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Research article

Solitonic behaviors in the coupled Drinfeld-Sokolov-Wilson system with fractional dynamics

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Abstract: I investigated soliton phenomena in a prominent nonlinear fractional partial differential equation (FPDE) namely the conformable coupled Drinfeld-Sokolov-Wilson system (CCDSWS) using a novel variant of the novel extended direct algebraic method (EDAM), namely r+mEDAM. The conformable fractional derivatives are used to generalize the model due to the memory and hereditary features that are inherent in the fractional dynamics. The model was initially transformed into a more manageable system of integer-order nonlinear ordinary differential equations (NODEs) through the implementation of complex transformation. The obtained system of NODEs is further transformed into a system of algebraic equations, which yields, by solving new plethora of soliton solutions for CCDSWS in the form of generalized trigonometrical, exponential hyperbolical, and rational functions. Moreover, we employed 2D, 3D, and contour graphics to show the behavior of acquired solitons, making it abundantly evident that the obtained solitons take the shape of kink, anti-kink, bright, dark, bright-dark, and bell-shaped kink solitons within the framework of CCDSWS. The results confirmed the efficiency of the presented approach in finding solitonic solutions, which in its turn expands knowledge of nonlinear FPDEs. The aimed to theoretical and application perspectives in fractional solitons applicable in areas such fluid mechanics, plasma physic, optical communications, etc.

Keywords: nonlinear fractional partial differential equations; coupled Drinfeld-Sokolov-Wilson system; r+mEDAM; complex transformation; conformable fractional derivatives; soliton
Mathematics Subject Classification: 34G20, 35A20, 35A22, 35R11

1. Introduction

Nonlinear fractional partial differential equations (NFPDEs) are commonly used in various of scientific fields because it presents stable modeling of such processes that cannot be modeled by linear equations [1–4]. They are used in physics to describe waves, turbulence, and behavior of fluids by providing an understanding of weather and particles. These equations assist engineers to analyze heat transfer, structural forces, and currents with the aim of coming up with efficient systems. NFPDEs are used to model many different processes in the life sciences such as infection transmission, population dynamics, and in neural networks. They are employed in the area of environmental science in estimations of all ecological transformations ranging from climate models to those of ecosystems. NFPDEs are an efficient and marked means of solving different actual problems in different branches of science, and they are the basis of modern scientific research [5–7].

Due to the limitations of numerical methods such as the Chebyshev wavelet method [8], the Elzaki transform decomposition method [9], and many researchers seek analytical solutions when solving NFPDEs because of challenges such as complexity of the computational requirements, errors and high computational costs. Analytical results provide the most basic of equations that can provide a lot of insight of the system without necessarily having to do a lot of computations. Some of these methods include the Adomian decomposition method (ADM) [10], Laplace ADM [11], the homotopy perturbation method [12], (G'/G)-expansion techniques [13], and the Natural transform method [14], which has made the studies and understanding of the mathematical models represented by NFPDEs much easier.

The study of soliton solutions, with reference to NFPDEs, has always been a concern for physicists and applied mathematicians. In order to describe and analyze of soliton behaviors in NFPDEs, some analytical methods have been proposed such as the (G'/G)-expansion approach [15], tan-cot function method [16], Sardar sub-equation method [17], Kudryashov method [18], sub-equation method [19], Khater method [20], exp-function method [21], extended direct algebraic method (EDAM) [22, 23], and others [24–26]. Among these analytical techniques for obtaining soliton solutions, EDAM is one of modern approaches for NFPDEs. This method involves a change of procedure in transforming NFPDEs to NODEs to be solved via series solutions. The resulting NODE is, in turn, utilized to derive an algebraic equation system which, upon solving, yields a large number of soliton solutions in generalized hyperbolic, rational, exponential, and trigonometric forms to the NFPDE. The said EDAM is quite remarkable for being highly efficient in terms of generating a larger number of soliton solution families.

The goal of this study is to deploy the r+mEDAM to investigate and classify soliton solutions for the CCDSWS. This model's mathematical formulation is described as [27, 28]:

$$\begin{aligned} &D_t^{\sigma} z + D_x^{\varsigma}(u^2) = 0, \\ &D_t^{\sigma} u - D_x^{\varsigma}(D_x^{\varsigma}(D_x^{\varsigma}(u))) + 3uD_t^{\sigma} z + 3zD_x^{\varsigma} u = 0, \end{aligned} \tag{1.1}$$

where $D_x^{\varsigma}(\cdot)$ and $D_t^{\sigma}(\cdot)$ are conformable fractional derivatives, and $u = u(x,t)$ and $z = z(x,t)$ are functions of both spatial variable x and temporal variable t that define the amplitudes of wave modes. Drinfeld and Sokolov demonstrated that this system is a subcategory of the four-times reduced Kadomtsev-Petviashvili model [29] and they included it in the construction of the general Drinfeld-Sokolov system [30]. Wilson then developed the coupled Drinfeld-Sokolov-Wilson system (CDSWS), which is crucial in modeling dispersive phenomena in water waves and fluid mechanics. Gravitational water flow dominated by shear stress, such as overland flows, flows through vegetation, dam breakdowns, and floods, has been successfully modeled using the diffusive wave approximations of the shallow water equations (SWEs). Furthermore, the nonlinear CCDSWS is used in the study of dusty plasmas.

Several researchers have used various methodologies to address the proposed model in the literature. As documented in [31], Arora and Kumar used the homotopy analysis method (HAM) to approximate solutions for the CDSWS. As described in [32], Usman et al. used the Lie Symmetry technique in conjunction with the Jacobi elliptic function method for the CDSWS. In a similar way, Singh et al; in [33] used an effective numerical algorithm that combined the HAM, the Sumudu transform approach, and homotopy polynomials to approximate solutions for a nonlinear fractional CDSWS. In our study, we intend to use the r+mEDAM to handle the reduced NODEs from CCDSWS, enabling us to arrive at solutions in the form of rational, periodic, hyperbolic, and exponential functions. The various solutions are kink, anti-kink, bright, dark, and bell-shaped kink waves.

The rest of the paper is structured as follows: The r+mEDAM is described in Section 2. In Section 3 the CCDSWS's soliton solutions. Section 4 includes a discussion and several graphics, and I concludes my research.

2. Methodology and material

In this section, I offer the definition of the conformable fractional derivative and describe the working methodology of proposed r+mEDAM.

The definition of conformable fractional derivative

The advantages of the conformable fractional derivative over other fractional derivative operators can be used to achieve explicit soliton solutions to NFPDEs. Interestingly, alternate formulations of fractional derivatives do not yield the soliton solution of Eq (1.1) because they violate the chain rule [34, 35]. The conformable fractional derivatives are incorporated into Eq (1.1). The ς-order conformable fractional derivative operator is defined as follows [36]:

$$D_{\xi}^{\varsigma}\mathrm{u}(\xi) = \lim_{y\to 0}\frac{\mathrm{u}(y\xi^{1-\varsigma}+\xi)-\mathrm{u}(\xi)}{y}, \quad \varsigma \in (0,1]. \tag{2.1}$$

The following characteristics of this derivative are used in this study:

$$D_{\xi}^{\varsigma}\xi^{m} = m\xi^{m-\varsigma}, \tag{2.2}$$

$$D_{\xi}^{\varsigma}(m_1\sigma(\xi) \pm m_2\rho(\xi)) = m_1 D_{\xi}^{\varsigma}(\sigma(\xi)) \pm m_2 D_{\xi}^{\varsigma}(\rho(\xi)), \tag{2.3}$$

$$D_{\xi}^{\varsigma}\zeta[\tau(\xi)] = \zeta_{\tau}'(\tau(\xi))D_{\xi}^{\varsigma}\tau(\xi), \tag{2.4}$$

where m, m_1, m_2 represent constants, whereas $\sigma(\xi)$, $\rho(\xi)$, $\zeta(\xi)$, and $\tau(\xi)$ are arbitrary differentiable functions. Additionally, the following theorem demonstrates how conformable fractional derivative complies with the chain rule, a crucial rule in the resolution of CKGEs.

Theorem 2.1. *Let $\zeta(\xi)$ and $\tau(\xi)$ are arbitrary differentiable functions then*

$$D_{\xi}^{\varsigma}\zeta[\tau(\xi)] = \zeta_{\tau}'(\tau(\xi))D_{\xi}^{\varsigma}\tau(\xi).$$

Proof. If the function τ is a constant in a neighborhood ξ_0, then $D_{\xi}^{\varsigma}\zeta(\tau(\xi_0)) = 0$. However, we make the following assumption about non-constant function τ in the vicinity of ξ_0. Here, find an $y > 0 \ni$

$\tau(\xi_1) \neq \tau(\xi_2)$ for any $\xi_1, \xi_2 \in (\xi_0 - y_0, \xi_0 + y_0)$. Thus, since the function τ is continuous at ξ_0, for $\xi_0 > a$, $\xi_0^\varsigma \neq a$ (where $a \geq 0$), we acquire,

$$\begin{aligned}
D_\xi^\varsigma(\zeta \circ \tau)(\xi_0) &= \lim_{y \to 0} \frac{\zeta(\tau(\xi_0 + y\xi_0^{-\varsigma}(\xi_0 - a))) - \zeta(\tau(\xi_0))}{y(1 - a\xi_0^{-\varsigma})} \\
&= \lim_{y \to 0} \frac{\zeta(\tau(\xi_0 + y\xi_0^{-\varsigma}(\xi_0 - a))) - \zeta(\tau(\xi_0))}{\tau(\xi_0 + y\Omega_0^{-\varsigma}(\xi_0 - a)) - \tau(\xi_0)} \cdot \frac{\tau(\xi_0 + y\Omega_0^{-\varsigma}(\xi_0 - a)) - \tau(\xi_0)}{y(1 - a\xi_0^{-\varsigma})} \\
&= \lim_{y_1 \to 0} \frac{\zeta(\tau(\xi_0) + y_1) - \zeta(\tau(\xi_0))}{y_1} \cdot \frac{\tau(\xi_0 + y\xi_0^{-\varsigma}(\xi_0 - a)) - \tau(\xi_0)}{y(1 - a\xi_0^{-\varsigma})} \\
&= \zeta'(\tau(\xi_0)) D_\xi^\varsigma(\tau)(\xi_0).
\end{aligned}$$

Thus, the conformable fractional derivative satisfies the chain rule. □

3. The methodology of r+mEDAM

In this section, I introduce the r+mEDAM technique. Consider the general NFPDE given as [22]:

$$E(u, uD_x^\varsigma u, D_t^\sigma u, D_x^\varsigma(D_x^\varsigma u) \ldots) = 0, \tag{3.1}$$

where $u = u(x,t)$.

To investigate Eq (3.1), we take the following steps:

(1) A variable transformation of the form $u(x,t) = U(\xi)$ (where ξ can be expressed in a variety of ways) is performed first, which transforms (3.1) into the following NODE:

$$F(U, UU', U''', \ldots) = 0, \tag{3.2}$$

where primes denote derivatives of U with respect to ξ in (3.2). Equation (3.2) may be integrated once or may be integrated n times to find the constant of integration.

(2) We propose that (3.2) has the following series form solution:

$$U(\xi) = \sum_{i=-N}^{N} k_i (r + \Psi(\xi))^i. \tag{3.3}$$

The parameters k_i (where $i = -N, \ldots, N$) are estimated, and $\Psi(\xi)$ satisfies the given ODE:

$$\Psi'(\xi) = \ln(\varrho)(j + k\Psi(\xi) + l(\Psi(\xi))^2), \tag{3.4}$$

where $\varrho \neq 0, 1$ and j, k, l are constants.

(3) In Eq (3.3), we can achieve a positive integer N through the achievement of a symbiotic balance between the maximum nonlinear term and the maximum order derivative.

(4) To build a polynomial expression in $\Psi(\xi)$ It is possible to substitute the found equation with respect to x for the equation received as a result of integration of (3.2) for $\Psi(\xi)$, and then to arrange all the terms of $\Psi(\xi)$ in the same manner. The coefficients of this derivation polynomial are then set to zero, and a system of nonlinear algebraic equations in k_i $(i = N, \ldots, N)$ and other related parameters are obtained.

(5) This system of these nonlinear algebraic equations is solved using the Maple software.

