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Research article

Singular expansion of the wave kernel and harmonic sums on Riemannian symmetric spaces of the non-compact type

Ali Hassani*

Department of Mathematics, Al-Leith University College, Umm Al-Qura University, Mecca 24231, Saudi Arabia

* **Correspondence:** Email: abhassani@uqu.edu.sa.

Abstract: The Mellin transform assigned to the convolution Poisson kernel on higher rank Riemannian symmetric spaces of the non-compact type is equal to the wave kernel. This makes it possible to determine the poles and to deduce the singular expansion of this kernel by using the zeta function techniques on compact and non-compact manifolds. As a consequence, we studied the harmonic sums associated with the wave kernel. In particular, we derived its asymptotic expansion near 0 according to the Mellin-converse correspondence rule.

Keywords: Riemannian symmetric spaces; Poisson kernel; wave kernel; Mellin transform; singular expansion; asymptotic expansion; harmonic sums; zeta function
Mathematics Subject Classification: 53C35, 53Z05, 22E30, 43A85

1. Introduction

In this paper, we pursue the study of the Mellin transform associated with the main evolution equations on general Riemannian d-dimensional non-compact symmetric spaces $\mathbb{X} = G/K$ of higher real rank initiated in [1] with the wave equation, where G denotes a semisimple Lie group, connected, non-compact, with a finite center, and K is a maximal compact subgroup of G. Here the version of the Mellin transform is defined as a function on the complex plane by the relation

$$f^*(\sigma) = \frac{1}{\Gamma(\sigma)} \int_0^\infty f(s) s^{\sigma-1} ds, \tag{1.1}$$

for some function f defined on the positive real axis, and Γ denotes the usual gamma function.
Generally, for some absolutely integrable function f on a finite interval $(s_1, s_2), 0 < s_1 < s_2 < \infty$,

which satisfies the following bounds

$$\underbrace{f(s) = O(s^{-c_1})}_{s\to 0}, \qquad \underbrace{f(s) = O(s^{-c_2})}_{s\to\infty}, \tag{1.2}$$

where c_1 and c_2 are positive constants such that $c_1 < c_2$, the Mellin transform f^* exists and it is an analytic function on the vertical strip $c_1 < \mathrm{Re}(\sigma) < c_2$, called a fundamental strip. In some cases, this strip may extend to a half-plane ($c_1 = -\infty$ or $c_2 = +\infty$) or to the whole complex σ-plane ($c_1 = -\infty$ and $c_2 = +\infty$).

The Mellin transform is closely connected with the Fourier and Laplace transforms. A direct way to invert Mellin's transformation (1.1) is to start from Fourier's inversion theorem so that the original function $f(s)$ under some conditions can be recovered from its Mellin transform by the inverse Mellin formula

$$f(s) = \int_{a-i\infty}^{a+i\infty} \Gamma(\sigma) f^*(\sigma) s^{-\sigma} d\sigma, \tag{1.3}$$

where the integration is along a vertical line through $c_1 < \mathrm{Re}(\sigma) = a < c_2$.
As always, several questions arise around the convergence and analyticity of the integral (1.1) outside the fundamental strip $c_1 < \mathrm{Re}(\sigma) < c_2$. Classically, assume that $f(s)$ admits as $s \to 0$ a finite asymptotic expansion of the form

$$f(s) = \sum_{q\in I,\ I \text{ finite}} \alpha_q s^q + O(s^{c_3}), \quad -c_3 < -q \le c_1. \tag{1.4}$$

Then for $c_1 < \mathrm{Re}(\sigma) < c_2$, the integral (1.1) can be decomposed as follows:

$$\begin{aligned} f^*(\sigma) &= \frac{1}{\Gamma(\sigma)} \int_0^1 f(s) s^{\sigma-1} ds + \frac{1}{\Gamma(\sigma)} \int_1^\infty f(s) s^{\sigma-1} ds \\ &= \frac{1}{\Gamma(\sigma)} \int_0^1 (f(s) - \sum_{q\in I} \alpha_q s^q) s^{\sigma-1} ds + \underbrace{\frac{1}{\Gamma(\sigma)} \int_0^1 \sum_{q\in I} \alpha_q s^{q+\sigma-1} ds}_{=\frac{1}{\Gamma(\sigma)} \sum_{q\in I} \frac{\alpha_q}{\sigma+q}} + \frac{1}{\Gamma(\sigma)} \int_1^\infty f(s) s^{\sigma-1} ds. \end{aligned}$$

The first integral on the right defines an analytic function on the strip $(-c_3, \infty)$, and the second for all $\sigma \in \mathbb{C}$, so that $f^*(\sigma)$ is continuable to a meromorphic function on the strip $(-c_3, c_2)$ with simple poles of residue $\frac{\alpha_q}{\Gamma(q)}$ at $\sigma = -q$, and no other singularities. The grouping of the poles appearing in the middle sum is called the *singular expansion* of $f^*(\sigma)$ in the strip $(-c_3, c_2)$ and is written

$$f^*(\sigma) \asymp \frac{1}{\Gamma(\sigma)} \sum_{q\in I} \frac{\alpha_q}{\sigma+q}. \tag{1.5}$$

In summary, there is a remarkable correspondence of coefficients in asymptotic expansion (1.4) of the original function $f(s)$ and in the singular expansion of its Mellin transform $f^*(\sigma)$ expressed by the rule

$$s^q \mapsto \frac{1}{\Gamma(\sigma)} \frac{1}{\sigma+q}. \tag{1.6}$$

More generally, even if the asymptotic expansion of $f(s)$ as $s \to 0$ contains terms of the form $s^q(\ln s)^k$, where the k are nonnegative integers, it is possible to obtain poles of order greater than 1 at $\sigma = -q$, since in the strip $(-c_3, c_2)$

$$(\frac{\partial}{\partial\sigma})^k \int_0^1 s^{q+\sigma-1} ds = \frac{(-1)^k k!}{(\sigma+q)^{k+1}}. \tag{1.7}$$

It should be noted that the reverse way is still valid. In other words, the singularities of the Mellin transform $f^*(\sigma)$ can encode under certain conditions an asymptotic expansion of the original function $f(s)$ as $s \to 0$. Namely, if f is a continuous function on $(0, \infty)$ with Mellin transform $f^*(\sigma)$ having a meromorphic continuation to the extended strip (c_4, c_2) with a finite number of poles and a sufficiently fast decrease (say $f^*(\sigma) = O(|\sigma|^{-r}),\ \ r > 1$ as $|\sigma| \to \infty$) such that the singular expansion of $f^*(\sigma)$ on (c_4, c_1) is written

$$f^*(\sigma) \asymp \sum_{q,k} \frac{c_{q,k}}{(\sigma-q)^k}, \tag{1.8}$$

then an asymptotic expansion of $f(s)$ as $s \to 0$ takes the form

$$f(s) = \sum_{q,k} c_{q,k} \frac{(-1)^{k-1}}{(k-1)!} s^{-q} (\ln s)^k + O(s^{-c_4}). \tag{1.9}$$

So, as in the direct way, the following converse correspondence rule holds:

$$\frac{A}{(\sigma-q)^{k+1}} \mapsto \frac{(-1)^k}{k!} s^{-q} (\ln s)^k. \tag{1.10}$$

More elements of the theory of the Mellin transform can be found, for example, in [2–4].

