Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(2): 4492–4503.
DOI: 10.3934/math.2025207
Received: 30 December 2024
Revised: 19 February 2025
Accepted: 25 February 2025
Published: 28 February 2025

Research article

Three weak solutions for degenerate weighted quasilinear elliptic equations with indefinite weights and variable exponents

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Abstract: This paper explores the multiplicity of weak solutions to a class of weighted elliptic problems with variable exponents, incorporating a Hardy term and a nonlinear indefinite source term. Using critical point theory applied to the associated energy functional, we establish the existence of at least three weak solutions under general assumptions on the weight function and the nonlinearity. This result has wide applicability, extending existing theories on quasilinear elliptic equations.

Keywords: variational methods; Hardy inequality; degenerate $p(x)$-Laplacian operators
Mathematics Subject Classification: 35J15, 35J20, 35J25

1. Introduction

The presence of singularities and degeneracies in elliptic equations introduces significant challenges in analyzing the behavior of solutions. These singularities, especially near the origin or boundary, can profoundly affect the properties of the operator, making solutions more sensitive to changes in the domain. For instance, when $1 < p < N$, it is known that $\tilde{u}/|y| \in L^p(\mathbb{R}^N)$ if $\tilde{u} \in W^{1,p}(\mathbb{R}^N)$, or $\tilde{u}/|y| \in L^p(\Omega)$ when $\tilde{u} \in W^{1,p}(\Omega)$, where Ω is a bounded domain (see Lemma 2.1 in [12] for further details). In this context, the solution under consideration is $\tilde{u}$, and such behavior leads to the development of Hardy-type inequalities, which are crucial for controlling the singularities of solutions near critical points, particularly when the equation includes singular potential terms (see, e.g., [1, 12, 17, 18, 20]).

Furthermore, the presence of an indefinite weight in the source term creates several challenges, mainly because it can change sign or behave irregularly. This complicates the application of standard methods for proving the existence of solutions, such as ensuring the necessary properties of the energy functional. The irregular behavior of the weight also makes it difficult to use common mathematical tools like Sobolev embeddings and variational methods. To overcome these difficulties, this manuscript employs a more flexible approach based on critical point theory [4], which allows establishing the existence of solutions despite the complexities introduced by the indefinite weight.

Finally, the degeneracy of differential operators, such as p-Laplacian or $p(x)$-Laplacian, when coupled with a weight function $\omega(x)$ inside the divergence, introduces additional complexity to the problem. The presence of $\omega(x)$, whether it is singular or merely bounded, requires a shift in the selection of appropriate functional spaces. Traditional Sobolev spaces like $W^{1,p}(\Omega)$ or $W^{1,p(x)}(\Omega)$ may no longer be adequate in such cases. To properly handle the singularities or degeneracies, it becomes necessary to consider alternative Sobolev spaces, such as $W^{1,p(x)}(\omega,\Omega)$ (see section 2 for the definition of $W^{1,p}(\omega,\Omega)$) , which are specifically designed to accommodate the weight function (see [6] for further details). The most recent contribution to the study of the p Laplacian in a bounded domain and in the whole space can be found in respectively in [5] and [3], furthermore, the degenerate p-Laplacian operator combined with a Hardy potential can be found in [16].

This paper tackles the challenges posed by degeneracy, Hardy-type singularities, and sign-changing source terms, which are common in applied mathematical models, by examining a class of weighted quasilinear elliptic Dirichlet problem involving a variable exponent $p(x)$ and an indefinite source term. The main objective is to prove the existence of three weak solutions, using a critical point theorem introduced by Bonanno and Moranno in [4] while accounting for the complexities introduced by the operator's degeneracy and the singularities in the equation.

