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Research article

Existence, uniqueness, and localization of positive solutions to nonlocal problems of the Kirchhoff type via the global minimum principle of Ricceri

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Abstract: The purpose of this paper is to demonstrate the existence and uniqueness of positive solutions to fractional p-Laplacian problems with discontinuous Kirchhoff-type functions. The crucial tools for getting these results are the uniqueness result of the Brézis–Oswald–type problem and the abstract global minimum principle. The primary features of this paper are the discontinuity of the Kirchhoff coefficient in $[0, \infty)$ and the localization of solutions.

Keywords: fractional p-Laplacian; Kirchhoff-type function; weak solution; uniqueness; global minima
Mathematics Subject Classification: 35B33, 35D30, 35J20, 35J60, 35J66

1. Introduction

This paper is dedicated to a Kirchhoff-type equation driven by a nonlocal fractional p-Laplacian as follows:

$$\begin{cases} M\left([\psi]_{s,p}^p\right)\mathcal{L}_p^s\psi(z) = g(z,\psi) & \text{in} \quad \Omega, \\ \psi > 0 & \text{in} \quad \Omega, \\ \psi = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega, \\ [\psi]_{s,p}^p \in J, \end{cases} \tag{P}$$

where $s \in (0,1)$, $p \in (1,+\infty)$, $sp < N$, $J \subseteq (0,+\infty)$ is an open interval, $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ is an open bounded set with Lipschitz boundary $\partial\Omega$, $[\psi]_{s,p}^p := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z,y)\,|\psi(z) - \psi(y)|^p dz\,dy$, M is an increasing Kirchhoff-type function on J, and a function g is nonnegative, which will be introduced later. Here, $\mathcal{L}_p^s$ is a nonlocal operator defined pointwise as follows:

$$\mathcal{L}_p^s\psi(z) = 2\int_{\mathbb{R}^N} K(z,y)\,|\psi(z)-\psi(y)|^{p-2}(\psi(z)-\psi(y))dy \quad \text{for all } z\in\mathbb{R}^N,$$

where a function $K:\mathbb{R}^N\times\mathbb{R}^N\to(0,+\infty)$ fulfills the following assumptions:

($\mathcal{K}$1) $\kappa K\in L^1(\mathbb{R}^N\times\mathbb{R}^N)$, where $\kappa(z,y)=\min\{|z-y|^p,1\}$;

($\mathcal{K}$2) There exist positive constants γ_0 and γ_1 with $\gamma_0\geq 1$ such that $\gamma_0\leq K(z,y)|z-y|^{N+sp}\leq\gamma_1$ for $z\neq y$ and for almost all $(z,y)\in\mathbb{R}^N\times\mathbb{R}^N$;

($\mathcal{K}$3) $K(y,z)=K(z,y)$ for all $(y,z)\in\mathbb{R}^N\times\mathbb{R}^N$.

When $K(z,y)=|z-y|^{-(N+sp)}$, $\mathcal{L}_p^s$ becomes the fractional p-Laplacian operator $(-\Delta)_p^s$ defined as follows:

$$(-\Delta)_p^s\,\psi(z) = 2\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(z)}\frac{|\psi(z)-\psi(y)|^{p-2}\,(\psi(z)-\psi(y))}{|z-y|^{N+sp}}\,dy, \quad z\in\mathbb{R}^N,$$

where $B_\varepsilon(z):=\{z\in\mathbb{R}^N:|z-y|\leq\varepsilon\}$.

Over the last few decades, fractional Sobolev spaces and their corresponding nonlocal equations have gained increasing attention because they can be corroborated as models for many physical phenomena arising from studies of Lévy processes, fractional quantum mechanics, optimization, image processing, thin obstacle problems, anomalous diffusion in plasma, American options, game theory, geophysical fluid dynamics, and frame propagation; see [6, 14, 24, 28, 34] for comprehensive studies and details on these topics.

The study of Kirchhoff-type problems, which was originally proposed by Kirchhoff [18], has a powerful background in various applications in physics and biology. For this reason, much attention has recently been given to the investigation of elliptic equations related to Kirchhoff coefficients; for example, see [15, 16, 25, 26, 29, 32] and the references therein. The authors of [11] discussed in detail the physical implications underlying the fractional Kirchhoff model. Particularly, by considering a truncation argument and the mountain pass theorem, the existence of nontrivial solutions to a nonlocal elliptic problem was obtained when an increasing and continuous Kirchhoff term M has the nondegenerate condition $\inf_{\xi\in[0,+\infty)} M(\xi)\geq\xi_0>0$, where ξ_0 is a constant; see also [30] and references therein. However, the existence of at least two different nontrivial solutions to the fractional p-Laplacian equations of the Schrödinger–Kirchhoff type was demonstrated in [32] when the nondegenerate continuous Kirchhoff function M fulfills the hypothesis:

(M1) There is $\delta\in[1,\frac{N}{N-sp})$ such that $\delta\mathcal{M}(\xi):=\delta\int_0^\xi M(\sigma)d\sigma\geq M(\xi)\xi$ for any $\xi\geq 0$, where $0<s<1$.

The assumption (M1) contains not only the classical example $M(\xi)=1+a\xi^\delta$ $(a\geq 0,\xi\geq 0)$ but the nonmonotonic cases. In this regard, nonlinear elliptic equations of Kirchhoff type involving (M1) have received widely remarkable attention; see [7, 15, 16, 19, 20, 35]. Considering these related papers, the functional $\mathcal{A}:W_K^{s,p}(\Omega)\to\mathbb{R}$ associated with the principal part in (P) is given by

$$\mathcal{A}(\psi)=\frac{1}{p}\mathcal{M}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}K(z,y)\,|\psi(z)-\psi(y)|^p\,dz\,dy\right)$$

for any $\psi \in W_K^{s,p}(\Omega)$, where a solution space $W_K^{s,p}(\Omega)$ will be introduced later. Then, in accordance with the fact that $M \in C([0, +\infty))$, it follows that $\mathcal{A} \in C^1(W_K^{s,p}(\Omega), \mathbb{R})$ and its Fréchet derivative is defined as

$$\langle \mathcal{A}'(\psi), \phi \rangle = M\left([\psi]_{s,p}^p\right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(z, y)\, |\psi(z) - \psi(y)|^{p-2} (\psi(z) - \psi(y))(\phi(z) - \phi(y)) dz\, dy$$

for any $\psi, \phi \in W_K^{s,p}(\Omega)$. Specifically, assumptions $M \in C([0, \infty))$ and (M1) play an effective role in deriving some topological properties of functionals $\mathcal{A}, \mathcal{A}'$ and the compactness condition of Palais–Smale-type for an energy functional related to (P), which are essential in using variational methods such as Ekeland variational principle, mountain pass theorem, and fountain theorem. But, many examples are eliminated from the continuity of the nondegenerate Kirchhoff function M in $[0, \infty)$. For example, let the Kirchhoff functions be defined by

$$M(\xi) = \tan \xi \quad \text{for} \quad 0 < \xi < \frac{\pi}{2}$$

and

$$M(\xi) = (\delta - \xi)^{-\ell} \quad \text{for} \quad \xi \in (-\infty, \delta), \quad \text{where} \quad \delta > 0,\ 0 < \ell < 1.$$