(6) Accounting for the unknown parameters and substituting them into Eq (3.3) together with the $\Psi(\xi)$ (the general solution of (3.4)) yields the soliton solutions for Eq (3.1). Using the generic solution of Eq (4.1), we can obtain the families of soliton solutions shown below:

Family 1. For $H < 0,\ l \neq 0$:

$$\Psi_1(\xi) = -\frac{k}{2l} + \frac{\sqrt{-H}\tan_\varrho\left(\frac{1}{2}\sqrt{-H}\xi\right)}{2l},$$

$$\Psi_2(\xi) = -\frac{k}{2l} - \frac{\sqrt{-H}\cot_\varrho\left(\frac{1}{2}\sqrt{-H}\xi\right)}{2l},$$

$$\Psi_3(\xi) = -\frac{k}{2l} + \frac{\sqrt{-H}\left(\tan_\varrho\left(\sqrt{-H}\xi\right) \pm \left(\sqrt{pq}\sec_\varrho\left(\sqrt{-H}\xi\right)\right)\right)}{2l},$$

$$\Psi_4(\xi) = -\frac{k}{2l} - \frac{\sqrt{-H}\left(\cot_\varrho\left(\sqrt{-H}\xi\right) \pm \left(\sqrt{pq}\csc_\varrho\left(\sqrt{-H}\xi\right)\right)\right)}{2l},$$

and

$$\Psi_5(\xi) = -\frac{k}{2l} + \frac{\sqrt{-H}\left(\tan_\varrho\left(\frac{1}{4}\sqrt{-H}\xi\right) - \cot_\varrho\left(\frac{1}{4}\sqrt{-H}\xi\right)\right)}{4c}.$$

Family 2. For $H > 0,\ l \neq 0$:

$$\Psi_6(\xi) = -\frac{k}{2l} - \frac{\sqrt{H}\tanh_\varrho\left(\frac{1}{2}\sqrt{H}\xi\right)}{2l},$$

$$\Psi_7(\xi) = -\frac{k}{2l} - \frac{\sqrt{H}\coth_\varrho\left(\frac{1}{2}\sqrt{H}\xi\right)}{2l},$$

$$\Psi_8(\xi) = -\frac{k}{2l} - \frac{\sqrt{H}\left(\tanh_\varrho\left(\sqrt{H}\xi\right) \pm \left(\sqrt{pq}sech_\varrho\left(\sqrt{H}\xi\right)\right)\right)}{2l},$$

$$\Psi_9(\xi) = -\frac{k}{2l} - \frac{\sqrt{H}\left(\coth_\varrho\left(\sqrt{H}\xi\right) \pm \left(\sqrt{pq}csch_\varrho\left(\sqrt{H}\xi\right)\right)\right)}{2l},$$

and

$$\Psi_{10}(\xi) = -\frac{k}{2l} - \frac{\sqrt{H}\left(\tanh_\varrho\left(\frac{1}{4}\sqrt{H}\xi\right) - \coth_\varrho\left(\frac{1}{4}\sqrt{H}\xi\right)\right)}{4l}.$$

Family 3. For $jl > 0$ and $k = 0$:

$$\Psi_{11}(\xi) = \sqrt{\frac{j}{l}}\tan_\varrho\left(\sqrt{jl}\xi\right),$$

$$\Psi_{12}(\xi) = -\sqrt{\frac{j}{l}}\cot_\varrho\left(\sqrt{jl}\xi\right),$$

$$\Psi_{13}(\xi) = \sqrt{\frac{j}{l}}\left(\tan_\varrho\left(2\sqrt{jl}\xi\right) \pm \left(\sqrt{pq}\sec_\varrho\left(2\sqrt{jl}\xi\right)\right)\right),$$

$$\Psi_{14}(\xi) = -\sqrt{\frac{j}{l}}\left(\cot_{\varrho}\left(2\ \sqrt{jl}\xi\right) \pm \left(\sqrt{pq}\csc_{\varrho}\left(2\ \sqrt{jl}\xi\right)\right)\right),$$

and

$$\Psi_{15}(\xi) = \frac{1}{2}\sqrt{\frac{j}{l}}\left(\tan_{\varrho}\left(\frac{1}{2}\sqrt{jl}\xi\right) - \cot_{\varrho}\left(\frac{1}{2}\sqrt{jl}\xi\right)\right).$$

Family 4. For $jl < 0$ and $k = 0$:

$$\Psi_{16}(\xi) = -\sqrt{-\frac{j}{l}}\tanh_{\varrho}\left(\sqrt{-jl}\xi\right),$$

$$\Psi_{17}(\xi) = -\sqrt{-\frac{j}{l}}\coth_{\varrho}\left(\sqrt{-jl}\xi\right),$$

$$\Psi_{18}(\xi) = -\sqrt{-\frac{j}{l}}\left(\tanh_{\varrho}\left(2\ \sqrt{-jl}\xi\right) \pm \left(i\sqrt{pq}sech_{\varrho}\left(2\ \sqrt{-jl}\xi\right)\right)\right),$$

$$\Psi_{19}(\xi) = -\sqrt{-\frac{j}{l}}\left(\coth_{\varrho}\left(2\ \sqrt{-jl}\xi\right) \pm \left(\sqrt{pq}csch_{\varrho}\left(2\ \sqrt{-jl}\xi\right)\right)\right),$$

and

$$\Psi_{20}(\xi) = -\frac{1}{2}\sqrt{-\frac{j}{l}}\left(\tanh_{\varrho}\left(\frac{1}{2}\sqrt{-jl}\xi\right) + \coth_{\varrho}\left(\frac{1}{2}\sqrt{-jl}\xi\right)\right).$$

Family 5. For $l = j$ and $k = 0$:

$$\Psi_{21}(\xi) = \tan_{\varrho}(j\xi),$$

$$\Psi_{22}(\xi) = -\cot_{\varrho}(j\xi),$$

$$\Psi_{23}(\xi) = \tan_{\varrho}(2\,j\xi) \pm \left(\sqrt{pq}\sec_{\varrho}(2\,j\xi)\right),$$

$$\Psi_{24}(\xi) = -\cot_{\varrho}(2\,j\xi) \pm \left(\sqrt{pq}\csc_{\varrho}(2\,j\xi)\right),$$

and

$$\Psi_{25}(\xi) = \frac{1}{2}\tan_{\varrho}\left(\frac{1}{2}j\xi\right) - \frac{1}{2}\cot_{\varrho}\left(\frac{1}{2}j\xi\right).$$

Family 6. For $l = -j$ and $k = 0$:

$$\Psi_{26}(\xi) = -\tanh_{\varrho}(j\xi),$$

$$\Psi_{27}(\xi) = -\coth_{\varrho}(j\xi),$$

$$\Psi_{28}(\xi) = -\tanh_{\varrho}(2\,j\xi) \pm \left(i\sqrt{pq}sech_{\varrho}(2\,j\xi)\right),$$

$$\Psi_{29}(\xi) = -\coth_{\varrho}(2\,j\xi) \pm \left(\sqrt{pq}csch_{\varrho}(2\,j\xi)\right),$$

and

$$\Psi_{30}(\xi) = -\frac{1}{2}\tanh_{\varrho}\left(\frac{1}{2}j\xi\right) - \frac{1}{2}\coth_{\varrho}\left(\frac{1}{2}j\xi\right).$$

Family 7. For $J = 0$:

$$\Psi_{31}(\xi) = -2\,\frac{j\,(k\xi \ln \varrho + 2)}{k^2 \xi \ln \varrho}.$$

Family 8. For $k = \varpi$, $j = s\varpi (s \neq 0)$ and $l = 0$:

$$\Psi_{32}(\xi) = \varrho^{\varpi \xi} - s.$$

Family 9. For $k = l = 0$:

$$\Psi_{33}(\xi) = j\xi \ln(\varrho).$$

Family 10. For $k = j = 0$:

$$\Psi_{34}(\xi) = -\frac{1}{l\xi \ln(\varrho)}.$$

Family 11. For $j = 0$, $k \neq 0$ and $l \neq 0$:

$$\Psi_{35}(\xi) = -\frac{pk}{l\left(\cosh_{\varrho}(k\xi) - \sinh_{\varrho}(k\xi) + p\right)},$$

and

$$\Psi_{36}(\xi) = -\frac{k\left(\cosh_{\varrho}(k\xi) + \sinh_{\varrho}(k\xi)\right)}{l\left(\cosh_{\varrho}(k\xi) + \sinh_{\varrho}(k\xi) + q\right)}.$$

Family 12. For $k = \varpi$, $l = s\varpi (s \neq 0)$ and $j = 0$:

$$\Psi_{37}(\xi) = \frac{p\varrho^{\xi\varpi}}{p - sq\varrho^{\xi\varpi}}.$$

Where $q, p > 0$ and are known as a deformation parameter, and $H = k^2 - 4jl$. The following is a description of the trigonometric and hyperbolic functions that are generalized:

$$\begin{aligned}
\sin_{\varrho}(\xi) &= \frac{p\Psi^{i\xi} - q\Psi^{-i\xi}}{2i}, \quad \cos_{\varrho}(\xi) = \frac{p\Psi^{i\xi} + q\Psi^{-i\xi}}{2},\\
\sec_{\varrho}(\xi) &= \frac{1}{\cos_{\varrho}(\xi)}, \quad \csc_{\varrho}(\xi) = \frac{1}{\sin_{\varrho}(\xi)},\\
\tan_{\varrho}(\xi) &= \frac{\sin_{\varrho}(\xi)}{\cos_{\varrho}(\xi)}, \quad \cot_{\varrho}(\xi) = \frac{\cos_{\varrho}(\xi)}{\sin_{\varrho}(\xi)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sinh_{\varrho}(\xi) &= \frac{p\Psi^{\xi} - q\Psi^{-\xi}}{2}, \quad \cosh_{\varrho}(\xi) = \frac{p\Psi^{\xi} + q\Psi^{-\xi}}{2},\\
sech_{\varrho}(\xi) &= \frac{1}{\cosh_{\varrho}(\xi)}, \quad csch_{\varrho}(\xi) = \frac{1}{\sinh_{\varrho}(\xi)},\\
\tanh_{\varrho}(\xi) &= \frac{\sinh_{\varrho}(\xi)}{\cosh_{\varrho}(\xi)}, \quad \coth_{\varrho}(\xi) = \frac{\cosh_{\varrho}(\xi)}{\sinh_{\varrho}(\xi)}.
\end{aligned}$$

4. Execution of r+mEDAM

In this section, we look at soliton solutions for CCDSWS with r+mEDAM. We begin by applying the wave transformation as follows:

$$u(x,t) = U(\xi), \quad z(x,t) = Z(\xi), \quad \xi = \kappa\frac{x^{\varsigma}}{\varsigma} - \omega\frac{t^{\sigma}}{\sigma}, \tag{4.1}$$

to achieve the soliton result for (1.1). The subsequent scheme of nonlinear ordinary differential equations is obtained by transforming (1.1)

$$\begin{aligned} &-\omega Z' + \kappa(U^2)' = 0, \\ &-\omega U' - \kappa^3 U''' + 3\kappa(UZ' + ZU') = 0. \end{aligned} \tag{4.2}$$

Integrating both equations in (4.2) with respect to ξ while keeping the integration constant equal to zero yields:

$$Z = \frac{\kappa}{\omega}U^2. \tag{4.3}$$

Putting (4.3) in the first part of (4.2) yields the single NODE shown below:

$$\kappa^3\omega U'' + \omega^2 U - 3\kappa^2 U^3 = 0. \tag{4.4}$$

Soliton solutions for CCDSWS

Establishing a homogeneous balance between the highest order derivative U'' and the nonlinear term U^3 yields $N+2 = 3N$, implying that $N = 1$. We obtain the following series type of result for (4.4) by substituting $N = 1$ in (3.3):

$$U(\xi) = \sum_{i=-1}^{1} k_i(r + \Psi(\xi))^i = k_{-1}(r + \Psi(\xi))^{-1} + k_0 + k_1(r + \Psi(\xi))^1. \tag{4.5}$$

Substituting (4.5) into (4.4) and grouping together the terms containing the same powers of $\Psi(\xi)$ we obtain the expression $\Psi(\xi)$. A set of nonlinear algebraic equations is obtained by setting each coefficient equal to zero. Applying Maple to investigate the scheme yields the following two cases of results:

Case 1.

$$\begin{aligned} &k_{-1} = \frac{\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(j - rk + r^2 l\right)}{\sqrt{3}}, k_0 = \frac{\sqrt{H}\,(-2\,lr + k)\,(\ln(\varrho))^2\,\kappa^2}{2\sqrt{3}}, \\ &k_1 = 0, \kappa = \kappa, \omega = \frac{\kappa^3\,(\ln(\varrho))^2\,H}{2}. \end{aligned} \tag{4.6}$$

Case 2.

$$k_{-1} = 0, k_0 = \frac{k_1\,(-2\,lr + k)}{2l}, k_1 = k_1, \kappa = \frac{k_1\sqrt[4]{3}}{\sqrt[4]{\Lambda}\ln(\varrho)\,l}, \omega = \frac{3^{\frac{3}{4}}\sqrt[4]{\Lambda}k_1}{2\ln(\varrho)\,l}. \tag{4.7}$$

Where $\Lambda = \frac{k_1{}^2 H}{l^2}$.

