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Research article

Multiplicityof solution for a singular problem involving the φ-Hilfer derivative and variable exponents

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Abstract: This paper dealt with the existence of multiple solutions for some singular $p(s)$-Laplacian problems involving the φ-Hilfer derivative. Precisely, we combined the variational method with the Nehari manifold to prove that such a problem admited two nontrivial solutions. An example was presented to illustrate the effectiveness of our main result.

Keywords: variational methods; Nehari manifold; Sobolev space with variable exponents; φ-Hilfer derivative
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1. Introduction

Fractional calculus is a generalization of traditional calculus to non-integer orders of differentiation and integration. Fractional calculus allows one to deal with derivatives and integrals of any real or complex order. This extension has led to the development of various mathematical and physical concepts which have found a wide range of applications in several disciplines. Fractional calculus is often used in modeling materials that exhibit both elastic and viscous properties, such as polymers and biological tissues, because viscoelastic models use fractional derivatives to describe hereditary effects which are not captured by classical integer-order models; see [3, 19]. In anomalous diffusion processes, fractional calculus can describe non-Fickian diffusion where the mean square displacement of particles follows a power law, which can applied to systems associated with supercooled liquids and with fractal-like materials; see [20]. Also, fractional calculus is used in signal processing to improve filter resolution, noise filtering, system identification, and image processing see [2, 17]. For other applications, interested readers can consult the papers [15, 27, 28]. Due to their importance in several fields, many authors concentrated on the development of these derivatives and used different methods to solve boundary value problems involving fractional derivatives; we cite, for example, the papers of Sousa et al. [16, 23, 24] (in these papers, the Nehari manifold method with some variational methods is contributed), Ghanmi and Horrigue [11] (Schauder fixed point theorem), Ghanmi et al. [12, 13] (Nehari manifold method and fibering maps analysis), Nouf et al. [18] (combination of the mountain pass theorem with variational technique), Hamza et al. [14] (combination of the mountain pass theorem with the Fountain theorem and its dual form), Elhoussain et al. [8] (combination of the variational method with critical point theorem), and Alsaedi and Ghanmi [1] (combination of the mountain pass theorem with its symmetric version). In particular, Nouf et al. [18] considered the following fractional problem:

$$\begin{cases} M(\omega(t))\, {}_tD_T^{\alpha,\psi}\left(\varphi_p\left({}_0D_t^{\alpha,\psi}\omega(t)\right)\right) = \lambda f(t,\omega(t)) + g(t,\omega(t)), t \in (0,T), \\ \omega(0) = \omega(T) = 0, \end{cases} \tag{1.1}$$

where ${}_tD_T^{\alpha,\psi}$ and ${}_0D_t^{\alpha,\psi}$ are the derivative operators in the sense of ψ-Riemann Liouville, the functions f, g are assumed to be positively homogeneous, and M is a power function. Very recently, Alsaedi and Ghanmi [1] studied the following problem:

$$\begin{cases} M(\varphi(s))\, \mathcal{D}_T^{\mu,\alpha,\varphi}\left(\Phi_p\left(\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)\right)\right) = \lambda g(s,\omega(s))) + f(s,\omega(s)), \quad s \in (0,T), \\ I_{0^+}^{\alpha(\alpha-1);\varphi}(0) = I_T^{\alpha(\alpha-1);\varphi}(T) = 0, \end{cases} \tag{1.2}$$

where f is a carathéodory function and g is a positively homogeneous function, $\mathcal{D}_T^{\mu,\alpha,\varphi}$ and $\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}$ are the right-sided and left-sided φ-Hilfer derivatives, and $I_{0^+}^{\alpha(\alpha-1);\varphi}$ and $I_T^{\alpha(\alpha-1);\varphi}$ are the left-sided and the right-sided φ-Riemann-Liouville fractional integrals.

In this paper, we continue to develop a fractional problem involving φ-Hilfer derivatives. Precisely, we study the existence of solutions for the following fractional problem:

$$\begin{cases} \mathcal{D}_T^{\mu,\alpha,\varphi}|\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)-2}\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s) = \lambda \frac{g(s)}{\omega^{\gamma(s)}} + f(s,\omega(s)), \quad s \in (0,T), \\ I_{0^+}^{\alpha(\alpha-1);\varphi}(0) = I_T^{\alpha(\alpha-1);\varphi}(T) = 0, \end{cases} \tag{1.3}$$

where λ is a positive parameter, $0 < \frac{1}{p^-} < \mu \leq 1$, $0 \leq \alpha \leq 1$, and the functions f, g, p, and γ are assumed to satisfy the following hypotheses:

(H_1) There exists a function r in $\mathcal{L}_+^\infty([0,T])$, such that

$$f(s,ty) = t^{r-1}f(s,y), \ \forall (s,t,y) \in [0,T]\times(0,\infty)\times\mathbb{R},$$

where $\mathcal{L}_+^\infty([0,T])$ is introduced later in Section 2.

(H_2) The functions p, γ are continuous on $[0,T]$ and satisfy:

$$0 < \gamma^- \leq \gamma^+ < 1 < p^- \leq p^+ < r^- \leq r^+ < \infty,$$

where for a given function σ, σ^+ and σ^- are defined by:

$$\sigma^+ = \sup_{s\in[0,T]} \sigma(s), \ \sigma^- = \inf_{s\in[0,T]} \sigma(s).$$

(H_3) g is a nonnegative measurable function satisfying

$$g \in \mathcal{L}^{\frac{\beta(\cdot)}{\beta(\cdot)+\gamma(.)-1}}([0,T]),$$

for some function β with $1 < \beta^- \leq \beta^+$.

