



Research article

Asymptotic stability of deterministic and stochastic prey-predator models with prey herd immigration

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Abstract: This paper aimed to study the influence of herd immigration in prey species on the stability of the prey–predator interaction, in which prey immigration is modeled as a herd movement for defensive purposes. A stochastic version of the model was formulated to incorporate the influence of random noises. Positivity and boundedness are discussed for both deterministic and stochastic models, which validate the model biologically. For the deterministic model, the local asymptotic stability of the feasible equilibrium points is discussed, and the Hopf bifurcation is exhibited with respect to an immigration factor. Using a suitable Lyapunov function, sufficient conditions for global asymptotic stability are established for deterministic and stochastic models. Numerical simulations are carried out to verify and clarify our analytical findings. It is demonstrated that increasing prey herd immigration rates stabilizes the systems. Numerical simulations of the stochastic system reveal that population density fluctuations grow more consistently as prey herd immigration increases; these simulations also exhibit diverse dynamics, including quasi-steady states and quasi-limit cycles. It is concluded that the immigration of prey herds improves the survival of both species in deterministic and stochastic systems. Thus, it may be beneficial for prey to immigrate in groups to support unstable systems.

Keywords: immigration; herd immigration; prey–predator system; asymptotic stability

Mathematics Subject Classification: 34D05, 37H30, 60H10, 92-10

1. Introduction

Interactions between species have attracted considerable attention in applied mathematics and mathematical ecology. The basic rule between living food and its eater has been understood and analyzed for nearly a century. Alfred James Lotka [1] and Vito Volterra [2] independently formulated the first model to describe the predation process between prey and predator, which was called the Lotka–Volterra model. A modified version of this model incorporates Verhulst or logistic growth to

take into account limited resources and also to avoid structural instability [3]. In order to understand and regulate predator–prey interactions, functional and numerical responses were defined, which represent a predator’s feeding rate and the changes in predator density with prey consumption. They form the central structure of predator–prey models. A prominent family of functional responses is Holling’s [4–6] predator–prey theory, which is heavily based on Holling’s Type II functional and numerical responses; see, for example, [7–9]. Although it would be pointless to attempt to create a generic model or a set of generic models that can be applied to all populations in different environments [10]. There is a continuous effort by ecological, biological, and mathematical researchers to discover how models can be used to analyze and forecast natural phenomena.

Dynamical systems in many natural environments cannot be predicted with deterministic laws, especially when ecological systems are viewed from the perspective of variability rather than equilibrium dynamics. Therefore, random noises are useful tools to describe the fluctuations that arise in these environments. Recently, the effect of random noise on predators’ and prey’s dynamic behaviors was investigated in several studies; see, for instance, [11–13].

By adding external stimuli that mimic real-life effects in nature, predator-prey models have been developed and studied to make them more realistic and have received special attention in recent years, such as the effect of prey refuges [14–16], the influence of the fear effect [17–20], the Allee effect [21–23], and immigration. Immigration is one of the most significant factors that impact prey and predator interactions. Recent studies have focused on “small immigration” or “constant immigration” concepts, which assume that species’ immigration to the environment is limited. It plays a crucial role in the dynamics of these models and especially in their stability, as shown in limited recent research articles [24, 25] in this context.

A social gathering of various creatures of the same most commonly known as a herd or pack. When a species exhibits herd behavior, its members act socially and collectively, and each one adopts a behavior that is consistent with the majority of the other members of the species (e.g., at a particular time, all of them are moving in the same direction). Herd behavior can occur for a variety of reasons. Various prey species can utilize herding as a defense mechanism against predators (a large aggregation of prey may deter predators from initiating an attack) [26]. The most significant contributions to this field were those of Ajraldi et al. [26] and Braza [27], who presented novel approaches to modeling this phenomenon. Subsequently, some recent studies have applied this behavior to model predator–prey interactions in terms of functional and numerical responses [28, 29]. The density of prey, predator, or both populations in functional and numerical responses is expressed by the square root.

In empirical predator–prey systems, stability has been observed in the interaction between spiders (prey) and spider wasps (predators) [5]. Stability plays a pivotal role in the study of qualitative behavior. Asymptotic stability is most commonly used in deterministic prey and predator models to find conditions for stability, which, mathematically, means stability and is locally attractive. Therefore, asymptotic stability will be investigated in deterministic and stochastic models in this research.

In recent years, some researchers have discussed the effects of immigration and herd behavior separately on prey and predator dynamics. To the best of our knowledge, herd immigration has not been taken into account in the literature on prey–predator models. In this study, and in contrast to the literature, we introduce a novel idea by incorporating the herd immigration factor into the prey–predator systems. Our model assumes that prey immigrate in herds for defensive reasons. The

purpose of this paper is to investigate how herd immigration in prey species affects the stability of prey–predator interactions. A stochastic version of the model is formulated to consider the effect of random noise. The present study aimed mainly to investigate the boundedness and asymptotic stability of deterministic and stochastic prey–predator models with herd immigration of the prey. The boundedness of these models validates them biologically. The asymptotic stability addresses a significant issue, so specific conditions of deterministic and stochastic models with herd immigration of the prey are analytically obtained, and they are numerically verified and illustrated. The article is organized as follows. In the following section, we introduce the deterministic model, and the positivity and boundedness of its solutions are established. The stability results for the equilibrium points and Hopf bifurcations of the deterministic model are provided in Section 3. In Section 4, the stochastic version of the model is presented. For the stochastic model, the existence and uniqueness of global positive solutions are discussed in Section 5. The conditions of global stability of the coexistence equilibrium point are derived for deterministic and stochastic models in Section 6. Numerical simulations are utilized to verify our theoretical results in Section 7. Section 8 contains a general conclusion of the article.

2. The deterministic model

We introduce a non-dimensional Holling Type II prey–predator model with prey herd immigration as follows:

$$\begin{cases} \frac{dN}{dt} = rN \left(1 - \frac{N}{k}\right) - \frac{\alpha NP}{1 + \alpha hN} + \mu \sqrt{N}, \\ \frac{dP}{dt} = -\delta P + \frac{\alpha \beta NP}{1 + \alpha hN}, \end{cases} \quad (2.1)$$

subject to

$$N(0) = N_0 \geq 0, \quad P(0) = P_0 \geq 0.$$

The density of the prey population at time t is $N(t)$, and that of the predator population is $P(t)$; r is the growth rate of prey, k is the system's carrying capacity, α is the catching rate of prey by a predator, δ is the rate of natural death of the predator population, β is the efficiency of converting consumed prey into the predator's birth population, and μ represents the immigration factor. The new term in model (2.1) is $\mu \sqrt{N}$, which represents the herd immigration of the prey population. For biological meaning, all parameters are considered to have positive values, and $0 \leq \mu \leq 1$.

Theorem 2.1. *System (2.1) with the initial conditions has all solutions in the interval $[0, \infty)$ and $N(t) \geq 0$, $P(t) \geq 0 \forall t \geq 0$.*

Proof. Since the equations of system (2.1) are continuous and locally Lipschitzian functions, if $\exists t_{max} \geq 0$ and $t_{max} < \infty$, then the system (2.1) has a unique solution $(N(t), P(t))$ on the interval $[0, t_{max})$ with the initial conditions. Through the equations of system (2.1) with the initial conditions, we can conclude the following:

$$N(t) = N(0) \exp \left(\int_0^t \left[r \left(1 - \frac{N(\eta)}{K}\right) - \frac{\alpha P(\eta)}{1 + \alpha hN(\eta)} + \frac{\mu}{\sqrt{N(\eta)}} \right] d\eta \right) \geq 0,$$

$$P(t) = P(0) \exp\left(\int_0^t \left[-\delta + \frac{\alpha\beta N(\eta)(\eta)}{1 + \alpha h N(\eta)}\right] d\eta\right) \geq 0.$$

Therefore, we have the solutions of $N(t) \geq 0$ and $P(t) \geq 0 \forall t \geq 0$. \square

Theorem 2.2. In system (2.1), all solutions that initiate in R_+^2 are ultimately bounded in the region $\omega = \left\{(N, P) \in R_+^2 : 0 \leq F \leq \frac{\gamma}{\rho} + \epsilon\right\}$, where $0 < \rho < \delta$, where $F(t) = N(t) + \frac{1}{\beta}P(t) > 0$ and $\rho > 0$.