These functions cannot be covered by any of the results known to date. Recently, to obtain at most one positive solution for the non-local problems with discontinuous Kirchhoff functions, Ricceri [33] discussed a new approach different from those of previous related studies [2, 10, 11, 16, 29, 32]. The author of [21] recently extended the result of [33] to elliptic equations involving p-Laplacian; see also the paper [22] for problems involving double-phase operators. The primary tools for getting these results in [21, 22] are the uniqueness results of the Brézis–Oswald-type problem based on [5] and the abstract global minimum principle in [33]. Especially, the Dìaz–Saa-type inequalities in [8, 9] play an essential role in attaining the uniqueness of a positive solution to equations examined in [21, 22]. In addition, inspired by previous studies [4, 27], the author of [23] determined the existence and uniqueness of a positive solution to nonlinear the Brézis–Oswald type equations involving the fractional Laplacian. For its application, the existence of at most one positive solution to Kirchhoff-type equations driven by the nonlocal fractional Laplacian has been investigated.

The primary aim of this paper is to derive the existence and uniqueness of positive solutions to the fractional p-Laplacian equations involving discontinuous Kirchhoff-type coefficients. In the application of the inequalities of Dìaz–Saa-type in [8, 9], the well-known Hopf boundary lemma is required to show that the quotient between solutions is contained in the L^∞-space. Though, solutions of fractional-order equations are generally singular at the boundary, making it difficult to work with their quotient between solutions, as Hopf's boundary lemma is not maintained. Hence, in distinction from previous studies [21, 22], the major difficulty of this paper is to derive that Brézis–Oswald-type problems involving the fractional p-Laplacian admit at most one positive weak solution. Based on previous studies [4, 17, 27], we overcome this difficulty by taking into account the discrete Picone inequality in [3, 12]. As far as we are aware, the Brézis–Oswald-type result to nonlinear elliptic problems with the Kirchhoff coefficient has not been studied much; we only know of one study [2, 23] in this direction. Recently, Biagi and Vecchi [2] obtained uniqueness results for Brézis–Oswald-type Laplacian problems with degenerate Kirchhoff functions M in $[0, \infty)$ when M is a continuous, nonnegative and nondecreasing function satisfying $M(\xi) > 0$ for every $\xi > 0$. But, our main result differs from that of [2] because we consider a discontinuous Kirchhoff function M in $[0, \infty)$ and solution localization. Although our result is based on previous work [23], problem (P) has more complex nonlinearities than [23] and thus requires a more fastidious analysis to be performed carefully.

The remainder of this paper is organized as follows: In Section 2, we present some essential preliminary knowledge of our considered function spaces to be utilized in this paper. In Section 3, we provide the variational framework associated with problem (P), and then, we will derive the existence and uniqueness results of positive solutions under suitable assumptions.

2. Preliminaries

For the convenience of the reader, in this section we shortly present some practical definitions and fundamental properties of the fractional Sobolev spaces that will be used in the presnt paper. Let $s \in (0, 1)$ and $p \in (1, \infty)$ be real numbers, and let p_s^* be the fractional critical Sobolev exponent, such that is

$$p_s^* := \begin{cases} \frac{Np}{N-sp} & \text{if } sp < N, \\ +\infty & \text{if } sp \geq N. \end{cases}$$

Let $\Omega \subset \mathbb{R}^N$ be an bounded open set with a smooth boundary. Let the fractional Sobolev space $W^{s,p}(\Omega)$ be defined as follows:

$$W^{s,p}(\Omega) := \left\{ \psi \in L^p(\Omega) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(z) - \psi(y)|^p}{|z - y|^{N+ps}} \, dz \, dy < +\infty \right\},$$

endowed with the norm

$$|\psi|_{W^{s,p}(\Omega)} := \left(|\psi|^p_{L^p(\Omega)} + |\psi|^p_{W^{s,p}(\mathbb{R}^N)} \right)^{\frac{1}{p}},$$

where

$$|\psi|^p_{L^p(\Omega)} := \int_\Omega |\psi(z)|^p \, dz \qquad \text{and} \qquad |\psi|^p_{W^{s,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(z) - \psi(y)|^p}{|z - y|^{N+ps}} \, dz \, dy.$$

Then, $W^{s,p}(\Omega)$ is a reflexive and separable Banach space. In addition, the space $C_0^\infty(\Omega)$ is dense in $W^{s,p}(\Omega)$ such that $W_0^{s,p}(\Omega) = W^{s,p}(\Omega)$ (see, e.g., [1, 28]).

Lemma 2.1. ([28]) *Let* $0 < s < 1$ *and* $1 < p < +\infty$. *Then, we have the continuous embeddings as follows:*

$$\begin{aligned} &W^{s,p}(\Omega) \hookrightarrow L^r(\Omega) && \text{for any } r \in [1, p_s^*], && \text{if } sp < N; \\ &W^{s,p}(\Omega) \hookrightarrow L^r(\Omega) && \text{for every } r \in [1, \infty), && \text{if } sp = N; \\ &W^{s,p}(\Omega) \hookrightarrow C_b^{0,\nu}(\Omega) && \text{for all } \nu < s - N/p, && \text{if } sp > N. \end{aligned}$$

Particularly, the embedding $W^{s,p}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ *is compact for any* $r \in [1, p_s^*)$.

