\lambda_1+s_1+m-k_1)(m-k_2)}(\zeta_1,z_2) \\
&\cdot\frac{1}{\lambda_1!z_1^{\lambda_1}}\Big[\log(1-z_1\overline{\zeta_1})+\sum_{\sigma_1=1}^{\lambda_1}\frac{z_1^{\sigma_1}\overline{\zeta_1}^{\sigma_1}}{\sigma_1}\Big]\frac{d\zeta_1}{\zeta_1},
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\partial_{\nu_2}\Big\{\sum_{l_2=0}^{k_2-1}\frac{1}{l_2!}[u_{(m-k_1)(l_2+m-k_2)}(z)-d_{(m-k_1)(l_2+m-k_2)}(z_2)]\bar{z}_2^{l_2}\Big\}\\
&=-\frac{1}{2\pi i}\int_{\partial D_1}\sum_{t_1=0}^{k_1-1}\frac{(-2)^{t_1}}{t_1!}\overline{\zeta_1}^{t_1}g_{(t_1+m-k_1)(m-k_2)}(\zeta_1,z_2)\log(1-z_1\overline{\zeta_1})\frac{d\zeta_1}{\zeta_1}\\
&\quad-\sum_{\lambda_1=1}^{k_1-1}\frac{1}{2\pi i}\int_{\partial D_1}\sum_{s_1=0}^{k_1-1-\lambda_1}\frac{(-2)^{s_1}}{s_1!}\overline{\zeta_1}^{s_1}g_{(\lambda_1+s_1+m-k_1)(m-k_2)}(\zeta_1,z_2)\frac{1}{\lambda_1!z_1^{\lambda_1}}\Big[\log(1-z_1\overline{\zeta_1})+\sum_{\sigma_1=1}^{\lambda_1}\frac{z_1^{\sigma_1}\overline{\zeta_1}^{\sigma_1}}{\sigma_1}\Big]\frac{d\zeta_1}{\zeta_1}.
\end{aligned}
\tag{4.6}
$$

Further, applying Lemma 2.5 on (4.6) for $z_2 \in \partial D_2$, there exists an analytic function about z_2, i.e., $u_{(m-k_1)(m-k_2)}(z)$, satisfying (4.6) and

$$u_{(m-k_1)(m-k_2)}(z_1,0)-d_{(m-k_1)(m-k_2)}(0)=b_{(m-k_1)(m-k_2)}(z_1)-d_{(m-k_1)(m-k_2)}(0),$$

if and only if (4.4) is satisfied for $\forall z_2 \in D_2$ (at the same time $z_1 \in D_1$). Moreover, $u_{(m-k_1)(m-k_2)}(z)$ is determined by

$$
\begin{aligned}
&u_{(m-k_1)(m-k_2)}(z)-d_{(m-k_1)(m-k_2)}(z_2)\\
&=b_{(m-k_1)(m-k_2)}(z_1)-d_{(m-k_1)(m-k_2)}(0)\\
&\quad-\frac{1}{2\pi i}\int_{\partial D_2}\sum_{t_2=0}^{k_2-1}\frac{(-2)^{t_2}}{t_2!}\overline{\zeta_2}^{t_2}\Big\{\frac{-1}{2\pi i}\int_{\partial D_1}\sum_{t_1=0}^{k_1-1}\frac{(-2)^{t_1}}{t_1!}\overline{\zeta_1}^{t_1}g_{(t_1+m-k_1)(t_2+m-k_2)}(\zeta)\log(1-z_1\overline{\zeta_1})\frac{d\zeta_1}{\zeta_1}\\
&\quad-\sum_{\lambda_1=1}^{k_1-1}\frac{1}{2\pi i}\int_{\partial D_1}\sum_{s_1=0}^{k_1-1-\lambda_1}\frac{(-2)^{s_1}}{s_1!}\overline{\zeta_1}^{s_1}g_{(\lambda_1+s_1+m-k_1)(t_2+m-k_2)}(\zeta)\\
&\quad\cdot\frac{1}{\lambda_1!z_1^{\lambda_1}}\Big[\log(1-z_1\overline{\zeta_1})+\sum_{\sigma_1=1}^{\lambda_1}\frac{z_1^{\sigma_1}\overline{\zeta_1}^{\sigma_1}}{\sigma_1}\Big]\frac{d\zeta_1}{\zeta_1}\Big\}\log(1-z_2\overline{\zeta_2})\frac{d\zeta_2}{\zeta_2}\\
&\quad-\sum_{\lambda_2=1}^{k_2-1}\frac{1}{2\pi i}\int_{\partial D_2}\sum_{s_2=0}^{k_2-1-\lambda_2}\frac{(-2)^{s_2}}{s_2!}\overline{\zeta_2}^{s_2}\Big\{\frac{-1}{2\pi i}\int_{\partial D_1}\sum_{t_1=0}^{k_1-1}\frac{(-2)^{t_1}}{t_1!}\overline{\zeta_1}^{t_1}g_{(t_1+m-k_1)(\lambda_2+s_2+m-k_2)}(\zeta)\\
&\quad\cdot\log(1-z_1\overline{\zeta_1})\frac{d\zeta_1}{\zeta_1}-\sum_{\lambda_1=1}^{k_1-1}\frac{1}{2\pi i}\int_{\partial D_1}\sum_{s_1=0}^{k_1-1-\lambda_1}\frac{(-2)^{s_1}}{s_1!}\overline{\zeta_1}^{s_1}g_{(\lambda_1+s_1+m-k_1)(\lambda_2+s_2+m-k_2)}(\zeta)\\
&\quad\cdot\frac{1}{\lambda_1!z_1^{\lambda_1}}\Big[\log(1-z_1\overline{\zeta_1})+\sum_{\sigma_1=1}^{\lambda_1}\frac{z_1^{\sigma_1}\overline{\zeta_1}^{\sigma_1}}{\sigma_1}\Big]\frac{d\zeta_1}{\zeta_1}\Big\}\frac{1}{\lambda_2!z_2^{\lambda_2}}\Big[\log(1-z_2\overline{\zeta_2})+\sum_{\sigma_2=1}^{\lambda_2}\frac{z_2^{\sigma_2}\overline{\zeta_2}^{\sigma_2}}{\sigma_2}\Big]\frac{d\zeta_2}{\zeta_2},
\end{aligned}
$$