Proof. For system (2.1), consider $(N(t), P(t))$ to be any solution. Let $F(t) \in C^1(R_+ \rightarrow R_+)$ as $F(t) = N(t) + \frac{1}{\beta}P(t) > 0$ and let $\rho > 0$ be a constant.

The derivative of F with respect to time (t) is

$$\frac{dF}{dt} = \frac{dN}{dt} + \frac{1}{\beta} \frac{dP}{dt}, \quad (2.2)$$

$$\frac{dF}{dt} + \rho F = rN \left(1 - \frac{N}{K}\right) - \frac{\alpha NP}{1 + h\alpha N} + \mu \sqrt{N} - \frac{\delta}{\beta} P + \frac{\alpha NP}{1 + h\alpha N} + \rho N + \frac{\rho P}{\beta}, \quad (2.3)$$

$$\frac{dF}{dt} + \rho F = rN - \frac{rN^2}{K} + \mu \sqrt{N} + \rho N - \frac{1}{\beta}(\delta - \rho)P. \quad (2.4)$$

The term $\mu \sqrt{N} + (r + \rho)N - \frac{rN^2}{K}$ has a maximum value, since the second derivative is negative; we can say γ . Now we select $0 < \rho < \delta$. Then (2.4) implies the following:

$$\frac{dF}{dt} + \rho F \leq \gamma. \quad (2.5)$$

Thus,

$$F(t) \leq \frac{\gamma}{\rho} + F(0) e^{-\rho t}.$$

$\limsup_{t \rightarrow \infty} F \leq \frac{\gamma}{\rho}$. As a result, all the solutions of (2.1) are contained within

$$\omega = \left\{(N, P) \in R_+^2 : 0 \leq F \leq \frac{\gamma}{\rho} + \epsilon\right\}. \quad (2.6)$$

Thus, the solution of system (2.1) is bounded. \square

3. Equilibrium points and local stability

The system (2.1) has three non-negative equilibrium solutions:

- The two populations are extinct or have a trivial equilibrium point, $E_0(N = 0, P = 0)$.
- There are only prey species, and predator species are extinct,

$$\tilde{E}_1(\tilde{N}, (\tilde{P})) = \left(\tilde{N} = \frac{1}{3} 2K + \frac{\sqrt[3]{2} K^2 r^2}{\sqrt[3]{27\mu^2 K^2 r^4 - K^3 r^6 + 3\sqrt{3} \sqrt{27\mu^4 K^4 r^8 - 4\mu^2 K^5 r^{10}}}} + \frac{\sqrt[3]{2} \sqrt[3]{27\mu^2 K^2 r^4 - K^3 r^6 + 3\sqrt{3} \sqrt{27\mu^4 K^4 r^8 - 4\mu^2 K^5 r^{10}}}}{K^2 r^2}, \tilde{P} = 0 \right).$$

- The two populations coexist, $E_2 = \hat{E}_2 = (\hat{N} = \frac{\delta}{\alpha\beta - \delta h\alpha}, \hat{P} = \frac{r}{\alpha} - \frac{r\delta}{K\alpha^2(\beta - \delta h)} + \frac{\mu\sqrt{\delta}\sqrt{\beta - h\delta}}{\sqrt{\delta}})$.

Lemma 3.1. *The trivial equilibrium point $E_0(N = 0, P = 0)$ and the predator-free equilibrium point $E_1(N = \tilde{N}, \tilde{P} = 0)$ exist without any conditions. The coexistence or interior point exists if the following conditions are satisfied:*

$$\beta > \delta h, \quad (3.1)$$

and

$$\frac{r}{\alpha} + \frac{\mu\sqrt{\delta}\sqrt{\beta - h\delta}}{\sqrt{\delta}} > \frac{r\delta}{K\alpha^2(\beta - \delta h)}. \quad (3.2)$$

By considering the Jacobian matrix, we can determine the local stability of these equilibrium points as follows:

$$J = \begin{bmatrix} r - \frac{2rN}{K} - \frac{\alpha P}{(1+\alpha hN)^2} + \frac{\mu}{2\sqrt{N}} & \frac{-\alpha N}{1+\alpha hN} \\ \frac{\alpha\beta P}{(1+\alpha hN)^2} & -\delta + \frac{\alpha\beta N}{1+\alpha hN} \end{bmatrix}.$$

3.1. Trivial equilibrium point

The Jacobian matrix is indeterminate at $E_0(N = 0, P = 0)$. Local stability cannot be studied directly using the usual method at this equilibrium point; we follow Braza's way [27]. Because $N(t)$ is bounded $\forall t \geq 0$, the system (2.1) becomes

$$\begin{cases} \frac{dN}{dt} \approx rN \left(1 - \frac{N}{K}\right) - \frac{\alpha NP}{1 + \alpha hN}, \\ \frac{dP}{dt} = -\delta P + \frac{\alpha\beta NP}{1 + \alpha hN}. \end{cases} \quad (3.3)$$

Theorem 3.2. *The trivial equilibrium point $E_0(N = 0, P = 0)$ is a saddle point.*

Proof. The Jacobian matrix of the trivial equilibrium point $E_0(N = 0, P = 0)$ is

$$J(E_0) = \begin{bmatrix} r & 0 \\ 0 & -\delta \end{bmatrix}. \quad (3.4)$$

From the matrix (3.4), the eigenvalues are $\lambda_1 = r$ and $\lambda_2 = -\delta$, so the trivial equilibrium point $E_0(N = 0, P = 0)$ is a saddle point. \square

3.2. Predator-free equilibrium point

Through the following theorem, we examine the local stability of the predator-free equilibrium point $E_1(N = \tilde{N}, \tilde{P} = 0)$:

Theorem 3.3. *The predator-free equilibrium point $\tilde{E}_1(N = \tilde{N}, \tilde{P} = 0)$ is locally asymptotically stable provided the following conditions hold:*

$$\frac{2r\tilde{N}}{K} > r + \frac{\mu}{2\sqrt{\tilde{N}}}, \quad (3.5)$$

$$\delta > \frac{\alpha\beta\tilde{N}}{1 + \alpha h\tilde{N}}. \quad (3.6)$$

Proof. The Jacobian matrix of the predator-free equilibrium point $\tilde{E}_1(N = \tilde{N}, \tilde{P} = 0)$ is

$$J(\tilde{E}_1) = \begin{bmatrix} r - \frac{2r\tilde{N}}{k} + \frac{\mu}{2\sqrt{\tilde{N}}} & \frac{-\alpha\tilde{N}}{1+\alpha h\tilde{N}} \\ 0 & -\delta + \frac{\alpha\beta\tilde{N}}{1+\alpha h\tilde{N}} \end{bmatrix}. \quad (3.7)$$

From the matrix (3.7), the eigenvalues are $\lambda_1 = r - \frac{2r\tilde{N}}{k} + \frac{\mu}{2\sqrt{\tilde{N}}}$ and $\lambda_2 = -\delta + \frac{\alpha\beta\tilde{N}}{1+\alpha h\tilde{N}}$. Consequently, the predator-free equilibrium point $\tilde{E}_1(N = \tilde{N}, P = 0)$ is locally asymptotically stable, provided the conditions (3.5) and (3.6) hold. \square

3.3. Coexistence equilibrium point

Using the following theorem, we investigate the local stability of the coexistence equilibrium point $E_2 = \hat{E}_2 = (N = \hat{N}, P = \hat{P})$.

