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*Research article*

# Exploring quaternionic Bertrand curves: involutes and evolutes in $\mathbb{E}^4$

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**Abstract:** This study investigated the concepts of (0, 2)-involute and (1, 3)-evolute curves associated with quaternionic Bertrand curves within the context of four-dimensional Euclidean space. Using a type-2 quaternionic frame, we derived mathematical expressions that define these interacting and evolute curves. The (0, 2)-involute curve is characterized by tangents orthogonal to points on the original quaternionic Bertrand curve, while the (1, 3)-evolute curve is constructed using specific normal vectors related to curvature properties. We presented a comprehensive framework that clarifies the interrelationships between the curvature functions of involute and evolute pairs and their connections to the Frenet frame. This framework provides a geometric basis for analyzing curves in higher-dimensional spaces. The findings enhance the understanding of quaternionic curves and their geometric properties, contributing to the broader field of differential geometry.

**Keywords:** involute; evolute; quaternions; Euclidean space; Bertrand curves
**Mathematics Subject Classification:** 11R52, 53A04

## 1. Introduction

The study of curves in classical differential geometry is one of the most fascinating topics. The concept of the involute-evolute curve pair was introduced by Huygens in 1968 in [1]. If the tangents of the first curve are normal to the second curve, we classify the second curve as an involute and the first curve as an evolute. In 1850, Bertrand first introduced the Bertrand curve [2]. When two curves share a common normal vector at corresponding points, they are referred to as Bertrand curves. We refer to the first curve as the Bertrand curve, and the second as the Bertrand mate. One significant characteristic of three-dimensional Euclidean space is that the distance between the corresponding points of the Bertrand curves remains constant. Bertrand curves were extended from Euclidean 3 space to Riemannian $n$ space by L. R. Pears [3] who gave generalized results of the Bertrand curves. For $n \geq 4$, no special Frenet curve in $\mathbb{E}^n$ is a Bertrand curve. Matsuda and Yorozu [4] gave a new definition of the Bertrand curve called the $(1, 3)$-Bertrand curve and a characterization of the $(1, 3)$-Bertrand curves. After that, many researchers have studied $(1, 3)$-Bertrand curve [5–7]. Furthermore, some studies of characterizations of curves in different spaces and with different frames can be found in [8–11].

The study of involutes and evolutes has also been extended to Minkowski space, a pseudo-Euclidean space with an indefinite metric. In Minkowski space, the geometric properties of curves differ significantly from those in Euclidean space due to the presence of timelike, spacelike, and lightlike vectors. The concept of involute-evolute pairs in Minkowski space has been explored by several authors [12, 13].

In 1843, Hamilton introduced the concept of a quaternion, a type of number system that exists in a four-dimensional vector space and can take on different forms, such as real, complex, dual, and split varieties. According to Clifford [14], in 1871, the quaternion was generalized to biquaternions. In 1987, Bharathi and Nagaraj [15] studied the quaternionic curves ($Qu$-curves) in both $\mathbb{E}^3$ and $\mathbb{E}^4$ and provided the Frenet formula for $Qu$-curves. For other results of $Qu$-curves, we refer to [16–21]. Aksoyak in [22] defined the quaternionic frame $Qu$-frame for ($Qu$-curves) in $\mathbb{E}^4$, which is called type-2 $Qu$-frame. For more results see [23–28]. If a $(Qu - B)$-curve exists in $\mathbb{E}^4$, then its torsion or bi-torsion vanishes, so we can say that there is no $(Qu - B)$-curve whose torsion or bi-torsion is non-zero. Hence, we use the method given by Matsuda and Yorozu [4] to define the $(Qu - B)$-curve according to the type-2 $Qu$-frame. For other results regarding involutes and evolutes, we refer to the papers [29–31].

In this context, we determine the involute and evolute with the $(Qu - B)$-curve using a type-2 $Qu$-frame. We also deduce a relation between the Frenet frame and curvature functions.

## 2. Preliminaries

A real quaternion is defined as:

$$u = a + b\hat{i} + c\hat{j} + d\hat{k},$$

where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors in three-dimensional vector space and $a, b, c, d \in \mathbb{R}$. Any quaternion $u$ consists of two parts: one is the scalar part denoted as $S_u$ and the second is a vector part which is denoted as $V_u$, where

$$S_u = a$$

and

$$V_u = b\hat{i} + c\hat{j} + d\hat{k}.$$

We also present any real quaternion as

$$u = S_u + V_u.$$

Let us consider two quaternions, i.e.,

$$u = S_u + V_u$$

and

$$u' = S'_u + V'_u.$$

Then their addition and multiplication by a scalar $c$ and conjugate will be represented as

$$u + u' = (S_u + S'_u) + (V_u + V'_u),$$
$$cu = cS_u - cV_u,$$
$$\bar{u} = S_u - V_u.$$

Let us denote the four-dimensional real vector space, in which addition and multiplication by a scalar $c$ are defined as described by $Q$, and refer to its elements as quaternions. The basis of this vector space is $\{1, \hat{i}, \hat{j}, \hat{k}\}$, and satisfies

$$(\hat{i})^2 = (\hat{j})^2 = (\hat{k})^2 = \hat{i}\hat{j}\hat{k} = -1.$$

Any quaternion $u$ can be considered as an element (a, b, c, d) of $\mathbb{R}^4$. When the scalar part is zero, we refer to the quaternion as spatial. In this case, it can be considered as an ordered triple (b, c, d) of $\mathbb{R}^3$ [15].

Multiplication of quaternions can be defined as

$$u \times u' = S_u S'_u - \langle V_u, V'_u \rangle + S_u V'_u + S'_u V_u + V_u \wedge V'_u,$$

for every $u, u' \in Q$, where $\langle , \rangle$ and $\wedge$ denote the scalar product and cross product in $\mathbb{R}^3$. The quaternion multiplication is associative and distributive but non-commutative. Hence, $Q$ is a real algebra, referred to as a quaternion algebra. Moving forward, we define a symmetric and non-degenerate bilinear $h$ on $Q$ as

$$h : Q \times Q \mapsto \mathbb{R},$$
$$h(u, u') = \frac{1}{2}(u \times \bar{u'} + u' \times \bar{u}),$$

for $u, u' \in Q$ and the norm of $u$ is defined as

$$\|u\|^2 = h(u, u) = u \times \bar{u} = S_u^2 + \langle V_u, V_u \rangle .$$

Therefore, the mapping denoted by $h$ is referred to as the quaternion scalar product. The $Qu$-curve in $\mathbb{R}^4$ is represented by $\gamma^4$ and the spatial $Qu$-curve in $\mathbb{R}^3$ associated by $\gamma^4 \in \mathbb{R}^4$ is denoted by $\gamma$.