which leads to (4.1).

Since $\partial_{\nu_2}\Big[\sum_{l_2=0}^{k_2-1}\frac{\bar{z}_2^{l_2}}{l_2!}u_{(m-k_1)(l_2+m-k_2)}(z)\Big]$ is analytic about z_1 and $u_{(m-k_1)(m-k_2)}(z)$ is analytic about z_2, then $u_{(m-k_1)(m-k_2)}(z)$ is analytic on D^2. Therefore, there exists an analytic function $u_{(m-k_1)(m-k_2)}(z)$ on D^2, determined by (4.1), satisfying (4.2) on the conditions of (4.3) and (4.4).

Theorem 4.2. *Let $\varphi, g_{\mu\nu} \in C(\partial_0 D^2)$ and $c_{\mu\nu} \in \mathbb{C}$. Let $b_{\mu\nu}(z_1)$ and $d_{\mu\nu}(z_2)$ be analytic functions about z_1 and z_2, respectively, with $b_{\mu\nu}(0) = d_{\mu\nu}(0) = c_{\mu\nu}$ $(1 \le \mu, \nu \le m-1,\ \ m \ge 2)$. Let $\lambda \in \mathbb{R} \setminus \{-1, 0, 1\}$. Then the problem*

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2} f(z) = \frac{\lambda-1}{4\lambda}\phi(z) + \frac{\lambda+1}{4\lambda}\bar{\phi}(z),\ \partial^m_{\bar{z}_1}\partial^m_{\bar{z}_2}\phi(z) = 0,\ z \in D^2,$$

with the conditions

$$\begin{cases} f(z) = \varphi(z),\ \partial\nu_1\partial\nu_2(\partial^{\mu}_{\bar{z}_1}\partial^{\nu}_{\bar{z}_2}\phi(z)) = g_{\mu\nu}(z),\ z \in \partial_0 D^2, \\ \partial^{\mu}_{\bar{z}_1}\partial^{\nu}_{\bar{z}_2}\phi(z_1, 0) = \sum\limits_{l_1=0}^{k_1-1} \frac{1}{l_1!} b_{(l_1+m-k_1)(m-k_2)}(z_1)\overline{z_1}^{l_1},\ z_1 \in D_1, \\ \partial^{\mu}_{\bar{z}_1}\partial^{\nu}_{\bar{z}_2}\phi(0, z_2) = \sum\limits_{l_2=0}^{k_2-1} \frac{1}{l_2!} d_{(m-k_1)(l_2+m-k_2)}(z_2)\overline{z_2}^{l_2},\ z_2 \in D_2, \\ \partial^{\mu}_{\bar{z}_1}\partial^{\nu}_{\bar{z}_2}\phi(0, 0) = c_{(m-k_1)(m-k_2)} \end{cases} \tag{4.7}$$

for $1 \le \mu, \nu, k_1, k_2 \le m-1$, has a unique solution if and only if (4.3) and (4.4) are satisfied. The solution is given by (3.1), where u_0 is determined by (3.2), and

$$\widetilde{\phi}(z) = \sum_{l_1, l_2=1}^{m-1} \frac{u_{l_1 l_2}(z)}{l_1! l_2!}\overline{z_1}^{l_1}\overline{z_2}^{l_2}, \tag{4.8}$$

in which $u_{l_1 l_2}(z)$ is determined by (4.1).