By in [1, formula (1.13)], the author asserts that the zeta function $\zeta_{G/K}$ on the symmetric space $\mathbb{X} = G/K$ appears as a certain limit of the Mellin transform of the Poisson kernel on this space. However, in the general case, it is well known that

$$\begin{aligned} \zeta(\sigma) &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \frac{1}{e^s - 1} s^{\sigma-1} ds, \ \ \mathrm{Re}(\sigma) > 1 \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty (\sum_{k\geq 1} e^{-ks}) s^{\sigma-1} ds \\ &= G^*_\zeta(\sigma), \end{aligned} \tag{1.11}$$

where $G_\zeta(s) = \sum_{k\geq 1} g(ks), \quad g(s) = e^{-s}, \ \ s > 0.$

So, the singular behavior of the zeta function (which is well known) comes from the asymptotic development of the original function $g(s) = e^{-s},\ \ s \to 0$, or more precisely of the function $G_\zeta(s) = \sum_{k\geq 1} g(ks)$, and the reverse way is still true as already indicated. Note that $G_\zeta(s)$ presents a special case of sums of the form

$$G(s) = \sum_{k\geq 1} \lambda_k g(\mu_k s), \ \ s > 0 \tag{1.12}$$

called harmonic sums, which are highly useful in the theory of Dirichlet series [5]. The λ_k and μ_k are being interpreted as amplitudes and frequencies, respectively, and $g(s)$ is called the base function. Those expressions arise in applications of combinatorial theory, especially in the evaluation of algorithms where the problem is to find the behavior of $G(s)$ when s tends to 0 or infinity. Motivated by these facts, we propose to study the asymptotic of $G(s)$, in the case where $g(s) = e^{-bs}p(s - it, x)$, where $b > 0$ is a real parameter and $\big(p(s - it, x)\big)_{s>0,\ t>0,\ x\in\mathbb{X}}$ denotes the Poisson kernel on Riemannian symmetric space $\mathbb{X} = G/K$ and whose Mellin transform equals the wave kernel noted $w(t, x, \sigma, b)$, $\sigma \in \mathbb{C}$.

Let Δ be the Laplace-Beltrami operator on $\mathbb{X} = G/K$. Using the inverse spherical Fourier transform, we consider the wave convolution kernel associated with Δ as a bi-K-invariant kernel on G expressed as follows:

$$w(t, x, \sigma, b) = \frac{1}{|W|} \int_{\mathfrak{a}^\star} e^{it\sqrt{|\lambda|^2+|\rho|^2}} (b + \sqrt{|\lambda|^2 + |\rho|^2})^{-\sigma} \varphi_\lambda(x) |c(\lambda)|^{-2} d\lambda, \tag{1.13}$$

for $t > 0$, $x \in G$ and $\sigma \in \mathbb{C}$, such that $\mathrm{Re}(\sigma) > d = \dim(\mathbb{X})$, and a real parameter $b > 0$, where $\mathfrak{a}^\star$ denotes a vector dual of a maximal abelian subspace $\mathfrak{a}$ in the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of the group G. Here, φ_λ denotes the spherical function on G of index $\lambda \in \mathfrak{a}^\star$ and $c(\lambda)$ is the Harish-Chandra c-function. $|W|$ denotes the cardinality of the Weyl group W associated with the root system Σ, and ρ is the half sum of positive roots counted with their multiplicities relative to $\mathfrak{a}$.

The kernel $w(t, x, \sigma, b)$ satisfies the wave equation $\frac{\partial^2}{\partial t^2} w(t, x, \sigma, b) = \Delta w(t, x, \sigma, b)$, and its spherical Fourier transform is given by

$$\widehat{w}(t, \lambda, \sigma, b) = e^{it\sqrt{|\lambda|^2+|\rho|^2}} (b + \sqrt{|\lambda|^2 + |\rho|^2})^{-\sigma}, \quad \lambda \in \mathfrak{a}^\star,$$

and it represents the convolution kernel of the operator $\mathcal{W}(t, \sigma, b) = e^{it\sqrt{-\Delta}}(b + \sqrt{-\Delta})^{-\sigma}$ on the symmetric space $\mathbb{X}$:

$$\mathcal{W}(t, \sigma, b) f(x) = f \star w(t, x, \sigma, b) = \int_G w(t, y^{-1}x, \sigma, b) f(y) dy.$$

According to the formula

$$a^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-ua} u^{\sigma-1} du \quad \forall a > 0, \tag{1.14}$$

we easily write the integral expression (1.13) differently:

$$\begin{aligned} w(t, x, \sigma, b) &= \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-bs} \underbrace{\left(\frac{1}{|W|} \int_{\mathfrak{a}^\star} e^{-(s-it)\sqrt{|\lambda|^2+|\rho|^2}} \varphi_\lambda(x) |c(\lambda)|^{-2} d\lambda \right)}_{p(s-it,x)} s^{\sigma-1} ds \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \underbrace{e^{-bs} p(s - it, x)}_{\widetilde{p}(s-it,x)} s^{\sigma-1} ds \\ &= \widetilde{p}^*(r - it, x)(\sigma), \end{aligned} \tag{1.15}$$

where $p(s - it, x)$ denotes the bi-K-invariant convolution kernel of the Poisson operator $\mathcal{P}_{s-it}$.