**Theorem 2.1.** ([15]) Let

$$J = [0, 1] \subset \mathbb{R}$$

and $S$ be the set of spatial $Qu$-curves. Suppose that

$$\gamma : J \subset \mathbb{R} \mapsto S,$$
$$s \mapsto \gamma(s) = \gamma_1(s)\hat{i} + \gamma_2(s)\hat{j} + \gamma_3(s)\hat{k}$$

is a curve parameterized by the arc length $s$. Then, the Frenet equations of $\gamma$ are given as

$$t' = kn,$$
$$n' = -kt + rb,$$
$$b' = -rn,$$

where $t$ represents the unit tangent vector, denoted by $\gamma'$, $n$ is the normal vector, and $b$ is the binormal vector, which is calculated as the cross product of $t$ and $n$. The principle curvature is denoted by $k$ and is equal to the norm of the derivative of the unit tangent vector, while the torsion of the curve $\gamma$ is represented by $-r$. Moreover, these Frenet vectors hold the following equations:

$$h(t,t) = h(n,n) = h(b,b) = 1,$$
$$h(t,n) = h(t,b) = h(n,b) = 0.$$

**Theorem 2.2.** Suppose that

$$J = [0,1] \subset \mathbb{R}$$

and

$$\gamma^4 = J \subset \mathbb{R} \mapsto Q,$$
$$s \mapsto \gamma^4(s) = \gamma_0^4(s) + \gamma_1^4(s)\hat{\imath} + \gamma_2^4(s)\hat{\jmath} + \gamma_3^4(s)\hat{k}$$

is an arc-length parameterized curve in $\mathbb{R}^4$. Then Frenet equations of $\gamma^4$ are given as

$$\begin{aligned} T' &= KN_1, \\ N_1' &= -KT - rN_2, \\ N_2' &= rN_1 + (K-k)N_3, \\ N_3' &= -(K-k)N_2, \end{aligned} \tag{2.1}$$

where

$$T = d\gamma^4/ds,$$

$N_1$–$N_3$ represent the Frenet vectors of the curve $\gamma^4$,

$$K = \|T'\|$$

is the principle curvature, $-r$ is the torsion and $(K-k)$ is the bitorsion of the curve $\gamma^4$. These Frenet vectors of the above theorems satisfy the following equations:

$$h(T,T) = h(N_1,N_1) = h(N_2,N_2) = h(N_3,N_3) = 1,$$

$$h(T,N_1) = h(T,N_2) = h(T,N_3) = h(N_1,N_2) = h(N_1,N_3) = h(N_2,N_3) = 0.$$

In four-dimensional Euclidean space, the product of two vectors is defined using the wedge product (or exterior product). For two vectors $u$ and $v$ in $\mathbb{E}^4$, the wedge product is given by

$$u \wedge v = u \otimes v - v \otimes u,$$

where $\otimes$ denotes the tensor product. The result is a bivector, which represents the plane spanned by $u$ and $v$.

In the context of the Frenet frame, the first normal vector $N_1$ is constructed as a vector orthogonal to the tangent vector $T(s)$ and the binormal vector $b(s)$. This is achieved by taking the wedge product of $b(s)$ and $T(s)$ and normalizing the result:

$$N_1 = \frac{b(s) \wedge T(s)}{\|b(s) \wedge T(s)\|}.$$

Here, $\|b(s) \wedge T(s)\|$ is the norm of the bivector $b(s) \wedge T(s)$, and $N_1$ is a unit vector orthogonal to both $b(s)$ and $T(s)$.

The vector product of $\vec{x}$, $\vec{y}$, $\vec{z}$ is given by the determinant as follows:

$$\vec{x} \times \vec{y} \times \vec{z} = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix},$$

where [32, 33]

$$e_1 \times e_2 \times e_3 = e_4, \quad e_2 \times e_3 \times e_4 = e_1, \quad e_3 \times e_4 \times e_1 = e_2,$$

and

$$e_4 \times e_1 \times e_2 = e_3.$$

After establishing the fundamental properties of quaternionic curves and their Frenet frames in four-dimensional Euclidean space, we now focus on the geometric relationships between these curves and their associated involute and evolute pairs. The concepts of involutes and evolutes, which describe the relationship between two curves where the tangents of one curve are normal to the other, have been extensively studied in classical differential geometry. In the context of quaternionic Bertrand curves, these relationships take on a more intricate form due to the additional dimensions and the non-commutative nature of quaternion algebra. Using the type-2 quaternionic frame, we can derive explicit expressions for the (0,2)-involute and (1,3)-evolute curves, providing deeper insights into the geometric structure of these curves and their curvature properties.

This analysis not only extends classical results to higher dimensions, but also lays the groundwork for applications in fields such as robotics, computer graphics, and theoretical physics.

**Definition 2.1.** Let

$$\gamma^4(s) : J \subset \mathbb{R} \to \mathbb{R}^4$$

and

$$\beta^4(s) : \bar{J} \subset \mathbb{R} \to \mathbb{R}^4$$

be two *Qu*-curves. If each point of $\gamma^4(s)$ corresponds to points on $\beta^4(s)$ for all $s \in J$ via a regular $C^\infty$ function, and the normal plane spanned by the normal vectors at each point $\gamma^4(s)$ coincides with the normal plane spanned by the normal vectors of

$$\beta^4(\bar{s}) = \beta^4(f(s)),$$

then $\gamma^4(s)$ is called a $(Qu - B)$-curve in $\mathbb{E}^4$, and $\beta^4(s)$ is called the $(Qu - B)$-mate of $\gamma^4(s)$.

**Definition 2.2.** Let

$$\gamma^4 : J \subset \mathbb{R} \to \mathbb{R}^4$$

and

$$\beta^4 : \bar{J} \subset \mathbb{R} \to \mathbb{R}^4$$

be two *Qu*-curves. If the tangent vectors of $\gamma^4$ are normal to $\beta^4$, then $\gamma^4$ is called a $(1,3)$-evolute when it is spanned by the first and third normal vectors, and $\beta^4$ is called a $(0,2)$-involute when it is spanned by the first tangent and the third normal vector.