Let us define the fractional Sobolev space $W_K^{s,p}(\mathbb{R}^N)$ as follows:

$$W_K^{s,p}(\mathbb{R}^N) := \left\{ \psi \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(z, y) |\psi(z) - \psi(y)|^p \, dz \, dy < +\infty \right\},$$

where $K : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0,+\infty)$ is a kernel function with the properties $(\mathcal{K}1)$–$(\mathcal{K}3)$. By the condition $(\mathcal{K}1)$, the function

$$(z,y) \mapsto K^{\frac{1}{p}}(z,y)(\psi(z)-\psi(y)) \in L^p(\mathbb{R}^N)$$

for any $\psi \in C_0^\infty(\mathbb{R}^N)$. We consider the problem (P) in the closed linear subspace defined by

$$X := \left\{\psi \in W_K^{s,p}(\mathbb{R}^N) : \psi(z) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\}$$

with respect to the norm

$$\|\psi\|_X := \left(\|\psi\|^p_{L^p(\Omega)} + [\psi]^p_{s,p}\right)^{\frac{1}{p}},$$

where

$$[\psi]^p_{s,p} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z,y)|\psi(z)-\psi(y)|^p\,dz\,dy.$$

In what follows, let $0 < s < 1$ and $1 < p < +\infty$ with $ps < N$ and let the kernel function $K : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0,\infty)$ ensure the assumptions $(\mathcal{K}1)$–$(\mathcal{K}3)$.

Lemma 2.2. ([35]) *If $\psi \in X$, then $\psi \in W^{s,p}(\Omega)$. Moreover,*

$$\|\psi\|_{W^{s,p}(\Omega)} \le \max\{1, \gamma_0^{-\frac{1}{p}}\}\|\psi\|_X,$$

where γ_0 is given in $(\mathcal{K}2)$.

From Lemmas 2.1 and 2.2, we can obtain the following consequence instantly.

Lemma 2.3. ([35]) *For $1 \le r \le p_s^*$ and for any $\psi \in X$, there exists a constant $C_0 = C_0(s,N,p) > 0$ such that*

$$\begin{aligned}\|\psi\|^p_{L^r(\Omega)} &\le C_0 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\psi(z)-\psi(y)|^p}{|z-y|^{N+ps}}dz\,dy \\ &\le \frac{C_0}{\gamma_0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z,y)|\psi(z)-\psi(y)|^p\,dz\,dy,\end{aligned}$$

where γ_0 is given in $(\mathcal{K}2)$. Consequently, the embedding $X \hookrightarrow L^r(\Omega)$ is continuous for any $r \in [1, p_s^*]$. In addition, the embedding

$$X \hookrightarrow\hookrightarrow L^r(\Omega)$$

is compact for $r \in (1, p_s^*)$.

3. Variational setting and main result

In this section, we introduce the variational setting corresponding to the problem (P). In addition, we present some useful auxiliary consequences and Ricceri's variational principle before delving into our main result.

Definition 3.1. *We say that $\psi \in X$ is called a weak solution of* (P) *if*

$$M\left([\psi]_{s,p}^{p}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}K(z,y)|\psi(z)-\psi(y)|^{p-2}(\psi(z)-\psi(y))(\varphi(z)-\varphi(y))\,dz\,dy$$
$$=\int_{\Omega}g(z,\psi)\varphi(y)\,dy$$

for any $\varphi \in X$.

Let us define the functional $\mathcal{A} : X \to \mathbb{R}$ as

$$\mathcal{A}(\psi) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}K(z,y)|\psi(z)-\psi(y)|^{p}\,dz. \tag{3.1}$$

Then, it is immediate to obtain that the functional $\mathcal{A} : X \to \mathbb{R}$ belongs to a class of $C^1(X,\mathbb{R})$, and its Fréchet derivative is

$$\langle \mathcal{A}'(\psi),\varphi\rangle = p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}K(z,y)\,|\psi(z)-\psi(y)|^{p-2}\,(\psi(z)-\psi(y))(\varphi(z)-\varphi(y))\,dz\,dy$$

for any $\psi,\varphi \in X$; see [32].

Lemma 3.2. *The functional $\mathcal{A}$ is convex and weakly lower semicontinuous on X.*

Proof. It is trivial that $\mathcal{A}$ is convex. Let $\{w_n\}$ be a sequence in X satisfying $w_n \rightharpoonup w$ in X as $n \to \infty$. Because $\mathcal{A}$ is convex and C^1-functional on X, we obtain

$$\mathcal{A}(w_n) \geq \langle \mathcal{A}'(w_n), w_n - w\rangle + \mathcal{A}(w).$$

Then, it is immediate that

$$\liminf_{n\to\infty}\mathcal{A}(w_n) \geq \mathcal{A}(w) + \liminf_{n\to\infty}\langle \mathcal{A}'(w_n), w_n - w\rangle$$
$$\geq \mathcal{A}(w).$$

Therefore, the conclusion holds. □

Meanwhile, $g : \Omega\times\mathbb{R} \to \mathbb{R}$ is assumed to verify the following conditions:

(G1) g satisfies a Carathéodory condition;

(G2) $0 \leq g(\cdot,\xi) \in L^\infty(\Omega)$ for every $\xi \geq 0$, and there is a constant $\rho_1 > 0$ such that

$$g(z,\xi) \leq \rho_1\left(1+|\xi|^{p-1}\right)$$

for all $\xi \geq 0$ and for almost everywhere $z \in \Omega$;

(G3) The function $\xi \mapsto \frac{g(z,\xi)}{\xi^{p-1}}$ is strictly decreasing in $(0,+\infty)$ for almost all $z \in \Omega$;

(G4) $\lim_{\xi\to+\infty}\frac{g(z,\xi)}{\xi^{p-1}} = 0$ and $\lim_{\xi\to 0^+}\frac{g(z,\xi)}{\xi^{p-1}} = +\infty$, uniformly in $z \in \Omega$.

Under hypothesis (G1), let us define the functional $\mathcal{B}_0 : X \to \mathbb{R}$ by

$$\mathcal{B}_0(\psi) := \int_\Omega G(z, \psi(z))\, dz$$

for any $\psi \in X$, where $G(z, \xi) = \int_0^\xi g(z, t)dt$. Thus, it is immediate to prove that $\mathcal{B}_0 \in C^1(X, \mathbb{R})$, and its Fréchet derivative is

$$\langle \mathcal{B}_0'(\psi), w \rangle = \int_\Omega g(z, \psi) w\, dz$$

for any $\psi, w \in X$. Next, we define the functional $\mathcal{J} : X \to \mathbb{R}$ by

$$\mathcal{J}(\psi) = \frac{1}{p}\mathcal{A}(\psi) - \lambda \mathcal{B}_0(\psi).$$

Then, the functional $\mathcal{J}$ belongs to $C^1(X, \mathbb{R})$, and its Fréchet derivative is

$$\langle \mathcal{J}'(\psi), \varphi \rangle = \frac{1}{p} \langle \mathcal{A}'(\psi), \varphi \rangle - \lambda \langle \mathcal{B}_0'(\psi), \varphi \rangle \quad \text{for any } \psi, \varphi \in X.$$

The following is a discrete version of the renowned Picone inequality; see [3, Proposition 4.2] and [12, Lemma 2.6] for a proof.