We show that the tools employed in the study of zeta functions on compact and non-compact manifolds [6–13] can be generalized to $w(t, x, \sigma, b) = \widetilde{p}^*(r - it, x)(\sigma)$. Using pointwise kernel estimates proved in [14–16], we shall see that $w(t, x, \sigma, b)$ is a well-defined analytic function on $\mathrm{Re}(\sigma) > d$, for every $x \in \mathbb{X} = G/K$. We prove that $w(t, x, \sigma, b)$ extends meromorphically to the entire complex plane

$\mathbb{C}$ with a finite number of simple poles on the real line at the points $(\sigma_j = d - j)_{j=0,\cdots,d-1}$, by using the short-time asymptotic expansion of the Poisson kernel

$$p(s-it,x) \sim c_d s^{-d} \sum_{j\geq 0} \alpha_j(t,x) s^j, \quad s \to 0^+,$$

and residues given by

$$\mathrm{res}(w(t,x,\sigma,b), \sigma_j = d-j) = \frac{c_d}{\Gamma(d-j)} \tilde{\alpha}_j(t,x,b), \;\; \forall\, x \in \mathbb{X} = G/K.$$

By the fundamental correspondence (1.6), we deduce the singular expansion of $w(t,x,\sigma,b)$

$$w(t,x,\sigma,b) \asymp \frac{c_d}{\Gamma(\sigma)} \sum_{j=0}^{d-1} \frac{\tilde{\alpha}_j(t,x,b)}{\sigma+(j-d)}, \;\; \sigma \in \mathbb{C}, \;\; t>0, \;\; x \in \mathbb{X} = G/K,$$

where

$$\tilde{\alpha}_j(t,x,b) = \sum_{l=0}^{j} \frac{(-1)^{j-l} b^{j-l}}{(j-l)!} \alpha_l(t,x), \;\; j \in \mathbb{N}_0.$$

As an application, we study the harmonic sums

$$G(t,x,s,b) = \sum_{k\geq 1} \tilde{p}(ks-it,x) = \sum_{k\geq 1} e^{-bks} p(ks-it,x), \;\; s>0, \;\; x \in \mathbb{X} = G/K.$$

In particular, we prove that the Mellin transformation performs a separation principle between the base function $\tilde{p}(s-it,x)$ and the amplitude $\lambda_k = 1$ and the frequencies $\mu_k = k$. More precisely, we state

$$G^*(t,x,\sigma,b) = \zeta(\sigma) w(t,x,\sigma,b), \;\; \mathrm{Re}(\sigma) > d, \;\; t>0, \;\; x \in \mathbb{X} = G/K,$$

which allows us to establish the singular expansion of the function $\Gamma(\sigma)G^*(t,x,\sigma,b)$

$$\Gamma(\sigma)G^*(t,x,\sigma,b) \asymp \frac{c_d \tilde{\alpha}_{d-1}(t,x,b)}{(\sigma-1)^2} + \frac{a_{-1}}{\sigma-1} + c_d \sum_{j=0}^{d-2} \frac{\zeta(d-j)\tilde{\alpha}_j(t,x,b)}{\sigma-(d-j)},$$

where $a_{-1} = \mathrm{res}(\Gamma(\sigma)G^*(t,x,\sigma,b), \sigma = 1)$. This translates by the converse correspondence rule (1.10) into the asymptotic expansion of $G(t,x,s,b)$ as $s \to 0$

$$G(t,x,s,b) \sim \frac{1}{s}(-c_d \tilde{\alpha}_{d-1}(t,x,b) \ln s + a_{-1}) + c_d \sum_{j=0}^{d-2} \zeta(d-j)\tilde{\alpha}_j(t,x,b) s^{j-d}.$$

This paper is organized as follows: After recalling some basic notations and reviewing little aspects of Spherical-Fourier analysis on non-compact symmetric spaces in Section 2, we prove the extension of $\sigma \mapsto w(t,x,\sigma,b)$ to a meromorphic function on the entire complex plane, and we deduce its singular expansion in Section 3. We devote Section 4 to the study of harmonic sums assigned to the wave kernel. Finally, we give some conclusions and comparative remarks about similar and possible future results in Section 5.

2. Preliminaries

We recall the basic notations of Fourier analysis on Riemannian symmetric spaces of the non-compact type. We refer to [17–19] for geometric properties and more details for harmonic analysis on these spaces.

Throughout this paper, the symbol $A \lesssim B$ between two positive expressions means that there is a positive constant C such that $A \leq CB$.

Let G be a non-compact connected semisimple Lie group with a finite center and K a maximal compact subgroup of G. Let $\mathfrak{g}$ (resp., $\mathfrak{k}$) be the Lie algebra of G (resp., K) and consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$. Here, $\mathfrak{p}$ is the (K-invariant) orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form $\langle .\,,.\rangle$ of $\mathfrak{g}$. This form induces a K-invariant scalar product on $\mathfrak{p}$, and, hence, a G-invariant Riemannian metric on the symmetric homogeneous manifold $\mathbb{X} = G/K$ whose tangent space at the origin eK is naturally identified with $\mathfrak{p}$. Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ and denote by $\mathfrak{a}^\star$ (respectively, $\mathfrak{a}^\star_{\mathbb{C}}$) the real (respectively, complex) vector dual of $\mathfrak{a}$. The Killing form of $\mathfrak{g}$ induces a scalar product on $\mathfrak{a}^\star$ and a $\mathbb{C}$-bilinear form on $\mathfrak{a}^\star_{\mathbb{C}}$. If $\lambda,\ \mu \in \mathfrak{a}^\star_{\mathbb{C}}$, let $H_\lambda \in \mathfrak{a}_{\mathbb{C}}$ be determined by $\lambda(H) = \langle H_\lambda\,, H\rangle,\ \ \forall H \in \mathfrak{a}_{\mathbb{C}}$, and put $\langle \lambda\,,\mu\rangle = \langle H_\lambda\,, H_\mu\rangle$. We put $|\lambda| = \langle \lambda\,,\lambda\rangle^{1/2}$ for $\lambda \in \mathfrak{a}^\star$. Denote by Σ the root system of $\mathfrak{g}$ relative to $\mathfrak{a}$. Let W be the Weyl group associated with Σ, and let m_λ denote the multiplicity of the root $\lambda \in \Sigma$. In particular, if $\mathfrak{a}^+$ denotes the positive Weyl chamber in $\mathfrak{a}$ corresponding to some fixed set Σ_+ of positive roots, we have the Cartan decomposition $G = K\exp(\overline{\mathfrak{a}^+})K$ of G, where $\overline{\mathfrak{a}^+}$ denotes the closure of $\mathfrak{a}^+$. Each element $x \in G$ is written uniquely as $x = k_1A(x)k_2$. We denote by $|x| = |A(x)|$ the norm defined on G. Let $\rho = \frac{1}{2}\sum_{\lambda\in\Sigma_+} m_\lambda\lambda$ and let d be the dimension of $\mathbb{X}$, and D be the dimension at infinity of $\mathbb{X}$. Consider $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, the Iwasawa decomposition of $\mathfrak{g}$, and the corresponding Iwasawa decomposition $G = K\exp(\mathfrak{a})N$ of G, where N is the analytic subgroup of G associated to the nilpotent subalgebra $\mathfrak{n}$. Denote by $H(x)$ the Iwasawa component of $x \in G$ in $\mathfrak{a}$. There is a basic estimate for this component (see [17, p. 476],)

$$|H(x)| \lesssim |x|, \tag{2.1}$$

and it is called Iwasawa projection. Finally, let $\ell = \dim(\mathfrak{a})$, the real dimension of $\mathfrak{a}$. By definition, ℓ is the real rank of G.