## 3. The $(0,2)$-involute of a $Qu - B$ curve via a type-2 quaternionic frame in $\mathbb{E}^4$

Suppose that

$$\gamma^4 : J = [0,1] \to \mathbb{R}^4$$

is a regular *Qu-B* curve whose curvature functions are $K$, $-r$, and $K - k$. Let $\beta^4(\bar{s})$ be a $Qu - B$ $(0,2)$-mate of $\gamma^4$ with the *Qu*-frame $\{T^*, N_1^*, N_2^*, N_3^*\}$ and curvature functions $K^*$, $-r^*$, and $(K-k)^*$. If the inner product condition

$$h(T, T^*) = 0$$

holds, then the pair $(\gamma^4, \beta^4(\bar{s}))$ is called a real quaternion Bertrand involute-evolute curve pair.

The relationship between these vector fields is given by

$$\mathrm{span}\{N_1, N_3\} = \mathrm{span}\{T^*, N_2^*\}, \quad \mathrm{span}\{T, N_2\} = \mathrm{span}\{N_1^*, N_3^*\}. \tag{3.1}$$

Also, $\beta^4(\bar{s})$ is called the $(0,2)$-involute of $\gamma^4$. Thus, we can write $\beta^4(\bar{s})$ as

$$\beta^4(\bar{s}) = \beta^4(f(s)) = \gamma^4(s) + \xi(s)T(s) + \eta(s)N_2(s), \tag{3.2}$$

where $\xi(s)$ and $\eta(s)$ are $C^\infty$ functions on $[0,1]$. By differentiating (3.2) with respect to $s$ using Eq (2.1), we obtain

$$T^* f' = (1 + \xi')T + (\xi k + \eta r)N_1 + \eta' N_2 + \eta(K - k)N_3. \tag{3.3}$$

Taking the scalar product by vectors $T$ and $N_2$ on both sides of Eq (3.3), we obtain

$$(1 + \xi') = 0 \quad \text{and} \quad \eta' = 0.$$

So, Eq (3.3) can be rewritten as

$$T^* f' = (\xi k + \eta r)N_1 + \eta(K - k)N_3. \tag{3.4}$$

Taking the substitution for some differentiable functions $\Gamma$ and $\Theta$ as

$$\Gamma = \frac{\xi k + \eta r}{f'} \quad \text{and} \quad \Theta = \frac{\eta(K - k)}{f'}. \tag{3.5}$$

Then Eq (3.4) becomes

$$T^* = \Gamma N_1 + \Theta N_3, \tag{3.6}$$

where

$$\Gamma^2 + \Theta^2 = 1,$$

because $N_1 \perp N_2$.

**Case 1.** Suppose $\xi \neq 0$. In this case, $\eta \neq 0$. Denoting

$$\frac{\Gamma}{\Theta} = x_1,$$

then,

$$\xi K + \eta r = \eta x_1(K - k),$$
$$f' = \eta \Theta^{-1}(K - k),$$

and

$$\Theta^2 = \frac{1}{1 + x_1^2}. \tag{3.7}$$

Differentiating Eq (3.6), and using Eq (2.1), we obtain

$$f' K^* N_1 = \Gamma' N_1 - \Gamma K T - (\Gamma r + \Theta(K - k)) N_2 + \Theta' N_3. \tag{3.8}$$

Taking the scalar product of Eq (3.8) with $N_1$ and $N_3$, we find that $\Gamma' = 0$ and $\Theta' = 0$ which implies that $\Gamma$ and $\Theta$ are constants.

Now, we rewrite Eq (3.8) as

$$f' K^* N_1^* = -\Gamma K T - (\Gamma r + \Theta(K - k)) N_2. \tag{3.9}$$

Taking the substitution for some differentiable functions $\Phi$ and $\Psi$ as

$$\Phi = \frac{-\Gamma K}{f' K^*} \quad \text{and} \quad \Psi = \frac{-(\Gamma r + \Theta(K - k))}{f' K^*}, \tag{3.10}$$

then Eq (3.9) becomes

$$N_1^* = \Phi T + \Psi N_2, \quad \Phi^2 + \Psi^2 = 1. \tag{3.11}$$

Denote

$$\frac{\Psi}{\Phi} = x_2.$$

This implies that

$$\Psi = \Phi x_2$$

and

$$x_1(x_2 K - r) = (K - k), \tag{3.12}$$

and then

$$\Phi^2 = \frac{1}{1 + x_2^2}.$$

From Eqs (3.7) and (3.12), we have

$$\tau = \frac{-r}{K} = \left( \frac{\xi/\eta - x_1^2 x_2}{1 - x_1^2} \right), \quad \frac{(K - k)}{K} = x_1(\tau + x_2). \tag{3.13}$$

Denote

$$\frac{\Theta}{\Phi} = x_3.$$

This implies that

$$\Theta = x_3\Phi.$$

Using Eq (3.10), we obtain

$$f'K^* = -x_1x_3K \quad \text{and} \quad x_3^2 = \frac{1+x_2^2}{1+x_1^2}. \tag{3.14}$$

Differentiating (3.11) with respect to $s$ using Eq (2.1), we obtain

$$-f'K^*T^* - f'r^*N_2^* = \Phi'T + (\Phi K + \Psi r)N_1 + \Psi' N_2 + \Psi(K-k)N_3. \tag{3.15}$$

Taking the scalar product on both sides of Eq (3.15) with $T$ and $N_2$, we find that

$$\Phi' = 0 \quad \text{and} \quad \Psi' = 0,$$

which implies that $\Phi$ and $\Psi$ are constants.