Theorem 3.4. *The coexistence equilibrium point $E_2 = \hat{E}_2 = (N = \hat{N} > 0, P = \hat{P} > 0)$ is locally asymptotically stable under the following condition:*

$$r + \frac{\mu}{2\sqrt{\hat{N}}} < \frac{2r\hat{N}}{K} + \frac{\alpha\hat{P}}{(1 + \alpha h\hat{N})^2}. \quad (3.8)$$

Proof. The Jacobian matrix of the coexistence equilibrium point $E_2 = \hat{E}_2 = (N = \hat{N}, P = \hat{P})$ is

$$J(E_2) = \begin{bmatrix} r - \frac{2r\hat{N}}{K} - \frac{\alpha\hat{P}}{(1+\alpha h\hat{N})^2} + \frac{\mu}{2\sqrt{\hat{N}}} & \frac{-\alpha\hat{N}}{1+\alpha h\hat{N}} \\ \frac{\alpha\beta\hat{P}}{(1+\alpha h\hat{N})^2} & -\delta + \frac{\alpha\beta\hat{N}}{1+\alpha h\hat{N}} \end{bmatrix}. \quad (3.9)$$

From the matrix (3.9)

$$J(E_2) = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

where

$$h_{11} = r - \frac{2r\hat{N}}{K} - \frac{\alpha\hat{P}}{(1 + \alpha h\hat{N})^2} + \frac{\mu}{2\sqrt{\hat{N}}},$$

$$h_{12} = \frac{-\alpha\hat{N}}{(1 + \alpha h\hat{N})^2},$$

$$h_{21} = \frac{-\alpha\hat{N}}{1 + \alpha h\hat{N}},$$

$$h_{22} = -\delta + \frac{\alpha\beta\hat{N}}{1 + \alpha h\hat{N}},$$

and the characteristic equation of $J(E_2)$ is

$$\lambda^2 - \text{tr}J(E_2)\lambda + \det J(E_2) = 0.$$

Since $h_{12} < 0$, $h_{21} > 0$, and $h_{22} = 0$, at the coexistence equilibrium point $\hat{E}_2 = (N = \hat{N}, P = \hat{P})$, $|J(\hat{N}, \hat{P})| > 0$ and $\text{Trace}(J(\hat{N}, \hat{P})) < 0$, if $h_{11} < 0$. Thus, the coexistence equilibrium point $\hat{E}_2 = (N = \hat{N}, P = \hat{P})$ is locally asymptotically stable under the condition (3.8). \square

Corollary 3.4.1. *The coexistence equilibrium point $E_2 = (\hat{N}, \hat{P})$ destabilizes if the condition (3.8) is violated.*

3.4. Hopf bifurcation

Hopf bifurcations occur when a system loses its stability and a periodic solution appears. A Hopf bifurcation of the coexistence equilibrium point is discussed in this section. The parameter μ is selected as the bifurcation parameter for studying the Hopf bifurcation.

Theorem 3.5. *System (2.1) has a Hopf bifurcation around the coexistence equilibrium point $(E_{cII}(\hat{N}, \hat{P}))$ at the threshold $\mu = \mu_H$, where $\mu_H = \frac{4r\hat{N}\sqrt{\hat{N}}}{K} - 2r\sqrt{\hat{N}} - \frac{2\alpha\sqrt{\hat{N}}\hat{P}}{(1+h\alpha\hat{N})^2} + 2\delta\sqrt{\hat{N}} + \frac{2\alpha\beta\sqrt{\hat{N}}\hat{N}}{1+h\alpha\hat{N}}$.*

Proof. The equilibrium point $E_2(\hat{N}, \hat{P})$ will be in an unstable case if $\mu > \mu_H$; then $\mu > \mu_H$ is the critical value where the stability of $E_2(\hat{N}, \hat{P})$ changes.

Assume that $\lambda = R(\mu) + T(\mu)i$ represents an eigenvalue of the Jacobian matrix $J((\hat{N}, \hat{P}); \mu_H)$. Thus, λ is a purely imaginary number. When $\mu = \mu_H$, a Hopf bifurcation occurs. We test these conditions by substituting m for m_H ; we can obtain

$$\text{Trace}(J((\hat{N}, \hat{P}); \mu_H)) = 0, \text{Det}(J((\hat{N}, \hat{P}); \mu_H)) > 0 \text{ and } \frac{d(\text{trace}(J((\hat{N}, \hat{P}); \mu_H)))}{d\mu} = \frac{1}{\sqrt{\hat{N}}} \neq 0.$$

Thus, a Hopf bifurcation occurs at the point $\mu = \mu_H$. \square

4. The stochastic model

There are many factors that can affect the environment randomly. There is an urgent need to model these using random models, especially in volatile environments. This section presents the stochastic formulation of system (2.1) to account for the impact of environmental noise as follows:

$$\begin{cases} dN = \left(rN \left(1 - \frac{N}{K} \right) - \frac{\alpha NP}{1 + \alpha hN} + \mu \sqrt{N} \right) dt + \xi_1 N dW_1, \\ dP = \left(-\delta P + \frac{\alpha \beta NP}{1 + \alpha hN} \right) dt + \xi_2 P dW_2, \end{cases} \quad (4.1)$$

subject to

$$N(0) = N_0 \geq 0, \quad P(0) = P_0 \geq 0,$$

where ξ_1 and ξ_2 indicate the strength noise parameters, while dW_1 and dW_2 represent standard Wiener processes. The general form of Itô's stochastic differential equation is shown as follows:

$$\begin{aligned} dZ_t &= \psi(Z_t, t)dt + \zeta(Z_t, t)dW, \\ Z_t(0) &= Z_0, \forall t \geq t_0, \end{aligned} \quad (4.2)$$

where $\psi : [(0, T) \times \mathbb{R}^+]^2 \rightarrow \mathbb{R}$ and $\sigma : [(0, T) \times \mathbb{R}^+]^2 \rightarrow \mathbb{R}$.

The differential operator L of Eq. (4.2) is defined by the following formula [30]:

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^n \psi(Z(t), t) \frac{\partial}{\partial Z_i} + \frac{1}{2} \sum_{i,j=1}^n [\zeta(Z(t), t), \zeta(Z(t), t)^T]_{ij} \frac{\partial^2}{\partial Z_i \partial Z_j}.$$

If L acts on a function $V \in C^{2,1}(\mathbb{R}_+^n, \mathbb{R}_+)$, then

$$LV(Z(t), t) = V_t(Z(t), t) + V_Z(Z(t), t) \psi(Z(t), t) + \frac{1}{2} \text{trace} \left[\zeta^T(Z(t), t) V_{ZZ}(Z(t), t) \zeta(Z(t), t) \right],$$

where $V_t = \frac{\partial V}{\partial t}$, $V_Z = \left(\frac{\partial V}{\partial Z_1}, \frac{\partial V}{\partial Z_2}, \dots, \frac{\partial V}{\partial Z_n} \right)$, $V_{ZZ} = \left(\frac{\partial^2 V}{\partial Z_i \partial Z_j} \right)_{n \times n}$. By Itô's formula, we can obtain $dV(Z(t), t) = LV(Z(t), t) dt + V_Z(Z(t), t) \zeta(Z(t), t) dW(t)$.

5. Properties of the solution

This section establishes the existence and uniqueness of global positive solutions, as well as proving the stochastic boundedness of the system (4.1), which aligns with biological feasibility.

5.1. The existence and uniqueness of global positive solutions

The system (4.1) can be written as follows:

$$\begin{cases} dN = N \left(r \left(1 - \frac{N}{K} \right) - \frac{\alpha P}{1 + \alpha h N} + \frac{\mu \sqrt{N}}{N} \right) dt + \xi_1 N dW_1, \\ dP = P \left(-\delta + \frac{\alpha \beta N}{1 + \alpha h N} \right) dt + \xi_2 P dW_2. \end{cases} \quad (5.1)$$

Theorem 5.1. *Given any initial value $(N_0, P_0) \in R_+^2$, there is a unique global positive solution $(N(t), P(t)) \in R_+^2$ of the stochastic model (4.1) almost surely for all $t \geq 0$ with a probability of one.*