Remark 1.1. *If F is the antiderivative of the function f with respect to the second variable and vanished at zero, then from* (H_1)*, we have*

$$F(s, ty) = t^r F(s, y),$$

$$uf(s, y) = rF(s, \omega(s))$$

and there exists $B > 0$ such that

$$|F(s, y)| \leqslant B\, |y|^r .$$

We note that singular fractional problems like (1.3) are highly specialized areas of fractional calculus. It builds on the existing theory of fractional derivatives, introducing an additional layer of flexibility through the φ-Hilfer derivative, and is useful for modeling systems with complex, memory-dependent behaviors, singularities, or other nontrivial dynamics. The study of such derivatives allows for more accurate and generalized modeling of real-world phenomena in physics, engineering, and applied mathematics. So, the novelty in our problem is that it contains a singular nonlinearity, so the associated functional energy is not Gateau differentiable and the direct variational method cannot be applied. Moreover, the exponents p and γ are variable, which means that the manipulation of the properties of the functional energy is more complicated. Since the functional energy is not of class C^1, we use the Nehari manifold method to prove the following theorem.

Theorem 1.2. *Under the hypotheses* (H_1)*–*(H_3)*, there exists $\lambda_* > 0$ such that for each $\lambda \in (0, \lambda_*)$, the problem* (1.3) *admits two nontrivial solutions.*

2. Mathematical background

This section gathers some basic ideas for the theory of variable exponent Lebesgue spaces. We also offer significant findings on the fractional and classical φ-Hilfer fractional derivative spaces. These results and other properties are available for interested readers in the papers [5, 7, 9, 10, 23].

Let $\mathcal{L}^{\infty}([0, T])$ be the space of all bounded functions on $[0, T]$, and put

$$\mathcal{L}^{\infty}_{+}([0, T]) = \Big\{p \in \mathcal{L}^{\infty}([0, T]) : p^{-} > 1\Big\}.$$

For $p \in \mathcal{L}^{\infty}_{+}([0, T])$, we define the space $\mathcal{L}^{p(s)}([0, T])$ by

$$\mathcal{L}^{p(s)}([0, T]) = \left\{\nu \in V([0, T]) : \int_0^T |\nu(s)|^{p(s)} ds < \infty\right\},$$

where $V([0, T])$ denotes the set of all measurable real-valued functions on $[0, T]$. We equip the space $\mathcal{L}^{p(s)}([0, T])$ with the norm

$$\|\nu\|_{p(s)} = \inf\left\{\lambda > 0 : \int_0^T \left|\frac{\nu(s)}{\lambda}\right|^{p(s)} ds \leq 1\right\}.$$

Remember that $\left(\mathcal{L}^{p(s)}([0, T]), \|\cdot\|_{p(s)}\right)$ is a reflexive and separable Banach space.

We recall from [6, 9] that if p and q are such that for any $s \in [0, T]$, we have $\frac{1}{p(s)} + \frac{1}{q(s)} = 1$, then for each $\nu \in \mathcal{L}^{p(s)}([0, T])$ and each $u \in \mathcal{L}^{q(s)}([0, T])$, one has

$$\Big| \int_{[0,T]} \nu(s)u(s)ds \Big| \leq \; 2\|\nu\|_{p(s)}\|u\|_{q(s)}. \tag{2.1}$$

Put

$$\varrho^{p(s)}(\nu) = \int_0^T |\nu(s)|^{p(s)} ds,$$

then we have the following result.

Proposition 2.1. *[6] Let $\nu \in \mathcal{L}^{p(s)}([0, T])$ and $\{\nu_r\}_{r\in \mathrm{N}} \subset \mathcal{L}^{p(s)}([0, T])$, then the following statements hold:*

(i) $\|\nu\|_{p(s)} < 1 \, (= 1, > 1) \Longleftrightarrow \varrho^{p(s)}(\nu) < 1 \, (= 1, > 1)$,

(ii) $\|\nu\|_{p(s)} > 1 \Longrightarrow \|\nu\|^{p^-}_{p(s)} \leq \varrho^{p(s)}(\nu) \leq \|\nu\|^{p^+}_{p(s)}$,

(iii) $\|\nu\|_{p(s)} < 1 \Longrightarrow \|\nu\|^{p^+}_{p(s)} \leq \varrho^{p(s)}(\nu) \leq \|\nu\|^{p^-}_{p(s)}$,

(iv) $\lim_{r\to\infty} \|\nu_r - \nu\|_{p(s)} = 0 \Longleftrightarrow \lim_{r\to\infty} \varrho^{p(s)}(\nu_r - \nu) = 0$.

Hereafter, φ is an increasing positive function on I with a continuous derivative $\varphi'(s) \neq 0$ over I, μ is a positive real integer, and $I := [b, c]$ denotes a finite or infinite interval of the real line $\mathbb{R}$. The left and right-sided fractional integrals of order μ of a function ω with respect to the function φ on I are defined by:

$$I^{\mu;\varphi}_{b^+}\,\omega(s) = \frac{1}{\Gamma(\mu)} \int_b^s \varphi'(u)(\varphi(s) - \varphi(u))^{\mu-1}\omega(u)d\mathrm{u}, \tag{2.2}$$

and

$$I^{\mu;\varphi}_{c^-}\,\omega(s) = \frac{1}{\Gamma(\mu)} \int_s^c \varphi'(u)(\varphi(u) - \varphi(s))^{\mu-1}\omega(u)d\mathrm{u}. \tag{2.3}$$

If r is an integer and μ is such that $r - 1 < \mu < r$, then the left and right-sided φ-Hilfer fractional derivative of order μ and type $0 \leq \alpha \leq 1$, are defined respectively, by:

$$\mathcal{D}^{\mu,\alpha;\varphi}_{b^+}\,\omega(s) = I^{\alpha(r-\mu);\varphi}_{b^+}\Big(\frac{1}{\varphi'(s)}\frac{d}{d\nu}\Big)^r I^{(1-\alpha)(r-\mu);\varphi}_{b^+}\,\omega(s), \tag{2.4}$$

and

$$\mathcal{D}^{\mu,\alpha;\varphi}_{c^-}\,\omega(s) = I^{\alpha(r-\mu);\varphi}_{c^-}\Big(-\frac{1}{\varphi'(s)}\frac{d}{d\nu}\Big)^r I^{(1-\alpha)(r-\mu);\varphi}_{c^-}\,\omega(s). \tag{2.5}$$

Remark 2.1. *The φ-Hilfer fractional derivative generalizes other fractional derivatives. In particular, we have:*

(i) If $\alpha = 0$, then the φ-Hilfer fractional derivative is reduced to the φ-Riemann Liouville fractional derivative. If in addition, $\varphi(x) = x$, we obtain the Riemann Liouville fractional derivative.