Proof. (4.8) follows that

$$\begin{cases} \partial\nu_1\partial\nu_2(\partial^{\mu}_{\bar{z}_1}\partial^{\nu}_{\bar{z}_2}\phi(z)) = g_{\mu\nu}(z),\ z \in \partial_0 D^2, \\ \partial^{\mu}_{\bar{z}_1}\partial^{\nu}_{\bar{z}_2}\phi(z_1, 0) = \sum\limits_{l_1=0}^{k_1-1} \frac{1}{l_1!} b_{(l_1+m-k_1)(m-k_2)}(z_1)\overline{z_1}^{l_1},\ z_1 \in D_1, \\ \partial^{\mu}_{\bar{z}_1}\partial^{\nu}_{\bar{z}_2}\phi(0, z_2) = \sum\limits_{l_2=0}^{k_2-1} \frac{1}{l_2!} d_{(m-k_1)(l_2+m-k_2)}(z_2)\overline{z_2}^{l_2},\ z_2 \in D_2, \\ \partial^{\mu}_{\bar{z}_1}\partial^{\nu}_{\bar{z}_2}\phi(0, 0) = c_{(m-k_1)(m-k_2)} \end{cases}$$

is equivalent to (4.2). By Theorem 4.1 and similar to the proof of Theorem 3.1, we obtain the desired conclusion.

Theorem 4.3. *Let $g \in L_1(D^2)$ and $\varphi_1, \varphi_2 \in C(\partial D^2)$. Then the problem*

$$\begin{cases} \partial_{\bar{z}_1}\partial_{\bar{z}_2} f(z) = g(z), f(z_1, 0) = \alpha_1(z_1), f(0, z_2) = \alpha_2(z_2),\ z_1 \in D_1,\ z_2 \in D_2, \\ \partial_{\nu_1} f(z) = \varphi_1(z), \partial_{\nu_2} f(z) = \varphi_2(z),\ z \in \partial_0 D^2 \end{cases} \tag{4.9}$$

with the compatibility condition

$$\alpha_1(0) = \alpha_2(0) = \alpha,\ \partial_{\nu_1}\varphi_2(z) = \partial_{\nu_2}\varphi_1(z) = \varphi(z)$$

has a unique solution

$$
\begin{aligned}
f(z) = {} & \alpha_1(z_1) + \alpha_2(z_2) - \alpha - \frac{1}{2\pi i}\int_{\partial D_2}[\varphi_2(z_1,\zeta_2) - \partial_{\nu_2}\alpha_2(\zeta_2)]\log(1 - z_2\overline{\zeta_2})\frac{d\zeta_2}{\zeta_2}\\
& + \frac{1}{2\pi i}\int_{\partial D_2}\Big\{\frac{1}{2\pi i}\int_{\partial D_1}[\zeta_2\varphi(\zeta) - \zeta_2^2\partial_{\zeta_2}\varphi_1(\zeta)]\log(1 - z_1\overline{\zeta_1})\frac{d\zeta_1}{\zeta_1}\\
& + \frac{1}{2\pi i}\int_{\partial D_1} g(\zeta)\log(1 - z_1\overline{\zeta_1})d\overline{\zeta_1} + \frac{1}{\pi}\int_{D_1} g(\zeta)\frac{z_1}{\zeta_1(\zeta_1 - z_1)}d\sigma_{\zeta_1}\Big\}\log(1 - z_2\overline{\zeta_2})d\overline{\zeta_2}\\
& + \frac{1}{\pi}\int_{D_2}\Big\{\frac{1}{2\pi i}\int_{\partial D_1}[\zeta_2\varphi(\zeta) - \zeta_2^2\partial_{\zeta_2}\varphi_1(\zeta)]\log(1 - z_1\overline{\zeta_1})\frac{d\zeta_1}{\zeta_1}\\
& + \frac{1}{2\pi i}\int_{\partial D_1} g(\zeta)\log(1 - z_1\overline{\zeta_1})d\overline{\zeta_1} + \frac{1}{\pi}\int_{D_1} g(\zeta)\frac{z_1}{\zeta_1(\zeta_1 - z_1)}d\sigma_{\zeta_1}\Big\}\frac{z_2}{\zeta_2(\zeta_2 - z_2)}d\sigma_{\zeta_2},
\end{aligned}
\tag{4.10}
$$