This manuscript explores the multiplicity of weak solutions to a weighted elliptic equations of the form:

$$\begin{cases} -\Delta_{p(x),a(x,u)}u + \frac{b(x)|u|^{q-2}u}{|x|^q} = \lambda k(x)|u|^{s(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where λ is a positive parameter, $1 < q < N$, and $\Omega \subset \mathbb{R}^N$ (with $N \geq 2$) is a bounded open subset with smooth boundary $\partial\Omega$. The function u is a solution to a weighted quasilinear elliptic equation involving a variable exponent $p(x) \in C_+(\overline{\Omega})$(see, the beginning of Section 2) and the nonlinear source term of the form $k(x)|u|^{s(x)-2}u$ which involves a weight function $k(x)$ and may exhibit singularities on Ω and can change sign, belongs to a nonstandard Lebesgue space $L^{\gamma(x)}(\Omega)$.

The operator $\Delta_{p(x),a(x,u)}u$ represents a nonlinear generalization of the classical Laplacian, defined by:

$$\Delta_{p(x),a(x,u)}u = \operatorname{div}\left(a(x,u)|\nabla u|^{p(x)-2}\nabla u\right),$$

here $a(x,u)$ denotes a Carathéodory function satisfying the inequality:

$$a_1\omega(x) \leq a(x,u) \leq a_2\omega(x),$$

with a_1, a_2 are two positive constants, the function $\omega(x)$ is assumed to belongs to the local Lebesgue space $L^1_{\text{loc}}(\Omega)$, and it satisfies additional growth conditions, such as $\omega^{-h(x)} \in L^1(\Omega)$, where $h(x)$ satisfies certain bounds related to the variable exponent $p(x)$. Specifically, we assume that

$$(\omega) \quad \omega^{-h(x)} \in L^1(\Omega), \quad \text{for} \quad h(x) \in C(\overline{\Omega}) \quad \text{and} \quad h(x) \in \left(\frac{N}{p(x)}, +\infty\right) \cap \left[\frac{1}{p(x)-1}, +\infty\right).$$

The nonlinearity in the equation involves the functions $k(x)$ and $s(x)$, which are assumed to satisfy the following inequality for almost every $x \in \Omega$

$$(k) \quad 1 < s(x) < p_h(x) < N < \gamma(x),$$

where $p_h(x) = \frac{h(x)p(x)}{h(x)+1}$.

2. Backgrounds and preliminary results

Set, $\mathcal{S}(\Omega)$, the space that contains all measurable functions in Ω and

$$C_+(\overline{\Omega}) = \{p(x) | p(x) \in C(\overline{\Omega}),\ p(x) > 1,\ \forall x \in \overline{\Omega}\},$$

$$p^+ = \max_{x \in \overline{\Omega}} p(x), \quad p^- = \min_{x \in \overline{\Omega}} p(x).$$

For $\tau > 0$, and $p(x) \in C_+(\overline{\Omega})$, we use the following notations

$$\tau^{\hat{p}} = \max\{\tau^{p^-},\ \tau^{p^+}\},\ \ \tau^{\check{p}} = \min\{\tau^{p^-},\ \tau^{p^+}\}.$$

In the sequel, we define the space $L^{p(x)}(\omega, \Omega)$ as follows

$$L^{p(x)}(\omega, \Omega) = \left\{ u \in \mathcal{S}(\Omega) \mid \int_\Omega \omega(x) |u(x)|^{p(x)}\, dx < \infty \right\},$$

where $p(x)$ is a variable exponent, and $\omega(x)$ is a weight function. The space is endowed with a Luxemburg-type norm, given by:

$$\|u\|_{L^{p(x)}(\omega,\Omega)} = \inf \left\{ \nu > 0 \mid \int_\Omega \omega(x) \left| \frac{u(x)}{\nu} \right|^{p(x)} dx \leq 1 \right\}.$$

Next, we define the corresponding variable exponent Sobolev space, which incorporates the variable exponent $p(x)$ in the functional setting.

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) :\ |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|\nabla u\|_{p(x)} + \|u\|_{p(x)},$$

where $\|\nabla u\|_{p(x)} = \||\nabla u|\|_{p(x)}, |\nabla u| = (\sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^2)^{\frac{1}{2}}, \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N}\right)$ is the gradient of u at $(x_1, x_2, ..., x_N)$.