(ii) If, $\alpha = 1$, then the φ-Hilfer fractional derivative is reduced to the φ-Caputo fractional derivative in addition, $\varphi(x) = x$, we obtain the Caputo fractional derivative.

Moreover, we have the following important results.

Lemma 2.2. *Let $\varphi(\cdot)$ be an increasing and positive monotone function on (a, b), having a continuous derivative $\varphi^{'}(\cdot) \neq 0$, on (a, b). If $0 < \mu \leq 1$ and $0 \leq \alpha \leq 1$, then*

$$\int_a^b \left(\mathcal{D}_{a^+}^{\mu,\alpha;\varphi} w(s)\right) \theta(s) ds = \int_a^b w(s)\varphi^{'}(s)\mathcal{D}_{a^+}^{\mu,\alpha;\varphi} \left(\frac{\theta(s)}{\varphi^{'}(s)}\right) ds$$

for any $w \in AC^1$ and $\theta \in C^1$ satisfying the boundary conditions $w(a) = w(b) = 0$, where AC^1 denotes the space of absolutely continuous functions with absolutely continuous derivatives.

Since the variational method will be used, it makes sense to begin by defining the working space, which is provided by

$$\mathscr{H}_{p(s)}^{\mu,\alpha,\varphi}([0, T]) = \left\{ \nu \in \mathcal{L}^{p(s)}([0, T]) : \ | \mathcal{D}_{0^+}^{\mu,\alpha;\varphi} \nu| \in \mathcal{L}^{p(s)}([0, T]) \right\}$$

and has the norm

$$\|\nu\|_{\mathscr{H}_{p(s)}^{\mu,\alpha,\varphi}([0,T])} = \|\nu\|_{p(s)} + \|\mathcal{D}_{0^+}^{\mu,\alpha,\varphi} \nu\|_{p(s)}.$$

Furthermore, we define $\mathscr{H}_{p(s),0}^{\mu,\alpha,\varphi}([0, T])$ as the closure of $C_0^\infty(\mathbb{R}^{\mathrm{N}})$ in $\mathscr{H}_{p(s)}^{\mu,\alpha,\varphi}([0, T])$, which can be equipped by the following equivalent norm

$$\|\nu\|_{\mathscr{H}_{p(s),0}^{\mu,\alpha,\varphi}([0,T])} := \|\mathcal{D}_{0^+}^{\mu,\alpha,\varphi} \nu\|_{p(s)},$$

Next, we recall the following results.

Proposition 2.2. *[24, 25] Assume that $p \in \mathcal{L}_+^\infty([0, T])$. The Banach spaces $\mathscr{H}_{p(s)}^{\mu,\alpha,\varphi}([0, T])$ and $\mathscr{H}_{p(s),0}^{\mu,\alpha,\varphi}([0, T])$ are reflexive and separable if $0 < \mu \leq 1$ and $0 \leq \alpha \leq 1$.*

After that, we gather three theorems needed for the main result's proof.

Theorem 2.3. *[21] Consider $[0, T] \subset \mathbb{R}^{\mathrm{N}}$ as an open bounded domain with Lipschitz boundary. Let p and t be two functions in $C(\overline{[0, T]})$, such that $p(s) > 1$ and $p(s) \leq t(s) \leq p^\star(s)$ for each $s \in \overline{[0, T]}$. Then, $\mathcal{L}^{t(s)}([0, T])$ is the continuous embedding from $\mathscr{H}_{p(s)}^{\mu,\alpha,\varphi}([0, T])$. Furthermore, this embedding is compact if $t(s) < p^\star(s)$, that is, if it is in $\overline{[0, T]}$.*

Theorem 2.4. *[6] Assume that $t, q \in \mathcal{L}_+^\infty([0, T])$ such that $q(s) \leq t(s)$ for every $s \in [0, T]$. Then, $\nu \in \mathcal{L}^{t(s)}([0, T])$, and there exists $q_0 \in [q^-, q^+]$ if $|\nu|^{q(s)} \in \mathcal{L}^{\frac{t(s)}{q(s)}}([0, T])$ such that*

$$\left\| |\nu|^{q(s)} \right\|_{\frac{t(s)}{q(s)}} = \|\nu\|_{t(s)}^{q_0}.$$

Proposition 2.5. *[26] Let $0 \leq \alpha \leq 1$, $1 < p(s) < \infty$, and $0 < \frac{1}{p(s)} < \mu < 1$. In $\mathscr{H}_{p(s)}^{\mu,\alpha,\varphi}([0, T], \mathbb{R})$, let $\{\nu_r\}_{r \in \mathrm{N}}$ be a sequence that converges weakly to ν. Then, in $C([0, T], \mathbb{R})$, ν_r converges strongly to ν.*

3. Nehari manifold analysis

In this section, we study the analysis of the functional energy in the Nehari manifold sets. So, we begin by remarking that if ω is a solution for problem (1.3), then for any $\kappa \in C_0^\infty((0,T),\mathbb{R}^N)$, we have

$$\int_0^T \mathcal{D}_T^{\mu,\alpha,\varphi}|\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)-2}\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)\kappa(s)ds = \lambda\int_0^T \frac{g(s)}{\omega^{\gamma(s)}}\kappa(s)ds + \int_0^T f(s,\omega(s))\kappa(s)ds. \tag{3.1}$$

On the other hand, from Lemma 2.2, we get

$$\int_0^T \mathcal{D}_T^{\mu,\alpha,\varphi}|\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)-2}\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)\kappa(s)ds = \int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)-2}\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)\varphi'(s)\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\left(\frac{\kappa(s)}{\varphi'(s)}\right)ds.$$