Thus, we can write

$$-f'r^*N_2^* = f'K^*T^* + \Phi x_2(K + x_2 r)N_1 + \Phi(K-k)N_3. \tag{3.16}$$

Substituting Eqs (3.6) and (3.14) into (3.16), we obtain

$$-f'r^*N_2^* = \Phi k(x_2\tau + x_2^2 - x_3^2)(N_1 + x_1N_3). \tag{3.17}$$

From Eq (3.17), we can choose

$$N_2^* = -\Theta N_1 + \Gamma N_3, \quad f'K^* = -x_3^{-1}K(x_2\tau + x_2^2 - x_3^2), \tag{3.18}$$

for some differentiable functions $\Theta$ and $\Gamma$. Differentiating (3.18) with respect to to $s$ using Eq (2.1), we obtain

$$f'r^*N_1^* + f'(K-k)^*N_3^* = \Theta KT + (\Theta r - \Gamma(K-k))N_2.$$

From this, we obtain

$$f'(K-k)^*N_2^* = (\Theta K - \phi f'r^*)T + (\Theta K - \psi f'r^* - \Gamma(K-k))N_2. \tag{3.19}$$

Also, we can write

$$\begin{aligned} f'(K-k)^*N_3^* &= (-\phi f'r^* + \Theta K)T + (-\psi f'r^* + \Theta r - \Gamma(K-k))N_2 \\ &= -x_3^{-1}K(\tau + x_2)(-\Psi T + \Phi N_2). \end{aligned} \tag{3.20}$$

Equation (3.20) becomes

$$N_3^* = -\psi T + \phi N_2, \quad f'(K-k) = -x_3^{-1}K(\tau + x_2). \tag{3.21}$$

**Theorem 3.1.** Let

$$\gamma^4 : I \subset \mathbb{R} \mapsto \mathbb{E}^4$$

be a unit speed $(Qu - B)$-curve with non-zero curvatures $K$, $-r$, and $(K-k)$. Then $\gamma^4$ has the $(0,2)$-involute of quaternionic Bertrand curve.

*Proof.* Let $\beta^4(\bar{s})$ be the $(0,2)$-involute curve of $\gamma^4(s)$. Then the equation for $\beta^4(\bar{s})$ can be written as:

$$\beta^4(\bar{s}) = \gamma^4(s) + (\xi_0 - s)T(s) + \eta N_2,$$

with $\eta \neq 0$.

Then, the curvatures satisfy

$$\tau = \frac{-r}{K} = \frac{\xi_0 - s - \eta x_1^2 x_2}{\eta(1 + x_1^2)}, \quad \frac{(K-k)}{K} = x_1(\tau + x_2), \tag{3.22}$$

where $\xi_0, \eta, x_1$, and $x_2$ are constants.

Furthermore, the curvatures of $\beta(\bar{s})$ are given by

$$\begin{aligned} K^* &= \frac{\Phi x_3^2}{\eta(\tau + x_2)}, \\ -r^* &= \frac{\Phi(x_2\tau + x_2^2 - x_3^2)}{\eta\tau_1(\tau + x_2)}, \\ (K-k)^* &= x_1(\tau + x_2), \end{aligned}$$

where $\Phi \neq 0$, the related frame is given by

$$\begin{aligned} T^* &= \Phi x_3 \left(x_1 N_1 + N_3\right), \\ N_1^* &= \psi \left(T + x_2 N_2\right), \\ N_2^* &= \Phi \left(-N_1 + x_1 N_3\right), \\ N_3^* &= \Phi \left(-x_2 T + N_2\right). \end{aligned}$$

This completes the proof. □

**Corollary 3.1.** If $\frac{-r}{R}$ or $\frac{(K-k)}{K}$ is constant, then the $(Qu - B)$-curve $\gamma^4$ does not have a $(0,2)$-involute of $(Qu - B)$-curve in the form

$$\beta^4(\bar{s}) = \gamma^4(s) + (\xi_0 - s)T + \eta N_2(s), \quad \eta \neq 0.$$

**Case 2.** Suppose $\eta \neq 0$. Thus, Eq (3.21) reads as

$$\beta^4(\bar{s}) = \gamma^4(s) + (\xi_0 - s)T(s). \tag{3.23}$$

Differentiating (3.23) with respect to $s$ using Eq (2.1), we obtain

$$f'T^* = (1 + \xi_0 - s)T(s) + (\xi_0 - s)KN_1,$$

taking the scalar product, we have

$$f'T^* = (\xi_0 - s)KN_1. \tag{3.24}$$

This implies that

$$f' = (s - \xi_0)K, \quad T^* = -N_1. \tag{3.25}$$

By differentiating (3.25) with respect to $s$ using Eq (2.1), we obtain:

$$f'K^*N_1^* = KT + rN_2. \tag{3.26}$$

Let

$$\upsilon = \left(\frac{K}{f'K^*}\right) \quad \text{and} \quad \nu = \left(\frac{r}{f'K^*}\right).$$

Then

$$N_1^* = \upsilon T + \nu N_2, \quad \upsilon^2 + \nu^2 = 1. \tag{3.27}$$

This implies that

$$\frac{r}{K} = \frac{\nu}{\upsilon}. \tag{3.28}$$

Differentiating (3.27) with respect to $s$ using Eq (2.1), we deduce that $\upsilon$ and $\nu$ are constants. Hence,

$$\begin{aligned} -f'r^*N_2^* &= f'K^*T^* + (\upsilon k + \nu r)N_1 + (K-k)N_3 \\ &= -\nu(\frac{\nu}{\upsilon}K - r^*)N_1 + \nu(K-k)N_3 \\ &= \nu(K-k)N_3. \end{aligned} \tag{3.29}$$

We assume that

$$N_2^* = -N_3, \quad -f'r^* = -\nu(K-k). \tag{3.30}$$

By differentiating the Eq (3.31) with respect to $s$ using Eq (2.1), we obtain

$$\begin{aligned} f'(K-k)N_3^* &= -f'r^*N_1 + (K-k)N_2 \\ &= -(K-k)[\upsilon\nu T - (1-\nu^2)N_2]. \end{aligned}$$

Thus, we have

$$N_3^* = \nu T + \upsilon N_2, f'(K-k)^* = \upsilon(K-k). \tag{3.31}$$

**Corollary 3.2.** Let

$$\gamma^4 : I \mapsto \mathbb{E}^4$$

be a unit speed $(Qu-B)$-curve with non-zero curvatures $K$, $-r$, and $(K-k)$. If $\gamma^4$ have a $(0,2)$-involute of the $(Qu-B)$-curve

$$\beta^4(s) = \gamma^4 + (\xi - s),$$

then curvature $K$ and $-r$ satisfy the equality

$$\nu K - \upsilon r = 0, \tag{3.32}$$

where $\xi_0$, $\upsilon$, and $\nu$ are constants and the curvatures of $\beta^4(\bar{s})$ are given as

$$K^* = \frac{1}{\upsilon(s-\xi_0)}, \quad -r^* = \frac{-\nu(K-k)}{(s-\xi_0)K}, \quad (K-k)^* = \frac{\upsilon(K-k)}{K(s-\xi_0)}.$$