We identify functions on $\mathbb{X} = G/K$ with functions on G, which are K-invariant on the right and, hence, bi-K-invariant functions on G with functions on $\mathbb{X}$, K-invariants on the left. If f is a sufficiently regular bi-K-invariant function on G, then its spherical-Fourier transform is a function on $\mathfrak{a}^\star_{\mathbb{C}}$ defined by

$$\widehat{f}(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)dx, \quad \lambda \in \mathfrak{a}^\star_{\mathbb{C}},$$

where φ_λ denotes the spherical function on G defined by

$$\varphi_\lambda(x) = \int_K e^{<i\lambda-\rho, H(xk)>}dk, \quad x \in G, \quad \lambda \in \mathfrak{a}^\star_{\mathbb{C}},$$

and it satisfies the basic estimates

$$|\varphi_\lambda(x)| \leq \varphi_0(x), \quad \forall x \in G, \quad \forall \lambda \in \mathfrak{a}^\star_{\mathbb{C}}.$$

The Plancherel and inversion formulas for the spherical transform are, respectively, given by

$$\int_G |f(x)|^2 dx = \frac{1}{|W|}\int_{\mathfrak{a}^\star} |\widehat{f}(\lambda)|^2 |c(\lambda)|^{-2} d\lambda,$$

$$f(x) = \tfrac{1}{|W|}\int_{\mathfrak{a}^\star} \varphi_\lambda(x)\widehat{f}(\lambda)|c(\lambda)|^{-2} d\lambda, \quad x \in G,$$

where $|W|$ denotes the order of the Weyl group W and $c(\lambda)$ is the Harish-Chandra c-function, and it satisfies the estimate

$$|c(\lambda)|^{-2} \lesssim (1+|\lambda|)^{d-\ell}, \tag{2.2}$$

together with all its derivatives.

3. Singular expansion of the wave kernel $w(t, x, \sigma, b)$

In this section, we use the following key pointwise estimates (see [14–16])

$$\forall\ t > 0,\ \forall\ x \in \mathbb{X} = G/K,\ |p(s - it, x)| \lesssim \begin{cases} s^{-d}, & 0 < s \leq 1; \\ s^{-D}, & s > 1; \end{cases} \tag{3.1}$$

to study the analyticity of the wave kernel $w(t, x, \sigma, b)$ given by the rule:

$$w(t, x, \sigma, b) = \frac{1}{\Gamma(\sigma)}\int_0^\infty \widetilde{p}(s - it, x)s^{\sigma-1} ds = \widetilde{p}^*(r - it, x)(\sigma), \tag{3.2}$$

for a complex number $\sigma \in \mathbb{C}$ such that $\mathrm{Re}(\sigma) > d$, and fixed $x \in \mathbb{X} = G/K$ and a real parameter $b > 0$, where $\widetilde{p}(s - it, x) = e^{-bs}p(s - it, x)$ denotes the bi-K-invariant convolution kernel of the Poisson operator $\mathcal{P}_{s-it}$.

3.1. *Meromorphic continuation of $\tilde{p}^*(r - it, x)(\sigma) = w(t, x, \sigma, b)$*

Theorem 3.1. *The function $\sigma \mapsto w(t, x, \sigma, b)$ extends meromorphically to $\mathbb{C}$ with simple poles located at $\sigma_j = d - j$; $j \in \mathbb{N}_0$, for all $x \in \mathbb{X}$ and for all positive real numbers $t > 0$.*

Proof. We divide the proof into three steps.
Step 1. The convergence of the integral given by Eq (3.2), where $\mathrm{Re}(\sigma) > d$, is easily handled. Clearly, using the pointwise estimates in Eq (3.1) , we have the following straightforward bound:

$$\int_0^\infty |e^{-bs}p(s - it, x)s^{\sigma-1}| ds \lesssim \int_0^1 s^{-d+\mathrm{Re}(\sigma)-1} ds + \int_1^\infty e^{-bs}s^{-D+\mathrm{Re}(\sigma)-1} ds,$$

and observe that the first integral converges for $\mathrm{Re}(\sigma) > d$. However, the second integral is controlled as follows:

$$\int_1^\infty e^{-bs}s^{-D+\mathrm{Re}(\sigma)-1} ds \lesssim \int_1^\infty e^{-bs/2} ds < \infty.$$

This concludes the absolute convergence for $\mathrm{Re}(\sigma) > d$.
Step 2. We check the holomorphy of $\tilde{p}^*(r - it, x)(\sigma) = w(t, x, \sigma, b)$. First, we split $\tilde{p}^*(r - it, x)(\sigma)$ into

a local part $\tilde{p}_0^*(r-it,x)(\sigma)$ and a part at infinity $\tilde{p}_\infty^*(r-it,x)(\sigma)$.
Write $\tilde{p}^*(r-it,x)(\sigma)=\tilde{p}_0^*(r-it,x)(\sigma)+\tilde{p}_\infty^*(r-it,x)(\sigma)$, where

$$\tilde{p}_0^*(r-it,x)(\sigma)=\frac{1}{\Gamma(\sigma)}\int_0^1 \tilde{p}(s-it,x)s^{\sigma-1}ds, \tag{3.3}$$

and

$$\tilde{p}_\infty^*(r-it,x)(\sigma)=\frac{1}{\Gamma(\sigma)}\int_1^\infty \tilde{p}(s-it,x)s^{\sigma-1}ds. \tag{3.4}$$

To treat the local part, consider the vertical band

$$S_{\beta,\gamma}=\{\sigma\in\mathbb{C}; d<\beta<\mathrm{Re}(\sigma)<\gamma\},\quad \forall 0<\beta<\gamma<\infty,$$

and let ε be a positive real number, and introduce the sequence $(\tilde{p}_{0,\varepsilon}^*(r-it,x)(\sigma))_{\varepsilon>0}$ of holomorphic functions on $S_{\beta,\gamma}$

$$\tilde{p}_{0,\varepsilon}^*(r-it,x)(\sigma)=\frac{1}{\Gamma(\sigma)}\int_\varepsilon^1 \tilde{p}(s-it,x)s^{\sigma-1}ds.$$