Lemma 3.3. *(Discrete Picone inequality). Let $p \in (1, +\infty)$ and let $a, b, c, d \in [0, +\infty)$, with $a, b > 0$. Then,*

$$\phi_p(a - b) \left[\frac{c^p}{a^{p-1}} - \frac{d^p}{b^{p-1}} \right] \le |c - d|^p, \tag{3.2}$$

where $\phi_p(\xi) = |\xi|^{p-2}\xi$ for $\xi \in \mathbb{R}$. Moreover, if the equality holds in (3.2)*, then*

$$\frac{a}{b} = \frac{c}{d}.$$

We prove a practical lemma that will be very usable hereinafter. For any $\varepsilon > 0$ and $\psi_j \in X$, define the truncation

$$\psi_{j,\varepsilon} := \min\{\psi_j, \varepsilon^{-1}\}. \tag{3.3}$$

Lemma 3.4. *Let $\psi_1, \psi_2 \in X$ with $\psi_1, \psi_2 \ge 0$ and set*

$$w := \frac{\psi_{2,\varepsilon}^p}{(\varepsilon + \psi_1)^{p-1}} - \psi_{1,\varepsilon},$$

where $\psi_{1,\varepsilon}, \psi_{2,\varepsilon}$ are as in (3.3)*. Then, we derive $w \in X$.*

Proof. Let $\varepsilon > 0$ be fixed. Because $\xi \mapsto \min\{|\xi|, \varepsilon^{-1}\}$ is 1-Lipschitz function, we assert

$$\left|\psi_{j,\varepsilon}(y) - \psi_{j,\varepsilon}(z)\right| \le \left|\psi_j(y) - \psi_j(z)\right| \quad \text{for} \quad j = 1, 2, \tag{3.4}$$

which implies that $\psi_{j,\varepsilon} \in X$. On account of the Lagrange theorem, we deduce that

$$|a^r - b^r| \le r\,|a - b| \max\{a^{r-1}, b^{r-1}\} \tag{3.5}$$

for every $r \geq 0$ and for any $a, b \geq 0$. Because $\varepsilon^{p-1} \leq (\varepsilon + \psi_{1,\varepsilon})^{p-1}$ and $\psi_{2,\varepsilon} \leq \frac{1}{\varepsilon}$, by considering (3.4) and (3.5), we have

$$\begin{aligned}
&\left|\frac{\psi_{2,\varepsilon}^{p}(z)}{(\varepsilon + \psi_1(z))^{p-1}} - \frac{\psi_{2,\varepsilon}^{p}(y)}{(\varepsilon + \psi_1(y))^{p-1}}\right| \\
&= \left|\frac{\psi_{2,\varepsilon}^{p}(z) - \psi_{2,\varepsilon}^{p}(y)}{(\varepsilon + \psi_1(z)n)^{p-1}} + \psi_{2,\varepsilon}^{p}(y)\frac{(\varepsilon + \psi_1(y))^{p-1} - (\varepsilon + \psi_1(z))^{p-1}}{(\varepsilon + \psi_1(z))^{p-1}(\varepsilon + \psi_1(y))^{p-1}}\right| \\
&\leq \frac{p}{\varepsilon^{2p-2}}\left|\psi_{2,\varepsilon}(z) - \psi_{2,\varepsilon}(y)\right| + \frac{1}{\varepsilon^{p}}\left|\frac{(\varepsilon + \psi_1(y))^{p-1} - (\varepsilon + \psi_1(z))^{p-1}}{(\varepsilon + \psi_1(z))^{p-1}(\varepsilon + \psi_1(y))^{p-1}}\right| \\
&\leq \frac{p}{\varepsilon^{2p-2}}\left|\psi_{2,\varepsilon}(z) - \psi_{2,\varepsilon}(y)\right| \\
&\quad + \frac{p-1}{\varepsilon^{p}}\max\left\{(\varepsilon + \psi_1(z))^{p-2}, (\varepsilon + \psi_1(y)n)^{p-2}\right\}\frac{|\psi_1(z) - \psi_1(y)|}{(\varepsilon + \psi_1(z))^{p-1}(\varepsilon + \psi_1(y))^{p-1}} \\
&\leq \frac{p}{\varepsilon^{2p-2}}|\psi_2(z) - \psi_2(y)| + \frac{p-1}{\varepsilon^{2p}}|\psi_1(z) - \psi_1(y)|
\end{aligned}$$

for every $p > 1$. Hence, the Gagliardo seminorm of w is finite. In addition, one has

$$\frac{\psi_{2,\varepsilon}^{p}}{(\varepsilon + \psi_1)^{p-1}} = \frac{\psi_{2,\varepsilon}^{p-1}}{(\varepsilon + \psi_1)^{p-1}}\psi_{2,\varepsilon} \leq \frac{1}{\varepsilon^{2p-2}}\psi_2;$$

thus,

$$\int_{\Omega}|w|^{p}\, dz \leq 2^{p-1}\left(\int_{\Omega}\left|\frac{\psi_{2,\varepsilon}^{p}}{(\varepsilon + \psi_1)^{p-1}}\right|^{p} dz + \int_{\Omega}\left|\psi_{1,\varepsilon}\right|^{p}\, dz\right) \leq C(\varepsilon, p)(|\psi_2|_{L^p(\Omega)} + |\psi_1|_{L^p(\Omega)}) < +\infty,$$

where $C(\varepsilon, p) > 0$. As a result, we arrive that $w \in X$. □

Definition 3.5. *Let X be a topological space. A function $h : X \to \mathbb{R}$ is inf-compact if the set $h^{-1}((-\infty, \xi])$ is compact for each $\xi \in \mathbb{R}$.*

Now, we present the uniqueness result of a nontrivial positive solution for the nonlocal fractional p-Laplacian problem of a Kirchhoff-type. To this end, we employ the abstract global minimum principle introduced by B. Ricceri [33], which plays a crucial role in obtaining our main result.