If $\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\left(\frac{\kappa(s)}{\varphi'(s)}\right) = \frac{1}{\varphi'(s)}\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\kappa(s),\ \forall s\in(0,T)$, then Eq (3.1) can be rewritten as

$$\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)-2}\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\kappa(s)ds = \lambda\int_0^T \frac{g(s)}{\omega^{\gamma(s)}}\kappa(s)ds + \int_0^T f(s,\omega(s))\kappa(s)ds.$$

Consider $\kappa = w$, which yields

$$\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds = \lambda\int_0^T g(s)\,\omega^{1-\gamma(s)}(s)ds + \int_0^T f(s,\omega(s))w(s)ds. \tag{3.2}$$

So, from Eq (3.2), we can define the functional energy $Z_\lambda : \mathscr{H}_{p(s)}^{\mu,\alpha;\varphi}([0,T]) \to R$, associated to problem (1.3) by:

$$Z_\lambda(\omega) = \int_0^T \frac{1}{p(s)}|\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds - \lambda\int_0^T \frac{g(s)\,|\omega(s)|^{1-\gamma(s)}}{1-\gamma(s)}ds - \int_0^T F(s,\omega(s))ds. \tag{3.3}$$

We note that the functional Z_λ is not of class C^1, which implies that we cannot use the direct variational method; moreover, it is not coercive in $\mathscr{H}_{p(s)}^{\mu,\alpha;\varphi}([0,T])$. So, we will work on the following set

$$K_\lambda = \left\{\omega\in\mathscr{H}_{p(s)}^{\mu,\alpha;\varphi} : \varrho^{p(s)}(\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega) = \lambda\int_0^T g(s)\,|\omega(s)|^{1-\gamma(s)}\,ds + \int_0^T r(s)F(s,\omega(s))ds\right\}. \tag{3.4}$$

Next, for $\omega\in K_\lambda$, we define $\xi_{\lambda,\omega} : [0,\infty)\to\mathbb{R}$ by:

$$\xi_{\lambda,\omega}(t) = \int_0^T \frac{t^{p(s)}}{p(s)}|\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds - \lambda\int_0^T \frac{t^{1-\gamma(s)}}{1-\gamma(s)}g(s)\,|\omega(s)|^{1-\gamma(s)}\,ds - \int_0^T t^{r(s)}F(s,\omega(s))ds.$$

Then, we can easily see that

$$\xi'_{\lambda,\omega}(t) = \int_0^T t^{p(s)-1}|\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds - \lambda\int_0^T t^{-\gamma(s)}g(s)\,|\omega(s)|^{1-\gamma(s)}\,ds - \int_0^T r(s)\,t^{r(s)-1}F(s,\omega(s))ds,$$

and

$$\xi''_{\lambda,\omega}(t) = \int_0^T (p(s)-1)t^{p(s)-2}|\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds + \lambda\int_0^T \gamma(s)t^{-\gamma(s)-1}g(s)\,|\omega(s)|^{1-\gamma(s)}\,ds$$

$$- \int_0^T r(s)(r(s)-1)t^{r(s)-2}F(s,\omega(s))ds,$$

Moreover, $\omega \in K_\lambda$ if, and only if, we have:

$$\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds - \lambda \int_0^T g(s)\,|\omega(s)|^{1-\gamma(s)}\,ds - \int_0^T r(s)F(s,\omega(s))ds = 0. \tag{3.5}$$

Lemma 3.1. *Assume that hypotheses* (H_2) *hold, then the functional* Z_λ *is coercive in* K_λ *and bounded below.*

Proof. Let $\omega \in K_\lambda$ with $\|\omega\| > 1$. Then, using (3.5), Proposition 2.1, hypothesis (H_3), and the Hölder inequality, we obtain

$$\begin{aligned}
Z_\lambda(\omega) &= \int_0^T \frac{1}{p(s)}|\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds - \lambda\int_0^T \frac{g(s)\,|\omega(s)|^{1-\gamma(s)}}{1-\gamma(s)}ds - \int_0^T F(s,\omega(s))ds \\
&\geq \frac{1}{p^+}\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds - \frac{\lambda}{1-\gamma^+}\int_0^T g(s)\,|\omega(s)|^{1-\gamma(s)}\,ds \\
&- \frac{1}{r^-}\int_0^T r(s)F(s,\omega(s))ds \\
&\geq \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds \\
&- \lambda\left(\frac{1}{1-\gamma^+} - \frac{1}{r^-}\right)\int_0^T g(s)\,|\omega(s)|^{1-\gamma(s)}\,ds \\
&\geq \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\|\omega\|^{p^-} - \lambda\left(\frac{1}{1-\gamma^+} - \frac{1}{r^-}\right)\|g\|_{\frac{\beta(\cdot)}{\beta(\cdot)+\gamma(\cdot)-1}}\|\omega\|^{1-\gamma^-}.
\end{aligned}$$

Since $r^- > p^+ \geq p^- > 1-\gamma^+$, then we deduce that $Z_\lambda(\omega) \to \infty$ as $\|\omega\| \to \infty$. This implies that Z_λ is coercive in K_λ and bounded below. □

To prove the multiplicity of solutions, we define the following sets:

$$K_\lambda^0 = \{\omega \in K_\lambda : \xi''_{\lambda,\omega}(1) = 0\},$$

$$K_\lambda^+ = \{\omega \in K_\lambda : \xi''_{\lambda,\omega}(1) > 0\},$$

and

$$K_\lambda^- = \{\omega \in K_\lambda : \xi''_{\lambda,\omega}(1) < 0\}.$$

Lemma 3.2. *Under assumptions* (H_1)*, the set* K_λ^0 *is empty, provided that* λ *is small enough.*