Taking into account case 1 and using (4.1), (4.3), and (4.5) together with the corresponding solution of (3.4), we construct the following families of soliton results for (1.1):

Family 1.1. When $H < 0,\ l \neq 0$:

$$\begin{aligned} u_{1,1}(x,t) &= \frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(-H + (-2\,lr + k)\ \sqrt{-H}\tan_\varrho\left(\frac{1}{2}\ \sqrt{-H}\xi\right)\right)}{2\,lr - k + \ \sqrt{-H}\tan_\varrho\left(\frac{1}{2}\ \sqrt{-H}\xi\right)}, \\ z_{1,1}(x,t) &= \frac{\kappa}{\omega}\left(\frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(-H + (-2\,lr + k)\ \sqrt{-H}\tan_\varrho\left(\frac{1}{2}\ \sqrt{-H}\xi\right)\right)}{2\,lr - k + \ \sqrt{-H}\tan_\varrho\left(\frac{1}{2}\ \sqrt{-H}\xi\right)}\right)^2, \end{aligned} \tag{4.8}$$

$$\begin{aligned} u_{1,2}(x,t) &= \frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(H + (-2\,lr + k)\ \sqrt{-H}\cot_\varrho\left(\frac{1}{2}\ \sqrt{-H}\xi\right)\right)}{-2\,lr + k + \ \sqrt{-H}\cot_\varrho\left(\frac{1}{2}\ \sqrt{-H}\xi\right)}, \\ z_{1,2}(x,t) &= \frac{\kappa}{\omega}\left(\frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(H + (-2\,lr + k)\ \sqrt{-H}\cot_\varrho\left(\frac{1}{2}\ \sqrt{-H}\xi\right)\right)}{-2\,lr + k + \ \sqrt{-H}\cot_\varrho\left(\frac{1}{2}\ \sqrt{-H}\xi\right)}\right)^2, \end{aligned} \tag{4.9}$$

$$\begin{aligned} u_{1,3}(x,t) &= \frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(H + (-2\,lr + k)\ \sqrt{-H}\cot_\varrho\left(\frac{1}{2}\ \sqrt{-H}\xi\right)\right)}{-2\,lr + k + \ \sqrt{-H}\cot_\varrho\left(\frac{1}{2}\ \sqrt{-H}\xi\right)}, \\ z_{1,3}(x,t) &= \frac{\kappa}{\omega}\left(\frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(H + (-2\,lr + k)\ \sqrt{-H}\cot_\varrho\left(\frac{1}{2}\ \sqrt{-H}\xi\right)\right)}{-2\,lr + k + \ \sqrt{-H}\cot_\varrho\left(\frac{1}{2}\ \sqrt{-H}\xi\right)}\right)^2, \end{aligned} \tag{4.10}$$

$$\begin{aligned} u_{1,4}(x,t) &= \frac{\frac{1}{\sqrt{3}}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(j - rk + r^2 l\right)}{\left(r - \frac{1}{2}\,\frac{k}{l} - \frac{1}{2}\,\frac{\sqrt{-H}\left(\cot_\varrho\left(\sqrt{-H}\xi\right) + \sqrt{pq}\csc_\varrho\left(\sqrt{-H}\xi\right)\right)}{l}\right)} + \frac{1}{2\sqrt{3}}\,\sqrt{H}\,(-2\,lr + k)\,(\ln(\varrho))^2\,\kappa^2, \\ z_{1,4}(x,t) &= \frac{\kappa}{\omega}\left(\frac{\frac{1}{\sqrt{3}}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(j - rk + r^2 l\right)}{\left(r - \frac{1}{2}\,\frac{k}{l} - \frac{1}{2}\,\frac{\sqrt{-H}\left(\cot_\varrho\left(\sqrt{-H}\xi\right) + \sqrt{pq}\csc_\varrho\left(\sqrt{-H}\xi\right)\right)}{l}\right)} + \frac{1}{2\sqrt{3}}\,\sqrt{H}\,(-2\,lr + k)\,(\ln(\varrho))^2\,\kappa^2\right)^2, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} u_{1,5}(x,t) &= \frac{\frac{1}{\sqrt{3}}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(j - rk + r^2 l\right)}{\left(r - \frac{1}{2}\,\frac{k}{l} + \frac{1}{4}\,\frac{\sqrt{-H}\left(\tan_\varrho\left(\frac{1}{4}\ \sqrt{-H}\xi\right) - \cot_\varrho\left(\frac{1}{4}\ \sqrt{-H}\xi\right)\right)}{l}\right)} + \frac{1}{2\sqrt{3}}\,\sqrt{H}\,(-2\,lr + k)\,(\ln(\varrho))^2\,\kappa^2, \\ z_{1,5}(x,t) &= \frac{\kappa}{\omega}\left(\frac{\frac{1}{\sqrt{3}}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(j - rk + r^2 l\right)}{\left(r - \frac{1}{2}\,\frac{k}{l} + \frac{1}{4}\,\frac{\sqrt{-H}\left(\tan_\varrho\left(\frac{1}{4}\ \sqrt{-H}\xi\right) - \cot_\varrho\left(\frac{1}{4}\ \sqrt{-H}\xi\right)\right)}{l}\right)} + \frac{1}{2\sqrt{3}}\,\sqrt{H}\,(-2\,lr + k)\,(\ln(\varrho))^2\,\kappa^2\right)^2. \end{aligned} \tag{4.12}$$

Family 1.2. When $H > 0,\ l \neq 0$:

$$
\begin{aligned}
u_{1,6}(x,t) &= \frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(H + (-2\,lr + k)\,\sqrt{H}\tanh_\varrho\left(\frac{1}{2}\,\sqrt{H}\xi\right)\right)}{-2\,lr + k + \sqrt{H}\tanh_\varrho\left(\frac{1}{2}\,\sqrt{H}\xi\right)},\\
z_{1,6}(x,t) &= \frac{\kappa}{\omega}\left(\frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(H + (-2\,lr + k)\,\sqrt{H}\tanh_\varrho\left(\frac{1}{2}\,\sqrt{H}\xi\right)\right)}{-2\,lr + k + \sqrt{H}\tanh_\varrho\left(\frac{1}{2}\,\sqrt{H}\xi\right)}\right)^2,
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
u_{1,7}(x,t) &= \frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(H + (-2\,lr + k)\,\sqrt{H}\coth_\varrho\left(\frac{1}{2}\,\sqrt{H}\xi\right)\right)}{-2\,lr + k + \sqrt{H}\coth_\varrho\left(\frac{1}{2}\,\sqrt{H}\xi\right)},\\
z_{1,7}(x,t) &= \frac{\kappa}{\omega}\left(\frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(H + (-2\,lr + k)\,\sqrt{H}\coth_\varrho\left(\frac{1}{2}\,\sqrt{H}\xi\right)\right)}{-2\,lr + k + \sqrt{H}\coth_\varrho\left(\frac{1}{2}\,\sqrt{H}\xi\right)}\right)^2,
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
u_{1,8}(x,t) &= \frac{\frac{1}{\sqrt{3}}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(j - rk + r^2 l\right)}{\left(r - \frac{1}{2}\,\frac{k}{l} - \frac{1}{2}\,\frac{\sqrt{H}\left(\tanh_\varrho\left(\sqrt{H}\xi\right) + \sqrt{-pq}sech_\varrho\left(\sqrt{H}\xi\right)\right)}{l}\right)} + \frac{1}{2\sqrt{3}}\,\sqrt{H}\,(-2\,lr + k)\,(\ln(\varrho))^2\,\kappa^2,\\
z_{1,8}(x,t) &= \frac{\kappa}{\omega}\left(\frac{\frac{1}{\sqrt{3}}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(j - rk + r^2 l\right)}{\left(r - \frac{1}{2}\,\frac{k}{l} - \frac{1}{2}\,\frac{\sqrt{H}\left(\tanh_\varrho\left(\sqrt{H}\xi\right) + \sqrt{-pq}sech_\varrho\left(\sqrt{H}\xi\right)\right)}{l}\right)} + \frac{1}{2\sqrt{3}}\,\sqrt{H}\,(-2\,lr + k)\,(\ln(\varrho))^2\,\kappa^2\right)^2,
\end{aligned}
\tag{4.15}
$$

$$
\begin{aligned}
u_{1,9}(x,t) &= \frac{\frac{1}{\sqrt{3}}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(j - rk + r^2 l\right)}{\left(r - \frac{1}{2}\,\frac{k}{l} - \frac{1}{2}\,\frac{\sqrt{H}\left(\coth_\varrho\left(\sqrt{H}\xi\right) + \sqrt{pq}csch_\varrho\left(\sqrt{H}\xi\right)\right)}{l}\right)} + \frac{1}{2\sqrt{3}}\,\sqrt{H}\,(-2\,lr + k)\,(\ln(\varrho))^2\,\kappa^2,\\
z_{1,9}(x,t) &= \frac{\kappa}{\omega}\left(\frac{\frac{1}{\sqrt{3}}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(j - rk + r^2 l\right)}{\left(r - \frac{1}{2}\,\frac{k}{l} - \frac{1}{2}\,\frac{\sqrt{H}\left(\coth_\varrho\left(\sqrt{H}\xi\right) + \sqrt{pq}csch_\varrho\left(\sqrt{H}\xi\right)\right)}{l}\right)} + \frac{1}{2\sqrt{3}}\,\sqrt{H}\,(-2\,lr + k)\,(\ln(\varrho))^2\,\kappa^2\right)^2,
\end{aligned}
\tag{4.16}
$$

and

$$
\begin{aligned}
u_{1,10}(x,t) &= \frac{\frac{1}{\sqrt{3}}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(j - rk + r^2 l\right)}{\left(r - \frac{1}{2}\,\frac{k}{l} - \frac{1}{4}\,\frac{\sqrt{H}\left(\tanh_\varrho\left(\frac{1}{4}\,\sqrt{H}\xi\right) - \coth_\varrho\left(\frac{1}{4}\,\sqrt{H}\xi\right)\right)}{l}\right)} + \frac{1}{2\sqrt{3}}\,\sqrt{H}\,(-2\,lr + k)\,(\ln(\varrho))^2\,\kappa^2,\\
z_{1,10}(x,t) &= \frac{\kappa}{\omega}\left(\frac{\frac{1}{\sqrt{3}}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(j - rk + r^2 l\right)}{\left(r - \frac{1}{2}\,\frac{k}{l} - \frac{1}{4}\,\frac{\sqrt{H}\left(\tanh_\varrho\left(\frac{1}{4}\,\sqrt{H}\xi\right) - \coth_\varrho\left(\frac{1}{4}\,\sqrt{H}\xi\right)\right)}{l}\right)} + \frac{1}{2\sqrt{3}}\,\sqrt{H}\,(-2\,lr + k)\,(\ln(\varrho))^2\,\kappa^2\right)^2.
\end{aligned}
\tag{4.17}
$$

Family 1.3. When $jl > 0$ and $k = 0$:

$$
\begin{aligned}
u_{1,11}(x,t) &= \frac{-\frac{1}{\sqrt{3}}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(-j + lr\sqrt{\frac{j}{l}}\tan_\varrho\left(\sqrt{lj}\xi\right)\right)}{\left(r + \sqrt{\frac{j}{l}}\tan_\varrho\left(\sqrt{lj}\xi\right)\right)},\\
z_{1,11}(x,t) &= \frac{\kappa}{\omega}\left(\frac{-\frac{1}{\sqrt{3}}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(-j + lr\sqrt{\frac{j}{l}}\tan_\varrho\left(\sqrt{lj}\xi\right)\right)}{\left(r + \sqrt{\frac{j}{l}}\tan_\varrho\left(\sqrt{lj}\xi\right)\right)}\right)^2,
\end{aligned}
\tag{4.18}
$$

$$u_{1,12}(x,t)=\frac{-\frac{1}{\sqrt{3}}\sqrt{H}\,(\ln(\varrho))^2\kappa^2\left(j+lr\sqrt{\frac{j}{l}}\cot_\varrho\left(\sqrt{lj}\xi\right)\right)}{\left(-r+\sqrt{\frac{j}{l}}\cot_\varrho\left(\sqrt{lj}\xi\right)\right)},$$
$$z_{1,12}(x,t)=\frac{\kappa}{\omega}\left(\frac{-\frac{1}{\sqrt{3}}\sqrt{H}\,(\ln(\varrho))^2\kappa^2\left(j+lr\sqrt{\frac{j}{l}}\cot_\varrho\left(\sqrt{lj}\xi\right)\right)}{\left(-r+\sqrt{\frac{j}{l}}\cot_\varrho\left(\sqrt{lj}\xi\right)\right)}\right)^2, \tag{4.19}$$

$$u_{1,13}(x,t)=\frac{-\frac{1}{\sqrt{3}}\sqrt{H}\,(\ln(\varrho))^2\kappa^2\left(-j\cos_\varrho\left(2\sqrt{lj}\xi\right)+lr\sqrt{\frac{j}{l}}\sin_\varrho\left(2\sqrt{lj}\xi\right)+lr\sqrt{\frac{j}{l}}\sqrt{pq}\right)}{\left(r\cos_\varrho\left(2\sqrt{lj}\xi\right)+\sqrt{\frac{j}{l}}\sin_\varrho\left(2\sqrt{lj}\xi\right)+\sqrt{\frac{j}{l}}\sqrt{pq}\right)},$$
$$z_{1,13}(x,t)=\frac{\kappa}{\omega}\left(\frac{-\frac{1}{\sqrt{3}}\sqrt{H}\,(\ln(\varrho))^2\kappa^2\left(-j\cos_\varrho\left(2\sqrt{lj}\xi\right)+lr\sqrt{\frac{j}{l}}\sin_\varrho\left(2\sqrt{lj}\xi\right)+lr\sqrt{\frac{j}{l}}\sqrt{pq}\right)}{\left(r\cos_\varrho\left(2\sqrt{lj}\xi\right)+\sqrt{\frac{j}{l}}\sin_\varrho\left(2\sqrt{lj}\xi\right)+\sqrt{\frac{j}{l}}\sqrt{pq}\right)}\right)^2, \tag{4.20}$$

$$u_{1,14}(x,t)=\frac{-\frac{1}{\sqrt{3}}\sqrt{H}\,(\ln(\varrho))^2\kappa^2\left(j\sin_\varrho\left(2\sqrt{lj}\xi\right)+lr\sqrt{\frac{j}{l}}\cos_\varrho\left(2\sqrt{lj}\xi\right)+lr\sqrt{\frac{j}{l}}\sqrt{pq}\right)}{\left(-r\sin_\varrho\left(2\sqrt{lj}\xi\right)+\sqrt{\frac{j}{l}}\cos_\varrho\left(2\sqrt{lj}\xi\right)+\sqrt{\frac{j}{l}}\sqrt{pq}\right)},$$
$$z_{1,14}(x,t)=\frac{\kappa}{\omega}\left(\frac{-\frac{1}{\sqrt{3}}\sqrt{H}\,(\ln(\varrho))^2\kappa^2\left(j\sin_\varrho\left(2\sqrt{lj}\xi\right)+lr\sqrt{\frac{j}{l}}\cos_\varrho\left(2\sqrt{lj}\xi\right)+lr\sqrt{\frac{j}{l}}\sqrt{pq}\right)}{\left(-r\sin_\varrho\left(2\sqrt{lj}\xi\right)+\sqrt{\frac{j}{l}}\cos_\varrho\left(2\sqrt{lj}\xi\right)+\sqrt{\frac{j}{l}}\sqrt{pq}\right)}\right)^2, \tag{4.21}$$