Clearly, we have the following estimate:

$$\begin{aligned}|\tilde{p}_0^*(r-it,x)(\sigma)-\tilde{p}_{0,\varepsilon}^*(r-it,x)(\sigma)| &= \frac{1}{|\Gamma(\sigma)|}\left|\int_0^\varepsilon \tilde{p}(s-it,x)s^{\sigma-1}ds\right| \\ &\lesssim \int_0^\varepsilon s^{-d+\mathrm{Re}(\sigma)-1}ds \\ &\leq \frac{1}{\beta-d}\varepsilon^{\beta-d}\xrightarrow[\varepsilon\to 0^+]{} 0 \text{ uniformly for every } \sigma\in S_{\beta,\gamma},\end{aligned}$$

which proves that the sequence $(\tilde{p}_{0,\varepsilon}^*(r-it,x)(\sigma))_{\varepsilon>0}$ converges uniformly with limit $\tilde{p}_0^*(r-it,x)(\sigma)$. Cauchy's theorem will guarantee that this limit is holomorphic.
Now, for s large, according to (3.1), we deduce that

$$|\tilde{p}(s-it,x)s^{\sigma-1}|\lesssim e^{-bs}s^{-D+\mathrm{Re}(\sigma)-1},$$

which ensures that the integral $\int_1^\infty \tilde{p}(s-it,x)s^{\sigma-1}ds$ converges uniformly on $\mathrm{Re}(\sigma)\leq R$, for each $R\in\mathbb{R}$. In particular, it converges uniformly on compact subsets of $\mathbb{C}$. It follows that the infinity part $\tilde{p}_\infty^*(r-it,x)(\sigma)$ represents an entire function of σ ([20, Ch. XII, Lemma 1.1, p. 308]).
Step 3. It remains to prove the meromorphic extension to the entire complex plane $\mathbb{C}$ of the function $\sigma\mapsto\tilde{p}^*(r-it,x)(\sigma)=w(t,x,\sigma,b)$ and to determine its poles. To do this, we proceed in the classical and usual way using an appropriate asymptotic expansion of the Poisson kernel.
To start, we state

$$p(s-it,x)\sim c_d s^{-d}\sum_{j\geq 0}\alpha_j s^j,\quad s\to 0^+, \tag{3.5}$$

where the coefficients $\alpha_j=\alpha_j(t,x)$ are to be calculated later. Here, c_d is a constant depending on d.
Given the Maclaurin series

$$e^{-bs}\sim\sum_{j\geq 0}\frac{(-1)^j b^j}{j!}s^j,\quad s\to 0^+, \tag{3.6}$$

the asymptotic expansion of the product is given by

$$\tilde{p}(s-it,x) = e^{-bs}p(s-it,x) \sim c_d s^{-d} \sum_{j\geq 0} \tilde{\alpha}_j s^j, \quad s \to 0^+, \tag{3.7}$$

where

$$\tilde{\alpha}_j = \tilde{\alpha}_j(t,x,b) = \sum_{l=0}^{j} \frac{(-1)^{j-l} b^{j-l}}{(j-l)!} \alpha_l(t,x), \quad j \in \mathbb{N}_0. \tag{3.8}$$

Now (3.7) is equivalent to asserting that for each nonnegative integer N, there exists a positive constant $C_N > 0$ such that

$$\forall x \in \mathbb{X}, \ \ \forall t > 0, \ \ |\ \tilde{p}(s-it,x) - c_d \sum_{j=0}^{N} \tilde{\alpha}_j s^{j-d} \ |\leq C_N s^{N+1-d}, \ \ 0 < s \leq 1. \tag{3.9}$$

Then we decompose $\tilde{p}^*(r-it,x)(\sigma)$ for given σ with $\mathrm{Re}(\sigma) > d$ in the form

$$\tilde{p}^*(r-it,x) = w(t,x,\sigma,b) = A_N(t,x,\sigma,b) + B_N(t,x,\sigma,b) + \tilde{p}^*_\infty(r-it,x)(\sigma), \tag{3.10}$$

where

- $A_N(t,x,\sigma,b) = \frac{1}{\Gamma(\sigma)} \int_0^1 (\tilde{p}(s-it,x) - c_d \sum_{j=0}^{N} \tilde{\alpha}_j s^{j-d}) s^{\sigma-1} dt,$
- $B_N(t,x,\sigma,b) = \frac{c_d}{\Gamma(\sigma)} \sum_{j=0}^{N} \tilde{\alpha}_j \int_0^1 s^{j-d+\sigma-1} ds = \dfrac{c_d}{\Gamma(\sigma)} \sum_{j=0}^{N} \dfrac{\tilde{\alpha}_j}{\sigma + (j-d)},$
- $\tilde{p}^*_\infty(r-it,x)(\sigma)$ is an entire function of σ given by Eq (3.4).

Thus, the exact argument in [1,13] shows that $A_N(t,x,\sigma,b)$ is uniformly convergent on $\mathrm{Re}(\sigma) > d-(N+1)+\varepsilon$ for every $\varepsilon > 0$. It follows that $A_N(t,x,\sigma,b)$ is a holomorphic function on $\mathrm{Re}(\sigma) > d-(N+1)$, and (3.10) provides the meromorphic continuation of $\tilde{p}^*(r-it,x) = w(t,x,\sigma,b)$ with simple poles at $(\sigma_j = d-j)_{j\in\mathbb{N}_0}$ appearing in the function $B_N(t,x,\sigma,b)$, with residue given by

$$\begin{aligned} \mathrm{res}(w(t,x,\sigma,b)), \sigma_j = d-j \quad &= \quad \frac{c_d}{\Gamma(d-j)} \tilde{\alpha}_j(t,x,b) \\ &= \quad \frac{c_d}{\Gamma(d-j)} \sum_{l=0}^{j} \frac{(-1)^{j-l} b^{j-l}}{(j-l)!} \alpha_l(x,\sigma), \ \ \forall\ x \in \mathbb{X} = G/K, \end{aligned} \tag{3.11}$$

which ends the proof of Theorem 3.1. □

Remark 3.2. *Notice that, if $j \geq d$, the integer $d-j$ is negative; therefore, $\frac{1}{\Gamma(d-j)} = 0$, and consequently res$(w(t,x,\sigma,b), \sigma_j) = 0$, which allows us to affirm that $w(t,x,\sigma,b)$ has a finite number of simple poles at the points $\sigma_j = d-j$, for $j = 0,1,\cdots,(d-1)$, exactly as in the case of the ζ function on a compact manifold M [11].*

Remark 3.3. *Generally, we cannot compare the dimension d of $\mathbb{X}$ and D, the dimension at infinity of $\mathbb{X}$ without specifying the geometric structure of $\mathbb{X}$. For example, if G complex, one has $d = D$, but when $\mathbb{X}$ has a normal real form, one has $d < D$, and in this case, the fundamental strip of $\tilde{p}^*(r-it,x)(\sigma) = w(t,x,\sigma,b)$ will be $d < Re(\sigma) < D$.*

Let us turn to the computations of the coefficients $\tilde{\alpha}_j = \tilde{\alpha}_j(t, x, b)$. To do this, we refer to the key decomposition (3.10), which allows us to calculate the values of $w(t, x, \sigma, b)$ for $\sigma = -k,\ \ k \in \mathbb{N}_0$ and to deduce an explicit expression of $\tilde{\alpha}_j$, or even more simply an expression of α_j.