Proof. Set $s(t) = \ln N(t)$, $q(t) = \ln P(t)$. We then get the following equations:

$$\begin{cases} ds(t) = \left(r - \frac{re^s}{k} - \frac{\alpha e^q}{1 + \alpha h e^s} - \frac{\mu \sqrt{e^s}}{e^s} - \frac{\xi_1^2}{2} \right) dt + \xi_1 dW_1, \\ dq(t) = \left(-\delta + \frac{\alpha \beta e^s}{1 + \alpha h e^s} - \frac{\xi_2^2}{2} \right) dt + \xi_2 dW_2, \end{cases} \quad (5.2)$$

where $s(0) = \ln N(0)$, $q(0) = \ln P(0)$. We know that the system (4.1) has a unique local solution $(s(t), q(t))$ on $t \in [0, \tau_e)$ since its coefficients satisfy the local Lipschitz condition. Here, τ_e is the explosion time, which indicates that $\lim_{t \rightarrow \tau_e} |s(t)| = \infty$ or $\lim_{t \rightarrow \tau_e} |q(t)| = \infty$. By applying Itô's formula, the system (4.1) has a unique local solution that $(N(t) = e^{s(t)}, P(t) = e^{q(t)}) \in R_+^2$ with any initial condition $N_0 > 0, P_0 > 0$. To show that the solution is global, we need to prove that $\tau_e = \infty$ almost surely. Consider $k_0 > 0$ to be large enough for $(N_0, P_0) \in E_{k_0} = [\frac{1}{k_0}, k_0] \times [\frac{1}{k_0}, k_0]$. Assuming that $k > k_0$, the stopping time is defined as follows:

$$\tau_k = \inf \left\{ t \in [0, \tau_e) : \min\{N(t), P(t)\} \leq \frac{1}{k} \text{ or } \max\{N(t), P(t)\} \geq k \right\},$$

where $\inf \emptyset = \infty$, and \emptyset is the empty set.

Let $\lim_{k \rightarrow \infty} \tau_k = \infty$, in which case $\tau_\infty \leq \tau_e$ a.s. by supposing $\tau_\infty = \infty$ a.s. Instead, if $\tau_\infty \neq \infty, T > 0$ and $\epsilon \in (0, 1)$ exist such that $\mathbb{P}(\tau_\infty \leq T) > \epsilon$. Consequently, there exists a constant $k \geq k_0$ such that $\mathbb{P}(\tau_k \leq T) \geq \epsilon$ for all $k \geq k_1$.

Next, define a C^2 -function $V : R_+^2 \rightarrow R_+, V(N, P) = (N - 1 - \ln N) + (P - 1 - \ln P)$.

On the basis of Itô's formula, one can conclude that

$$\begin{aligned} dV = & \left(rN - \frac{rN^2}{k} - \frac{\alpha NP}{1 + \alpha h N} + \mu \sqrt{N} - r + \frac{rN}{k} + \frac{\alpha P}{1 + \alpha h N} - \frac{\mu \sqrt{N}}{N} - \delta P + \frac{\alpha \beta NP}{1 + \alpha h N} \right. \\ & \left. - \delta P + \frac{\alpha \beta NP}{1 + \alpha h N} + \delta - \frac{\alpha \beta N}{1 + \alpha h N} + \frac{\xi_1^2}{2} + \frac{\xi_2^2}{2} \right) dt + \xi_1 (N - 1) dW_1 + \xi_2 (P - 1) dW_2, \end{aligned}$$

$$dV \leq \left(rN + \mu \sqrt{N} - r + \frac{rN}{k} + \frac{\alpha P}{1 + \alpha hN} + \frac{\alpha \beta NP}{1 + \alpha hN} + \delta + \frac{\xi_1^2}{2} + \frac{\xi_2^2}{2} \right) dt + \xi_1(N-1)dW_1 + \xi_2(P-1)dW_2.$$

Based on Lemma 4.1 of Dalal et al. [31], for $u_i \in R_+$,

$$u_i \leq 2(u_i + 1 - \ln u_i) - (4 - 2 \ln 2) \leq 2(u_i + 1 - \ln u_i),$$

Due to this, we have the following inequalities:

$$\left(r + \frac{r}{k} \right) N + \mu \sqrt{N} + \left(\frac{\alpha}{1 + \alpha hN} + \frac{\alpha \beta N}{1 + \alpha hN} \right) P - r + \delta + \frac{\xi_1^2}{2} + \frac{\xi_2^2}{2} \leq \left(r + \frac{r}{k} \right) N + \mu \sqrt{N} + \left(\alpha + \frac{\beta}{h} \right) P - r + \delta + \frac{\xi_1^2}{2} + \frac{\xi_2^2}{2},$$

which is

$$\left(r + \frac{r}{k} \right) N + \mu \sqrt{N} + \left(\alpha + \frac{\beta}{h} \right) P \leq 2 \left(r + \frac{r}{k} \right) (N + 1 - \ln N) + 2\mu(\sqrt{N} + 1 - \ln \sqrt{N}) + 2 \left(\alpha + \frac{\beta}{h} \right) (P + 1 - \ln P).$$

Let $A_3 = \max\{A_1, 2A_2\}$, where $A_1 = r + \delta + \frac{\xi_1^2}{2} + \frac{\xi_2^2}{2}$, $A_2 = \max\{r + \frac{r}{k}, \alpha + \frac{\beta}{h}\}$. Thus,

$$dV \leq A_3(1 + V(N, P))dt + \xi_1(N-1)dW_1 + \xi_2(P-1)dW_2.$$

On the basis of the stopping time τ_k and because $T > 0$ is a constant, as defined in [30], integrating both sides will yield the following expectation:

$$\begin{aligned} EV(N(\tau_k \wedge T), P(\tau_k \wedge T)) &\leq V(N(0), P(0)) + E \int_0^{\tau_k \wedge T} A_3[1 + V(N, P)] ds, \\ &\leq V(N(0), P(0)) + A_3T + A_3 \int_0^{\tau_k \wedge T} EV(N, P) ds, \\ &\leq V(N(0), P(0)) + A_3T + e^{A_3T} = Q_0. \end{aligned}$$

As a result, we obtain $V(N(\tau_k \wedge T), P(\tau_k \wedge T)) \geq (k - 1 - \ln k) \wedge (\frac{1}{k} - 1 - \ln \frac{1}{k})$. Therefore, it can be deduced that

$$Q_0 \geq E[1_{\Omega_t(\theta)} V(N(\tau_k \wedge T), P(\tau_k \wedge T))] \geq \varepsilon([k - 1 - \ln k \wedge \frac{1}{k} - 1 - \ln \frac{1}{k}]),$$

where $1_{\Omega_t(\theta)}$ is an indicator function. In this case, the hypothesis is contradicted. Thus, the proof is complete. \square

5.2. Stochastic boundedness

Theorem 5.2. *If it is given any initial value $(N_0, P_0) \in \mathbb{R}_+^2$, then the solutions of system (4.1) are stochastically ultimately bounded.*

Proof. We define the Lyapunov functions $g_1 = e^t N^i$ and $g_2 = e^t P^i$, respectively, for $i \in (0, 1)$. Using Itô's formula, we calculate

$$dg_1(N) = e^t N^i dt + e^t \left[i N^i dN + \frac{1}{2} N(N-1) N^{i-2} (dN)^2 \right],$$

$$= e^t N^i dt + i e^t N^i \left(r - \frac{rN}{k} - \frac{\alpha P}{1 + \alpha h N} + \frac{\mu \sqrt{N}}{N} \right) dt + i e^t \xi_1 dW_1 + \frac{1}{2} N(N-1) e^t N^{i-2} \left(r - \frac{rN}{k} - \frac{\alpha P}{1 + \alpha h N} + \frac{\mu \sqrt{N}}{N} \right)^2 dt,$$

and

$$dg_2(P) = e^t P^i dt + e^t \left[i P^i dP + \frac{1}{2} P(P-1) P^{i-2} (dP)^2 \right],$$

$$= e^t P^i dt + i e^t P^i \left(-\delta + \frac{\alpha \beta N}{1 + \alpha h N} \right) dt + i e^t \xi_2 dW_2 + \frac{1}{2} P(P-1) e^t P^{i-2} \left(-\delta + \frac{\alpha \beta N}{1 + \alpha h N} \right)^2 dt.$$