if and only if

$$
\begin{aligned}
& \frac{1}{2\pi i}\int_{\partial D_1}[z_2\varphi(\zeta_1, z_2) - z_2^2\partial_{z_2}\varphi_1(\zeta_1, z_2)]\frac{d\zeta_1}{(1 - \bar{z}_1\zeta_1)\zeta_1} + \frac{1}{2\pi i}\int_{\partial D_1} g(\zeta_1, z_2)\frac{d\overline{\zeta_1}}{1 - \bar{z}_1\zeta_1}\\
= {} & \frac{-1}{\pi}\int_{D_1} g(\zeta_1, z_2)\frac{\bar{z}_1}{(1 - \bar{z}_1\zeta_1)^2}d\sigma_{\zeta_1}
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
& \frac{1}{2\pi i}\int_{\partial D_2}[\varphi_2(z_1,\zeta_2) - \partial_{\nu_2}\alpha_2(\zeta_2)]\frac{d\zeta_2}{(1 - \bar{z}_2\zeta_2)\zeta_2}\\
= {} & \frac{1}{2\pi i}\int_{\partial D_2}\Big\{\frac{1}{2\pi i}\int_{\partial D_1}[\zeta_2\varphi(\zeta) - \zeta_2^2\partial_{\zeta_2}\varphi_1(\zeta)]\log(1 - z_1\overline{\zeta_1})\frac{d\zeta_1}{\zeta_1}\\
& + \frac{1}{2\pi i}\int_{\partial D_1} g(\zeta)\log(1 - z_1\overline{\zeta_1})d\overline{\zeta_1} + \frac{1}{\pi}\int_{D_1} g(\zeta)\frac{z_1}{\zeta_1(\zeta_1 - z_1)}d\sigma_{\zeta_1}\Big\}\frac{d\overline{\zeta_2}}{1 - \bar{z}_2\zeta_2}\\
& + \frac{1}{\pi}\int_{D_2}\Big\{\frac{1}{2\pi i}\int_{\partial D_1}[\zeta_2\varphi(\zeta) - \zeta_2^2\partial_{\zeta_2}\varphi_1(\zeta)]\log(1 - z_1\overline{\zeta_1})\frac{d\zeta_1}{\zeta_1}\\
& + \frac{1}{2\pi i}\int_{\partial D_1} g(\zeta)\log(1 - z_1\overline{\zeta_1})d\overline{\zeta_1} + \frac{1}{\pi}\int_{D_1} g(\zeta)\frac{z_1}{\zeta_1(\zeta_1 - z_1)}d\sigma_{\zeta_1}\Big\}\frac{\bar{z}_2}{(1 - \bar{z}_2\zeta_2)^2}d\sigma_{\zeta_2}.
\end{aligned}
\tag{4.12}
$$

Proof. (4.9) follows $\partial_{\nu_1}(\partial_{\nu_2} f(z)) = \varphi(z)$, that is

$$\partial_{\nu_1}[z_2\partial_{z_2} f(z) + \bar{z}_2\partial_{\bar{z}_2} f(z)] = \varphi(z),$$

which is equivalent to

$$\partial_{\nu_1}[\partial_{\bar{z}_2} f(z)] = z_2\varphi(z) - z_2^2\partial_{z_2}\partial_{\nu_1} f(z) = z_2\varphi(z) - z_2^2\partial_{z_2}\varphi_1(z).$$

Applying Lemma 2.6 on the problem

$$\partial_{\bar{z}_1}[\partial_{\bar{z}_2} f(z)] = g(z),\ \partial_{\nu_1}[\partial_{\bar{z}_2} f(z)] = z_2\varphi(z) - z_2^2\partial_{z_2}\varphi_1(z),\ \partial_{\bar{z}_2} f(0, z_2) = \partial_{\bar{z}_2}\alpha_2(z_2),$$

we get that the unique solution $\partial_{\bar{z}_2} f(z)$ is determined by

$$\begin{aligned}\partial_{\bar{z}_2} f(z) = \partial_{\bar{z}_2}\alpha_2(z_2) - \frac{1}{2\pi i}\int_{\partial D_1}[z_2\varphi(\zeta_1, z_2) - z_2^2\partial_{z_2}\varphi_1(\zeta_1, z_2)]\log(1 - z_1\overline{\zeta_1})\frac{d\zeta_1}{\zeta_1} \\ - \frac{1}{2\pi i}\int_{\partial D_1} g(\zeta_1, z_2)\log(1 - z_1\overline{\zeta_1})d\overline{\zeta_1} - \frac{1}{\pi}\int_{D_1} g(\zeta_1, z_2)\frac{z_1}{\zeta_1(\zeta_1 - z_1)}d\sigma_{\zeta_1}\end{aligned} \tag{4.13}$$

if and only if (4.11) is satisfied.