Theorem 3.4. *The coefficients* $(\tilde{\alpha}_j(t, x, b))_{t>0,\ x\in\mathbb{X}}$ *are given by*

$$\tilde{\alpha}_{d+k}(t, x, b) = c_d^{-1}\frac{(-1)^k}{k!}w(t, x, -k, b),\ \ k \in \mathbb{N}_0. \tag{3.12}$$

Proof. For $k \in \mathbb{N}_0$ choose $N > d + k$ in Eq. (3.10), which implies that $-k > d - N > d - (N + 1)$. It follows that $A_N(t, x, \sigma, b)$ is holomorphic at $\sigma = -k$, and since $\frac{1}{\Gamma(-k)} = 0$, we obtain $A_N(t, x, -k, b) = 0$, and it is the same $\tilde{p}^*_\infty(r-it, x)(-k) = 0$. Now all that remains is to determine the value of $B_N(t, x, -k, b)$. Write

$$B_N(t, x, -k, b) = \underbrace{\frac{c_d}{\Gamma(-k)}\sum_{j=0, j\neq d+k}^{N}\frac{\tilde{\alpha}_j}{-k + (j - d)}}_{=0} + f(\sigma)_{/\sigma=-k}, \tag{3.13}$$

where

$$f(\sigma) = \frac{c_d}{\Gamma(\sigma)}\frac{\tilde{\alpha}_{d+k}}{\sigma + k}. \tag{3.14}$$

Knowing that the only singularities of $B_N(t, x, \sigma, b)$ are at the points $(\sigma_j = d - j)_{j=0,1,\cdots,(d-1)}$ and that

$$\lim_{\sigma\to-k} f(\sigma) = \frac{c_d\tilde{\alpha}_{d+k}}{\mathrm{res}(\Gamma(\sigma), -k)} = \frac{c_d\tilde{\alpha}_{d+k}}{\frac{(-1)^k}{k!}},$$

we conclude that f is holomorphic at $\sigma = -k$ and $f(-k) = \frac{c_d\tilde{\alpha}_{d+k}}{\frac{(-1)^k}{k!}}$. Eq (3.12) follows directly. □

Remark 3.5. *The results presented here are in agreement with general ζ-theory, but without any discussion on the parity of the space-dimension.*

3.2. Singular expansion of the wave kernel $w(t, x, \sigma, b)$

For the reader's benefit, we recall the definition of the singular expansion of a meromorphic function.

Definition 3.6. *(Singular expansion) Let* $f(\sigma)$ *be meromorphic in* Q *with* P *including all the poles of* $f(\sigma)$ *in* Q*. A singular expansion of* $f(\sigma)$ *in* Q *is a formal sum of singular elements of* $f(\sigma)$ *at all points of* P*.*
When S *is a singular expansion of* $f(\sigma)$ *in* Q*, we write*

$$f(\sigma) \asymp S, \quad \sigma \in Q.$$

According to the previous paragraph and considering this last definition, it is easy to obtain the following result.

Theorem 3.7. *The singular expansion of the wave kernel* $w(t, x, \sigma, b)$ *is given by*

$$w(t, x, \sigma, b) \asymp \frac{c_d}{\Gamma(\sigma)}\sum_{j=0}^{d-1}\frac{\tilde{\alpha}_j(t, x, b)}{\sigma + (j - d)},\ \ \sigma \in \mathbb{C},\ \ t > 0,\ \ x \in \mathbb{X} = G/K. \tag{3.15}$$

Remark 3.8. *It should be noted that the function $B_N(t,x,\sigma,b)$ in the decomposition* (3.10) *contains not only a singular part, but also a regular part. Indeed, since we chose $N > d + k$ (therefore large enough) in the previous calculation, $B_N(t,x,\sigma,b)$ is written*

$$B_N(t,x,\sigma,b) = \frac{c_d}{\Gamma(\sigma)} \sum_{j=0}^{d-1} \frac{\tilde{\alpha}_j(t,x,b)}{\sigma + (j-d)} + \frac{c_d}{\Gamma(\sigma)} \sum_{j=d}^{N} \frac{\tilde{\alpha}_j(t,x,b)}{\sigma + (j-d)}.$$

Thus, a passage to the limit ($N \to \infty$) without worrying about convergence problems of the second sum, and by (3.12)*, we obtain the desired regular part*

$$\frac{1}{\Gamma(\sigma)} \sum_{j\geq 0} \frac{(-1)^j}{j!} \frac{w(t,x,-j,b)}{\sigma + j}.$$

4. Harmonic sums

As an application, we shall now see the analysis of harmonic sum assigned to the wave kernel on Riemannian symmetric spaces $\mathbb{X} = G/K$. In particular, we derive a singular expansion of the Mellin transform of this sum, and we deduce an asymptotic expansion as $s \to 0$ of this sum using the converse correspondence rule (1.10). First, we introduce the general definition of harmonic sum, which will be involved in the asymptotic analysis of wave kernel. We refer to [21] for more details about sums of this type.

Definition 4.1. *A sum of the form*

$$G(s) = \sum_{k\geq 1} \lambda_k g(\mu_k s), \quad s > 0 \tag{4.1}$$

is called a harmonic sum. The λ_k are the amplitudes, the μ_k are the frequencies, and $g(s)$ is called the base function. The Dirichlet series of the harmonic sum is the sum given by

$$\Lambda(\sigma) = \sum_{k\geq 1} \lambda_k \mu_k^{-\sigma}, \quad \sigma \in \mathbb{C}. \tag{4.2}$$

Here, we consider $\lambda_k = 1$, $\mu_k = k$ and $g(s) = \tilde{p}(s - it, x) = e^{-bs} p(s - it, x)$ for $s > 0$ and $x \in \mathbb{X} = G/K$. In this case, $\Lambda(\sigma)$ coincides with the zeta function $\zeta(\sigma) = \sum_{k\geq 1} \frac{1}{k^\sigma}$ and

$$G(t,x,s,b) = \sum_{k\geq 1} \tilde{p}(ks - it, x) = \sum_{k\geq 1} e^{-bks} p(ks - it, x). \tag{4.3}$$

We start by calculating the Mellin transform of $G(t,x,s,b)$.

Theorem 4.2. *The Mellin transform of $G(t,x,s,b)$ equals*

$$G^*(t,x,\sigma,b) = \zeta(\sigma) w(t,x,\sigma,b), \quad t > 0, \quad x \in \mathbb{X} = G/K, \tag{4.4}$$

with fundamental strip (d,∞).