and

$$u_{1,15}(x,t)=\frac{-\frac{1}{\sqrt{3}}\sqrt{H}\,(\ln(\varrho))^2\kappa^2\left(2j\cos_\varrho\left(\frac{1}{2}\sqrt{lj}\xi\right)\sin_\varrho\left(\frac{1}{2}\sqrt{lj}\xi\right)-lr\sqrt{\frac{j}{l}}+2lr\sqrt{\frac{j}{l}}\left(\cos_\varrho\left(\frac{1}{2}\sqrt{lj}\xi\right)\right)^2\right)}{\left(-2r\cos_\varrho\left(\frac{1}{2}\sqrt{lj}\xi\right)\sin_\varrho\left(\frac{1}{2}\sqrt{lj}\xi\right)-\sqrt{\frac{j}{l}}+2\sqrt{\frac{j}{l}}\left(\cos_\varrho\left(\frac{1}{2}\sqrt{lj}\xi\right)\right)^2\right)},$$
$$z_{1,15}(x,t)=\frac{\kappa}{\omega}\left(\frac{\sqrt{H}\,(\ln(\varrho))^2\kappa^2\left(2j\cos_\varrho\left(\frac{1}{2}\sqrt{lj}\xi\right)\sin_\varrho\left(\frac{1}{2}\sqrt{lj}\xi\right)-lr\sqrt{\frac{j}{l}}+2lr\sqrt{\frac{j}{l}}\left(\cos_\varrho\left(\frac{1}{2}\sqrt{lj}\xi\right)\right)^2\right)}{-\sqrt{3}\left(-2r\cos_\varrho\left(\frac{1}{2}\sqrt{lj}\xi\right)\sin_\varrho\left(\frac{1}{2}\sqrt{lj}\xi\right)-\sqrt{\frac{j}{l}}+2\sqrt{\frac{j}{l}}\left(\cos_\varrho\left(\frac{1}{2}\sqrt{lj}\xi\right)\right)^2\right)}\right)^2. \tag{4.22}$$

Family 1.4. When $jl > 0$ and $k = 0$:

$$u_{1,16}(x,t)=\frac{-\frac{1}{\sqrt{3}}\sqrt{H}\,(\ln(\varrho))^2\kappa^2\left(j+lr\sqrt{-\frac{j}{l}}\tanh_\varrho\left(\sqrt{-lj}\xi\right)\right)}{\left(-r+\sqrt{-\frac{j}{l}}\tanh_\varrho\left(\sqrt{-lj}\xi\right)\right)},$$
$$z_{1,16}(x,t)=\frac{\kappa}{\omega}\left(\frac{-\frac{1}{\sqrt{3}}\sqrt{H}\,(\ln(\varrho))^2\kappa^2\left(j+lr\sqrt{-\frac{j}{l}}\tanh_\varrho\left(\sqrt{-lj}\xi\right)\right)}{\left(-r+\sqrt{-\frac{j}{l}}\tanh_\varrho\left(\sqrt{-lj}\xi\right)\right)}\right)^2, \tag{4.23}$$

$$u_{1,17}(x,t) = \frac{-\frac{1}{\sqrt{3}}\sqrt{H}\left(\ln\left(\varrho\right)\right)^{2}\kappa^{2}\left(j + lr\sqrt{-\frac{j}{l}}\coth_{\varrho}\left(\sqrt{-lj}\xi\right)\right)}{\left(-r + \sqrt{-\frac{j}{l}}\coth_{\varrho}\left(\sqrt{-lj}\xi\right)\right)},$$
$$z_{1,17}(x,t) = \frac{\kappa}{\omega}\left(\frac{-\frac{1}{\sqrt{3}}\sqrt{H}\left(\ln\left(\varrho\right)\right)^{2}\kappa^{2}\left(j + lr\sqrt{-\frac{j}{l}}\coth_{\varrho}\left(\sqrt{-lj}\xi\right)\right)}{\left(-r + \sqrt{-\frac{j}{l}}\coth_{\varrho}\left(\sqrt{-lj}\xi\right)\right)}\right)^{2}, \tag{4.24}$$

$$u_{1,18}(x,t) = \frac{-\frac{1}{\sqrt{3}}\sqrt{H}\left(\ln\left(\varrho\right)\right)^{2}\kappa^{2}\left(j\cosh_{\varrho}\left(2\sqrt{-lj}\xi\right) + lr\sqrt{-\frac{j}{l}}\sinh_{\varrho}\left(2\sqrt{-lj}\xi\right) + lr\sqrt{-\frac{j}{l}}\sqrt{-pq}\right)}{\left(-r\cosh_{\varrho}\left(2\sqrt{-lj}\xi\right) + \sqrt{-\frac{j}{l}}\sinh_{\varrho}\left(2\sqrt{-lj}\xi\right) + \sqrt{-\frac{j}{l}}\sqrt{-pq}\right)},$$
$$z_{1,18}(x,t) = \frac{\kappa}{\omega}\left(\frac{-\frac{1}{\sqrt{3}}\sqrt{H}\left(\ln\left(\varrho\right)\right)^{2}\kappa^{2}\left(j\cosh_{\varrho}\left(2\sqrt{-lj}\xi\right) + lr\sqrt{-\frac{j}{l}}\sinh_{\varrho}\left(2\sqrt{-lj}\xi\right) + lr\sqrt{-\frac{j}{l}}\sqrt{-pq}\right)}{\left(-r\cosh_{\varrho}\left(2\sqrt{-lj}\xi\right) + \sqrt{-\frac{j}{l}}\sinh_{\varrho}\left(2\sqrt{-lj}\xi\right) + \sqrt{-\frac{j}{l}}\sqrt{-pq}\right)}\right)^{2}, \tag{4.25}$$

$$u_{1,19}(x,t) = \frac{-\frac{1}{\sqrt{3}}\sqrt{H}\left(\ln\left(\varrho\right)\right)^{2}\kappa^{2}\left(j\sinh_{\varrho}\left(2\sqrt{-lj}\xi\right) + lr\sqrt{-\frac{j}{l}}\cosh_{\varrho}\left(2\sqrt{-lj}\xi\right) + lr\sqrt{-\frac{j}{l}}\sqrt{pq}\right)}{\left(-r\sinh_{\varrho}\left(2\sqrt{-lj}\xi\right) + \sqrt{-\frac{j}{l}}\cosh_{\varrho}\left(2\sqrt{-lj}\xi\right) + \sqrt{-\frac{j}{l}}\sqrt{pq}\right)},$$
$$z_{1,19}(x,t) = \frac{\kappa}{\omega}\left(\frac{-\frac{1}{\sqrt{3}}\sqrt{H}\left(\ln\left(\varrho\right)\right)^{2}\kappa^{2}\left(j\sinh_{\varrho}\left(2\sqrt{-lj}\xi\right) + lr\sqrt{-\frac{j}{l}}\cosh_{\varrho}\left(2\sqrt{-lj}\xi\right) + lr\sqrt{-\frac{j}{l}}\sqrt{pq}\right)}{\left(-r\sinh_{\varrho}\left(2\sqrt{-lj}\xi\right) + \sqrt{-\frac{j}{l}}\cosh_{\varrho}\left(2\sqrt{-lj}\xi\right) + \sqrt{-\frac{j}{l}}\sqrt{pq}\right)}\right)^{2}, \tag{4.26}$$

and

$$u_{1,20}(x,t) = \frac{-\frac{1}{\sqrt{3}}\sqrt{H}\left(\ln\left(\varrho\right)\right)^{2}\kappa^{2}\left(2\,j\Omega + 2\,lr\sqrt{-\frac{j}{l}}\left(\cosh_{\varrho}\left(\frac{1}{2}\sqrt{-lj}\xi\right)\right)^{2} - lr\sqrt{-\frac{j}{l}}\right)}{\left(-2\,r\Omega + 2\sqrt{-\frac{j}{l}}\left(\cosh_{\varrho}\left(\frac{1}{2}\sqrt{-lj}\xi\right)\right)^{2} - \sqrt{-\frac{j}{l}}\right)},$$
$$z_{1,20}(x,t) = \frac{\kappa}{\omega}\left(\frac{-\frac{1}{\sqrt{3}}\sqrt{H}\left(\ln\left(\varrho\right)\right)^{2}\kappa^{2}\left(2\,j\Omega + 2\,lr\sqrt{-\frac{j}{l}}\left(\cosh_{\varrho}\left(\frac{1}{2}\sqrt{-lj}\xi\right)\right)^{2} - lr\sqrt{-\frac{j}{l}}\right)}{\left(-2\,r\Omega + 2\sqrt{-\frac{j}{l}}\left(\cosh_{\varrho}\left(\frac{1}{2}\sqrt{-lj}\xi\right)\right)^{2} - \sqrt{-\frac{j}{l}}\right)}\right)^{2}, \tag{4.27}$$

where

$$\Omega = \cosh_{\varrho}\left(\frac{1}{2}\sqrt{-lj}\xi\right)\sinh_{\varrho}\left(\frac{1}{2}\sqrt{-lj}\xi\right).$$

Family 1.5. When $l = j$ and $k = 0$:

$$u_{1,21}(x,t) = -\frac{1}{\sqrt{3}}\frac{\sqrt{H}\left(\ln\left(\varrho\right)\right)^{2}\kappa^{2}j\left(-1 + r\tan_{\varrho}\left(j\xi\right)\right)}{r + \tan_{\varrho}\left(j\xi\right)},$$
$$z_{1,21}(x,t) = \frac{\kappa}{\omega}\left(-\frac{1}{\sqrt{3}}\frac{\sqrt{H}\left(\ln\left(\varrho\right)\right)^{2}\kappa^{2}j\left(-1 + r\tan_{\varrho}\left(j\xi\right)\right)}{r + \tan_{\varrho}\left(j\xi\right)}\right)^{2}, \tag{4.28}$$

$$u_{1,22}(x,t) = -\frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(1 + r\cot_\varrho(j\xi)\right)}{-r + \cot_\varrho(j\xi)},$$
$$z_{1,22}(x,t) = \frac{\kappa}{\omega}\left(-\frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(1 + r\cot_\varrho(j\xi)\right)}{-r + \cot_\varrho(j\xi)}\right)^2, \tag{4.29}$$

$$u_{1,23}(x,t) = -\frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(-\cos_\varrho(2j\xi) + r\sin_\varrho(2j\xi) + r\sqrt{pq}\right)}{r\cos_\varrho(2j\xi) + \sin_\varrho(2j\xi) + \sqrt{pq}},$$
$$z_{1,23}(x,t) = \frac{\kappa}{\omega}\left(-\frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(-\cos_\varrho(2j\xi) + r\sin_\varrho(2j\xi) + r\sqrt{pq}\right)}{r\cos_\varrho(2j\xi) + \sin_\varrho(2j\xi) + \sqrt{pq}}\right)^2, \tag{4.30}$$

$$u_{1,24}(x,t) = -\frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(\sin_\varrho(2j\xi) + r\cos_\varrho(2j\xi) + r\sqrt{pq}\right)}{-r\sin_\varrho(2j\xi) + \cos_\varrho(2j\xi) + \sqrt{pq}},$$
$$z_{1,24}(x,t) = \frac{\kappa}{\omega}\left(-\frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(\sin_\varrho(2j\xi) + r\cos_\varrho(2j\xi) + r\sqrt{pq}\right)}{-r\sin_\varrho(2j\xi) + \cos_\varrho(2j\xi) + \sqrt{pq}}\right)^2, \tag{4.31}$$

and

$$u_{1,25}(x,t) = \frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(2\cos_\varrho\left(\frac{1}{2}j\xi\right)\sin_\varrho\left(\frac{1}{2}j\xi\right) - r + 2r\left(\cos_\varrho\left(\frac{1}{2}j\xi\right)\right)^2\right)}{2r\cos_\varrho\left(\frac{1}{2}j\xi\right)\sin_\varrho\left(\frac{1}{2}j\xi\right) + 1 - 2\left(\cos_\varrho\left(\frac{1}{2}j\xi\right)\right)^2},$$
$$z_{1,25}(x,t) = \frac{\kappa}{\omega}\left(\frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(2\cos_\varrho\left(\frac{1}{2}j\xi\right)\sin_\varrho\left(\frac{1}{2}j\xi\right) - r + 2r\left(\cos_\varrho\left(\frac{1}{2}j\xi\right)\right)^2\right)}{2r\cos_\varrho\left(\frac{1}{2}j\xi\right)\sin_\varrho\left(\frac{1}{2}j\xi\right) + 1 - 2\left(\cos_\varrho\left(\frac{1}{2}j\xi\right)\right)^2}\right)^2. \tag{4.32}$$