In addition, (4.13) is equivalent to

$$\begin{aligned}\partial_{\bar{z}_2}[f(z) - \alpha_2(z_2)] = -\frac{1}{2\pi i}\int_{\partial D_1}[z_2\varphi(\zeta_1, z_2) - z_2^2\partial_{z_2}\varphi_1(\zeta_1, z_2)]\log(1 - z_1\overline{\zeta_1})\frac{d\zeta_1}{\zeta_1} \\ - \frac{1}{2\pi i}\int_{\partial D_1} g(\zeta_1, z_2)\log(1 - z_1\overline{\zeta_1})d\overline{\zeta_1} - \frac{1}{\pi}\int_{D_1} g(\zeta_1, z_2)\frac{z_1 d\sigma_{\zeta_1}}{\zeta_1(\zeta_1 - z_1)}.\end{aligned} \tag{4.14}$$

In view of

$$\partial_{\nu_2}[f(z) - \alpha_2(z_2)] = \varphi_2(z) - \partial_{\nu_2}\alpha_2(z_2), \quad f(z_1, 0) - \alpha_2(0) = \alpha_1(z_1) - \alpha, \tag{4.15}$$

applying Lemma 2.6, we obtain the unique solution of the problem (4.14) with the conditions (4.15) is (4.10) if and only if (4.12) is satisfied. So we get the desired conclusion.

Theorem 4.4. *Let $\varphi, g_{\mu\nu} \in C(\partial_0 D^2)$ and $c_{\mu\nu} \in \mathbb{C}$. Let $b_{\mu\nu}(z_1)$ and $d_{\mu\nu}(z_2)$ be analytic functions about z_1 and z_2, respectively, with $b_{\mu\nu}(0) = d_{\mu\nu}(0) = c_{\mu\nu}$ ($1 \le \mu, \nu \le m-1$, $m \ge 2$). Let $\lambda \in \mathbb{R} \setminus \{-1, 0, 1\}$. Then the problem*

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2} f(z) = \frac{\lambda - 1}{4\lambda}\phi(z) + \frac{\lambda + 1}{4\lambda}\bar{\phi}(z), \; \partial_{\bar{z}_1}^m\partial_{\bar{z}_2}^m\phi(z) = 0, \; z \in D^2, \tag{4.16}$$

under the conditions

$$\begin{cases} \partial_{\nu_1} f(z) = \varphi_1(z), \;\; \partial_{\nu_2} f(z) = \varphi_2(z), \;\; \partial\nu_1\partial\nu_2(\partial_{\bar{z}_1}^{\mu}\partial_{\bar{z}_2}^{\nu}\phi(z)) = g_{\mu\nu}(z), \;\; z \in \partial_0 D^2, \\ f(z_1, 0) = \alpha_1(z_1), \;\; \partial_{\bar{z}_1}^{\mu}\partial_{\bar{z}_2}^{\nu}\phi(z_1, 0) = \sum\limits_{l_1=0}^{k_1-1}\frac{1}{l_1!}b_{(l_1+m-k_1)(m-k_2)}(z_1)\overline{z_1}^{l_1}, \;\; z_1 \in D_1, \\ f(0, z_2) = \alpha_2(z_2), \;\; \partial_{\bar{z}_1}^{\mu}\partial_{\bar{z}_2}^{\nu}\phi(0, z_2) = \sum\limits_{l_2=0}^{k_2-1}\frac{1}{l_2!}d_{(m-k_1)(l_2+m-k_2)}(z_2)\overline{z_2}^{l_2}, \;\; z_2 \in D_2, \\ \partial_{\bar{z}_1}^{\mu}\partial_{\bar{z}_2}^{\nu}\phi(0, 0) = c_{(m-k_1)(m-k_2)} \end{cases} \tag{4.17}$$

with $1 \le \mu, \nu, k_1, k_2 \le m-1$ and

$$\alpha_1(0) = \alpha_2(0) = \alpha, \quad \partial_{\nu_1}\varphi_2(z) = \partial_{\nu_2}\varphi_1(z) = \varphi(z)$$