Proof. We apply the dominated convergence theorem two times. The first is dedicated to proving the convergence of $G(t,x,s,b)$, while the second aims to obtain the separation formula (4.4).
Consider the partial sums

$$G_n(t,x,s,b) = \sum_{k=1}^{n} \tilde{p}(ks - it, x) = \sum_{k=1}^{n} e^{-bks} p(ks - it, x), \ n \geq 1.$$

A straightforward computation using a change of variables in the definition of $G_n^*(t,x,\sigma,b)$ and the inversion formula (1.3) gives

$$\begin{aligned} G_n(t,x,s,b) &= \int_{a-i\infty}^{a+i\infty} \Gamma(\sigma) G_n^*(t,x,\sigma,b) s^{-\sigma} d\sigma \\ &= \int_{a-i\infty}^{a+i\infty} \underbrace{\Gamma(\sigma) \sum_{k=1}^{n} \frac{1}{k^\sigma} \tilde{p}^*(r - it, x)(\sigma) s^{-\sigma}}_{f_n(\sigma)} d\sigma, \quad a > d. \end{aligned} \tag{4.5}$$

On the one hand, it is easy to see that

$$\tilde{p}^*(r - it, x)(\sigma) = O(1),$$

since $\tilde{p}(s - it, x) = O(s^{-d})$ as $s \to 0$ and the integration is along the vertical line $\mathrm{Re}(\sigma) = a$.
Then, this fact combined with the following well-known bounds

$$\Gamma(\sigma) \sim \sqrt{2\pi} |\mathrm{Im}(\sigma)|^{\mathrm{Re}(\sigma)-1/2} e^{-\frac{\pi}{2}|\mathrm{Im}(\sigma)|}, \ \ \mathrm{Im}(\sigma) \to \infty, \tag{4.6}$$

and

$$|\zeta(\sigma)| \lesssim |\mathrm{Im}(\sigma)|^{1-\varepsilon}, \ \varepsilon < 1, \tag{4.7}$$

allow to dominate the sequence $f_n(\sigma)$ by an integrable function. More precisely, we obtain

$$|f_n(\sigma)| \lesssim |\mathrm{Im}(\sigma)|^{1/2-\varepsilon+a} e^{-\frac{\pi}{2}|\mathrm{Im}(\sigma)|} s^{-a}. \tag{4.8}$$

Hence the convergence of $G(t,x,s,b)$ by the dominated convergence theorem. Moreover, one has

$$G(t,x,s,b) = \lim_{n\to\infty} G_n(t,x,s,b) = \int_{a-i\infty}^{a+i\infty} \Gamma(\sigma)\zeta(\sigma) w(t,x,\sigma,b) s^{-\sigma} d\sigma. \tag{4.9}$$

Now, it is clear by (4.8) that $G_n(t,x,s,b) = O(s^{-a})$ and that $G(t,x,s,b) = O(s^{-a})$, which makes it possible to apply the dominated convergence theorem to obtain via the definition of G^* and interchange integral and limits

$$G^*(t,x,\sigma,b) = \zeta(\sigma) w(t,x,\sigma,b), \ \ \mathrm{Re}(\sigma) > d, \ x \in \mathbb{X} = G/K,$$

which ends the proof of the theorem. □

Remark 4.3. *A general proof of the previous theorem consists of taking into account a base function of fast decrease (see [21]), in order to use the dominated convergence theorem. Here we avoided this data and took advantage of Stirling's formula* (4.6) *to control the Gamma function present in the inversion formula* (4.5).

Theorem 4.4. *The asymptotic expansion of harmonic sum $G(t, x, s, b)$ as $s \to 0$ is given by*

$$G(t, x, s, b) \sim \frac{1}{s}(-c_d\tilde{\alpha}_{d-1}(t, x, b) \ln s + a_{-1}) + c_d \sum_{j=0}^{d-2} \zeta(d - j)\tilde{\alpha}_j(t, x, b)s^{j-d}, \tag{4.10}$$

where $a_{-1} = res(\Gamma(\sigma)G^*(t, x, \sigma, b), \sigma = 1)$.

Proof. First, notice that there are singularities at $(\sigma_j = d - j)_{j=0,1,\cdots,d-1}$. Globally, the poles of $G^*(t, x, \sigma, b)$ are a double pole at $\sigma = 1$ and simple at $(\sigma_j = d - j)_{j=0,1,\cdots,d-2}$.
According to the separation formula (4.4), we obtain the singular expansion

$$\Gamma(\sigma)G^*(t, x, \sigma, b) \asymp \frac{c_d\tilde{\alpha}_{d-1}(t, x, b)}{(\sigma - 1)^2} + \frac{a_{-1}}{\sigma - 1} + c_d \sum_{j=0}^{d-2} \frac{\zeta(d - j)\tilde{\alpha}_j(t, x, b)}{\sigma - (d - j)}. \tag{4.11}$$

Again from the bound (4.8), we conclude that $\sigma^r\Gamma(\sigma)G^*(t, x, \sigma, b) = O(1)$ as $|\sigma| \to \infty$, for all $r > 1$. In other words, $\Gamma(\sigma)G^*(t, x, \sigma, b)$ is of fast decrease.
Now all the ingredients are ready to be able to apply the correspondence rule $\frac{A}{(\sigma-q)^{k+1}} \mapsto \frac{(-1)^k}{k!} s^{-q}(\ln s)^k$ in singular expansion (4.11), and the asymptotic expansion is deduced directly. □

Remark 4.5. *It is not difficult to see that $\Gamma(\sigma)G^*(t, x, \sigma, b)$ admits a meromorphic continuation to $\mathbb{C}$ and satisfies the conditions of the converse correspondence rule; thus, a complete asymptotic expansion of $G(t, x, s, b)$ results.*

Note that it is possible to express certain coefficients $\tilde{\alpha}_{d+k}(t, x, b)$ via the transform $G^*(t, x, \sigma, b)$. More precisely, one has:

Theorem 4.6. *The coefficients $(\tilde{\alpha}_{d+2k+1}(t, x, b))_{t>0,\ x\in\mathbb{X}}$ are given by*

$$\tilde{\alpha}_{d+2k+1}(t, x, b) = c_d^{-1} \frac{(2k + 2)G^*(t, x, -2k - 1, b)}{(2k + 1)!B_{2k+2}}, \quad k \in \mathbb{N}_0, \tag{4.12}$$

where B_k denotes the Bernoulli numbers.