Theorem 3.6. *Let X be a topological space, and let $\mathcal{A} : X \to \mathbb{R}$, with $\mathcal{A}^{-1}(0) \neq \emptyset$ and $\mathcal{B} : X \to \mathbb{R}$ being two functions such that, for each $\gamma > 0$, the function $\gamma\mathcal{A} - \mathcal{B}$ is lower semicontinuous, inf-compact, and has a unique global minimum. Moreover, assume that $\mathcal{B}$ has no global maxima in X. Further, let $J \subseteq (0, +\infty)$ be an open interval and $M : J \to \mathbb{R}$ be an increasing function with $M(J) = (0, +\infty)$. There exists a unique $\tilde{u} \in X$ such that $\mathcal{A}(\tilde{u}) \in J$ and*

$$M(\mathcal{A}(\tilde{u}))\mathcal{A}(\tilde{u}) - \mathcal{B}(\tilde{u}) = \inf_{u \in X}(M(\mathcal{A}(\tilde{u}))\mathcal{A}(u) - \mathcal{B}(u)).$$

If each assumption of Theorem 3.6 is satisfied, we derive our main result. The fundamental idea of the proof of the uniqueness of positive solutions to problem (P) follows from the paper [4, 27]; see also [23].

Theorem 3.7. *Assume that an open interval $J \subseteq (0,+\infty)$ exists such that $M(J) = (0,+\infty)$ and the restriction of M to J is increasing. Let $g : \Omega \times [0,+\infty) \to (0,+\infty)$ be a function satisfying conditions* (G1)–(G4) *and $g(z,0) = 0$ for almost every $z \in \Omega$. Then, problem (P) has a unique positive weak solution $\widetilde{w}$, which is the unique global minimum in X of the functional*

$$\psi \mapsto \frac{1}{p} M\Big([\widetilde{w}]^p_{s,p}\Big) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(z,y) \left|\psi(z) - \psi(y)\right|^p \, dz\, dy - \int_\Omega \Big(\int_0^{\psi^+(z)} g(z,t)\, dt \Big) dz,$$

where $\psi^+ := \max\{\psi, 0\}$

Proof. First, extend g to $\mathbb{R}$, putting $g(z,\xi) = 0$ for all $\xi < 0$. To utilize Theorem 3.6, consider $\mathcal{A}$ given in (3.1) and define $\mathcal{B}$ by

$$\mathcal{B}(\psi) := p \int_\Omega G(z, \psi^+(z))\, dz$$

for any $\psi \in X$. The functional $\mathcal{B}$ belongs to a class of $C^1(X,\mathbb{R})$ with derivatives given by

$$\langle \mathcal{B}'(\psi), w \rangle = p \int_\Omega g(z,\psi) w(z)\, dz$$

for any $\psi, w \in X$. Moreover, owing to the fact that g has subcritical growth, the functional $\mathcal{B}$ is sequentially weakly continuous on X. Fix $\eta > 0$. Then, Lemma 3.2 implies the sequentially weakly lower semicontinuity of functional $\eta\mathcal{A} - \mathcal{B}$ on X. Choose

$$\epsilon \in \left(0, \frac{\eta(C_0 + \gamma_0)}{2C_0}\right),$$

where γ_0 and C_0 are given in Lemma 2.3. Because $\lim_{\xi \to +\infty} \frac{G(z,\xi)}{\xi^p} = 0$, there exists a positive real number $C_\varepsilon > 0$ satisfying

$$G(z,\xi) \leq \frac{\varepsilon}{p} |\xi|^p + \frac{C_\varepsilon}{p} \tag{3.6}$$

for almost everywhere $z \in \Omega$ and for any $\xi \in \mathbb{R}$. Hence, we obtain

$$\mathcal{B}(\psi) \leq \varepsilon \int_\Omega |\psi(z)|^p \, dz + C_\varepsilon \text{meas}(\Omega),$$

where meas(Ω) means the Lebesgue measure of Ω on $\mathbb{R}^N$. Using this, Lemma 2.3, (3.6) and the definition of the X-norm, we derive that

$$\begin{aligned}
\eta\mathcal{A}(\psi) - \mathcal{B}(\psi) &\geq \eta \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(z,y) \left|\psi(z) - \psi(y)\right|^p \, dz\, dy \\
&\quad - \varepsilon \int_\Omega |\psi(z)|^p \, dz - C_\varepsilon \text{meas}(\Omega) \\
&\geq \eta \left(\frac{1}{2} + \frac{\gamma_0}{2C_0} \right) \|\psi\|_X^p - \varepsilon \int_\Omega |\psi(z)|^p \, dz - C_\varepsilon \text{meas}(\Omega) \\
&\geq \left(\frac{\eta(C_0 + \gamma_0)}{2C_0} - \epsilon \right) \|\psi\|_X^p - C_\varepsilon \text{meas}(\Omega)
\end{aligned}$$

for any $\psi \in X$. Thus, owing to the choice of ε, we infer

$$\lim_{|u|\to+\infty}(\eta\mathcal{A}(u) - \mathcal{B}(u)) = +\infty.$$

This, together with the reflexivity of X and the Eberlein–Smulyan theorem, yields that the sequentially weakly lower semicontinuous functional $\eta\mathcal{A} - \mathcal{B}$ is weakly inf-compact. Now, we claim that it has a unique global minimum in X. As we know, its critical points are exactly the weak solutions to the problem

$$\begin{cases} \mathcal{L}_p^s\psi(z) = \frac{1}{\eta}g(z,\psi) & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

where $\psi \in X$ is said to be a weak solution of problem (3.7) if

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z,y)\,|\psi(z)-\psi(y)|^{p-2}\,(\psi(z)-\psi(y))(\phi(z)-\phi(y))\,dz\,dy = \frac{1}{\eta}\int_\Omega g(z,\psi)\phi\,dz \tag{3.8}$$

for any $\phi \in X$.

Let us define the energy functional $\mathcal{J} : X \to \mathbb{R}$ as

$$\mathcal{J}(\psi) := \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z-y)|\psi(z)-\psi(y)|^p\,dz\,dy - \frac{1}{\eta}\int_{\mathbb{R}^N} G(z,\psi)\,dz, \quad \psi \in X,$$

and let the modified energy functional $\widetilde{\mathcal{J}} : X \to \mathbb{R}$ be defined by

$$\widetilde{\mathcal{J}}(\psi) := \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z-y)|\psi(z)-\psi(y)|^p\,dz\,dy - \frac{1}{\eta}\int_{\mathbb{R}^N} G^+(z,\psi)\,dz, \quad \psi \in X,$$

where

$$G^+(z,\tau) := \int_0^\tau g^+(z,\xi)\,d\xi \;\text{ and }\; g^+(z,\tau) := \begin{cases} g(z,\tau), & \tau \ge 0, \\ 0, & \tau < 0 \end{cases}$$

for any $\tau \in \mathbb{R}$ and for almost everywhere $z \in \mathbb{R}^N$. In compliance with Lemma 3.2 and the argument above, the functional $\widetilde{\mathcal{J}}$ is also coercive and sequentially weakly lower semicontinuous on X. From this, there is an element $\psi_0 \in X$ satisfying

$$\widetilde{\mathcal{J}}(\psi_0) = \inf\{\widetilde{\mathcal{J}}(\psi) : \psi \in X\}.$$