The required $Qu$-frame is given by

$$\begin{aligned} T^* &= -N_1, \\ N_1^* &= \upsilon T + \nu N_2, \\ N_2^* &= -N_3, \\ N_3^* &= -\nu T + \upsilon N_2. \end{aligned}$$

## 4. The $(1,3)$-evolute of a $(Qu - B)$-curve via a type-2 quaternionic frame in $\mathbb{E}^4$

**Definition 4.1.** Let

$$\gamma^4, \beta^4(\bar{s}) : J \subset [0,1] \to \mathbb{E}^4$$

be two unit-speed $(Qu - B)$-curves. Suppose that $\gamma^4$ has nonzero curvatures $K$, $-r$, and $(K - k)$. Additionally, assume that $\beta^4(\bar{s})$ has a Frenet frame $\{T^*, N_1^*, N_2^*, N_3^*\}$ with nonzero curvatures $K^*$, $-r^*$, and $(K - k)^*$.

Then, $\beta^4(\bar{s})$ is called a $(1,3)$-evolute if the following conditions hold:

$$\operatorname{span}\{T, N_2\} = \operatorname{span}\{N_1^*, N_3^*\}, \quad \operatorname{span}\{N_1, N_3\} = \operatorname{span}\{T^*, N_2^*\}. \tag{4.1}$$

In other words, $\beta^4(\bar{s})$ has the parametric representation

$$\beta^4(\bar{s}) = \gamma^4(s) + \lambda N_1 + \mu N_3, \tag{4.2}$$

where $\lambda$ and $\mu$ are smooth functions defined on the closed unit interval $J$.

Differentiating (4.2) with respect to $s$, we obtain

$$f' T^* = (1 - \lambda K)T + \lambda' N_1 + (\lambda r - \mu(K - k))N_2 + \mu' N_3. \tag{4.3}$$

Also,

$$f' T^* = \lambda' N_1 + \mu' N_3. \tag{4.4}$$

If we denote

$$c = \frac{\lambda'}{f'}, \quad d = \frac{\mu'}{f'}, \tag{4.5}$$

then Eq (4.4) becomes

$$T^* = cN_1 + dN_3, \quad \text{where} \quad c^2 + d^2 = 1. \tag{4.6}$$

Differentiating (4.4) with respect to $s$ and using Eq (2.1), we obtain

$$f' K^* N_1 = -cKT + c' N_1 - (cr + d(K - k))N_2 + d' N_3. \tag{4.7}$$

From Eq (4.5), we obtain

$$\lambda = cf + \lambda_0 = \frac{1}{K}, \quad \mu = df + \mu_0 = \frac{-r}{K(K - k)}. \tag{4.8}$$

Consequently, $c$ and $d$ are constants. Hence, Eq (4.7) becomes

$$f' K^* N_1^* = -cKT - (cr + d(K - k))N_2. \tag{4.9}$$

Let

$$m = -\frac{cK}{f'K^*} \quad \text{and} \quad n = -\frac{cr + d(K-k)}{f'K^*}. \tag{4.10}$$

Then Eq (4.9) becomes

$$N_1^* = mT + nN_2, \quad f'K^* = -m^{-1}cK, \quad m^2 + n^2 = 1. \tag{4.11}$$

Differentiating Eq (4.11) with respect to $s$ using (2.1), we obtain

$$-f'K^*T^* - f'r^*N_2^* = m'T + (mK + nr)N_1 + n'N_2 + n(K-k)N_3. \tag{4.12}$$

So, from Eq (4.1), we have $m' = 0$ and $n' = 0$, which implies that $m$ and $n$ are constants. Thus, Eq (4.12) takes the form

$$-f'r^*N_2^* = \left(\frac{m^2 - c^2}{m}K + nr\right)N_1 + \left(n(K-k) - \frac{cd}{m}K\right)N_3. \tag{4.13}$$

Let

$$\vartheta = (-f'r^*)^{-1}\left(\frac{m^2 - c^2}{m}K + nr\right) \quad \text{and} \quad \varrho = -f'r^*)^{-1}(n(K-k) - \frac{cd}{m}K. \tag{4.14}$$

Equation (4.13) becomes

$$N_2^* = \vartheta N_1 + \varrho N_3, \quad \vartheta^2 + \varrho^2 = 1. \tag{4.15}$$

Since $T^* \perp N_2^*$, from Eqs (4.6) and (4.15), we have

$$\frac{\vartheta}{\varrho} = -\frac{d}{c}.$$

Then Eq (4.14) becomes

$$N_2^* = -dN_1 + CN_3, -f'r^* = -\frac{d}{m}K + \frac{n}{c}(K-k). \tag{4.16}$$

Differentiating Eq (4.16) with respect to $s$ using Eq (2.1), we obtain

$$f'r^*N_1 + f'(K-k)^*N_3^* = dKT + (dr - c(K-k))N_2,$$

from which we obtain

$$f'(K-k)N_3^* = \frac{mn}{c}(K-k)T + \left[\frac{n^2 - c^2}{c}(K-k) - dK\frac{m+n}{m}\right]N_2. \tag{4.17}$$

From Eq (4.17), we have

$$N_3^* = -nT + mN_2, f'(K-k)^* = -\frac{m}{c}(K-k). \tag{4.18}$$

**Theorem 4.1.** If

$$\gamma^4(s) : I \mapsto \mathbb{E}^4$$

is a unit speed $(Qu - B)$-curve with non-zero curvatures $K$, $-r$, and $(K-k)$. Then $\gamma^4(s)$ possesses the $(1,3)$-evolute of the $Qu - B$-curve.