has a unique solution if and only if (4.3), (4.4) and

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\partial D_1}[z_2\varphi(\zeta_1, z_2) - z_2^2\partial_{z_2}\varphi_1(\zeta_1, z_2)]\frac{d\zeta_1}{(1 - \bar{z}_1\zeta_1)\zeta_1} \\ &= \frac{1}{\pi}\int_{D_1}\Big[\frac{\lambda - 1}{4\lambda}\phi_{\zeta_1}(\zeta_1, z_2) + \frac{\lambda + 1}{4\lambda}\overline{\phi_{\overline{\zeta_1}}(\zeta_1, z_2)}\Big]\frac{d\sigma_{\zeta_1}}{1 - \bar{z}_1\zeta_1},\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial D_2}[\varphi_2(z_1,\zeta_2)-\partial_{\nu_2}\alpha_2(\zeta_2)]\frac{d\zeta_2}{(1-\bar{z}_2\zeta_2)\zeta_2}\\
=&\frac{1}{2\pi^2 i}\int_{\partial D_1}\int_{D_2}\log(1-z_1\overline{\zeta_1})[\varphi(\zeta)+\zeta_2\varphi_{\zeta_2}(\zeta)-2\zeta_2\partial_{\zeta_2}\varphi_1(\zeta)-\zeta_2^2\partial_{\zeta_2}^2\varphi_1(\zeta)]\frac{d\sigma_{\zeta_2}}{\bar{z}_2\zeta_2-1}\frac{d\zeta_1}{\zeta_1}\\
&+\frac{1}{2\pi^2 i}\int_{\partial D_1}\int_{D_2}\log(1-z_1\overline{\zeta_1})\Big[\frac{\lambda-1}{4\lambda}\phi_{\zeta_2}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\phi_{\overline{\zeta_2}}(\zeta)}\Big]\frac{d\sigma_{\zeta_2}}{\bar{z}_2\zeta_2-1}d\overline{\zeta_1}\\
&+\frac{1}{\pi^2}\int_{D_1}\int_{D_2}\Big[\frac{\lambda-1}{4\lambda}\phi_{\zeta_2}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\phi_{\overline{\zeta_2}}(\zeta)}\Big]\frac{d\sigma_{\zeta_2}}{\bar{z}_2\zeta_2-1}\frac{z_1 d\sigma_{\zeta_1}}{\zeta_1(\zeta_1-z_1)}
\end{aligned}\tag{4.19}$$

are satisfied. The solution is given by (4.10) where $g(\zeta)$ is replaced by $\frac{\lambda-1}{4\lambda}\phi(\zeta)+\frac{\lambda+1}{4\lambda}\bar{\phi}(\zeta)$, and

$$\phi(z)=\sum_{l_1,l_2=1}^{m-1}\frac{u_{l_1l_2}(z)}{l_1!l_2!}\overline{z_1}^{l_1}\overline{z_2}^{l_2},$$

in which $u_{l_1l_2}(z)$ is determined by (4.1).

Proof. Applying Theorems 4.1 and 4.3, the problem (4.16) with conditions (4.17) has a unique solution (4.10) if and only if (4.3), (4.4), (4.11), and (4.12) are satisfied, where $g(\zeta)$ is replaced by $\frac{\lambda-1}{4\lambda}\phi(\zeta)+\frac{\lambda+1}{4\lambda}\bar{\phi}(\zeta)$. To obtain the specific representations of the solvable conditions, we need the following equations.

By the Gauss formula, we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial D_2}\frac{[\zeta_2\varphi(\zeta)-\zeta_2^2\partial_{\zeta_2}\varphi_1(\zeta)]}{1-\bar{z}_2\zeta_2}d\overline{\zeta_2}=\frac{-1}{\pi}\int_{D_2}\partial_{\zeta_2}\frac{[\zeta_2\varphi(\zeta)-\zeta_2^2\partial_{\zeta_2}\varphi_1(\zeta)]}{1-\bar{z}_2\zeta_2}d\sigma_{\zeta_2}\\
=&\frac{-1}{\pi}\int_{D_2}\Big\{[\zeta_2\varphi(\zeta)-\zeta_2^2\partial_{\zeta_2}\varphi_1(\zeta)]\frac{\bar{z}_2}{(1-\bar{z}_2\zeta_2)^2}\\
&+\frac{1}{1-\bar{z}_2\zeta_2}[\varphi(\zeta)+\zeta_2\varphi_{\zeta_2}(\zeta)-2\zeta_2\partial_{\zeta_2}\varphi_1(\zeta)-\zeta_2^2\partial_{\zeta_2}^2\varphi_1(\zeta)]\Big\}d\sigma_{\zeta_2},
\end{aligned}$$

which follows

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial D_2}\frac{[\zeta_2\varphi(\zeta)-\zeta_2^2\partial_{\zeta_2}\varphi_1(\zeta)]}{1-\bar{z}_2\zeta_2}d\overline{\zeta_2}+\frac{1}{\pi}\int_{D_2}[\zeta_2\varphi(\zeta)-\zeta_2^2\partial_{\zeta_2}\varphi_1(\zeta)]\frac{\bar{z}_2}{(1-\bar{z}_2\zeta_2)^2}d\sigma_{\zeta_2}\\
=&\frac{1}{\pi}\int_{D_2}\frac{1}{\bar{z}_2\zeta_2-1}[\varphi(\zeta)+\zeta_2\varphi_{\zeta_2}(\zeta)-2\zeta_2\partial_{\zeta_2}\varphi_1(\zeta)-\zeta_2^2\partial_{\zeta_2}^2\varphi_1(\zeta)]d\sigma_{\zeta_2}.
\end{aligned}\tag{4.20}$$