Now, we show that it is possible to assume that $\psi_0 \ge 0$. To this end, we assume that ψ_0 is sign-changing. Taking Lemma 3.4 into account, we know $\psi_0^+ \in X$ and thus $\widetilde{\mathcal{J}}(\psi_0) \le \widetilde{\mathcal{J}}(\psi_0^+)$. Because $\widetilde{\mathcal{J}}(\psi) = \mathcal{J}(\psi)$ when $\psi(z) \ge 0$ for almost everywhere $z \in \Omega$, we assert

$$\begin{aligned} \widetilde{\mathcal{J}}(\psi_0^+) &= \mathcal{J}(\psi_0^+) \\ &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z,y)\left|\psi_0^+(z)-\psi_0^+(y)\right|^p\,dz\,dy - \frac{1}{\eta}\int_\Omega G(z,\psi_0^+)\,dz \\ &\le \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z,y)\,|\psi_0(z)-\psi_0(y)|^p\,dz\,dy - \frac{1}{\eta}\int_\Omega G(z,\psi_0^+)\,dz \end{aligned}$$

$$= \widetilde{\mathcal{J}}(\psi_0).$$

Therefore, ψ_0^+ is a nonnegative solution to problem (3.7). For simplicity, let us write directly ψ_0 instead of ψ_0^+. Let us claim $\psi_0 > 0$. As $\psi_0(z) \geq 0$ for almost everywhere $z \in \mathbb{R}^N$, we know that either $\psi_0(z) > 0$ or $\psi_0(z) = 0$ for almost everywhere $z \in \mathbb{R}^N$. Indeed, let us assume that $\psi_0 \not\equiv 0$ in Ω. Then it is enough to prove that $\psi_0 \not\equiv 0$ in all connected components of Ω. Assume to the contrary that there exists a connected component Λ of Ω such that $\psi_0(z) = 0$ for almost everywhere $z \in \Lambda$. Let us take any nonnegative function $\omega \in C_0^\infty(\Lambda)$ as a test function in (3.8). Then, since g is a nonnegative function and ψ_0 is a nonnegative solution of (3.7), we have

$$\begin{aligned}
0 &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z,y)\,|\psi_0(z)-\psi_0(y)|^{p-2}\,(\psi_0(z)-\psi_0(y))(\omega(z)-\omega(y))\,dz\,dy \\
&\quad - \frac{1}{\eta}\int_\Omega g(z,\psi_0)\omega(z)\,dz \\
&\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z,y)\,|\psi_0(z)-\psi_0(y)|^{p-2}\,(\psi_0(z)-\psi_0(y))(\omega(z)-\omega(y))\,dz\,dy \\
&= 2\int_\Lambda\int_{\Lambda^c} K(z,y)\,|\psi_0(z)-\psi_0(y)|^{p-2}\,(\psi_0(z)-\psi_0(y))(\omega(z)-\omega(y))\,dz\,dy \\
&= -2\int_\Lambda\int_{\Lambda^c} K(z,y)(\psi_0(z))^{p-1}\omega(y)\,dz\,dy.
\end{aligned}$$

From this, we infer that $\psi_0(z) = 0$ for almost everywhere $z \in \Lambda^c$, that is $\psi_0(z) = 0$ for almost everywhere $z \in \mathbb{R}^N$. This yields a contradiction to the fact that $\psi_0(z) \neq 0$ for almost everywhere $z \in \Omega$.

Therefore, to show $\psi_0 > 0$, it suffices to prove that $\widetilde{\mathcal{J}}(\psi_0) < 0$. Now, with consideration for Lemma 2.1 in [13], let us fix any nonnegative function $\varrho \in X$, with $\varrho = 0$ on $\partial\Omega$, such that

$$\eta_1 \int_\Omega |\varrho(z)|^p\,dz = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z-y)\ |\varrho(z)-\varrho(y)|^p\ dz\,dy,$$

where η_1 is a positive eigenvalue that can be characterized as

$$\eta_1 = \min_{\{\varrho\in X\,:\,|\varrho|_{L^p(\Omega)}=1\}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z-y)\ |\varrho(z)-\varrho(y)|^p\ dz\,dy.$$

In light of Theorem 3.2 in [13], we assert that $\varrho \in L^\infty(\mathbb{R}^N)$. Let $\alpha_0 \in L^\infty(\Omega)$ with $\alpha_0 > 0$ and let $\kappa_0 \in (0, |\alpha_0|_{L^\infty(\Omega)})$ be fixed. Then, the set

$$\Omega_{\kappa_0} := \{z \in \Omega : \alpha_0(z) \geq \kappa_0\}$$

has a positive measure. Furthermore, fix $\Re > 0$ so that

$$\Re > \frac{\eta\eta_1 \int_\Omega |\varrho(z)|^p\ dz}{\kappa_0 \int_{\Omega_{\kappa_0}} |\varrho(z)|^p\ dz}.$$

From the first condition in (G4), we can choose a constant $\xi_0 > 0$ satisfying

$$\frac{G(z,\xi)}{\xi^p} \geq \frac{\alpha_0(z)\Re}{p}$$

for any $\xi \in (0, \xi_0]$, and for almost everywhere $z \in \Omega$. Then, for small enough $\varepsilon > 0$, we get

$$\begin{aligned}\frac{1}{\eta}\int_{\Omega}\frac{G(z,\varepsilon\varrho)}{\varepsilon^p}\,dz &\geq \frac{\Re}{p\eta}\int_{\Omega}\alpha_0(z)\,|\varrho(z)|^p\,dz\\ &\geq \frac{\Re\kappa_0}{p\eta}\int_{\Omega_{\kappa_0}}|\varrho(z)|^p\,dz\\ &> \frac{\eta_1}{p}\int_{\Omega}|\varrho(z)|^p\,dz\\ &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}K(z-y)\,|\varrho(z)-\varrho(y)|^p\,dz\,dy = \frac{1}{p}[\varrho]^p_{s,p}. \end{aligned} \tag{3.9}$$

Hence, using (3.9), we conclude that

$$[\varrho]^p_{s,p} - \frac{p}{\eta}\int_{\mathbb{R}^N}\frac{G(z,\varepsilon\varrho)}{\varepsilon^p}\,dz < 0$$

for any $\varepsilon > 0$ sufficiently small, which implies $\mathcal{J}(\varepsilon\varrho) < 0$, as required. In consequence, problem (3.7) has a positive solution for any $\eta > 0$. In particular, this also implies that 0 is not a global minimum of $\eta\mathcal{A} - \mathcal{B}$.