Proof. Assume that

$$0 < \lambda < \frac{r^- - p^+}{(\gamma^- + r^+ - 1)\|g\|_{\frac{\alpha(.)}{\alpha(.)+\gamma(.)-1}}}\left(\frac{p^- - 1 + \gamma^-}{Br^+(\gamma^- + r^+ - 1)}\right)^{\frac{p^-+\gamma^--1}{r^+-p^-}},$$

and suppose otherwise that $K_\lambda^0 \neq \emptyset$. Let ω be a nontrivial function in K_λ^0. Since the proofs are similar for $\|\omega\| \leq 1$ and $\|\omega\| \geq 1$, then, we prove the result only for $\|\omega\| \geq 1$. Using Eq (3.5) and the definition of K_λ^0, we get

$$\begin{aligned}
0 = \xi''_{\lambda,\omega}(1) &= \int_0^T (p(s)-1)|\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds + \lambda\int_0^T \gamma(s)g(s)\,|\omega(s)|^{1-\gamma(s)}\,ds \\
&- \int_0^T r(r-1)F(s,\omega(s))ds \\
&\leq (p^+-1)\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds + \lambda\gamma^+\int_0^T g(s)\,|\omega(s)|^{1-\gamma(s)}\,ds \\
&- (r^--1)\int_0^T r(s)F(s,\omega(s))ds \\
&\leq (p^+-r^-)\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds \\
&+ \lambda\,(\gamma^+ + r^- - 1)\int_0^T g(s)\,|\omega(s)|^{1-\gamma(s)}\,ds.
\end{aligned}$$

Since $r^- > p^+$, we obtain

$$(r^- - p^+)\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds \leq \lambda\,(\gamma^- + r^+ - 1)\int_0^T g(s)\,|\omega(s)|^{1-\gamma(s)}\,ds.$$

Now, from Proposition 2.1 and the Hölder inequality, we get

$$\begin{aligned}
(r^- - p^+)\,\|\omega\|^{p^-} &\leq (r^- - p^+)\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds \\
&\leq \lambda\,(\gamma^- + r^+ - 1)\int_0^T g(s)\,|\omega(s)|^{1-\gamma(s)}\,ds \\
&\leq \lambda\,(\gamma^- + r^+ - 1)\,\|g\|_{\frac{\beta(\cdot)}{\beta(\cdot)+\gamma(.)-1}}\,\||\omega|^{1-\gamma}\|_{\frac{\beta(\cdot)}{1-\gamma(.)}} \\
&\leq \lambda\,(\gamma^- + r^+ - 1)\,\|g\|_{\frac{\beta(\cdot)}{\beta(\cdot)+\gamma(.)-1}}\,\|\omega\|^{1-\gamma^-}.
\end{aligned}$$

Therefore, we deduce

$$\|\omega\| \leq \left(\frac{\lambda(\gamma^- + r^+ - 1)\|g\|_{\frac{\beta(\cdot)}{\beta(\cdot)+\gamma(.)-1}}}{r^- - p^+}\right)^{\frac{1}{p^- + \gamma^- - 1}}. \tag{3.6}$$

Similarly, we have

$$\begin{aligned}
0 &= \xi''_{\lambda,\omega}(1) \\
&= \int_0^T (p(s)-1)|\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds + \lambda\int_0^T \gamma(s)g(s)\,|\omega(s)|^{1-\gamma(s)}\,ds - \int_0^T r(r-1)F(s,\omega(s))ds \\
&\geq (p^- - 1)\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds + \gamma^-\left(\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds - \int_0^T r(s)F(s,\omega(s))ds\right) \\
&- (r^+ - 1)\int_0^T r(s)F(s,\omega(s))ds
\end{aligned}$$

$$\geq (p^- - 1 + \gamma^-) \int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)} ds - r^+ (\gamma^- + r^+ - 1) \int_0^T F(s, \omega(s)) ds.$$

So, we get

$$(p^- - 1 + \gamma^-) \int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)} ds \leq r^+ (\gamma^- + r^+ - 1) \int_0^T F(s, \omega(s)) ds.$$

Now, from Remark 1.1 and Proposition 2.1, we conclude that

$$(p^- - 1 + \gamma^-) \|\omega\|^{p^-} \leq Br^+ (\gamma^- + r^+ - 1) \|\omega\|^{r^+}.$$

Therefore

$$\|\omega\| \geq \left(\frac{p^- - 1 + \gamma^-}{Br^+ (\gamma^- + r^+ - 1)} \right)^{\frac{1}{r^+ - p^-}}. \tag{3.7}$$

Finally, by combining Eq (3.6) with Eq (3.7), one has

$$\lambda \geq \frac{r^- - p^+}{(\gamma^- + r^+ - 1)\|g\|_{\frac{\beta(\cdot)}{\beta(\cdot)+\gamma(.)-1}}} \left(\frac{p^- - 1 + \gamma^-}{Br^+ (\gamma^- + r^+ - 1)} \right)^{\frac{p^- + \gamma^- - 1}{r^+ - p^-}} := \lambda_0,$$

which is a contradiction. Hence, K_λ^0 is empty. □

Lemma 3.3. *For each $\omega \in K_\lambda$ there exist $t_1 > 0$ and $t_2 > 0$, such that $t_1 \omega \in K_\lambda^+$ and $t_2 \omega \in K_\lambda^-$, provided that λ is small enough.*

Proof. Let $\omega \in K_\lambda$, and $t > 0$. Then, we know that

$$\xi'_{\lambda,\omega}(t) = \int_0^T t^{p(s)-1} |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)} ds - \lambda \int_0^T t^{-\gamma(s)} g(s) |\omega(s)|^{1-\gamma(s)} ds - \int_0^T r(s) t^{r(s)-1} F(s, \omega(s)) ds.$$

Set

$$A = \int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)} ds, \ B = \int_0^T g(s) |\omega(s)|^{1-\gamma(s)} ds, \ C = \int_0^T r(s) F(s, \omega(s)) ds.$$