Thus,

$$Lg_1 = e^t N^i dt + i e^t N^i \left(r - \frac{rN}{k} - \frac{\alpha P}{1 + \alpha h N} + \frac{\mu \sqrt{N}}{N} \right) + \frac{1}{2} N(N-1) e^t N^{i-2} \left(r - \frac{rN}{k} - \frac{\alpha P}{1 + \alpha h N} + \frac{\mu \sqrt{N}}{N} \right)^2,$$

$$= e^t N^i + i e^t N^i \left(r - \frac{rN}{k} - \frac{\alpha P}{1 + \alpha h N} + \frac{\mu \sqrt{N}}{N} \right) + \frac{1}{2} N(N-1) e^t N^{i-2} \left(r - \frac{rN}{k} - \frac{\alpha P}{1 + \alpha h N} + \frac{\mu \sqrt{N}}{N} \right)^2,$$

From Theorem 2.2, we get

$$\leq e^t M_1(i), \quad (5.3)$$

and

$$Lg_2 = e^t P^i dt + i e^t P^i \left(-\delta + \frac{\alpha \beta N}{1 + \alpha h N} \right) + \frac{1}{2} P(P-1) e^t P^{i-2} \left(-\delta + \frac{\alpha \beta N}{1 + \alpha h N} \right)^2.$$

We can conclude that

$$\leq e^t M_2(i). \quad (5.4)$$

In this case, $\lim_{t \rightarrow \infty} \sup \mathbb{E} N^i \leq M_1(i) \leq +\infty$ and $\lim_{t \rightarrow \infty} \sup \mathbb{E} P^i \leq M_2(i) \leq +\infty$. Furthermore, $H(t) = (N(t), P(t)) \in \mathbb{R}_+^2$, so note that

$$|H(t)|^i \leq (2 \max\{N^2(t), P^2(t)\})^i.$$

Now, we set $i = \frac{1}{2}$, $C_1 > 0$ exists, such that

$$\lim_{t \rightarrow \infty} \mathbb{E} |H(t)|^{\frac{1}{2}} \leq C_1.$$

For any $\epsilon > 0$, set $C_2 = \frac{C_1^2}{\epsilon}$. Then, by Chebyshev's inequality,

$$\mathbb{P}\{|H(t)| > C_2\} \leq \frac{\mathbb{E}|H(t)|^{\frac{1}{2}}}{C_2^{\frac{1}{2}}}.$$

Consequently,

$$\limsup_{t \rightarrow \infty} \mathbb{P}\{|H(t)| > C_2\} \leq \frac{C_1}{C_2^{\frac{1}{2}}} = \epsilon.$$

This completes the proof. \square

6. The global asymptotic stability of the coexistence equilibrium point

Many researchers in biological and mathematical fields are interested in the long-term behavior of species. We focus our attention on this important point, which expresses the coexistence of prey with their predators. We prove the global stability of the stochastic model as follows:

Theorem 6.1. *If, recalling condition (3.8), the following condition holds*

$$\left(\sqrt{K} + \sqrt{\hat{N}}\right) + \frac{\alpha^2 h}{(1 + h\alpha\hat{N})(1 + h\alpha K)} (|\zeta\hat{N} - \hat{P}K|) < \frac{r}{K} (K - \hat{N})^2 - \frac{1}{2}\hat{N}\xi_1^2 - \frac{1}{2\beta}\hat{P}\xi_2^2, \quad (6.1)$$

then, the coexistence equilibrium point is globally asymptotically stable almost surely with probability one, i.e., for any initial condition $(N_0, P_0) \in \mathbb{R}_+^2$, the solution of model (4.1) has the property

$$\limsup_{t \rightarrow \infty} N(t) = \hat{N}, \quad \limsup_{t \rightarrow \infty} P(t) = \hat{P},$$

a.s.

Proof. Consider the Lyapunov functions

$$V(N, P) = \left(N - \hat{N} - \hat{N} \ln\left(\frac{N}{\hat{N}}\right)\right) + C \left(N - \hat{P} - \hat{P} \ln\left(\frac{P}{\hat{P}}\right)\right),$$

where C is a positive constant that will be determined later.

We apply Itô's formula, so we get

$$\begin{aligned} LV(N, P) &= \left[1 - \frac{\hat{N}}{N}\right] dN + \frac{1}{2} \frac{\hat{N}}{N^2} (dN)^2 + C \left(\left[1 - \frac{\hat{P}}{P}\right] dP + \frac{1}{2} \frac{\hat{P}}{P^2} (dP)^2\right), \\ &= (N - \hat{N}) \left[\left(r\left(1 - \frac{N}{K}\right) - \frac{\alpha P}{1 + h\alpha N} + \frac{\mu}{\sqrt{N}}\right) dt + \xi_1 dW_1\right] + \frac{1}{2} \hat{N} \xi_1^2 dt \\ &\quad + (P - \hat{P}) \left[\left(-\delta + \frac{\alpha\beta N}{1 + \alpha hN}\right) dt + \xi_2 dW_2\right] + \frac{1}{2} \hat{P} \xi_2^2 dt, \\ &= (N - \hat{N}) \left[\left(\frac{-rN}{K} - \frac{\alpha P}{1 + h\alpha N} - \frac{r\hat{N}}{K} + \frac{\alpha\hat{P}}{1 + h\alpha\hat{N}} + \frac{\mu}{\sqrt{N}} - \frac{\mu}{\sqrt{\hat{N}}}\right) dt + \xi_1 dW_1\right] + \frac{1}{2} \hat{N} \xi_1^2 dt \\ &\quad + (P - \hat{P}) \left[\left(\frac{\alpha\beta N}{1 + \alpha hN} - \frac{\alpha\beta\hat{N}}{1 + \alpha h\hat{N}}\right) dt + \xi_2 dW_2\right] + \frac{1}{2} \hat{P} \xi_2^2 dt, \end{aligned}$$

$$\begin{aligned}
&= (N - \hat{N}) \left[\left(\frac{-rN}{K} - \frac{\alpha P(1 + h\alpha\hat{N})}{(1 + h\alpha N)(1 + h\alpha\hat{N})} - \frac{r\hat{N}}{K} + \frac{\alpha\hat{P}(1 + h\alpha N)}{(1 + h\alpha N)(1 + h\alpha\hat{N})} + \frac{\mu}{\sqrt{N}} - \frac{\mu}{\sqrt{\hat{N}}} + \frac{1}{2}\hat{N}\xi_1^2 \right) dt\xi_1 dW_1 \right] \\
&+ C(P - \hat{P}) \left[\left(\frac{\alpha\beta N(1 + h\alpha\hat{N})}{1 + \alpha hN} - \frac{\alpha\beta\hat{N}(1 + h\alpha N)}{1 + \alpha h\hat{N}} + \frac{1}{2}\hat{P}\xi_2^2 \right) dt + \xi_2 dW_2 \right], \\
&= \left[-\frac{r}{K}(N - \hat{N})^2 - \frac{\alpha}{(1 + h\alpha\hat{N})(1 + h\alpha N)}(N - \hat{N})(P - \hat{P}) + \frac{\alpha^2 h}{(1 + h\alpha\hat{N})(1 + h\alpha N)}(P\hat{N} - \hat{P}N) \right. \\
&+ \frac{\mu(N - \hat{N})}{(\sqrt{N} - \sqrt{\hat{N}})} + \frac{C\alpha\beta}{(1 + h\alpha\hat{N})(1 + h\alpha N)}(N - \hat{N})(P - \hat{P}) + \frac{1}{2}\hat{N}\xi_1^2 + \frac{1}{2}\hat{P}\xi_2^2 \left. \right] dt \\
&+ \xi_1(N - \hat{N})dW_1 + \xi_2(P - \hat{P})dW_2.
\end{aligned}$$