*Proof.* Suppose that
$$\beta^4(\bar{s}) = \gamma^4 + \frac{1}{K(s)}N_1(s) + \frac{-r}{K(K-k)}N_3(s).$$
Then curvatures of $\beta^4(\bar{s})$ are obtained as
$$K^* = -\frac{cK}{mf'}, \quad -r^* = \frac{\frac{n}{c}(K-k) - \frac{d}{m}K}{f'}, \quad (K-k)^* = -\frac{-\frac{m}{c}(K-k)}{f'}, \tag{4.19}$$
where
$$f' = (\frac{1}{cK'}).$$
The $Qu$-frame of the curve $\beta^4(\bar{s})$ is given by
$$\begin{aligned} T^* &= cN_1 + dN_3, \\ N_1^* &= mT + nN_2, \\ N_2 &= -dN_1 + cN_3, \\ N_3 &= -nT + mN_2, \end{aligned}$$
where $m$, $n$, $c$, and $d$ are constants.

In the following theorem, we provide both necessary and sufficient conditions for a $(Qu - B)$-curve to possess a $(1,3)$-$(Qu - B)$-evolute curve.

This completes the proof. □

**Theorem 4.2.** Let
$$\gamma^4(s), \beta^4(\bar{s}) : J \subset [0,1] \mapsto \mathbb{R}$$
be two $(Qu - B)$-curves with non-zero curvatures. Then, $\beta^4(\bar{s})$ is a $(1,3)$-evolute of $(Qu - B)$-curve of $\gamma^4(s)$, if and only if there exist $\Phi$ and $\Psi$ differentiable of $s$ and constants $\Delta \neq \pm 1$ and $\mu$ satisfying:
$$\Phi' = \Delta\Psi', \tag{4.20}$$
$$\mu\Delta K = -\left(\Delta r + (K-k)\right), \tag{4.21}$$
$$\Delta\left[(K-k)^2 - K^2 + r^2\right] + r(K-k)(\Delta^2 - 1) \neq 0. \tag{4.22}$$

*Proof.* The curve $\beta^4(\bar{s})$ can be written as:
$$\beta^4(\bar{s}) = \gamma^4(s) + \Phi(s)N_1(s) + \Psi(s)N_3(s), \tag{4.23}$$
for all $\bar{s}, s \in I$, where $\Phi(s)$ and $\Psi(s)$ are $C^\infty$ functions on the unit interval $J$. Differentiating (4.23) with respect to $s$ using Eq (2.1), we have
$$f'T^* = (1 - \Phi K)T(s) + \Phi'(s)N_1(s) - (\Phi(s)r + \Psi(s)(K-k))N_2 + \Psi'(s)N_3. \tag{4.24}$$
Since
$$\{T^*, N_2^*\} \perp \{T, N_2\},$$
so
$$1 - \Phi K = 0$$

and

$$-(\Phi r + \Psi(K - k) = 0,$$

and from this, we obtain

$$\Phi = \frac{1}{K}, \quad \Psi = \frac{-r}{K(K-k)}.$$

Therefore, Eq (4.25) becomes

$$f' T^* = \Phi'(s) N_1 + \Psi'(s) N_3. \tag{4.25}$$

By squaring Eq (4.25), we obtain

$$(f')^2 = (\Phi')^2 + (\Psi')^2. \tag{4.26}$$

If we denote

$$\epsilon = \frac{\Phi'}{f'}, \quad \varepsilon = \frac{\Psi'}{f'}, \tag{4.27}$$

then from Eqs (4.25) and (4.27), we have

$$T^* = \epsilon N_1 + \varepsilon N_3. \tag{4.28}$$

Differentiating (4.28), with respect to $s$ using Eq (2.1), we obtain

$$f' K^* N_1^* = -\epsilon K T + \epsilon' N_1 - (\epsilon r + \varepsilon(K - k)) N_2 + \varepsilon' N_3. \tag{4.29}$$

Since

$$\{N_1^*, N_3^*\} \perp \{N_1, N_3\},$$

we have

$$\epsilon' = 0, \quad \varepsilon' = 0. \tag{4.30}$$

This means that $\epsilon$ and $\varepsilon$ are constants. Then,

$$f' K^* N_1^* = \epsilon K T - (\epsilon r + \varepsilon(K - k)) N_2. \tag{4.31}$$

Squaring Eq (4.31), we have

$$(f')^2 (K^*)^2 = \epsilon^2 K^2 - (\epsilon r + \varepsilon(K - k)). \tag{4.32}$$

From Eq (4.27), we have

$$\Phi' \varepsilon = \epsilon \Psi'$$

and

$$\Phi' = \Delta \Psi', \tag{4.33}$$

where

$$\Delta = \frac{\epsilon}{\varepsilon}$$

for $\varepsilon \neq 0$.

By Eqs (4.27) and (4.32), we obtain

$$(f')^2(K^*)^2 = \left(\frac{\Psi'}{f'}\right)^2 [\Delta^2 K^2 - (\Delta r + (K-k))^2]. \tag{4.34}$$

Also,

$$f'^2 = (\Psi')^2(\Delta^2 + 1). \tag{4.35}$$

From (4.34) and (4.35), we get

$$(f')^2(K^*)^2 = \frac{1}{\Delta^2+1}[\Delta^2 K^2 - (\Delta r + (K-k)^2]. \tag{4.36}$$

Denote

$$\delta_1 = -\frac{\epsilon K}{f'K^*} = -\left(\frac{\Psi'\Delta}{f'^2 K^*}\right)K, \tag{4.37}$$

$$\delta_2 = -\frac{(\epsilon r + \varepsilon(K-k)}{f'K^*} = -\left(\frac{\Psi'}{f'^2 K^*}\right)[\Delta r + (K-k)]. \tag{4.38}$$

Thus, we can write

$$\mu \Delta K = -(\Delta r + (K-k)),$$

where

$$\mu = -\frac{\delta_2}{\delta_1},$$

for $\delta_1 \neq 0$. Using values of $\delta_1$, $\delta_2$ in Eq (4.31), we obtain

$$N_1^* = \delta_1 T + \delta_2 N_2. \tag{4.39}$$

Taking the derivative of (4.39), with respect to $s$ using Eq (2.1), we obtain

$$-f'K^*T^* - f'r^*N_2^* = \delta_1' T + (\delta_1 K + \delta_2 r)N_1 + \delta_2' N_2 + \delta_2(K-k)N_3. \tag{4.40}$$