If $t > 1$, then we have

$$At^{p^- - 1} - \lambda Bt^{-\gamma^-} - Ct^{r^+ - 1} \leq \xi'_{\lambda,\omega}(t) \leq At^{p^+ - 1} - \lambda Bt^{-\gamma^+} - Ct^{r^- - 1},$$

and if $0 < t < 1$, then we have

$$At^{p^+ - 1} - \lambda Bt^{-\gamma^+} - Ct^{r^- - 1} \leq \xi'_{\lambda,\omega}(t) \leq At^{p^- - 1} - \lambda Bt^{-\gamma^-} - Ct^{r^+ - 1}.$$

Now, we define the function $m_\pm$ on $(0, \infty)$ by:

$$m_\pm(t) = At^{p^\pm - 1} - \lambda Bt^{-\gamma^\pm} - Ct^{r^\mp - 1}.$$

It is clear that

$$m_\pm(t) = 0 \Longleftrightarrow t^{-\gamma^\pm} (g_\pm(t) - \lambda B) = 0 \Longleftrightarrow g_\pm(t) = \lambda B, \tag{3.8}$$

where

$$g_\pm(t) = At^{p^\pm+\gamma^\pm-1} - Ct^{r^\mp+\gamma^\pm-1}.$$

The function $g_\pm$ has a unique maximum point $t_\pm > 0$, which is given by

$$t_\pm = \left(\frac{A(p^\pm + \gamma^\pm - 1)}{C(r^\mp + \gamma^\pm - 1)}\right)^{\frac{1}{r^\mp - p^\pm}}.$$

The fact that $\lim_{t\to 0} g_\pm(t) = 0$ and $\lim_{t\to\infty} g_\pm(t) = -\infty$, implies that $g_\pm(t_\pm) > 0$. So, we put

$$\lambda_1 = \frac{g_\pm(t_\pm)}{B},$$

and we take $\lambda < \lambda_1$. From the variation of the function $g_\pm$ and the fact that

$$0 < \lambda B < g_\pm(t_\pm),$$

we deduce the existence of $0 < T^1_\pm < t_\pm < T^2_\pm$, such that

$$g_\pm(T^1_\pm) = g_\pm(T^2_\pm) = \lambda B,\ g'_\pm(T^1_\pm) > 0,\ \text{and}\ g'_\pm(T^2_\pm) < 0.$$

From Eq (3.8), we have

$$m_\pm(T^1_\pm) = m_\pm(T^2_\pm) = 0.$$

Since for all $t > 0$, $\xi'_{\lambda,\omega}(t)$ is between m_+ and m_-, and since both equations $m_+(t) = 0$ and $m_-(t) = 0$ have two solutions, then we deduce the existence of $0 < t_1 < t_2 < \infty$, such that for all $\lambda \in (0, \lambda_1)$, we have

$$\xi'_{\lambda,\omega}(t_1) = \xi'_{\lambda,\omega}(t_2) = 0,\ t_1\,\omega \in K^+_\lambda\ \text{and}\ t_2\,\omega \in K^-_\lambda.$$

□

4. Proof of the main result

In this section, we will prove the main result of this paper (Theorem 1.2). For this, we assume that $0 < \lambda < \min(\lambda_0, \lambda_1)$, which implies that the above lemmas hold. We begin this section by proving the following lemma.

Lemma 4.1. *If $0 < \lambda < \min(\lambda_0, \lambda_1)$, and if (H_1)–(H_3) are satisfied, then the following statements hold:*

(i) There exists $\omega^+_\kappa \in K^+_\lambda$, such that

$$\inf_{\omega\in K^+_\lambda} Z_\lambda(\omega) = Z_\lambda(\omega^+_\kappa) = c^+_\lambda < 0.$$

(ii) There exists $\omega^-_\kappa \in K^-_\lambda$, such that

$$\inf_{\omega\in K^-_\lambda} Z_\lambda(\omega) = Z_\lambda(\omega^-_\kappa) = c^-_\lambda > 0.$$

Proof. Since the proof of case (ii) is very similar to the one in (i), then we prove only the statement (i). We know that Z_λ is bounded below on K_λ, and so on K_λ^+. Then, there exists a minimizing sequence $\{\omega_n^+\}$ on K_λ^+ such that

$$\lim_{n\to\infty} Z_\lambda(\omega_n^+) = \inf_{\omega\in K_\lambda^+} Z_\lambda(\omega) = c_\lambda^+.$$

Since Z_λ is coercive, $\{\omega_n^+\}$ is bounded in $\mathscr{H}_{p(s)}^{\mu,\alpha;\varphi}([0,T])$. So, there exist $\omega_\kappa^+ \in \mathscr{H}_{p(s)}^{\mu,\alpha;\varphi}([0,T])$, and a subsequence still denoted by $\{\omega_n^+\}$, such that

$$\begin{cases} \omega_n^+ \rightharpoonup \omega_\kappa^+, & \text{weakly in } \mathscr{H}_{p(s)}^{\mu,\alpha;\varphi}([0,T]), \\ \omega_n^+ \to \omega_\kappa^+, & \text{strongly in } \mathcal{L}^r([0,T]), \\ \omega_n^+ \to \omega_\kappa^+, & \text{a.e. in } [0,T]. \end{cases}$$

We begin by remarking that from [22, Theorem 2.3], we have

$$\lim_{n\to\infty} \int_0^T g(s) \left|\omega_n^+(s)\right|^{1-\gamma(s)} ds = \int_0^T g(s) \left|\omega_\kappa^+(s)\right|^{1-\gamma(s)} ds.$$

Moreover, from Remark 1.1 and the compact embedding, we have

$$\lim_{n\to\infty} \int_0^T F(s,\omega_n^+(s))ds = \int_0^T F(s,\omega_\kappa^+(s))ds.$$