Family 1.6. When $l = -j$ and $k = 0$:

$$u_{1,26}(x,t) = \frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(-1 + r\tanh_\varrho(j\xi)\right)}{-r + \tanh_\varrho(j\xi)},$$
$$z_{1,26}(x,t) = \frac{\kappa}{\omega}\left(\frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(-1 + r\tanh_\varrho(j\xi)\right)}{-r + \tanh_\varrho(j\xi)}\right)^2, \tag{4.33}$$

$$u_{1,27}(x,t) = \frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(-1 + r\coth_\varrho(j\xi)\right)}{-r + \coth_\varrho(j\xi)},$$
$$z_{1,27}(x,t) = \frac{\kappa}{\omega}\left(\frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(-1 + r\coth_\varrho(j\xi)\right)}{-r + \coth_\varrho(j\xi)}\right)^2, \tag{4.34}$$

$$u_{1,28}(x,t) = \frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(-\cosh_\varrho(2j\xi) + r\sinh_\varrho(2j\xi) + r\sqrt{-pq}\right)}{-r\cosh_\varrho(2j\xi) + \sinh_\varrho(2j\xi) + \sqrt{-pq}},$$
$$z_{1,28}(x,t) = \frac{\kappa}{\omega}\left(\frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(-\cosh_\varrho(2j\xi) + r\sinh_\varrho(2j\xi) + r\sqrt{-pq}\right)}{-r\cosh_\varrho(2j\xi) + \sinh_\varrho(2j\xi) + \sqrt{-pq}}\right)^2, \tag{4.35}$$

$$
\begin{aligned}
u_{1,29}(x,t) &= \frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(-\sinh_\varrho(2\,j\xi)+r\cosh_\varrho(2\,j\xi)+r\sqrt{pq}\right)}{-r\sinh_\varrho(2\,j\xi)+\cosh_\varrho(2\,j\xi)+\sqrt{pq}},\\
z_{1,29}(x,t) &= \frac{\kappa}{\omega}\left(\frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(-\sinh_\varrho(2\,j\xi)+r\cosh_\varrho(2\,j\xi)+r\sqrt{pq}\right)}{-r\sinh_\varrho(2\,j\xi)+\cosh_\varrho(2\,j\xi)+\sqrt{pq}}\right)^2,
\end{aligned}
\tag{4.36}
$$

and

$$
\begin{aligned}
u_{1,30}(x,t) &= \frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(2\cosh_\varrho\left(\frac{1}{2}\,j\xi\right)\sinh_\varrho\left(\frac{1}{2}\,j\xi\right)-2\,r\left(\cosh_\varrho\left(\frac{1}{2}\,j\xi\right)\right)^2+r\right)}{2\,r\cosh_\varrho\left(\frac{1}{2}\,j\xi\right)\sinh_\varrho\left(\frac{1}{2}\,j\xi\right)-2\left(\cosh_\varrho\left(\frac{1}{2}\,j\xi\right)\right)^2+1},\\
z_{1,30}(x,t) &= \frac{\kappa}{\omega}\left(\frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j\left(2\cosh_\varrho\left(\frac{1}{2}\,j\xi\right)\sinh_\varrho\left(\frac{1}{2}\,j\xi\right)-2\,r\left(\cosh_\varrho\left(\frac{1}{2}\,j\xi\right)\right)^2+r\right)}{2\,r\cosh_\varrho\left(\frac{1}{2}\,j\xi\right)\sinh_\varrho\left(\frac{1}{2}\,j\xi\right)-2\left(\cosh_\varrho\left(\frac{1}{2}\,j\xi\right)\right)^2+1}\right)^2.
\end{aligned}
\tag{4.37}
$$

Family 1.7. When $H = 0$:

$$
\begin{aligned}
u_{1,31}(x,t) &= \frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(2\,k^2\xi\,\ln(\varrho)\,j-k^3\xi\,\ln(\varrho)\,r+4\,lrj\,(B)-2\,kj\,(B)\right)}{rk^2\xi\,\ln(\varrho)-2\,j\,(B)},\\
z_{1,31}(x,t) &= \frac{\kappa}{\omega}\left(\frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\left(2\,k^2\xi\,\ln(\varrho)\,j-k^3\xi\,\ln(\varrho)\,r+4\,lrj\,(B)-2\,kj\,(B)\right)}{rk^2\xi\,\ln(\varrho)-2\,j\,(B)}\right)^2,
\end{aligned}
\tag{4.38}
$$

where $B = k\xi\,\ln(\varrho) + 2$.

Family 1.8. When $k = \varpi$, $j = s\varpi (s \neq 0)$ and $l = 0$:

$$
\begin{aligned}
u_{1,32}(x,t) &= \frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\varpi\,(\ln(\varrho))^2\,\kappa^2\left(s-r+\varrho^{\varpi\xi}\right)}{r+\varrho^{\varpi\xi}-s},\\
z_{1,32}(x,t) &= \frac{\kappa}{\omega}\left(\frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\varpi\,(\ln(\varrho))^2\,\kappa^2\left(s-r+\varrho^{\varpi\xi}\right)}{r+\varrho^{\varpi\xi}-s}\right)^2.
\end{aligned}
\tag{4.39}
$$

Family 1.9. When $k = l = 0$:

$$
\begin{aligned}
u_{1,33}(x,t) &= \frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j}{r+j\xi\,\ln(\varrho)},\\
z_{1,33}(x,t) &= \frac{\kappa}{\omega}\left(\frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 j}{r+j\xi\,\ln(\varrho)}\right)^2.
\end{aligned}
\tag{4.40}
$$

Family 1.10. When $j = k = 0$:

$$
\begin{aligned}
u_{1,34}(x,t) &= \frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}lr\,(\ln(\varrho))^2\,\kappa^2}{rl\xi\,\ln(\varrho)-1},\\
z_{1,34}(x,t) &= \frac{\kappa}{\omega}\left(\frac{1}{3}\,\frac{\sqrt{3}\,\sqrt{H}lr\,(\ln(\varrho))^2\,\kappa^2}{rl\xi\,\ln(\varrho)-1}\right)^2.
\end{aligned}
\tag{4.41}
$$

Family 1.11. When $j = 0$, $k \neq 0$ and $l \neq 0$:

$$u_{1,35}(x,t) = -\frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 k\left(lr\cosh_\varrho(k\xi) - lr\sinh_\varrho(k\xi) - lrp + pk\right)}{lr\cosh_\varrho(k\xi) - lr\sinh_\varrho(k\xi) + lrp - pk},$$
$$z_{1,35}(x,t) = \frac{\kappa}{\omega}\left(-\frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 k\left(lr\cosh_\varrho(k\xi) - lr\sinh_\varrho(k\xi) - lrp + pk\right)}{lr\cosh_\varrho(k\xi) - lr\sinh_\varrho(k\xi) + lrp - pk}\right)^2, \tag{4.42}$$

and

$$u_{1,36}(x,t) = \frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 k\left(lrq + (k-lr)\cosh_\varrho(k\xi) + (k-lr)\sinh_\varrho(k\xi)\right)}{-lrq + (k-lr)\cosh_\varrho(k\xi) + (k-lr)\sinh_\varrho(k\xi)},$$
$$z_{1,36}(x,t) = \frac{\kappa}{\omega}\left(\frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2 k\left(lrq + (k-lr)\cosh_\varrho(k\xi) + (k-lr)\sinh_\varrho(k\xi)\right)}{-lrq + (k-lr)\cosh_\varrho(k\xi) + (k-lr)\sinh_\varrho(k\xi)}\right)^2. \tag{4.43}$$

Family 1.12. When $k = \varpi$, $l = s\varpi (s \neq 0)$ and $j = 0$:

$$u_{1,37}(x,t) = -\frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\varpi\left(rp - rs\varrho^{\varpi\xi} + 2\,rsp\varrho^{\varpi\xi} - p\varrho^{\varpi\xi}\right)}{rp - rs\varrho^{\varpi\xi} + p\varrho^{\varpi\xi}},$$
$$z_{1,37}(x,t) = \frac{\kappa}{\omega}\left(-\frac{1}{6}\,\frac{\sqrt{3}\,\sqrt{H}\,(\ln(\varrho))^2\,\kappa^2\varpi\left(rp - rs\varrho^{\varpi\xi} + 2\,rsp\varrho^{\varpi\xi} - p\varrho^{\varpi\xi}\right)}{rp - rs\varrho^{\varpi\xi} + p\varrho^{\varpi\xi}}\right)^2, \tag{4.44}$$

where

$$\xi = \kappa\frac{x^\varsigma}{\varsigma} - \frac{\kappa^3(\ln(\varrho))^2 H}{2}\frac{t^\sigma}{\sigma}.$$

Now, taking into account case 2 and utilizing (4.1), (4.3), and (4.5) together with the corresponding solution of (3.4), we construct the following families of soliton solutions for (1.1).

Family 2.1. When $H < 0$, $l \neq 0$:

$$u_{2,1}(x,t) = \frac{1}{2}\,\frac{k_1\sqrt{-H}\tan_\varrho\left(\frac{1}{2}\sqrt{-H}\xi\right)}{l},$$
$$z_{2,1}(x,t) = \frac{\kappa}{\omega}\left(\frac{1}{2}\,\frac{k_1\sqrt{-H}\tan_\varrho\left(\frac{1}{2}\sqrt{-H}\xi\right)}{l}\right)^2, \tag{4.45}$$

$$u_{2,2}(x,t) = -\frac{1}{2}\,\frac{k_1\sqrt{-H}\cot_\varrho\left(\frac{1}{2}\sqrt{-H}\xi\right)}{l},$$
$$z_{2,2}(x,t) = \frac{\kappa}{\omega}\left(-\frac{1}{2}\,\frac{k_1\sqrt{-H}\cot_\varrho\left(\frac{1}{2}\sqrt{-H}\xi\right)}{l}\right)^2, \tag{4.46}$$

$$u_{2,3}(x,t) = \frac{1}{2}\,\frac{k_1\sqrt{-H}\left(\sin_\varrho\left(\sqrt{-H}\xi\right) + \sqrt{pq}\right)}{\cos_\varrho\left(\sqrt{-H}\xi\right)l},$$
$$z_{2,3}(x,t) = \frac{\kappa}{\omega}\left(\frac{1}{2}\,\frac{k_1\sqrt{-H}\left(\sin_\varrho\left(\sqrt{-H}\xi\right) + \sqrt{pq}\right)}{\cos_\varrho\left(\sqrt{-H}\xi\right)l}\right)^2, \tag{4.47}$$

$$u_{2,4}(x,t) = -\frac{1}{2}\frac{k_1\sqrt{-H}\left(\cos_\varrho\left(\sqrt{-H}\xi\right)+\sqrt{pq}\right)}{\sin_\varrho\left(\sqrt{-H}\xi\right)l},$$
$$z_{2,4}(x,t) = \frac{\kappa}{\omega}\left(-\frac{1}{2}\frac{k_1\sqrt{-H}\left(\cos_\varrho\left(\sqrt{-H}\xi\right)+\sqrt{pq}\right)}{\sin_\varrho\left(\sqrt{-H}\xi\right)l}\right)^2, \tag{4.48}$$

and

$$u_{2,5}(x,t) = -\frac{1}{4}\frac{k_1\sqrt{-H}\left(-1+2\left(\cos_\varrho\left(\frac{1}{4}\sqrt{-H}\xi\right)\right)^2\right)}{\cos_\varrho\left(\frac{1}{4}\sqrt{-H}\xi\right)\sin_\varrho\left(\frac{1}{4}\sqrt{-H}\xi\right)l},$$
$$z_{2,5}(x,t) = \frac{\kappa}{\omega}\left(-\frac{1}{4}\frac{k_1\sqrt{-H}\left(-1+2\left(\cos_\varrho\left(\frac{1}{4}\sqrt{-H}\xi\right)\right)^2\right)}{\cos_\varrho\left(\frac{1}{4}\sqrt{-H}\xi\right)\sin_\varrho\left(\frac{1}{4}\sqrt{-H}\xi\right)l}\right)^2. \tag{4.49}$$

Family 2.2. When $H > 0,\ l \neq 0$:

$$u_{2,6}(x,t) = -\frac{1}{2}\frac{k_1\sqrt{H}\tanh_\varrho\left(\frac{1}{2}\sqrt{H}\xi\right)}{l},$$
$$z_{2,6}(x,t) = \frac{\kappa}{\omega}\left(-\frac{1}{2}\frac{k_1\sqrt{H}\tanh_\varrho\left(\frac{1}{2}\sqrt{H}\xi\right)}{l}\right)^2, \tag{4.50}$$

$$u_{2,7}(x,t) = -\frac{1}{2}\frac{k_1\sqrt{H}\coth_\varrho\left(\frac{1}{2}\sqrt{H}\xi\right)}{l},$$
$$z_{2,7}(x,t) = \frac{\kappa}{\omega}\left(-\frac{1}{2}\frac{k_1\sqrt{H}\coth_\varrho\left(\frac{1}{2}\sqrt{H}\xi\right)}{l}\right)^2, \tag{4.51}$$

$$u_{2,8}(x,t) = -\frac{1}{2}\frac{k_1\sqrt{H}\left(\sinh_\varrho\left(\sqrt{H}\xi\right)+\sqrt{-pq}\right)}{\cosh_\varrho\left(\sqrt{H}\xi\right)l},$$
$$z_{2,8}(x,t) = \frac{\kappa}{\omega}\left(-\frac{1}{2}\frac{k_1\sqrt{H}\left(\sinh_\varrho\left(\sqrt{H}\xi\right)+\sqrt{-pq}\right)}{\cosh_\varrho\left(\sqrt{H}\xi\right)l}\right)^2, \tag{4.52}$$

$$u_{2,9}(x,t) = -\frac{1}{2}\frac{k_1\sqrt{H}\left(\cosh_\varrho\left(\sqrt{H}\xi\right)+\sqrt{pq}\right)}{\sinh_\varrho\left(\sqrt{H}\xi\right)l},$$
$$z_{2,9}(x,t) = \frac{\kappa}{\omega}\left(-\frac{1}{2}\frac{k_1\sqrt{H}\left(\cosh_\varrho\left(\sqrt{H}\xi\right)+\sqrt{pq}\right)}{\sinh_\varrho\left(\sqrt{H}\xi\right)l}\right)^2, \tag{4.53}$$

and

$$u_{2,10}(x,t) = \frac{1}{4}\frac{k_1\sqrt{H}}{\cosh_\varrho\left(\frac{1}{4}\sqrt{H}\xi\right)\sinh_\varrho\left(\frac{1}{4}\sqrt{H}\xi\right)l},$$
$$z_{2,10}(x,t) = \frac{\kappa}{\omega}\left(\frac{1}{4}\frac{k_1\sqrt{H}}{\cosh_\varrho\left(\frac{1}{4}\sqrt{H}\xi\right)\sinh_\varrho\left(\frac{1}{4}\sqrt{H}\xi\right)l}\right)^2. \tag{4.54}$$