Next, we prove that problem (3.7) admits at most one positive solution for any $\eta > 0$. Let ψ_1 and ψ_2 be two weak positive solutions of (3.7). For any $\varepsilon > 0$, we define the truncations $\psi_{j,\varepsilon}$ as in (3.3) for $j = 1, 2$. Let us define the functions

$$\omega_{1,\varepsilon} := \frac{\psi^p_{2,\varepsilon}}{(\varepsilon+\psi_1)^{p-1}} - \psi_{1,\varepsilon}$$

and

$$\omega_{2,\varepsilon} := \frac{\psi^p_{1,\varepsilon}}{(\varepsilon+\psi_2)^{p-1}} - \psi_{2,\varepsilon}.$$

In accordance with Lemma 3.4, we assert that $\omega_{j,\varepsilon} \in X$ for $j = 1, 2$. Now, set

$$\phi_p(\xi) := |\xi|^{p-2}\,\xi.$$

Considering the weak formulation (3.8) of ψ_j, by choosing $\phi = \omega_{j,\varepsilon}$ for $j = 1, 2$, one has

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}K(z,y)\,\phi_p\,(\psi_1(z)-\psi_1(y))\,(\omega_{1,\varepsilon}(z)-\omega_{1,\varepsilon}(y))\,dz\,dy\\ &\qquad = \frac{1}{\eta}\int_{\Omega}g(z,\psi_1)\omega_{1,\varepsilon}(z)\,dz\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}K(z,y)\,\phi_p\,(\psi_2(z)-\psi_2(y))\,(\omega_{2,\varepsilon}(z)-\omega_{2,\varepsilon}(y))\,dz\,dy\\ &\qquad = \frac{1}{\eta}\int_{\Omega}g(z,\psi_2)\omega_{2,\varepsilon}(z)\,dz.\end{aligned} \tag{3.11}$$

Adding the above two equations (3.10) and (3.11) and utilizing the fact that

$$\phi_p\left(\psi_j(z)-\psi_j(y)\right)=\phi_p\left((\varepsilon+\psi_j n)(z)-(\varepsilon+\psi_j)(y)\right)\quad \text{for}\quad j=1,2,$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z,y)\,\phi_p\left((\varepsilon+\psi_1)(z)-(\varepsilon+\psi_1)(y)\right)\left(\frac{\psi_{2,\varepsilon}^p}{(\varepsilon+\psi_1)^{p-1}}(z)-\frac{\psi_{2,\varepsilon}^p}{(\varepsilon+\psi_1)^{p-1}}(y)\right)dz\,dy\\
&-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z,y)\,\phi_p\left(\psi_1(z)-\psi_1(y)\right)\left(\psi_{1,\varepsilon}(z)-\psi_{1,\varepsilon}(y)\right)dz\,dy\\
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z,y)\,\phi_p\left((\varepsilon+\psi_2)(z)-(\varepsilon+\psi_2)(y)\right)\left(\frac{\psi_{1,\varepsilon}^p}{(\varepsilon+\psi_2)^{p-1}}(z)-\frac{\psi_{1,\varepsilon}^p}{(\varepsilon+\psi_2)^{p-1}}(y)\right)dz\,dy\\
&-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z,y)\,\phi_p\left(\psi_2(z)-\psi_2(y)\right)\left(\psi_{2,\varepsilon}(z)-\psi_{2,\varepsilon}(y)\right)dz\,dy\\
&=\frac{1}{\eta}\left(\int_\Omega\left[g(z,\psi_1)\left(\frac{\psi_{2,\varepsilon}^p}{(\varepsilon+\psi_1)^{p-1}}-\psi_{1,\varepsilon}\right)+g(z,\psi_2)\left(\frac{\psi_{1,\varepsilon}^p}{(\varepsilon+\psi_2)^{p-1}}-\psi_{2,\varepsilon}\right)\right]dz\right).
\end{aligned}\tag{3.12}$$

Now, according to the fact that $\xi\to\min\{|\xi|,\varepsilon^{-1}\}$ is 1-Lipschitz function and the discrete Picone inequality in Lemma 3.3, we derive

$$\phi_p\left((\varepsilon+\psi_1)(z)-(\varepsilon+\psi_1)(y)\right)\left(\frac{\psi_{2,\varepsilon}^p}{(\varepsilon+\psi_1)^{p-1}}(z)-\frac{\psi_{2,\varepsilon}^p}{(\varepsilon+\psi_1)^{p-1}}(y)\right)\le|\psi_2(z)-\psi_2(y)|^p$$

and

$$\phi_p\left((\varepsilon+\psi_2)(z)-(\varepsilon+\psi_2)(y)\right)\left(\frac{\psi_{1,\varepsilon}^p}{(\varepsilon+\psi_2)^{p-1}}(z)-\frac{\psi_{1,\varepsilon}^p}{(\varepsilon+\psi_2)^{p-1}}(y)\right)\le|\psi_1(z)-\psi_1(y)|^p\,.$$

Because $\psi_{j,\varepsilon}\to\psi_j$ as $\varepsilon\to 0$ for $j=1,2$, by taking to the limit in (3.12) and applying the Fatou Lemma in the first and third terms as well as using the Lebesgue dominated convergence theorem for all the other terms, one has

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z,y)\,\phi_p\left(\psi_1(z)-\psi_1(y)\right)\left(\frac{\psi_2^p}{\psi_1^{p-1}}(z)-\frac{\psi_2^p}{\psi_1^{p-1}}(y)\right)dz\,dy\\
&-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z,y)\;|\psi_1(z)-\psi_1(y)|^p\;dz\,dy\\
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z,y)\,\phi_p\left(\psi_2(z)-\psi_2(y)\right)\left(\frac{\psi_1^p}{\psi_2^{p-1}}(z)-\frac{\psi_1^p}{\psi_2^{p-1}}(y)\right)dz\,dy\\
&-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(z,y)\;|\psi_2(z)-\psi_2(y)|^p\;dz\,dy\\
&\ge\frac{1}{\eta}\left(\int_\Omega g(z,\psi_1)\left(\frac{\psi_2^p}{\psi_1^{p-1}}-\psi_1\right)+g(z,\psi_2)\left(\frac{\psi_1^p}{\psi_2^{p-1}}-\psi_2\right)dz\right)\\
&=-\frac{1}{\eta}\int_\Omega\left(\frac{g(z,\psi_1)}{\psi_1^{p-1}}-\frac{g(z,\psi_2)}{\psi_2^{p-1}}\right)(\psi_1^p-\psi_2^p)\,dz.
\end{aligned}\tag{3.13}$$