Now, we have

$$dV = L(V)dt + \xi_1(N - \hat{N})dW_1 + \xi_2(P - \hat{P})dW_2,$$

where

$$\begin{aligned}
L(V) &= -\frac{r}{K}(N - \hat{N})^2 - \frac{\alpha}{(1 + h\alpha\hat{N})(1 + h\alpha N)}(N - \hat{N})(P - \hat{P}) + \frac{\alpha^2 h}{(1 + h\alpha\hat{N})(1 + h\alpha N)}(P\hat{N} - \hat{P}N) \\
&+ \frac{\mu(N - \hat{N})}{(\sqrt{N} - \sqrt{\hat{N}})} + \frac{C\alpha\beta}{(1 + h\alpha\hat{N})(1 + h\alpha N)}(N - \hat{N})(P - \hat{P}) + \frac{1}{2}\hat{N}\xi_1^2 + C\frac{1}{2}\hat{P}\xi_2^2 dt \\
&+ \xi_1(N - \hat{N})dW_1 + C\xi_2(P - \hat{P})dW_2, \\
&\leq -\frac{r}{K}(N - \hat{N})^2 - \frac{\alpha}{(1 + h\alpha\hat{N})(1 + h\alpha N)}(N - \hat{N})(P - \hat{P}) + \frac{\alpha^2 h}{(1 + h\alpha\hat{N})(1 + h\alpha N)}(|P\hat{N} - \hat{P}N|) \\
&+ \frac{\mu(N - \hat{N})}{(\sqrt{N} - \sqrt{\hat{N}})} + \frac{C\alpha\beta}{(1 + h\alpha\hat{N})(1 + h\alpha N)}(N - \hat{N})(P - \hat{P}) + \frac{1}{2}\hat{N}\xi_1^2 + C\frac{1}{2}\hat{P}\xi_2^2 dt \\
&+ \xi_1(N - \hat{N})dW_1 + C\xi_2(P - \hat{P})dW_2.
\end{aligned}$$

If we choose $C = \frac{1}{\beta}$, we have

$$\leq -\frac{r}{K}(K - \hat{N})^2 + \frac{\alpha^2 h}{(1 + h\alpha\hat{N})(1 + h\alpha K)}(|\zeta\hat{N} - \hat{P}K|) + \frac{\mu(K - \hat{N})}{(\sqrt{K} - \sqrt{\hat{N}})} + \frac{1}{2}\hat{N}\xi_1^2 + \frac{1}{2\beta}\hat{P}\xi_2^2.$$

If $(\sqrt{K} + \sqrt{\hat{N}}) + \frac{\alpha^2 h}{(1 + h\alpha\hat{N})(1 + h\alpha K)}(|\zeta\hat{N} - \hat{P}K|) < \frac{r}{K}(K - \hat{N})^2 - \frac{1}{2}\hat{N}\xi_1^2 - \frac{1}{2\beta}\hat{P}\xi_2^2$, then $L(V) < 0$ along all trajectories. As a result, the proof is completed. \square

Remark. Random noises contribute to the destabilization of the model (4.1), and this is indicated by the condition (6.1). This is consistent with ecological explanations for the existence of random noises.

Through Theorem (6.1), we can conclude the global stability of the deterministic model (2.1) after removing the stochastic terms as follows:

Theorem 6.2. *If, recalling (3.8), the following condition holds*

$$\left(\sqrt{K} + \sqrt{\hat{N}}\right) + \frac{\alpha^2 h}{(1 + h\alpha\hat{N})(1 + h\alpha K)} (|\zeta\hat{N} - \hat{P}K|) < \frac{r}{K} (K - \hat{N})^2, \quad (6.2)$$

then, the coexistence equilibrium point of the corresponding deterministic model (2.1) is globally asymptotically stable.

Proof. This proof is similar to that of Theorem 6.1 and has been omitted. \square

7. Numerical simulations

To illustrate our results obtained in the previous sections, some numerical simulations are performed in this section. In both deterministic and stochastic versions of the model, steady-state and periodic dynamics are presented. In addition, the effects of herd immigration on the dynamics of both systems (2.1) and (4.1) are presented in this section. Moreover, different dynamics of the models are shown in the stochastic model. For all figures in this section, two kinds of figures are presented to depict the dynamic behaviors of all cases considered, which are (a) time series subfigures that show the density of prey and predator populations as time progresses and (b) phase portrait trajectory subfigures that explain the general shape of the dynamic behaviors. We select the parameters and initial conditions of the systems (2.1) and (4.1) as $r = 1$, $k = 4$, $\alpha = 1.5$, $h = 0.5$, $\delta = 0.65$, $\beta = 1$, $N(0) = 0.6$, and $P(0) = 0.3$. However, the parameters of the immigration factor and the strength of the noise are changed.

7.1. Numerical simulation of the deterministic model

For the system (2.1) (i.e., the deterministic model), when increasing the immigration factor parameter, changes in the dynamic behaviors are observed, as seen in Figures 1–3. Figure 1a illustrates that the density of populations fluctuates regularly with time, and Figure 1b demonstrates that the dynamic behavior is periodic and that it starts from the initial values and spirals out to form a large limit cycle, and the trajectory paths approach the axes. Figure 2a shows that the density of populations oscillates regularly with time but the oscillations are smaller than the oscillations in Figure 1a, so the dynamic behavior in this case is a limit cycle as shown in Figure 2b, but the limit cycle becomes smaller and the trajectories are further away from the axes than in Figure 1b. Figure 3a shows that the density of populations oscillates with time, but then stabilizes at the equilibrium point ($\hat{N} \approx 0.642$, $\hat{P} \approx 1.938$). In this case, the dynamic behavior is in a steady state, as shown in Figure 3b. From Figures 1–3, we note that herd immigration plays an important role in stabilizing the deterministic system. This satisfies condition (6.2) of Theorem 6.2.

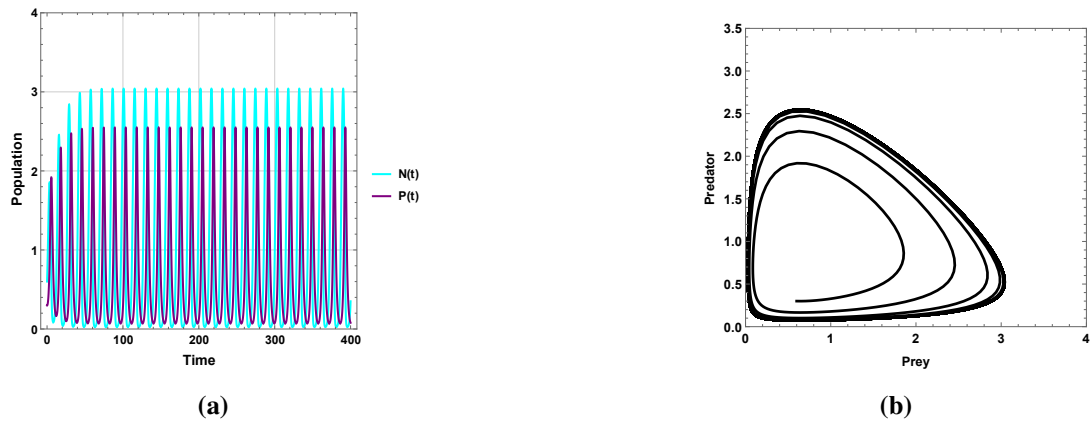


Figure 1. Dynamics of the system (2.1) when $\mu = 0$: (a) Time series; (b) phase portrait trajectories.

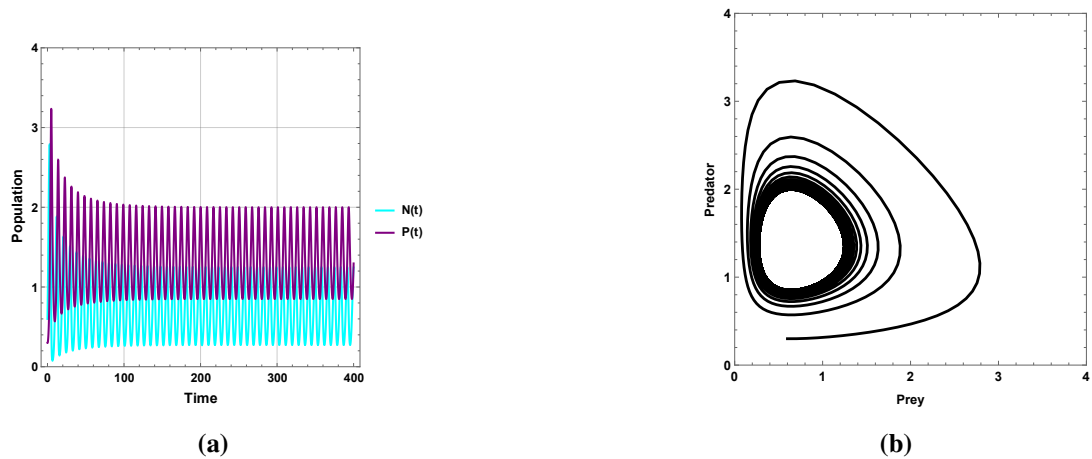


Figure 2. Dynamics of the system (2.1) when $\mu = 0.4$: (a) Time series; (b) phase portrait trajectories.