Family 2.3. When $jl > 0$ and $k = 0$:

$$\begin{aligned} u_{2,11}(x,t) &= k_1 \sqrt{\frac{j}{l}} \tan_\varrho\Big(\sqrt{jl}\xi\Big), \\ z_{2,11}(x,t) &= \frac{\kappa}{\omega}\Big(k_1 \sqrt{\frac{j}{l}} \tan_\varrho\Big(\sqrt{jl}\xi\Big)\Big)^2, \end{aligned} \tag{4.55}$$

$$\begin{aligned} u_{2,12}(x,t) &= -k_1 \sqrt{\frac{j}{l}} \cot_\varrho\Big(\sqrt{jl}\xi\Big), \\ z_{2,12}(x,t) &= \frac{\kappa}{\omega}\Big(-k_1 \sqrt{\frac{j}{l}} \cot_\varrho\Big(\sqrt{jl}\xi\Big)\Big)^2, \end{aligned} \tag{4.56}$$

$$\begin{aligned} u_{2,13}(x,t) &= \frac{k_1 \sqrt{\frac{j}{l}}\left(\sin_\varrho\left(2\sqrt{jl}\xi\right) + \sqrt{pq}\right)}{\left(\cos_\varrho\left(2\sqrt{jl}\xi\right)\right)}, \\ z_{2,13}(x,t) &= \frac{\kappa}{\omega}\left(\frac{k_1 \sqrt{\frac{j}{l}}\left(\sin_\varrho\left(2\sqrt{jl}\xi\right) + \sqrt{pq}\right)}{\left(\cos_\varrho\left(2\sqrt{jl}\xi\right)\right)}\right)^2, \end{aligned} \tag{4.57}$$

$$\begin{aligned} u_{2,14}(x,t) &= \frac{-k_1 \sqrt{\frac{j}{l}}\left(\cos_\varrho\left(2\sqrt{jl}\xi\right) + \sqrt{pq}\right)}{\left(\sin_\varrho\left(2\sqrt{jl}\xi\right)\right)}, \\ z_{2,14}(x,t) &= \frac{\kappa}{\omega}\left(\frac{-k_1 \sqrt{\frac{j}{l}}\left(\cos_\varrho\left(2\sqrt{jl}\xi\right) + \sqrt{pq}\right)}{\left(\sin_\varrho\left(2\sqrt{jl}\xi\right)\right)}\right)^2, \end{aligned} \tag{4.58}$$

and

$$\begin{aligned} u_{2,15}(x,t) &= \frac{-\frac{1}{2} k_1 \sqrt{\frac{j}{l}}\left(-1 + 2\left(\cos_\varrho\left(\frac{1}{2}\sqrt{jl}\xi\right)\right)^2\right)}{\left(\cos_\varrho\left(\frac{1}{2}\sqrt{jl}\xi\right)\right)\left(\sin_\varrho\left(\frac{1}{2}\sqrt{jl}\xi\right)\right)}, \\ z_{2,15}(x,t) &= \frac{\kappa}{\omega}\left(\frac{-\frac{1}{2} k_1 \sqrt{\frac{j}{l}}\left(-1 + 2\left(\cos_\varrho\left(\frac{1}{2}\sqrt{jl}\xi\right)\right)^2\right)}{\left(\cos_\varrho\left(\frac{1}{2}\sqrt{jl}\xi\right)\right)\left(\sin_\varrho\left(\frac{1}{2}\sqrt{jl}\xi\right)\right)}\right)^2. \end{aligned} \tag{4.59}$$

Family 2.4. When $jl > 0$ and $k = 0$:

$$\begin{aligned} u_{2,16}(x,t) &= -k_1 \sqrt{-\frac{j}{l}} \tanh_\varrho\Big(\sqrt{-jl}\xi\Big), \\ z_{2,16}(x,t) &= \frac{\kappa}{\omega}\Big(-k_1 \sqrt{-\frac{j}{l}} \tanh_\varrho\Big(\sqrt{-jl}\xi\Big)\Big)^2, \end{aligned} \tag{4.60}$$

$$\begin{aligned} u_{2,17}(x,t) &= -k_1 \sqrt{-\frac{j}{l}} \coth_\varrho\Big(\sqrt{-jl}\xi\Big), \\ z_{2,17}(x,t) &= \frac{\kappa}{\omega}\Big(-k_1 \sqrt{-\frac{j}{l}} \coth_\varrho\Big(\sqrt{-jl}\xi\Big)\Big)^2, \end{aligned} \tag{4.61}$$

$$\begin{aligned}
u_{2,18}(x,t) &= \frac{-k_1\sqrt{-\frac{j}{l}}\left(\sinh_\varrho\left(2\sqrt{-jl}\xi\right)+\sqrt{-pq}\right)}{\left(\cosh_\varrho\left(2\sqrt{-jl}\xi\right)\right)},\\
z_{2,18}(x,t) &= \frac{\kappa}{\omega}\Bigg(\frac{-k_1\sqrt{-\frac{j}{l}}\left(\sinh_\varrho\left(2\sqrt{-jl}\xi\right)+\sqrt{-pq}\right)}{\left(\cosh_\varrho\left(2\sqrt{-jl}\xi\right)\right)}\Bigg)^2,
\end{aligned} \tag{4.62}$$

$$\begin{aligned}
u_{2,19}(x,t) &= \frac{-k_1\sqrt{-\frac{j}{l}}\left(\cosh_\varrho\left(2\sqrt{-jl}\xi\right)+\sqrt{pq}\right)}{\left(\sinh_\varrho\left(2\sqrt{-jl}\xi\right)\right)},\\
z_{2,19}(x,t) &= \frac{\kappa}{\omega}\Bigg(\frac{-k_1\sqrt{-\frac{j}{l}}\left(\cosh_\varrho\left(2\sqrt{-jl}\xi\right)+\sqrt{pq}\right)}{\left(\sinh_\varrho\left(2\sqrt{-jl}\xi\right)\right)}\Bigg)^2,
\end{aligned} \tag{4.63}$$

and

$$\begin{aligned}
u_{2,20}(x,t) &= \frac{-\frac{1}{2}k_1\sqrt{-\frac{j}{l}}\left(2\left(\cosh_\varrho\left(1/2\sqrt{-jl}\xi\right)\right)^2-1\right)}{\left(\cosh_\varrho\left(\frac{1}{2}\sqrt{-jl}\xi\right)\right)\left(\sinh_\varrho\left(\frac{1}{2}\sqrt{-jl}\xi\right)\right)},\\
z_{2,20}(x,t) &= \frac{\kappa}{\omega}\Bigg(\frac{-\frac{1}{2}k_1\sqrt{-\frac{j}{l}}\left(2\left(\cosh_\varrho\left(1/2\sqrt{-jl}\xi\right)\right)^2-1\right)}{\left(\cosh_\varrho\left(\frac{1}{2}\sqrt{-jl}\xi\right)\right)\left(\sinh_\varrho\left(\frac{1}{2}\sqrt{-jl}\xi\right)\right)}\Bigg)^2.
\end{aligned} \tag{4.64}$$

Family 2.5. When $l = j$ and $k = 0$:

$$\begin{aligned}
u_{2,21}(x,t) &= k_1\tan_\varrho(j\xi),\\
z_{2,21}(x,t) &= \frac{\kappa}{\omega}\Big(k_1\tan_\varrho(j\xi)\Big)^2,
\end{aligned} \tag{4.65}$$

$$\begin{aligned}
u_{2,22}(x,t) &= -k_1\cot_\varrho(j\xi),\\
z_{2,22}(x,t) &= \frac{\kappa}{\omega}\Big(-k_1\cot_\varrho(j\xi)\Big)^2,
\end{aligned} \tag{4.66}$$

$$\begin{aligned}
u_{2,23}(x,t) &= \frac{k_1\left(\sin_\varrho(2j\xi)+\sqrt{pq}\right)}{\cos_\varrho(2j\xi)},\\
z_{2,23}(x,t) &= \frac{\kappa}{\omega}\Bigg(\frac{k_1\left(\sin_\varrho(2j\xi)+\sqrt{pq}\right)}{\cos_\varrho(2j\xi)}\Bigg)^2,
\end{aligned} \tag{4.67}$$

$$\begin{aligned}
u_{2,24}(x,t) &= -\frac{k_1\left(\cos_\varrho(2j\xi)+\sqrt{pq}\right)}{\sin_\varrho(2j\xi)},\\
z_{2,24}(x,t) &= \frac{\kappa}{\omega}\Bigg(-\frac{k_1\left(\cos_\varrho(2j\xi)+\sqrt{pq}\right)}{\sin_\varrho(2j\xi)}\Bigg)^2,
\end{aligned} \tag{4.68}$$

and

$$u_{2,25}(x,t) = -\frac{1}{2}\,\frac{k_1\left(-1+2\left(\cos_\varrho\left(\frac{1}{2}\,j\xi\right)\right)^2\right)}{\cos_\varrho\left(\frac{1}{2}\,j\xi\right)\sin_\varrho\left(\frac{1}{2}\,j\xi\right)},$$
$$z_{2,25}(x,t) = \frac{\kappa}{\omega}\Bigg(-\frac{1}{2}\,\frac{k_1\left(-1+2\left(\cos_\varrho\left(\frac{1}{2}\,j\xi\right)\right)^2\right)}{\cos_\varrho\left(\frac{1}{2}\,j\xi\right)\sin_\varrho\left(\frac{1}{2}\,j\xi\right)}\Bigg)^2. \tag{4.69}$$

Family 2.6. When $l = -j$ and $k = 0$:

$$u_{2,26}(x,t) = -k_1\tanh_\varrho(j\xi), \qquad z_{2,26}(x,t) = \frac{\kappa}{\omega}\Big(-k_1\tanh_\varrho(j\xi)\Big)^2, \tag{4.70}$$

$$u_{2,27}(x,t) = -k_1\coth_\varrho(j\xi), \qquad z_{2,27}(x,t) = \frac{\kappa}{\omega}\Big(-k_1\coth_\varrho(j\xi)\Big)^2, \tag{4.71}$$

$$u_{2,28}(x,t) = -\frac{k_1\left(\sinh_\varrho(2\,j\xi)+\sqrt{-pq}\right)}{\cosh_\varrho(2\,j\xi)},$$
$$z_{2,28}(x,t) = \frac{\kappa}{\omega}\Bigg(-\frac{k_1\left(\sinh_\varrho(2\,j\xi)+\sqrt{-pq}\right)}{\cosh_\varrho(2\,j\xi)}\Bigg)^2, \tag{4.72}$$

$$u_{2,29}(x,t) = -\frac{k_1\left(\cosh_\varrho(2\,j\xi)+\sqrt{pq}\right)}{\sinh_\varrho(2\,j\xi)},$$
$$z_{2,29}(x,t) = \frac{\kappa}{\omega}\Bigg(-\frac{k_1\left(\cosh_\varrho(2\,j\xi)+\sqrt{pq}\right)}{\sinh_\varrho(2\,j\xi)}\Bigg)^2, \tag{4.73}$$

and

$$u_{2,30}(x,t) = -\frac{1}{2}\,\frac{k_1\left(2\left(\cosh_\varrho\left(\frac{1}{2}\,j\xi\right)\right)^2-1\right)}{\cosh_\varrho\left(\frac{1}{2}\,j\xi\right)\sinh_\varrho\left(\frac{1}{2}\,j\xi\right)},$$
$$z_{2,30}(x,t) = \frac{\kappa}{\omega}\Bigg(-\frac{1}{2}\,\frac{k_1\left(2\left(\cosh_\varrho\left(\frac{1}{2}\,j\xi\right)\right)^2-1\right)}{\cosh_\varrho\left(\frac{1}{2}\,j\xi\right)\sinh_\varrho\left(\frac{1}{2}\,j\xi\right)}\Bigg)^2. \tag{4.74}$$

Family 2.7. When $j = k = 0$:

$$u_{2,31}(x,t) = -\frac{k_1}{l\xi\,\ln(\varrho)}, \qquad z_{2,31}(x,t) = \frac{\kappa}{\omega}\Big(-\frac{k_1}{l\xi\,\ln(\varrho)}\Big)^2. \tag{4.75}$$

Family 2.8. When $j = 0$, $k \neq 0$ and $l \neq 0$:

$$u_{2,32}(x,t) = \frac{1}{2}\,\frac{k_1k\left(\cosh_\varrho(k\xi)-\sinh_\varrho(k\xi)-p\right)}{l\left(\cosh_\varrho(k\xi)-\sinh_\varrho(k\xi)+p\right)},$$
$$z_{2,32}(x,t) = \frac{\kappa}{\omega}\Bigg(\frac{1}{2}\,\frac{k_1k\left(\cosh_\varrho(k\xi)-\sinh_\varrho(k\xi)-p\right)}{l\left(\cosh_\varrho(k\xi)-\sinh_\varrho(k\xi)+p\right)}\Bigg)^2, \tag{4.76}$$

and

$$u_{2,33}(x,t) = -\frac{1}{2}\,\frac{k_1 k\left(\cosh_{\varrho}(k\xi) + \sinh_{\varrho}(k\xi) - q\right)}{l\left(\cosh_{\varrho}(k\xi) + \sinh_{\varrho}(k\xi) + q\right)},$$
$$z_{2,33}(x,t) = \frac{\kappa}{\omega}\left(-\frac{1}{2}\,\frac{k_1 k\left(\cosh_{\varrho}(k\xi) + \sinh_{\varrho}(k\xi) - q\right)}{l\left(\cosh_{\varrho}(k\xi) + \sinh_{\varrho}(k\xi) + q\right)}\right)^2. \tag{4.77}$$