In addition,

$$\frac{1}{2\pi i}\int_{\partial D_2}\frac{\phi(\zeta)}{1-\bar{z}_2\zeta_2}d\overline{\zeta_2}=\frac{-1}{\pi}\int_{D_2}\Big[\phi(\zeta)\frac{\bar{z}_2}{(1-\bar{z}_2\zeta_2)^2}+\frac{\phi_{\zeta_2}(\zeta)}{1-\bar{z}_2\zeta_2}\Big]d\sigma_{\zeta_2}$$

follows

$$\frac{1}{2\pi i}\int_{\partial D_2}\frac{\phi(\zeta)}{1-\bar{z}_2\zeta_2}d\overline{\zeta_2}+\frac{1}{\pi}\int_{D_2}\Big[\phi(\zeta)\frac{\bar{z}_2}{(1-\bar{z}_2\zeta_2)^2}d\sigma_{\zeta_2}=\frac{1}{\pi}\int_{D_2}\frac{\phi_{\zeta_2}(\zeta)}{\bar{z}_2\zeta_2-1}d\sigma_{\zeta_2}.\tag{4.21}$$

Similarly, we obtain

$$\frac{1}{2\pi i}\int_{\partial D_2}\frac{\overline{\phi(\zeta)}}{1-\bar{z}_2\zeta_2}d\overline{\zeta_2}+\frac{1}{\pi}\int_{D_2}\Big[\overline{\phi(\zeta)}\frac{\bar{z}_2}{(1-\bar{z}_2\zeta_2)^2}d\sigma_{\zeta_2}=\frac{1}{\pi}\int_{D_2}\frac{\overline{\phi_{\overline{\zeta_2}}(\zeta)}}{\bar{z}_2\zeta_2-1}d\sigma_{\zeta_2}. \tag{4.22}$$

Using (4.21) and replacing $g(\zeta_1,z_2)$ in (4.11) by $\frac{\lambda-1}{4\lambda}\phi(\zeta_1,z_2)+\frac{\lambda+1}{4\lambda}\bar{\phi}(\zeta_1,z_2)$, we get (4.18). Using (4.20)–(4.22) and replacing $g(\zeta)$ in (4.12) by $\frac{\lambda-1}{4\lambda}\phi(\zeta)+\frac{\lambda+1}{4\lambda}\bar{\phi}(\zeta)$, we get