Proof. First, we know that

$$\zeta(-k) = (-1)^k \frac{B_{k+1}}{k + 1}, \quad k \in \mathbb{N}_0, \tag{4.13}$$

and that

$$\zeta(-2k) = 0, \quad B_{2k+1} = 0, \quad k \geq 1, \tag{4.14}$$

which implies that $G^*(t, x, -2k, b) = 0, \ \forall\, k \geq 1$.
For the odd negative integers, by (3.12) we write

$$w(t, x, -2k - 1, b) = c_d(-1)^{2k+1}(2k + 1)!\tilde{\alpha}_{d+2k+1}(t, x, b), \quad k \in \mathbb{N}_0.$$

Let us introduce this in the equation $G^*(t, x, -2k - 1, b) = \zeta(-2k - 1)w(t, x, -2k - 1, b)$ well combined with (4.13), and the desired result is simply deduced. □

Remark 4.7. *Notice that Bernoulli numbers can appear in the singular expansion of* $\Gamma(\sigma)G^*(t,x,\sigma,b)$ *given by formula* (4.11)*. In fact, the function* $\phi(s) = \frac{1}{e^s-1}$ *admits an expansion near* $s = 0$

$$\frac{1}{e^s - 1} = \sum_{k\geq -1} \frac{B_{k+1}}{(k+1)!} s^k,$$

and by definition, its Mellin transform equals $\zeta(\sigma)$*, which implies the singular expansion*

$$\Gamma(\sigma)\zeta(\sigma) \asymp \sum_{k\geq -1} \frac{B_{k+1}}{(k+1)!} \frac{1}{\sigma + k}, \quad \sigma \in \mathbb{C},$$

leading to writing (4.11) *differently, showing the numbers* B_k*, since one has* $\Gamma(\sigma)G^*(t,x,\sigma,b) = \Gamma(\sigma)\zeta(\sigma)w(t,x,\sigma,b)$*.*

Remark 4.8. *Consider the function*

$$\psi(t,x,s,b) = \int_0^\infty \frac{1}{e^{s/u} - 1} \tilde{p}(u - it, x) \frac{du}{u}, \quad s > 0, \quad t > 0, \quad x \in \mathbb{X} = G/K.$$

It is easy to see that

$$\psi^*(\sigma) = \Gamma(\sigma)G^*(t,x,\sigma,b), \quad Re(\sigma) > d.$$

Thus, $\psi(t,x,s,b)$ *can be considered as the integral version of the harmonic series* $G(t,x,s,b)$*. Consequently, it has the same asymptotic expansion* (4.10)*, and its Mellin transform has the same singular expansion* (4.11)*.*

5. Conclusions

The results presented in this paper will be helpful in understanding the Mellin transform associated with the main evolution equations in Riemannian symmetric spaces of the non-compact type. We proved that the wave kernel $\sigma \mapsto w(t,x,\sigma,b) = \tilde{p}^*(r - it, x)(\sigma)$ extends meromorphically to the entire complex plane $\mathbb{C}$ with a finite number of simple poles on the real line, and we derived its singular expansion. In addition, we stated the singular expansion of the Mellin transform $\sigma \mapsto G^*(t,x,\sigma,b)$ of the harmonic sums $G(t,x,s,b)$ to deduce its asymptotic expansion near $s \to 0$ by the converse correspondence rule.

In [1] we have studied the Mellin transform

$$w^*(z,x,\sigma,b) = \frac{1}{\Gamma(z)} \int_0^\infty e^{-bt} w(t,x,\sigma) t^{z-1} dt,$$

for $z, \sigma \in \mathbb{C}$ with $\mathrm{Re}(\sigma) > d$, and a fixed element $x \in \mathbb{X} = G/K$ and a real parameter $b > 0$, where $w(t,x,\sigma)$ denotes the wave kernel on $\mathbb{X} = G/K$ defined by

$$w(t,x,\sigma) = \frac{1}{|W|} \int_{\mathfrak{a}^\star} e^{it\sqrt{|\lambda|^2+|\rho|^2}} (|\lambda|^2 + |\rho|^2)^{-\sigma/2} \varphi_\lambda(x) |c(\lambda)|^{-2} d\lambda,$$

and we have obtained similar results as in subsection 3.1. In short, we can afford to configure the two studies, the previous and the current one, as follows:

$$\tilde{w}(t,x,\sigma) = e^{-bt} w(t,x,\sigma) \to_{t\to z} \tilde{w}^*(z,x,\sigma,b),$$

$$\tilde{p}(s-it,x)=e^{-bs}p(s-it,x)\rightarrow_{s\to\sigma}\tilde{p}^*(r-it,x)(\sigma)=w(t,x,\sigma,b).$$

Thanks to our results furnished in Theorems 3.7, 4.2, and 4.4, we can elaborate on the same ones about singular expansion and harmonic sums. More precisely, if $d=2p+1$ is odd, we prove the following further results about the Mellin transform $w^*(z,x,\sigma,b)$:

• Singular expansion:

$$\tilde{w}^*(z,x,\sigma,b)\asymp\frac{c_d}{\Gamma(z)}\sum_{j=0}^{\frac{d-3}{2}}\frac{\tilde{a}_j(x,\sigma,b)}{z+(j-\frac{d-1}{2})},\ \ \mathrm{Re}(\sigma)>d,\ \ x\in\mathbb{X}=G/K.$$

• Harmonic sums:

$$H(t,x,\sigma,b)=\sum_{k\geq1}e^{-bkt}w(kt,x,\sigma),\ \ x\in\mathbb{X}=G/K,$$

$$H^*(z,x,\sigma,b)=\zeta(z)w^*(z,x,\sigma,b),\ \ \mathrm{Re}(z)>\frac{d-1}{2},$$

$$\Gamma(z)H^*(z,x,\sigma,b)\asymp\frac{c_d\tilde{a}_{\frac{d-3}{2}}(x,\sigma,b)}{(z-1)^2}+\frac{a_{-1}}{z-1}+c_d\sum_{j=0}^{\frac{d-5}{2}}\frac{\zeta(\frac{d-1}{2}-j)\tilde{a}_j(x,\sigma,b)}{z-(\frac{d-1}{2}-j)},$$

$$H(t,x,\sigma,b)\sim\frac{1}{t}(-c_d\tilde{a}_{\frac{d-3}{2}}(x,\sigma,b)\ln t+a_{-1})+c_d\sum_{j=0}^{\frac{d-5}{2}}\zeta(\frac{d-1}{2}-j)\tilde{a}_j(x,\sigma,b)t^{j-\frac{d-1}{2}},\ \ t\to0,$$

where $a_{-1}=\mathrm{res}(\Gamma(z)H^*(z,x,\sigma,b),z=1)$.

We hope to return to all the questions proposed here in the special case when G is a complex semisimple Lie group, since the Harish-Chandra c-function and the spherical function φ_λ have elementary expressions, which can be a decisive factor to better express the coefficients $\alpha_j(t,x,b)$.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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