Family 2.9. When $k = \varpi$, $l = s\varpi (s \neq 0)$ and $j = 0$:

$$u_{2,34}(x,t) = -\frac{1}{2}\,\frac{\varpi_1\left(p - s\varrho^{\varpi\xi} + 2\,sp\varrho^{\varpi\xi}\right)}{s\,(-p + s\varrho^{\varpi\xi})},$$
$$z_{2,34}(x,t) = \frac{\kappa}{\omega}\left(-\frac{1}{2}\,\frac{\varpi_1\left(p - s\varrho^{\varpi\xi} + 2\,sp\varrho^{\varpi\xi}\right)}{s\,(-p + s\varrho^{\varpi\xi})}\right)^2, \tag{4.78}$$

where

$$\xi = \frac{k_1\sqrt[4]{3}}{\sqrt[4]{\Lambda}\ln(\varrho)\,l}\frac{x^{\varsigma}}{\varsigma} - \frac{3^{\frac{3}{4}}\sqrt[4]{\Lambda}k_1}{2\ln(\varrho)\,l}\frac{t^{\sigma}}{\sigma}.$$

5. Discussion and graphs

In this section, I delve into the soliton solutions discovered during our CCDSWS research. These soliton solutions are derived using a novel r+mEDAM, which enables us to fully appreciate the CCDSWS's intricate dynamics. Visual displays effectively depict a variety of soliton behaviors, particularly kink, anti-kink, bright, dark, bright-dark, and bell-shaped kink solitons.

Five distinct soliton types exist in the CCDSWS's realm, namely kink, anti-kink, bright, dark, bright-dark, and bell-shaped kink solitons, which manifest in dispersive phenomena within water waves and fluid mechanics. These soliton types demonstrate the diversity of wave behavior and characteristics, which contributes to a comprehensive understanding of dispersive phenomena in various physical systems such as water waves and fluid mechanics. Solitons form as a result of the delicate balance of nonlinear and dispersive effects in wave equations, resulting in the formation of stable, localized wave patterns. Solitons are the result of nonlinear interactions between wave components that maintain their shape and amplitude over long distances. Each of these soliton types has its own set of characteristics, such as anti-kink, and the reverse of a kink, soliton is a monotonous shift in the opposing direction, from an upper asymptotic phase to a lower one. It can depict backward-moving formations or reversal shifts in the structure's parameters. In contrast, a kink soliton is a confined, monotonic wave that joins two different asymptotic states and usually moves across the medium from one stable condition to another. The smooth shift between two states of equilibrium may be represented by a kink soliton, which may be connected to variations in the system's physical characteristics across spatial domains, especially optical brilliance or charge density. A localized dip (a smaller amount of amplitude) in wave strength encircled by an uninterrupted backdrop of higher amplitude is what defines a dark soliton. This kind of kink soliton, which is frequently associated with phase displacements or wave shortfalls in nonlinear media, is a localized loss of wave energy or amplitude. A bright soliton, on the other hand, is a localized wave with a peak intensity higher than the background, typically appearing in systems with a focusing or attractive nonlinearity. This soliton type may represent regions of localized energy or wave amplitude enhancement, potentially describing high-intensity optical pulses or coherent structures in the system. Combining the characteristics of bright solitons and kinks, a bell-shaped kink has a symmetric bell-like contour and a smooth localized topology. Energy outbursts or shifts with spatial symmetries may be modeled by this kind of kink soliton, which depicts energy or intensity changes with a symmetric and concentrated profile. Last, a bright-dark soliton is an amalgamated form in which a backdrop of dark soliton is integrated with a bright soliton. The complicated interaction between energy concentration and depletion may be reflected by bright-dark soliton, which might represent simultaneous enhancement and repression of wave patterns in linked fields. These solitons are essential components in the study of fluid mechanics and wave physics, exemplifying a wide range of important wave phenomena.

Figure 1 shows the dynamical behaviors of soliton solutions $u_{1,10}$ and $z_{1,10}$ respectively described in (4.17) for $j = 1, k = 10, l = 3, r = 10, \varrho = \mathrm{e}, \kappa = 0.005$. Using the same parameter values, the 2D graphs are constructed simultaneously for $t = 0$. Overall, the graphs for $u_{1,10}$ represent dark-bright soliton profiles while the graphs for $z_{1,10}$ represent bright soliton profiles.

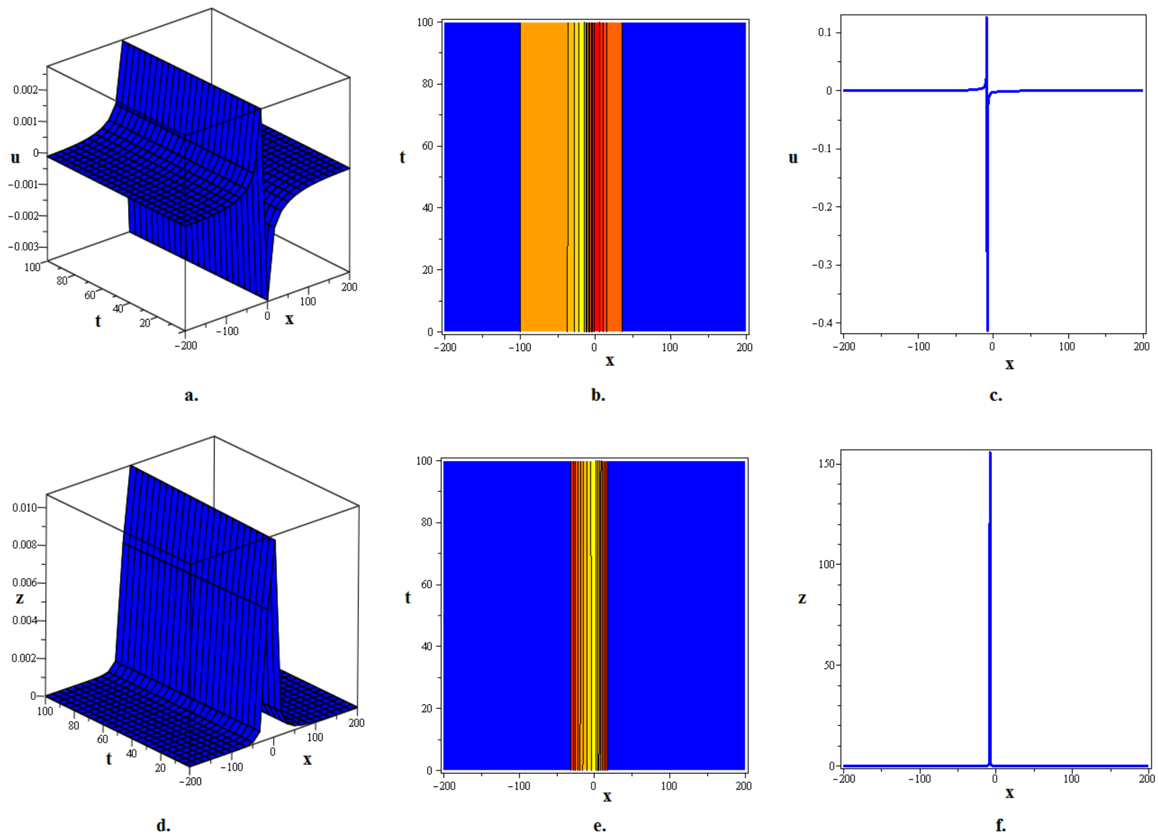

Figure 1. The dynamical behaviors of soliton solutions $u_{1,10}$ and $z_{1,10}$, respectively, described in (4.17) are depicted for $j = 1, k = 10, l = 3, r = 10, \varrho = \mathrm{e}, \kappa = 0.005$. Using the same parameter values, the 2D graphs are constructed simultaneously for $t = 0$.

Figure 2 shows the dynamical behaviors of soliton solutions $u_{1,21}$ and $z_{1,21}$, respectively, described in (4.28) for $j = 5, k = 0, l = 5, r = 0, \varrho = \mathrm{e}, \kappa = 0.0001$. Using the same parameter values, the 2D graphs are constructed simultaneously for $t = 10$. Overall, the graphs for $u_{1,21}$ represent dark-bright soliton profiles while the graphs for $z_{1,21}$ represent bright soliton profiles.

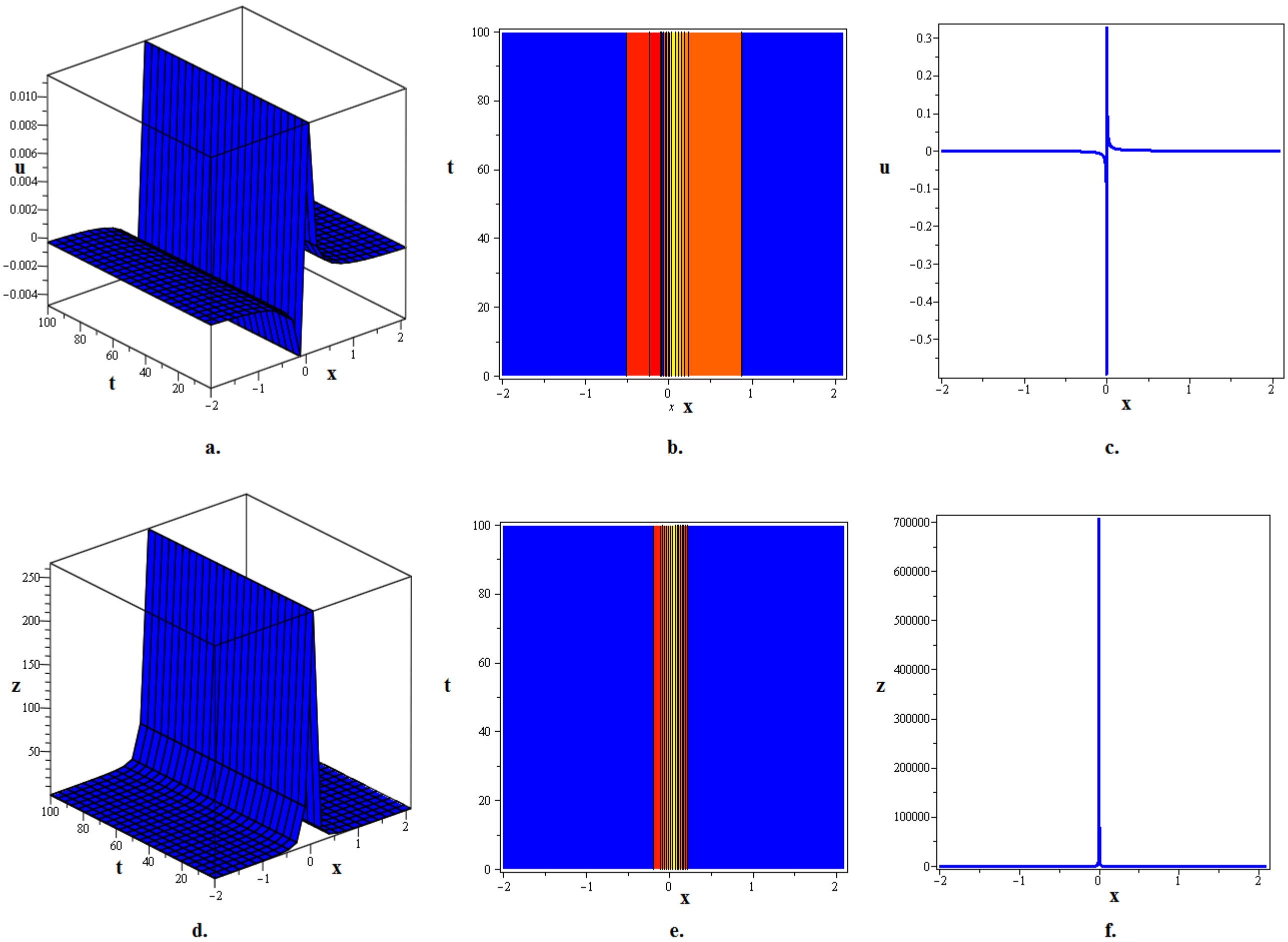

Figure 2. The dynamical behaviors of soliton solutions $u_{1,21}$ and $z_{1,21}$, respectively, described in (4.28) are depicted for $j = 5, k = 0, l = 5, r = 0, \varrho = \mathrm{e}, \kappa = 0.0001$. Using the same parameter values, the 2D graphs are constructed simultaneously for $t = 10$.

Figure 3 shows the dynamical behaviors of soliton solutions $u_{1,32}$ and $z_{1,32}$, respectively, described in (4.39) for $j = 6, k = 3, l = 0, \varpi = 3, s = 2, r = 1, \varrho = 2, \kappa = 0.00015$. Using the same parameter values, the 2D graphs are constructed simultaneously for $t = 20$. Overall, the graphs for $u_{1,32}$ represent dark-bright soliton profile and $z_{1,32}$ represent bright soliton profiles.