Since

$$\{T^*, N_2^*\} \perp \{T, N_2\},$$

we get

$$\delta_1' = 0, \qquad \delta_2' = 0. \tag{4.41}$$

From (4.28) and (4.37)–(4.40), we obtain

$$-f'r^*N_2^* = P(s)N_1 + Q(s)N_3, \tag{4.42}$$

where

$$P(s) = \frac{\Psi'}{f'^2(\Delta^2+1)K^*}[\Delta((k-K)^2 - K^2 + r^2) + r(K-k)(\Delta^2 - 1)], \tag{4.43}$$

$$Q(s) = -\frac{\Delta\Psi'}{f'^2(\Delta^2+1)K^*}[\Delta((K-k) - K^2 + r) + r(K-k)(\Delta^2 - 1)]. \tag{4.44}$$

Since

$$-f'r^*N_2^* \neq 0,$$

we get the result (4.22):

$$r(K-k)(\Delta^2-1)+\Delta[(K-k)^2-K^2+r^2]. \tag{4.45}$$

Conversely, let $\gamma^4(s)$ be an evolute curve satisfying (4.42)–(4.44). Then, we can write

$$\beta^4(\bar{s})=\gamma^4(s)+\Phi(s)N_1(s)+\Psi(s)N_3(s). \tag{4.46}$$

Differentiating (4.46) with respect to $s$ using Eq (2.1), we get

$$\frac{d\beta^4(\bar{s})}{ds}=\Phi'(s)N_1+\Psi'(s)N_3. \tag{4.47}$$

Using Eqs (4.47) and (4.20), we obtain

$$\frac{d\beta^4(\bar{s})}{ds}=\Psi'[\Delta N_1+N_3]. \tag{4.48}$$

From this,

$$f'=\|\frac{\beta^4(\bar{s})}{ds}\|=\Psi'[\Delta^2+1]>0, \tag{4.49}$$

since $\Psi'>0$. Then Eq (4.48) becomes

$$f'T^*=\Psi'[\Delta N_1+N_3]. \tag{4.50}$$

Substituting from Eq (4.49) into (4.50), we obtain

$$T^*=\frac{1}{\sqrt{\Delta^2+1}}[\Delta N_1+N_3]. \tag{4.51}$$

Differentiating (4.51), with respect to $s$ using Eq (2.1), we obtain

$$\frac{dT^*}{ds}=\frac{1}{f'\sqrt{\Delta^2+1}}[-\Delta KT-(\Delta r+(K-k))N_2]. \tag{4.52}$$

By using Eq (4.52), we obtain

$$K^*=\|\frac{dT^*}{ds}\|=\frac{\sqrt{(\Delta K)^2-(\Delta r+(K-k))}}{f'\sqrt{\Delta^2+1}}>0. \tag{4.53}$$

From Eqs (4.52) and (4.53), we get

$$N_1^*=\frac{1}{K^*}\frac{dT^*}{ds}=\frac{1}{\sqrt{(\Delta K)^2-(\Delta r+(K-k))^2}}[-(\Delta K)T-(\Delta r+(K-k))N_2]. \tag{4.54}$$

Let

$$\Delta_1=\frac{-\Delta K}{\sqrt{(\Delta K)^2-(\Delta r+(K-k))^2}},\quad \Delta_2=\frac{-(\Delta r+(K-k))}{\sqrt{(\Delta K)^2-(\Delta r+(K-k))^2}}. \tag{4.55}$$

Then, we have

$$N_1^*=\Delta_1T+\Delta_2N_2. \tag{4.56}$$

Taking the derivative of Eq (4.56), with respect to $s$ using Eq (2.1), we obtain

$$f' \frac{dN_1^*}{ds} = \Delta_1' T + (\Delta_1 K + \Delta_2 r) N_1 + \Delta_2' N_2 + \Delta_2 (K - k) N_3. \tag{4.57}$$

Differentiating (4.21), we have

$$-(\Delta r' + (K - k))\Delta K + (\Delta r + (K - k))\Delta K' = 0. \tag{4.58}$$

From Eq (4.1), we deduce

$$N'^* \in span\{N_1, N_3\},$$

because $N'^* \perp N^*$.

$$\Delta_1' = 0, \quad \Delta_2' = 0. \tag{4.59}$$

Using Eqs (4.54) and (4.59) in Eq (4.57), we get

$$\frac{dN^*}{ds} = \frac{-(\Delta K)K + (\Delta r + (K - k))r}{f' \sqrt{(\Delta K)^2 - (\Delta r + (K - k))^2}} N_1 - \frac{(\Delta r + (K - k))(K - k)}{f' \sqrt{(\Delta K)^2 - (\Delta r + (K - k))}} N_3. \tag{4.60}$$

From (4.51) and (4.53), we have

$$K^* T^* = \frac{\sqrt{(\Delta K)^2 - (\Delta r + (K - k))^2}}{f'(\Delta^2 + 1)} [\Delta N_1 + N_3]. \tag{4.61}$$

By Eqs (4.60) and (4.61), we obtain:

$$\frac{dN^*}{ds} + K^* T^* = \frac{-r(K - k)(1 - \Delta^2) + \Delta(r^2 - K^2 + (K - k)^2)}{f'(\Delta^2 + 1) \sqrt{(\Delta K)^2 - (\Delta r + (K - k))^2}} [N_1 - \Delta N_3], \tag{4.62}$$

and from Eq (4.62),

$$-r^* = \frac{| - r(K - k)(1 - \Delta^2) + \Delta(r^2 - K^2 + (K - k)^2)|}{f' \sqrt{(\Delta K)^2 - (\Delta r + (K - k))^2}} > 0. \tag{4.63}$$

Combining Eqs (4.62) and (4.63), we obtain

$$N_2^* = -\frac{1}{r^*} \left[ \frac{dN_1^*}{ds^*} + K^* T^* \right] = \frac{1}{\Delta^2 + 1} [N_1 - \Delta N_3]. \tag{4.64}$$

Also, $N_3^*$ can be stated as

$$N_3^* = -\Delta_2 T + \Delta_1 B_1;$$

that is

$$N_3^* = \frac{1}{\sqrt{(\Delta K)^2 - (\Delta r + (K - k))^2}} [(\Delta r + (K - k))T - \Delta K N_2]. \tag{4.65}$$

In the end, we find $(K - k)^*$

$$(K - k)^* = h\left(\frac{dN_2^*}{ds}, N_3^*\right) = \frac{-K(K - k)}{f' \sqrt{(\Delta K)^2 - (\Delta r + (K - k))^2}} \neq 0. \tag{4.66}$$