Now, we shall prove that $\omega_n^+ \to \omega_\kappa^+$ in $\mathscr{H}_{p(s)}^{\mu,\alpha;\varphi}$. Otherwise, suppose that $\omega_n^+ \nrightarrow \omega_\kappa^+$ in $\mathscr{H}_{p(s)}^{\mu,\alpha;\varphi}$. So, from the above equations, we must have

$$\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega_\kappa^+(s)|^{p(s)}ds < \lim_{n\to\infty}\inf \int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega_n^+(s)|^{p(s)}ds. \tag{4.1}$$

For $\omega \in K_\lambda^+$, we have

$$\begin{aligned} Z_\lambda(\omega) \le & \frac{1}{p^-}\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds - \frac{\lambda}{1-\gamma^-}\int_0^T g(s)\,|\omega(s)|^{1-\gamma(s)}\,ds \\ & - \frac{1}{r^+}\int_0^T r(s)F(s,\omega(s))ds. \end{aligned} \tag{4.2}$$

Moreover, from the definition of K_λ^+, for each $\omega \in K_\lambda^+$, we have

$$(p^- - 1)\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds + \lambda\gamma^-\int_0^T g(s)\,|\omega(s)|^{1-\gamma(s)}\,ds - (r^+ - 1)\int_0^T r(s)F(s,\omega(s))ds > 0. \tag{4.3}$$

Now, by multiplying Eq (3.5) by γ^- and by adding it to Eq (4.3), we deduce that

$$\int_0^T r(s)F(s,\omega(s))ds < \left(\frac{p^- - 1 + \gamma^-}{r^+ - 1 + \gamma^-}\right)\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds. \tag{4.4}$$

On the other hand, using (3.5) together with (4.2), we obtain

$$Z_\lambda(\omega) \le \left(\frac{1}{p^-} - \frac{1}{1-\gamma^-}\right)\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds + \left(\frac{1}{1-\gamma^-} - \frac{1}{r^+}\right)\int_0^T r(s)F(s,\omega(s))ds. \tag{4.5}$$

So, by combining Eq (4.4) with Eq (4.5), one has

$$\begin{aligned} Z_\lambda(\omega) &< \left(\frac{1-\gamma^- - p^-}{p^-(1-\gamma^-)}\right)\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}\,ds \\ &\quad + \left(\frac{r^+ - 1 + \gamma^-}{r^+(1-\gamma^-)}\right)\left(\frac{p^- - 1 + \gamma^-}{r^+ - 1 + \gamma^-}\right)\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds \\ &< \left(\frac{1-\gamma^- - p^-}{p^-(1-\gamma^-)}\right) + \left(\frac{p^- - 1 + \gamma^-}{r^+(1-\gamma^-)}\right)\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)}ds \\ &< -\frac{(r^+ - p^-)(p^- + \gamma^- - 1)}{p^- r^+(1-\gamma^-)}\,\|\omega\|^{p^+} < 0. \end{aligned} \tag{4.6}$$

This means that

$$c_\lambda^+ = \inf_{\omega\in K_\lambda^+} Z_\lambda(\omega) < 0. \tag{4.7}$$

Now, from Eq (3.5) and Lemma 3.1, we have

$$Z_\lambda(\omega_n^+) \geq \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega_n^+(s)|^{p(s)}ds + \lambda\left(\frac{1}{r^-} - \frac{1}{1-\gamma^+}\right)\int_0^T g(s)\left|\omega_n^+(s)\right|^{1-\gamma(s)}ds.$$

So, from Eq (4.1), we obtain

$$\begin{aligned} c_\lambda^+ = \lim_{n\to\infty} Z_\lambda(\omega_n^+) &\geq \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\liminf_{n\to\infty}\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega_n^+(s)|^{p(s)}ds \\ &\quad + \lambda\left(\frac{1}{r^-} - \frac{1}{1-\gamma^+}\right)\lim_{n\to\infty}\int_0^T g(s)\left|\omega_n^+(s)\right|^{1-\gamma(s)}ds \\ &> \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega_\kappa^+(s)|^{p(s)}ds \\ &\quad + \lambda\left(\frac{1}{r^-} - \frac{1}{1-\gamma^+}\right)\int_0^T g(s)\left|\omega_\kappa^+(s)\right|^{1-\gamma(s)}ds \\ &\geq \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\min\left(\left\|\omega_\kappa^+\right\|^{p^+}, \left\|\omega_\kappa^+\right\|^{p^-}\right) \\ &\quad + \lambda\left(\frac{1}{r^-} - \frac{1}{1-\gamma^+}\right)\|g\|_{\frac{\beta(\cdot)}{\beta(\cdot)+\gamma(.)-1}}\max\left(\|\omega_\kappa^+\|^{1-\gamma^+}, \|\omega_\kappa^+\|^{1-\gamma^-}\right). \end{aligned}$$

Since $p^- > 1-\gamma^+$, then we get

$$c_\lambda^+ = \inf_{\omega\in K_\lambda^+} Z_\lambda(\omega) > 0,$$

which contradicts Eq (5.1). So, ω_n^+ converges strongly to ω_κ^+. This implies that

$$Z_\lambda(\omega_\kappa^+) = \lim_{n\to\infty} Z_\lambda(\omega_n^+) = \inf_{\omega\in K_\lambda^+} Z_\lambda(\omega).$$

Finally, to deduce that ω_κ^+ is a minimizer for Z_λ on K_λ^+, it suffices to prove that $\omega_\kappa^+ \in K_\lambda^+$. Indeed, since for any integer n we have $\omega_n^+ \in K_\lambda^+$, then we have

$$\xi'_{\lambda,\omega_n^+}(1) = \int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega_n^+(s)|^{p(s)}ds - \lambda\int_0^T g(s)\left|\omega_n^+(s)\right|^{1-\gamma(s)}ds - \int_0^T r(s)F(s,\omega_n^+(s))ds = 0,$$

and

$$\xi''_{\lambda,\omega_n^+}(1) = \int_0^T (p(s)-1)|\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega_n^+(s)|^{p(s)}ds + \lambda\int_0^T \gamma(s)g(s)\left|\omega_n^+(s)\right|^{1-\gamma(s)}ds$$
$$-\int_0^T r(s)(r(s)-1)F(s,\omega_n^+(s))ds > 0.$$