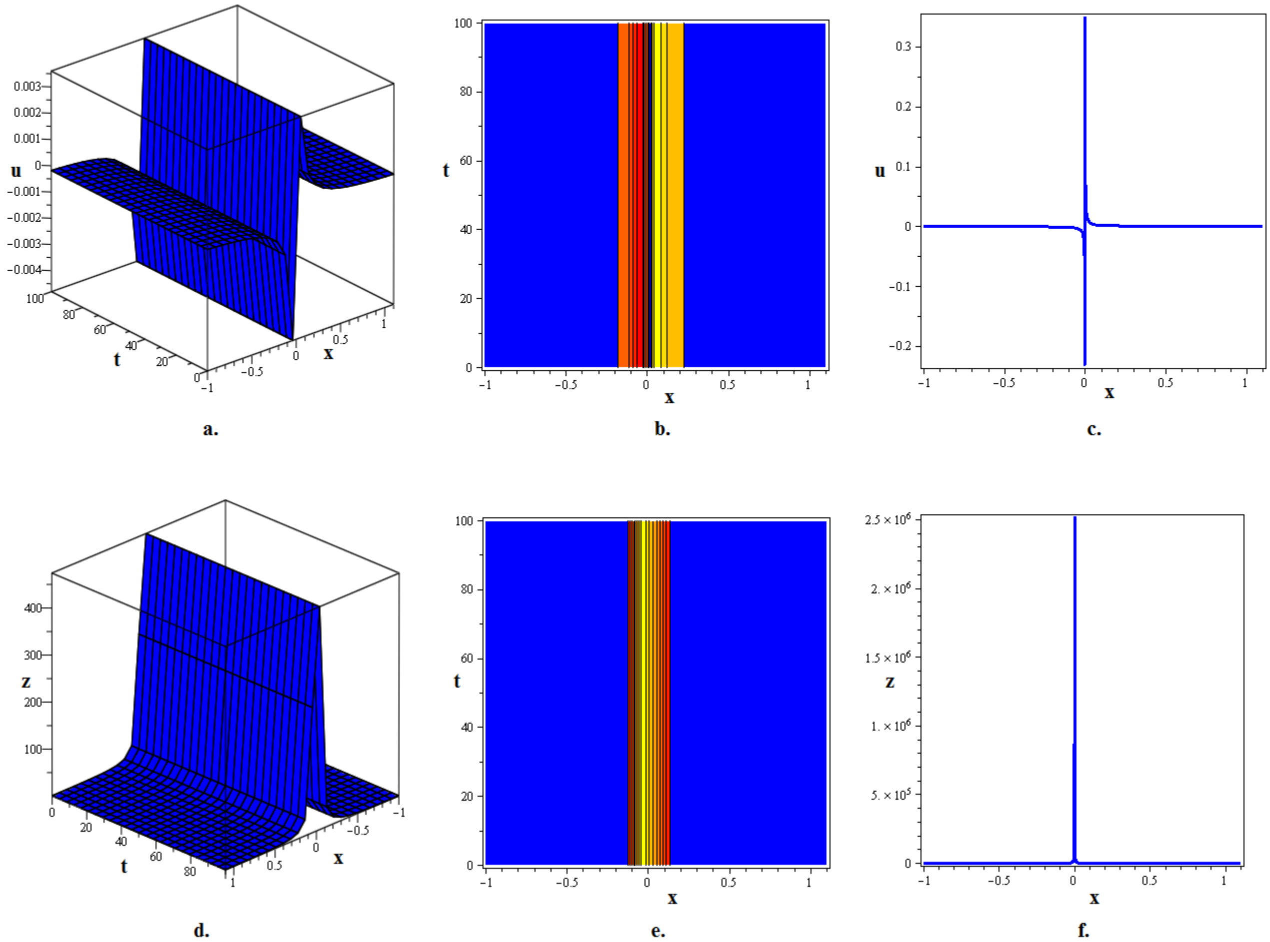

Figure 3. The dynamical behaviors of soliton solutions $u_{1,32}$ and $z_{1,32}$, respectively, described in (4.39) are depicted for $j = 6, k = 3, l = 0, \varpi = 3, s = 2, r = 1, \varrho = 2, \kappa = 0.00015$. Using the same parameter values, the 2D graphs are constructed simultaneously for $t = 20$.

Figure 4 shows the dynamical behaviors of soliton solutions $u_{2,16}$ and $z_{2,16}$, respectively, described in (4.60) for $j = 30, k = 0, k_1 = 0.00075, l = -20, r = 100, \varrho = \mathrm{e}$. Using the same parameter values, the 2D graphs are constructed simultaneously for the $t = 100$. Overall, the graphs for $u_{2,16}$ represent anti-kink soliton profiles while the graphs for $z_{2,16}$ represent bell-shape kink soliton profiles.

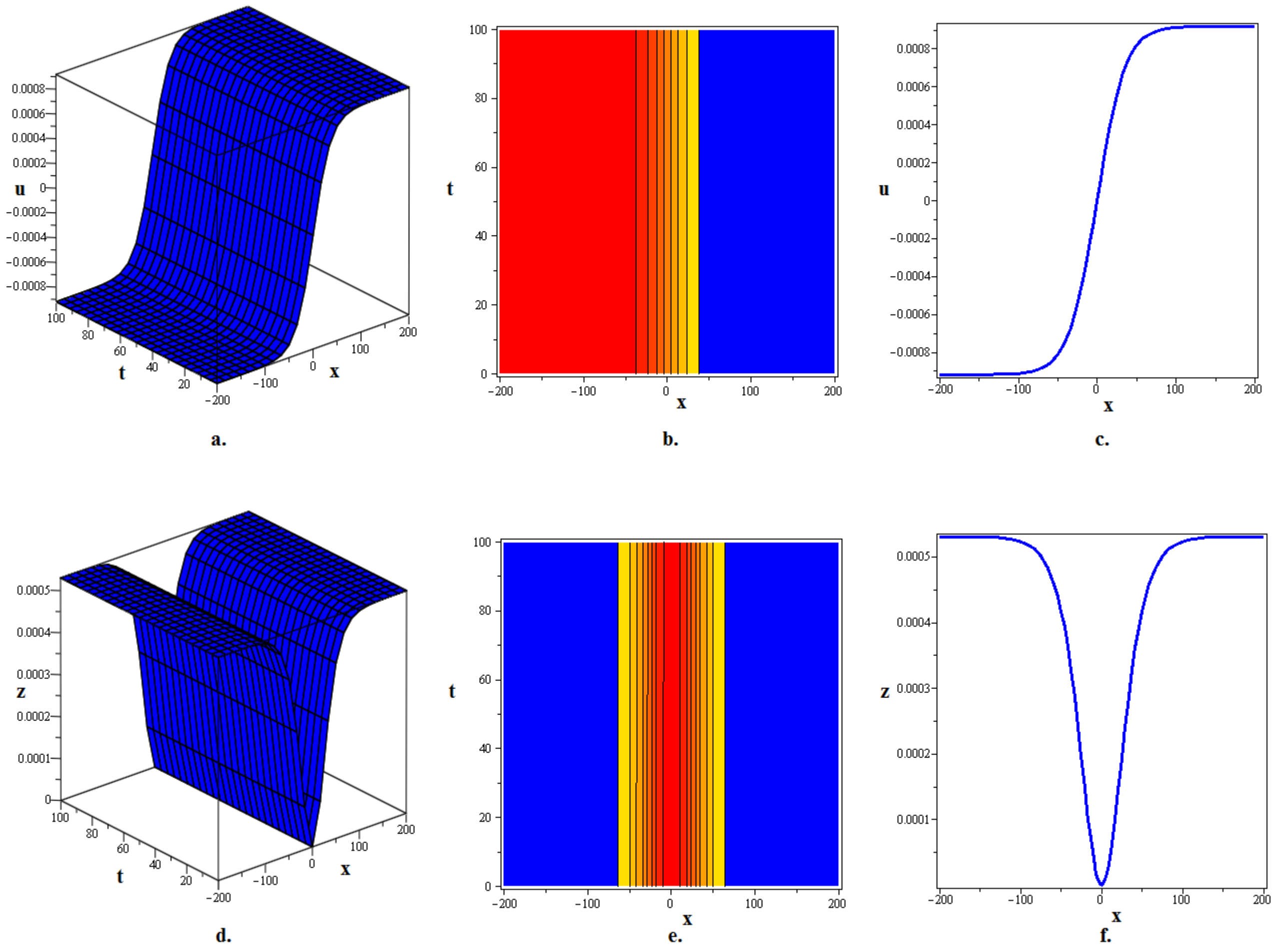

Figure 4. The dynamical behaviors of soliton solutions $u_{2,16}$ and $z_{2,16}$, respectively, described in (4.60) are depicted for $j = 30, k = 0, k_1 = 0.00075, l = -20, r = 100, \varrho = \mathrm{e}$. Using the same parameter values, the 2D graphs are constructed simultaneously for the $t = 100$.

Figure 5 exhibits the dynamical behaviors of soliton solutions $u_{2,33}$ and $z_{2,33}$, respectively, described in (4.77) for $j = 0, k_1 = 0.010, k = 2, p = 1, l = 30, r = 25, \varrho = 3$. Using the same parameter values, the 2D graph are constructed simultaneously for $t = 30$. Overall, the graphs for $u_{2,33}$ represent kink soliton profiles while the graphs for $z_{2,33}$ represent bell-shape kink soliton profiles.

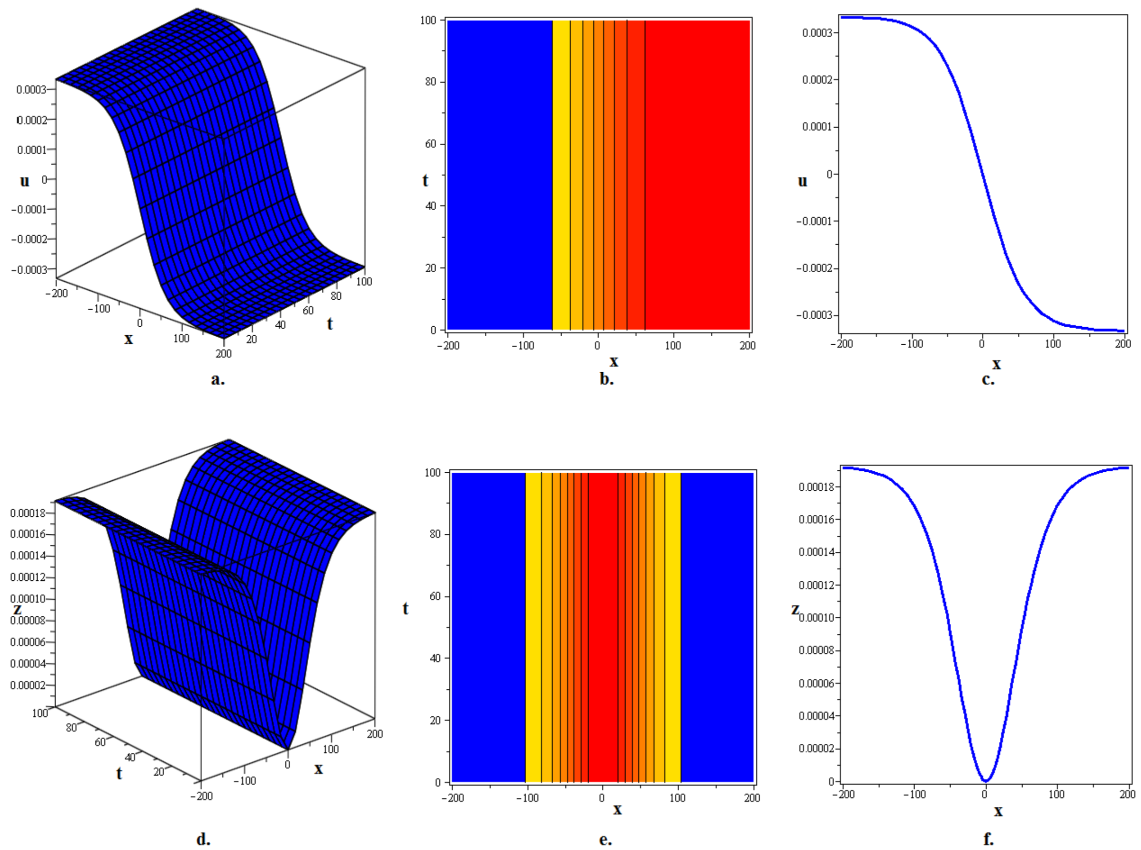

Figure 5. The dynamical behaviors of soliton solutions $u_{2,33}$ and $z_{2,33}$, respectively, described in (4.77) are depicted for $j = 0, k_1 = 0.010, k = 2, p = 1, l = 30, r = 25, \varrho = 3$. Using the same parameter values, the 2D graph are constructed simultaneously for $t = 30$.

6. Conclusions

In this exploration, r+mEDAM, a generalized version of mEDAM, is used to explore and analyse novel soliton solutions in the context of the nonlinear CCDSWS. This transformative technique reformulates the model into a set of nonlinear equations that are then solved using Maple software, yielding a large number of soliton solutions such as kink, anti-kink, bright, dark, bright-dark, and bell-shaped kink soliton solutions. New families of generalized functions, such as generalized trigonometric, hyperbolic, rational, and exponential functions, are among these solutions. Some soliton solutions are visually represented by appropriately tuning the constant parameters via graphical analysis and comparison, enabling a deeper understanding of real-world physical phenomena. This method is highly regarded as a scientific method that is both compatible and effective for examining various nonlinear mathematical models in engineering and physics, specifically those that address real-world problems.

Use of Generative-AI tools declaration

The author declare he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

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