Denote, by

$$W^{1,p(x)}(\omega, \Omega) = \{u \in L^{p(x)}(\Omega) : \omega^{\frac{1}{p(x)}} |\nabla u| \in L^{p(x)}(\Omega)\}$$

the weighted Sobolev space and by $W_0^{1,p(x)}(\omega, \Omega)$ as the closure of $C_0^\infty(\Omega)$ in the space $W^{1,p(x)}(\omega, \Omega)$ endowed with the norm

$$\|u\| = \inf \left\{ \nu > 0 : \int_\Omega \omega(x) \left|\frac{\nabla u(x)}{\nu}\right|^{p(x)} dx \leq 1 \right\}.$$

Lemma 2.1. *[8] If $p_1(x), p_2(x) \in C_+(\overline{\Omega})$ such that $p_1(x) \le p_2(x)$ a.e. $x \in \Omega$, then there exists the continuous embedding $W^{1,p_2(x)}(\Omega) \hookrightarrow W^{1,p_1(x)}(\Omega)$.*

Proposition 2.1 ([9]) *For $p(x) \in C_+(\overline{\Omega}), u, u_n \in L^{p(x)}(\Omega)$, one has*

$$\min\{\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}\} \le \int_\Omega |u(x)|^{p(x)} dx \le \max\{\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}\}.$$

Let $0 < d(x) \in S(\Omega)$, and define the space

$$L^{p(x)}(d,\Omega) := L^{p(x)}_{d(x)}(\Omega) = \left\{ u \in S(\Omega) \mid \int_\Omega d(x)|u(x)|^{p(x)}\, dx < \infty \right\},$$

where $p(x)$ is a variable exponent, and $d(x)$ is a weight function. The space is equipped with a Luxemburg-type norm, defined by

$$\|u\|_{L^{p(x)}_{d(x)}(\Omega)} = \|u\|_{(p(x),d(x))} := \inf\left\{ \nu > 0 \mid \int_\Omega d(x)\left|\frac{u(x)}{\nu}\right|^{p(x)} dx \le 1 \right\}.$$

Proposition 2.2 ([10]) *If $p \in C_+(\overline{\Omega})$. Then*

$$\min\{\|u\|_{(p(x),d(x))}^{p^-}, \|u\|_{(p(x),d(x))}^{p^+}\} \le \int_\Omega d(x)|u(x)|^{p(x)} dx \le \max\{\|u\|_{(p(x),d(x))}^{p^-}, \|u\|_{(p(x),d(x))}^{p^+}\}$$

for every $u \in L^{p(x)}_{d(x)}(\Omega)$ and for a.e. $x \in \Omega$.

Combining Proposition 2.1 with Proposition 2.2, one has

Lemma 2.2. *Let*

$$\rho_\omega(u) = \int_\Omega \omega(x)\left|\nabla u(x)\right|^{p(x)} dx.$$

For $p \in C_+(\overline{\Omega}), u \in W^{1,p(x)}(\omega,\Omega)$, we have

$$\min\{\|u\|^{p^-}, \|u\|^{p^+}\} \le \rho_\omega(u) \le \max\{\|u\|^{p^-}, \|u\|^{p^+}\}.$$

From Proposition 2.4 of [20], if (ω) holds, $W^{1,p(x)}(\omega,\Omega)$ is a reflexive separable Banach space.

From Theorem 2.11 of [15], if (ω) holds, the following embedding

$$W^{1,p(x)}(\omega,\Omega) \hookrightarrow W^{1,p_h(x)}(\Omega) \tag{2.1}$$

is continuous, where

$$p_h(x) = \frac{p(x)h(x)}{h(x)+1} < p(x).$$

Combining (2.1) with Proposition 2.7 and Proposition 2.8 in [11], we get the following embedding

$$W^{1,p(x)}(\omega,\Omega) \hookrightarrow L^{r(x)}(\Omega)$$

is continuous, where

$$1 \le r(x) \le p_h^*(x) = \frac{Np_h(x)}{N - p_h(x)} = \frac{Np(x)h(x)}{Nh(x) + N - p(x)h(x)}.$$

Furthermore, the following embedding

$$W^{1,p(x)}(\omega,\Omega) \hookrightarrow\hookrightarrow L^{t(x)}(\Omega)$$

is compact, when $1 \le t(x) < p_h^*(x)$.