By letting n tend to infinity in the last equations, we deduce that $\xi'_{\lambda,\omega_\kappa^+}(1) = 0$ and $\xi''_{\lambda,\omega_\kappa^+}(1) \geq 0$, which means that $\omega_\kappa^+ \in K_\lambda^+ \cup K_\lambda^0 = K_\lambda^+$, and this finishes the proof. □

Lemma 4.2. *Under hypotheses of Lemma 4.1, we have*

(i) If $\omega \in K_\lambda^+$, then there exist a continuous function δ^+ and $t^+ > 0$ such that $\delta^+(0) = 1$, $\delta^+(s) \to 1$ as $s \to 0$ and for each $|s| > t^+$, we have

$$\delta^+(s)(\omega + s\psi) \in K_\lambda^+,\ \forall\psi \in \mathscr{H}_{p(s)}^{\mu,\alpha;\varphi}([0,T]).$$

(ii) If $\omega \in K_\lambda^-$, then there exist a continuous function δ^- and $t^- > 0$ such that $\delta^-(0) = 1$, $\delta^-(s) \to 1$ as $s \to 0$ and for each $|s| > t^-$, we have

$$\delta^-(s)(\omega + s\psi) \in K_\lambda^+,\ \forall\psi \in \mathscr{H}_{p(s)}^{\mu,\alpha;\varphi}([0,T]).$$

Proof. The proof is very similar to the one in Chung and Ghanmi [4], so we omit it here. □

Proof of Theorem 1.2. We begin the proof by remarking that from Lemma 4.1, the functions ω_κ^+ and ω_κ^- are local minimizers for Z_λ in K_λ^+ and K_λ^-, respectively. On the other hand, from the definition of the sets K_λ^+ and K_λ^-, we see that the functions ω_κ^+ and ω_κ^- are distinct and nontrivial.
Now, let $\psi \in \mathscr{H}_{p(s)}^{\mu,\alpha;\varphi}([0,T])$, then from Lemma 4.2, we have

$$0 \leq \frac{Z_\lambda\left(\delta^+(s)(\omega_\kappa^\pm + s\psi)\right) - Z_\lambda(\omega_\kappa^\pm)}{s}\ \forall|s| < t^\pm.$$

By letting s tend to zero in the last inequality, we obtain

$$0 \leq \int_0^T |\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega_\kappa^\pm(s)|^{p(s)-2}\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega_\kappa^\pm(s)\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\psi(s)ds - \lambda\int_0^T \frac{g(s)}{(\omega_\kappa^\pm)^{\gamma(s)}}\psi(s)ds - \int_0^T f(s,\omega_\kappa^\pm(s))\psi(s)ds.$$

Since ψ is arbitrary, then we can replace the function ψ by $-\psi$ in the last inequality, which means that the last inequality becomes equality that is, $\omega_\kappa^\pm$ is a weak solution for problem (1.3). This finishes the proof of Theorem 1.2. □

5. Example

In this section, we present an example to illustrate the validity of our main result.

Examples. *Let $\alpha \in [0,1]$ and let p, r and γ be three continuous functions on $[0,T]$, such that*

$$0 < \gamma^- \leq \gamma^+ < 1 < p^- \leq p^+ < r^- \leq r^+ < \infty. \tag{5.1}$$

Let g be a nonnegative measurable function on $[0, T]$ *such that*

$$g \in \mathcal{L}^{\frac{\beta(\cdot)}{\beta(\cdot)+\gamma(.)-1}}([0, T]), \tag{5.2}$$

where β *is such that* $1 < \beta^- \leq \beta^+$.

We consider the following problem:

$$\begin{cases} \mathcal{D}_T^{\mu,\alpha,\varphi}|\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s)|^{p(s)-2}\mathcal{D}_{0^+}^{\mu,\alpha,\varphi}\omega(s) = \lambda\frac{g(s)}{\omega^{\gamma(s)}} + a(s)|u(s)|^{r(s)-2}u(s), \quad s \in (0, T), \\ I_{0^+}^{\alpha(\alpha-1);\varphi}(0) = I_T^{\alpha(\alpha-1);\varphi}(T) = 0, \end{cases} \tag{5.3}$$

where λ *is a positive parameter,* $\mu \in (\frac{1}{p^-}, 1]$, *and a is a measurable bounded function on* $[0, T]$.

It is easy to see that the continuity of the functions p, r, *and* γ *together with Eq* (5.1) *implies that hypothesis* (H_2) *is satisfied. On the other hand, one can see that the function* $f(s,t) = a(s)|t|^{r(s)-2}t$ *is positively homogeneous of degree* $r - 1$, *which implies that hypothesis* (H_1) *is also satisfied. Finally, Eq* (5.2) *implies that hypothesis* (H_2) *is satisfied. Hence, all hypotheses of Theorem 1.2 hold. This implies that problem 5.3 admits two nontrivial solutions, provided that* λ *is small enough.*

6. Conclusions

In this paper, we studied a singular problem involving the $p(\cdot)$-Laplace operator and the φ-Hilfer fractional derivative. More precisely, the question of existing solutions is transformed to the question of finding critical points to the functional energy. After that, the fibering map is defined and studied in disjoint sets called Nehari manifold sets. this study yields to the existence of two critical points for the functional energy. Since the functional energy is singular, to prove that these critical points are weak solutions to the studied problem, the implicit functions theorem is used.

In the next paper, I will generalize this study to the double-phase problem as well as the same problem with variable exponent.

Author contributions

Wafa M. Shammakh: Conceptualization, Writing-review and editing, Funding acquisition; Raghad D. Alqarni: Writing-review and editing; Hadeel Z. Alzumi: Writing-review and editing, Funding acquisition; Abdeljabbar Ghanmi: Conceptualization, Resources. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no competing interests.

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