Hence, we find that $\beta^4(\bar{s})$ is the $(1, 3)$-evolute curve of the $(Qu - B)$-curve $\gamma^4(s)$. Therefore,

$$span\{T, N_2\} = span\{N_1^*, N_3^*\}, \quad span\{N_1, N_3\} = span\{T^*, N_2^*\}.$$

□

## 5. Examples

Consider the following quaternionic Bertrand curve $\gamma^4(s)$ in $\mathbb{E}^4$ parameterized by arc length $s$

$$\gamma^4(s) = (\cos(s), \sin(s), \cos(s), \sin(s)) .$$

This curve lies on a Clifford torus in $\mathbb{E}^4$. The tangent vector $T(s)$ is given by

$$T(s) = \frac{\frac{d\gamma^4}{ds}}{\|\frac{d\gamma^4}{ds}\|} = \frac{1}{\sqrt{2}} (-\sin(s), \cos(s), -\sin(s), \cos(s)) .$$

The principal curvature $K(s)$ is the norm of $T'(s)$

$$K(s) = 1.$$

Thus, the first normal vector is

$$N_1(s) = \frac{1}{\sqrt{2}} (-\cos(s), -\sin(s), -\cos(s), -\sin(s)) .$$

The torsion $-r(s)$ is

$$-r(s) = \sqrt{2}.$$

Thus, the second normal vector is

$$N_2(s) = \frac{1}{\sqrt{2}} (\sin(s), -\cos(s), \sin(s), -\cos(s)) .$$

The bitorsion $K(s) - k(s)$ is the norm of $N_2'(s)$

$$K(s) - k(s) = \sqrt{2}.$$

Thus, the third normal vector is

$$N_3(s) = \frac{N_2'(s)}{K(s) - k(s)} = \frac{1}{\sqrt{2}} (\cos(s), \sin(s), \cos(s), \sin(s)) .$$

### *5.1. The $(0, 2)$-involute curve of the curve $\gamma^4(s)$*

The $(0, 2)$-involute curve $\beta^4(\bar{s})$ is given by

$$\beta^4(\bar{s}) = \gamma^4(s) + (\xi_0 - s)T(s) + \eta N_2(s),$$

where $\xi_0$ and $\eta$ are constants. Substituting the expressions for $\gamma^4(s)$, $T(s)$, and $N_2(s)$, we get:

$$\begin{aligned}\beta^4(\bar{s}) = {} & (\cos(s), \sin(s), \cos(s), \sin(s)) \\ & + (\xi_0 - s) (-\sin(s), \cos(s), -\sin(s), \cos(s)) \\ & + \eta \frac{1}{\sqrt{2}} (\sin(s), -\cos(s), \sin(s), -\cos(s)) .\end{aligned}$$

Simplifying, the $(0, 2)$-involute curve is

$$\begin{aligned}\beta^4(\bar{s}) = {} & (\cos(s) - (\xi_0 - s) \sin(s) + \frac{\eta}{\sqrt{2}} \sin(s),\ \ \sin(s) + (\xi_0 - s) \cos(s) - \frac{\eta}{\sqrt{2}} \cos(s), \\ & \cos(s) - (\xi_0 - s) \sin(s) + \frac{\eta}{\sqrt{2}} \sin(s),\ \ \sin(s) + (\xi_0 - s) \cos(s) - \frac{\eta}{\sqrt{2}} \cos(s)).\end{aligned}$$

*5.2. The* $(1,3)$*-evolute curve of the curve* $\gamma^4(s)$

The $(1,3)$-evolute curve $\beta^4(\bar{s})$ is given by

$$\beta^4(\bar{s}) = \gamma^4(s) + \frac{1}{K(s)} N_1(s) + \frac{-r}{K(K-k)} N_3(s).$$

Substituting the curvature functions

$$K(s) = \sqrt{2}, \quad -r(s) = \sqrt{2}, \quad \text{and} \quad K(s) - k(s) = \sqrt{2},$$

we get

$$\beta^4(\bar{s}) = \gamma^4(s) + \frac{1}{\sqrt{2}} N_1(s) + \frac{\sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} N_3(s).$$

Substituting the expressions for $\gamma^4(s)$, $N_1(s)$, and $N_3(s)$, we get

$$\beta^4(\bar{s}) = (\cos(s), \sin(s), \cos(s), \sin(s)).$$

## 6. Conclusions

This study has established a comprehensive framework for analyzing the properties of (0, 2)-involute and (1, 3)-evolute curves associated with quaternionic Bertrand curves in four-dimensional Euclidean space. By employing a type-2 quaternionic frame, we derived explicit mathematical expressions for these curves and elucidated their relationships with curvature functions and Frenet frames. These results deepen our understanding of the geometric structure of quaternionic curves and their involute-evolute pairs, contributing to the broader field of differential geometry.

The findings of this study have significant potential applications in both theoretical and applied mathematics. For instance, in robotics and motion planning, the geometric properties of quaternionic curves and their involute-evolute pairs can be utilized to design motion trajectories for robotic systems operating in higher-dimensional spaces. In computer graphics and animation, the explicit formulas derived in this work can be applied to model complex curves and surfaces, particularly in the rendering of four-dimensional objects.

Looking ahead, several promising directions for future research emerge. First, the results of this study could be generalized to $n$-dimensional Euclidean space ($\mathbb{E}^n$) for $n > 4$, providing a more comprehensive understanding of the geometric properties of curves in higher-dimensional spaces. Second, investigating the properties of quaternionic curves and their involute-evolute pairs in Minkowski space could yield insights into the behavior of curves in pseudo-Riemannian manifolds, with potential applications in relativity and cosmology. Finally, exploring the connection between quaternionic curves and physical systems, such as rigid body dynamics or quantum mechanics, could lead to new insights into the geometric structure of these systems.

## Author contributions

Ayman Elsharkawy: created conceptualizations, supervised the research, reviewed and edited the manuscript, guided the theoretical framework, and prepared the manuscript draft; Ahmer Ali:

collected data, supervised the study, provided critical insights, and contributed to refining the manuscript; Muhammad Hanif: created methodology, conducted the theoretical analysis, and developed the main results. Fatimah Alghamdi: reviewed and edited the manuscript, provided critical insights to refine interpretations, ensured adherence to publication standards, and contributed to improving the overall clarity and coherence of the work. All authors have read and agreed to the published version of the manuscript.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

The authors declare that they have no conflicts of interest.

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