Using Lemma 3.3 on the left-hand side of (3.13), we obtain

$$\int_\Omega \left(\frac{g(z,\psi_1)}{\psi_1^{p-1}} - \frac{g(z,\psi_2)}{\psi_2^{p-1}} \right) (\psi_1^p - \psi_2^p)\, dz \geq 0.$$

Hence, because the function $\xi \mapsto \frac{g(z,\xi)}{\xi^{p-1}}$ is decreasing in $(0, +\infty)$, we obtain that $\psi_1 = \psi_2$. Therefore, we ensure that problem (3.7) possesses at most one positive solution. As a result, we derive that $\eta\mathcal{A} - \mathcal{B}$ admits a unique global minimum in X, since otherwise, in consideration of [31, Corollary 1], it would have at least three critical points. Because 0 is not a global minimum for $\eta\mathcal{A} - \mathcal{B}$, the global minimum of this functional is consistent with its only nonzero critical point.

Finally, let us show that $\mathcal{B}$ has no global maxima. Assume to the contrary that $\widehat{\psi} \in X$ is a global maximum of $\mathcal{B}$. Obviously, we know $\mathcal{B}(\widehat{\psi}) > 0$. Thus, since g is nonnegative, it follows from (G3) that the set

$$\Gamma := \{z \in \Omega : g(z, \widehat{\psi}(z)) > 0\}$$

has a positive measure. Let us fix a closed set $\mathcal{P} \subset \Gamma$ of positive measures. Let $\varrho \in X$ be such that $\varrho \geq 0$ and $\varrho(z) = 1$ for almost everywhere $z \in \mathcal{P}$. Then, we obtain

$$\int_\Omega g(z, \widehat{\psi}(z))\varrho(z)\, dz \geq \int_{\mathcal{P}} g(z, \widehat{\psi}(z))\, dz > 0,$$

and so $\mathcal{B}'(\widehat{\psi}) \neq 0$, which is a contradiction.

Hence, each assumption of Theorem 3.6 is satisfied. Therefore, there exists a unique $\widetilde{w} \in X$, with $[\widetilde{w}]_{s,p}^p \in J$, such that

$$\begin{aligned} &M\left([\widetilde{w}]_{s,p}^p\right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(z,y)\, |\widetilde{w}(z) - \widetilde{w}(y)|^p \, dz\, dy - p \int_\Omega G(z, \widetilde{w}^+(z))\, dz \\ &= \inf_{\psi \in X} \left\{ M\left([\widetilde{w}]_{s,p}^p\right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(z,y)\, |\psi(z) - \psi(y)|^p \, dz\, dy - p \int_\Omega G(z, \psi^+(z))\, dz \right\}. \end{aligned}$$

Consequently, from what seen above, problem (P) possesses the unique positive weak solution $\widetilde{w}$.

□

4. Conclusions

This paper is devoted to deriving the existence and uniqueness of positive solutions to fractional p-Laplacian problems involving discontinuous Kirchhoff-type functions. The main tools for obtaining these results are the uniqueness results of the Brézis–Oswald-type based on [5] and the abstract global minimum principle in [33]. Particularly, based on previous studies [4, 27], we obtain the existence of at most one positive weak solution to the fractional p-Laplacian equations of the Brézis–Oswald type by employing the discrete Picone inequality in [3, 12]. But, our condition (G4) can be considered a special case of that of [2, 27] since the nonlinear term g satisfies the following assumption:

$$\beta_0(z) = \lim_{\xi \to 0^+} \frac{g(z,\xi)}{\xi^{p-1}} \qquad \text{and} \qquad \beta_\infty(z) = \lim_{\xi \to +\infty} \frac{g(z,\xi)}{\xi^{p-1}}.$$

Let us define $\Lambda_1\left(\mathcal{L}_p^s-\beta_0\right)$ and $\Lambda_1\left(\mathcal{L}_p^s-\beta_\infty\right)$ as

$$\Lambda_1\left(\mathcal{L}_p^s-\beta_0\right)=\inf_{\psi\in X}\left\{[\psi]_{s,p}^p-\int_\Omega\beta_0(z)\,|\psi(z)|^p\;dz\;:\;|\psi|_{L^p(\Omega)}=1\right\}$$

and

$$\Lambda_1\left(\mathcal{L}_p^s-\beta_0\right)=\inf_{\psi\in X}\left\{[\psi]_{s,p}^p-\int_\Omega\beta_\infty(z)\,|\psi(z)|^p\;dz\;:\;|\psi|_{L^p(\Omega)}=1\right\}.$$

If $\Lambda_1\left(\mathcal{L}_p^s-\beta_0\right)<0<\Lambda_1\left(\mathcal{L}_p^s-\beta_0\right)$ in place of (G4) holds, then analogous arguments such as those in [27] implies that problem (3.7) admits at most one positive solution for any $\eta>0$. Consequently, explicit modifications of the proof of Theorem 3.7 yield the same consequences concerning problem (P) when $\Lambda_1\left(\mathcal{L}_p^s-\beta_0\right)<0<\Lambda_1\left(\mathcal{L}_p^s-\beta_0\right)$ in place of (G4) is supposed.

Additionally, a new research direction is the investigation of the Brézis–Oswald type fractional p-Laplacian problems involving Hardy potentials:

$$\begin{cases} M\left([\psi]_{s,p}^p\right)\mathcal{L}_p^s\psi(z)=\mu\frac{|\psi|^{p-2}\psi}{|z|^p}+\lambda g(z,\psi) & \text{in} \quad \Omega,\\ \psi>0 & \text{in} \quad \Omega,\\ \psi=0 & \text{on} \quad \mathbb{R}^N\setminus\Omega, \end{cases} \tag{4.1}$$

where $p\in(1,p_s^*)$, $\mu\in(-\infty,\mu^*)$ for a positive constant μ^*. When $\mu\neq0$, the classical variational approach is not applicable because of the appearance of the term $\mu|\psi|^{p-2}\psi|z|^{-p}$. The reason is that the Hardy inequality ensures that only the embedding $W_0^{s,p}(\Omega)\hookrightarrow L^p(\Omega,|z|^{-p})$ is continuous but not compact. Hence, the situation with $\mu\neq0$ would be much more delicate than the situation in the present paper because of the lack of compactness. To the best of our belief, there are no results concerning the localization, existence, and uniqueness of positive solutions to problem (4.1).

Author contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

All the authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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