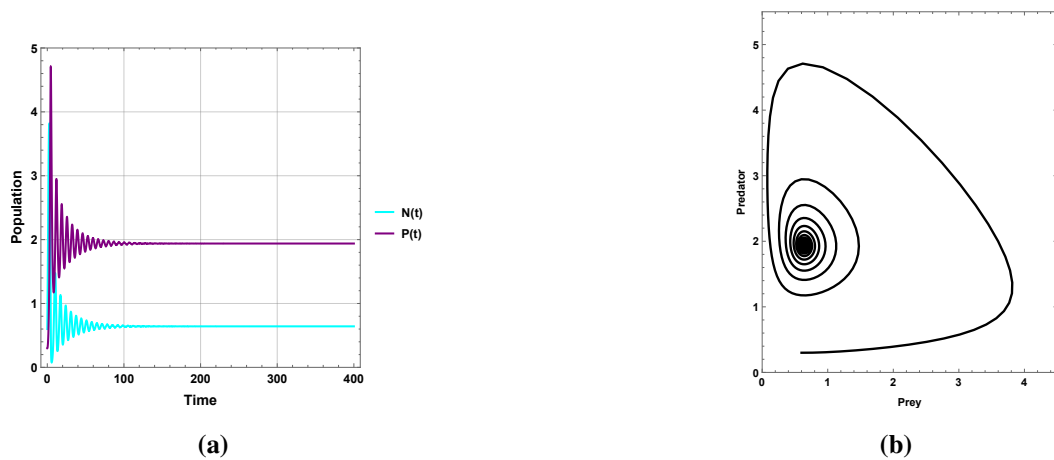


Figure 3. Dynamics of the system (2.1) when $\mu = 0.9$: (a) Time series; (b) phase portrait trajectories.

7.2. Numerical simulation of the stochastic model

Starting from the fourth figure, random noises are taken into account with the variation in the strength of these effects, which is represented by the model (4.1). When we assume small random noise strength values $\xi_1 = 0.1$ and $\xi_2 = 0.1$, Figure 4a illustrates a strong similarity to Figure 3a, with a slight fluctuation in the population density. Figure 4b reveals that the dynamic behavior is steady state, as in Figure 3b. When we increase the values of the random noise strength to $\xi_1 = 0.5$ and $\xi_2 = 0.5$, the density of the populations oscillates in a slightly irregular shape, as demonstrated in Figure 5a; the dynamic behavior is a quasi-steady state as depicted in Figure 5b. This is consistent with the condition (6.1) of Theorem 6.1. Under the assumption that the values of random noise strength are more than in the previous case, namely $\xi_1 = 1.0$ and $\xi_2 = 1.0$, Figure 6a indicates that the density of the populations oscillates more irregularly than in the previous case (i.e., Figure 5a), and Figure 6b shows that the dynamic behavior is a quasi-limit cycle. However, when we assume that the values of random noise strength are higher than in the previous cases, the density of the populations fluctuates more irregularly than in the previous cases, as illustrated in Figure 7a; the dynamic behavior is a quasi-limit cycle, as shown in Figure 7b. This is consistent with Corollary 3.4.1. As in the deterministic system, herd immigration stabilizes the stochastic system. This is explained from a biological point of view by the fact that the herd immigration benefits prey species as a defense mechanism against predators that are reluctant to attack them, so this leads to reduced predation [26, 32]; this can help stabilize unstable systems. In deterministic and stochastic systems, it is concluded that prey herd immigration enhances the survival of both species, whether through stabilizing the dynamic behaviors or reducing the fluctuations in the densities and making them more regular.

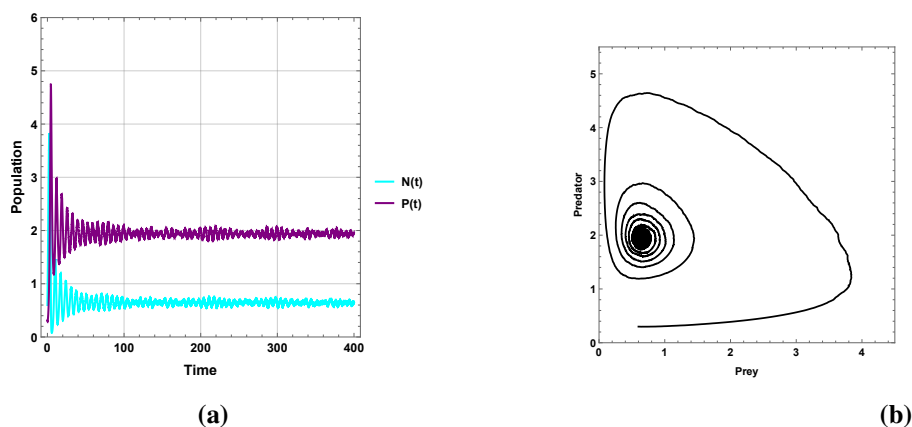


Figure 4. Dynamics of the system (4.1) when $\mu = 0.9$, $\xi_1 = 0.1$ and $\xi_2 = 0.1$: (a) Time series; (b) phase portrait trajectories.

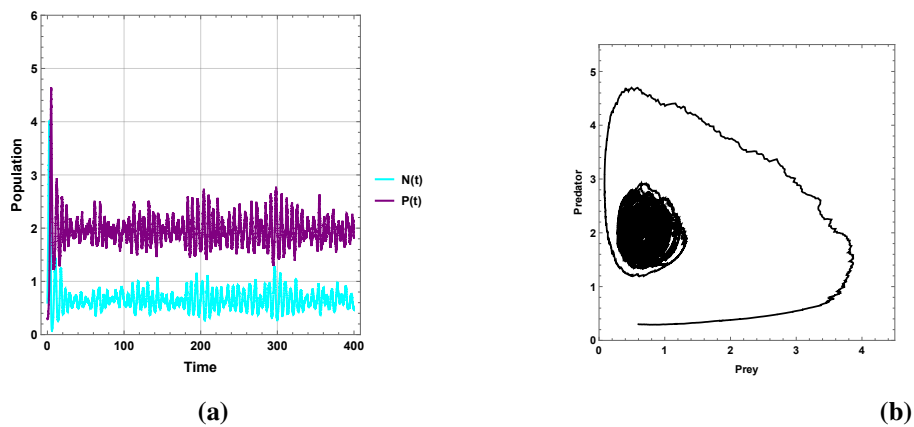


Figure 5. Dynamics of the system (4.1) when $\mu = 0.9$, $\xi_1 = 0.5$ and $\xi_2 = 0.5$: (a) Time series; (b) phase portrait trajectories.

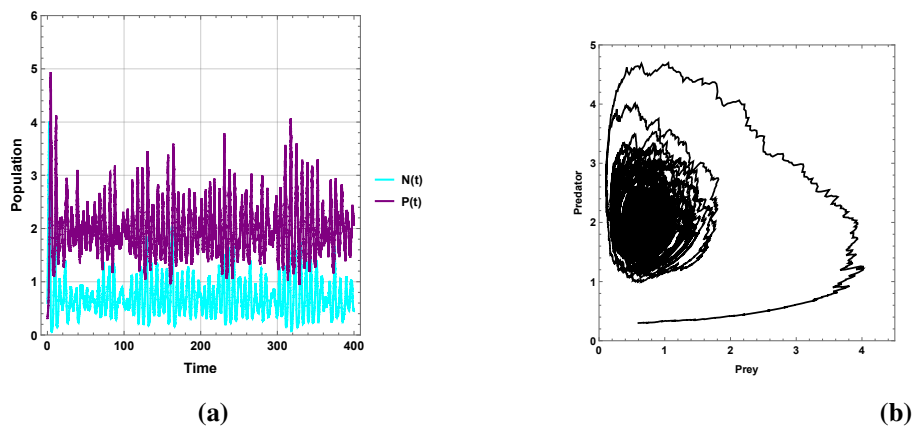


Figure 6. Dynamics of the system (4.1) when $\mu = 0.9$, $\xi_1 = 1.0$ and $\xi_2 = 1.0$: (a) Time series; (b) phase portrait trajectories.