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial D_2}[\varphi_2(z_1,\zeta_2)-\partial_{\nu_2}\alpha_2(\zeta_2)]\frac{d\zeta_2}{(1-\bar{z}_2\zeta_2)\zeta_2}\\
=&\frac{1}{2\pi i}\int_{\partial D_2}\Big\{\frac{1}{2\pi i}\int_{\partial D_1}[\zeta_2\varphi(\zeta)-\zeta_2^2\partial_{\zeta_2}\varphi_1(\zeta)]\log(1-z_1\overline{\zeta_1})\frac{d\zeta_1}{\zeta_1}\\
&+\frac{1}{2\pi i}\int_{\partial D_1}g(\zeta)\log(1-z_1\overline{\zeta_1})d\overline{\zeta_1}+\frac{1}{\pi}\int_{D_1}g(\zeta)\frac{z_1}{\zeta_1(\zeta_1-z_1)}d\sigma_{\zeta_1}\Big\}\frac{d\overline{\zeta_2}}{1-\bar{z}_2\zeta_2}\\
&+\frac{1}{\pi}\int_{D_2}\Big\{\frac{1}{2\pi i}\int_{\partial D_1}[\zeta_2\varphi(\zeta)-\zeta_2^2\partial_{\zeta_2}\varphi_1(\zeta)]\log(1-z_1\overline{\zeta_1})\frac{d\zeta_1}{\zeta_1}\\
&+\frac{1}{2\pi i}\int_{\partial D_1}g(\zeta)\log(1-z_1\overline{\zeta_1})d\overline{\zeta_1}+\frac{1}{\pi}\int_{D_1}g(\zeta)\frac{z_1}{\zeta_1(\zeta_1-z_1)}d\sigma_{\zeta_1}\Big\}\frac{\bar{z}_2}{(1-\bar{z}_2\zeta_2)^2}d\sigma_{\zeta_2}\\
=&\frac{1}{2\pi i}\int_{\partial D_1}\log(1-z_1\overline{\zeta_1})\Big\{\frac{1}{2\pi i}\int_{\partial D_2}[\zeta_2\varphi(\zeta)-\zeta_2^2\partial_{\zeta_2}\varphi_1(\zeta)]\frac{d\overline{\zeta_2}}{1-\bar{z}_2\zeta_2}\\
&+\frac{1}{\pi}\int_{D_2}[\zeta_2\varphi(\zeta)-\zeta_2^2\partial_{\zeta_2}\varphi_1(\zeta)]\frac{\bar{z}_2d\sigma_{\zeta_2}}{(1-\bar{z}_2\zeta_2)^2}\Big\}\frac{d\zeta_1}{\zeta_1}\\
&+\frac{1}{2\pi i}\int_{\partial D_1}\log(1-z_1\overline{\zeta_1})\Big\{\frac{1}{2\pi i}\int_{\partial D_2}g(\zeta)\frac{d\overline{\zeta_2}}{1-\bar{z}_2\zeta_2}+\frac{1}{\pi}\int_{D_2}g(\zeta)\frac{\bar{z}_2d\sigma_{\zeta_2}}{(1-\bar{z}_2\zeta_2)^2}\Big\}d\overline{\zeta_1}\\
&+\frac{1}{\pi}\int_{D_1}\Big\{\frac{1}{2\pi i}\int_{\partial D_2}g(\zeta)\frac{d\overline{\zeta_2}}{1-\bar{z}_2\zeta_2}+\frac{1}{\pi}\int_{D_2}g(\zeta)\frac{\bar{z}_2d\sigma_{\zeta_2}}{(1-\bar{z}_2\zeta_2)^2}\Big\}\frac{z_1d\sigma_{\zeta_1}}{\zeta_1(\zeta_1-z_1)}\\
=&\frac{1}{2\pi i}\int_{\partial D_1}\log(1-z_1\overline{\zeta_1})\Big\{\frac{1}{\pi}\int_{D_2}[\varphi(\zeta)+\zeta_2\varphi_{\zeta_2}(\zeta)-2\zeta_2\partial_{\zeta_2}\varphi_1(\zeta)-\zeta_2^2\partial_{\zeta_2}^2\varphi_1(\zeta)]\frac{d\sigma_{\zeta_2}}{\bar{z}_2\zeta_2-1}\Big\}\frac{d\zeta_1}{\zeta_1}\\
&+\frac{1}{2\pi i}\int_{\partial D_1}\log(1-z_1\overline{\zeta_1})\Big\{\frac{1}{\pi}\int_{D_2}\Big[\frac{\lambda-1}{4\lambda}\phi_{\zeta_2}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\phi_{\overline{\zeta_2}}(\zeta)}\Big]\frac{d\sigma_{\zeta_2}}{\bar{z}_2\zeta_2-1}\Big\}d\overline{\zeta_1}\\
&+\frac{1}{\pi}\int_{D_1}\Big\{\frac{1}{\pi}\int_{D_2}\Big[\frac{\lambda-1}{4\lambda}\phi_{\zeta_2}(\zeta)+\frac{\lambda+1}{4\lambda}\overline{\phi_{\overline{\zeta_2}}(\zeta)}\Big]\frac{d\sigma_{\zeta_2}}{\bar{z}_2\zeta_2-1}\Big\}\frac{z_1d\sigma_{\zeta_1}}{\zeta_1(\zeta_1-z_1)},
\end{aligned}$$

which leads to (4.19).

Remark 4.1. *Dirichlet problems and Neumann problems are two typical types of boundary value problems. The conclusions obtained in this paper have enriched the research on boundary value problems for bi-polyanalytic functions. With the methods used in this paper, we can discuss other complex partial differential equation problems for bi-polyanalytic functions. For example, it would be interesting to discuss more complex mixed boundary value problems for bi-polyanalytic functions that simultaneously satisfy multiple boundary conditions, such as Schwarz boundary conditions, Riemann-Hilbert boundary conditions, Neumann boundary conditions and other boundary conditions. However,* *the corresponding boundary value problems of polyanalytic functions need to be investigated first. Besides that, with the methods used in this paper, we can also solve some complex partial differential equation problems (homogeneous or non-homogeneous) in higher-dimensional complex spaces.*

5. Conclusions

We first discuss a kind of boundary value problem for polyanalytic functions with Schwarz conditions on the bicylinder. On this basis, with the help of the properties of the singular integral operators as well as the Cauchy-Pompeiu formula on the unit disc, we investigate a type of boundary value problem with Dirichlet boundary conditions and a type of mixed boundary value problems of higher order for bi-polyanalytic functions on the bicylinder and obtain the specific representations of the solutions. In addition, we discuss a system of complex partial differential equations with respect to polyanalytic functions with Neumann boundary conditions. On this foundation, we obtain the solutions to Neumann boundary value problems for bi-polyanalytic functions on the bicylinder. The conclusions in this paper provide effective methods for discussing other boundary value problems of inhomogeneous complex partial differential equations of higher order in spaces of several complex variables.

Author Contributions

Yanyan Cui: Conceptualization, Project administration, Writing original draft, Writing–review and editing; Chaojun Wang: Investigation, Writing original draft, Writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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