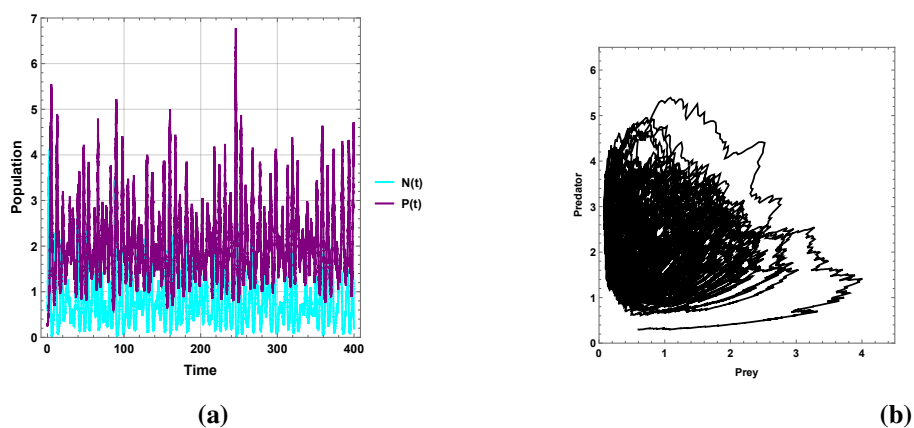


Figure 7. Dynamics of the system (4.1) when $\mu = 0.9$, $\xi_1 = 1.9$ and $\xi_2 = 1.9$: (a) Time series; (b) phase portrait trajectories.

Table 1. Summary comparison of the main results for deterministic and stochastic models.

Theoretical side		
The property	Deterministic model	Stochastic model
Positivity	Satisfied (Thm. 2.1)	Satisfied (Thm. 5.1)
Boundedness	Satisfied (Thm. 2.2)	Satisfied (Thm. 5.2)
Global asymptotic stability	Satisfied (Thm. 6.1)	Satisfied (Thm. 6.2)
Numerical side		
Dynamical behavior with the herd immigration parameter (μ) and random noise parameters (ξ_1 and ξ_2)		
The values of parameters	Deterministic model	Stochastic model
$\mu = 0, \xi_1 = 0, \text{ and } \xi_2 = 0$	Limit cycle with a large cycle (Figure 1)	Limit cycle with a large cycle (Figure 1)
$\mu = 0.4, \xi_1 = 0, \text{ and } \xi_2 = 0$	Limit cycle with a small cycle (Figure 2)	Limit cycle with a small cycle (Figure 2)
$\mu = 0.9, \xi_1 = 0, \text{ and } \xi_2 = 0$	Steady state (Figure 3)	Steady state (Figure 3)
$\mu = 0.9, \xi_1 = 0.1, \text{ and } \xi_2 = 0.1$	Not applicable	Steady-state (Figure 4)
$\mu = 0.9, \xi_1 = 0.5, \text{ and } \xi_2 = 0.5$	Not applicable	Quasi-steady state (Figure 5)
$\mu = 0.9, \xi_1 = 0.9, \text{ and } \xi_2 = 0.9$	Not applicable	Quasi-limit cycle (Figure 6)
$\mu = 0.9, \xi_1 = 1.9, \text{ and } \xi_2 = 1.9$	Not applicable	Quasi-limit cycle (Figure 7)

8. Conclusions

In recent years, some mathematicians have used the immigration factor in prey–predator systems to study its effects on dynamic behaviors. This factor stabilizes these systems, as shown, for instance in [25,33]. In prey–predator systems, herd behavior is considered a protective factor because individual members behave as a group. Separately, immigration and herd behavior have recently attracted some researchers due to their significance and effect on dynamic behaviors. This work proposes integrating the immigration factor and herd behavior to form a more realistic prey–predator model. In this system, the solutions remain positive forever and are ultimately bounded under specific conditions. Thus, the system behaves biologically well. The local asymptotic stability of equilibrium points is examined.

In addition, a stochastic version of this model has been formulated to take into account the influence of random noise. Using a suitable Lyapunov function, we obtained the conditions of the global asymptotic stability of the coexistence equilibrium point for stochastic and deterministic models (Theorems 6.1 and 6.2, respectively). Numerical simulations were used to explain the theoretical findings. The Table 1 shows a summary comparison of the main results for deterministic and stochastic models. Figure 8 presents an overall framework diagram that shows the significance and relationships of the models, theorems, and simulations. The numerical results showed that prey herd immigration has a positive role in stabilizing the systems. In a stochastic system, the numerical simulations show that the fluctuations in the density of populations become more regular with increased herd immigration; also, these simulations display rich dynamics, such as quasi-steady states

and quasi-limit cycles. Stabilizing the dynamic behaviors, making them more regular, and reducing density fluctuations enhance the survival of both species. This assumption plays an important role in boosting species' survival in environments where species' are vulnerable to extinction.

From an ecological standpoint, the effects of prey immigration on prey–predator systems may be explained by the support of these systems, known as the rescue phenomenon [34–36]; however, this study suggests that supporting unstable systems can be achieved through the immigration of prey in the form of herds.

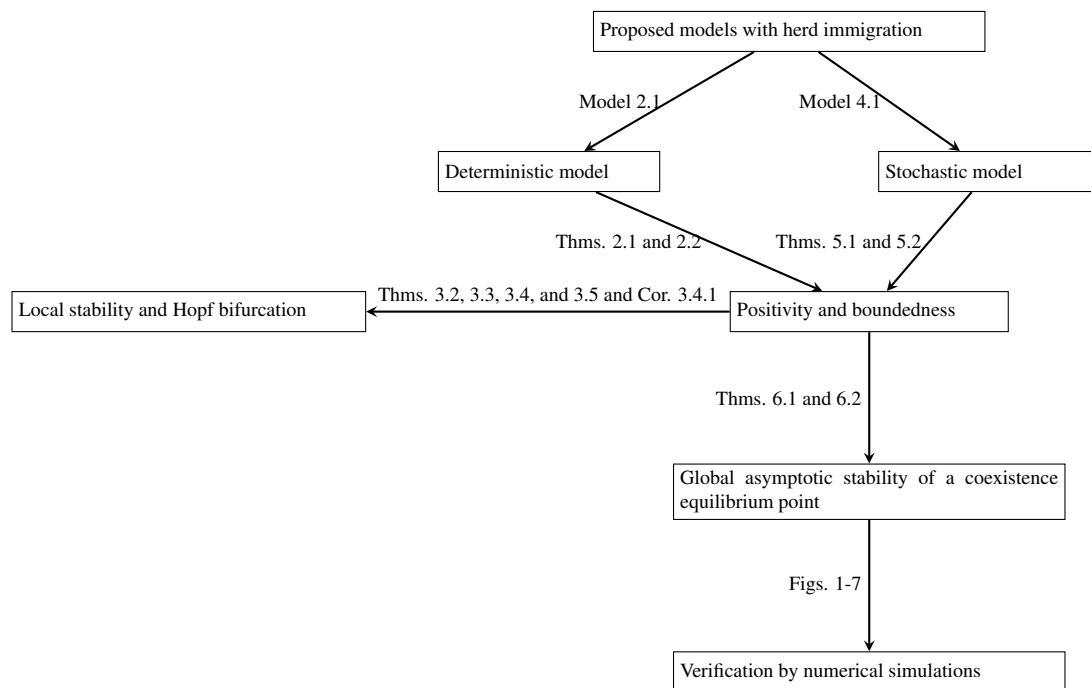


Figure 8. Framework diagram showing the significance and relationships of the models, theorems, and simulations.

Use of Generative-AI tools declaration

The author declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author extends the appreciation to the Deanship of Postgraduate Studies and Scientific Research at Majmaah University for funding this research work through the project number (R-2025-1600).

Conflict of interest

All authors declare no conflicts of interest in this paper.

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