In what follows, and for any $p(x) \in C_+(\overline{\Omega})$, let us denote by $p'(x) := \frac{p(x)}{p(x)-1}$, the conjugate exponent of $p(x)$.

Remark 2.1. Under Condition (k), one has

- $1 < \beta(x) < p_h^*(x)$ for almost every $x \in \Omega$, where $\beta(x) := \frac{\gamma(x)s(x)}{\gamma(x)-s(x)}$, consequently

$$W^{1,p(x)}(\omega,\Omega) \hookrightarrow\hookrightarrow L^{\beta(x)}(\Omega)$$

  is compact.
- $1 < \alpha(x) < p_h^*(x)$ for almost every $x \in \Omega$, where $\alpha(x) = \gamma'(x)s(x)$, consequently

$$W^{1,p(x)}(\omega,\Omega) \hookrightarrow\hookrightarrow L^{\alpha(x)}(\Omega)$$

  is compact.

Lemma 2.3 (Hölder type inequality [2, 11])**.** *Let $p_1, p_2, t \ge 1$ three functions that belong in $\mathcal{S}(\Omega)$ such that*

$$\frac{1}{t(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}, \quad \textit{for almost every } x \in \Omega.$$

If $f \in L^{p_1(x)}(\Omega)$ and $g \in L^{p_2(x)}(\Omega)$, then $fg \in L^{t(x)}(\Omega)$, moreover

$$\|fg\|_{t(x)} \le 2\|f\|_{p_1(x)}\|g\|_{p_2(x)}.$$

Similarly, if $\frac{1}{t(x)} + \frac{1}{p_1(x)} + \frac{1}{p_2(x)} = 1$, for a.e. $x \in \Omega$, then

$$\int_\Omega |f(x)g(x)h(x)|dx \le 3\|f\|_{t(x)}\|g\|_{p_1(x)}\|h\|_{p_2(x)}.$$

Lemma 2.4 ([7])**.** *Let $r_1(x)$ and $r_2(x)$ be measurable functions such that $r_1(x) \in L^\infty(\Omega)$, and $1 \le r_1(x)r_2(x) \le \infty$, for a.e. $x \in \Omega$. Let $w \in L^{r_2(x)}(\Omega)$, $w \ne 0$. Then*

$$\|w\|^{\check{r}_1}_{r_1(x)r_2(x)} \le \||w|^{p(x)}\|_{r_2(x)} \le \|w\|^{\hat{p}}_{r_1(x)r_2(x)}.$$

Let's define the functional $\mathcal{I}_\lambda \colon W_0^{1,p(x)}(\omega,\Omega) \to \mathbb{R}$ as

$$\mathcal{I}_\lambda(u) := \mathcal{L}(u) - \lambda\mathcal{M}(u),$$

where

$$\mathcal{L}(u) := \int_\Omega \frac{a(x,u)}{p(x)}|\nabla u|^{p(x)}dx + \frac{1}{q}\int_\Omega \frac{b(x)|u|^q}{|x|^q}dx, \tag{2.2}$$

and

$$\mathcal{M}(u) := \int_\Omega \frac{1}{s(x)}k(x)|u|^{s(x)}dx. \tag{2.3}$$

It is noted that, based on Remark 2.1 and Lemma 2.4, the aforementioned functionals are both well-defined and continuously Gâteaux differentiable (see [14] for further details). The Gâteaux derivatives are as follows

$$\langle \mathcal{L}'(u), v\rangle = \int_\Omega a(x,u)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla v\,dx + \int_\Omega \frac{b(x)|u|^{q-2}uv}{|x|^q}\,dx,$$

and

$$\langle \mathcal{M}'(u), v\rangle = \int_\Omega k(x)|u|^{s(x)-2}uv\,dx.$$

Furthermore, $\mathcal{M}'(u)$ is compact in the dual space $(W_0^{1,p(x)}(\omega,\Omega))^*$ (see [14]).

$u \in W_0^{1,p(x)}(\omega,\Omega)$ is said to be a weak solution of the problem (1.1) if, the following holds for every $v \in W_0^{1,p(x)}(\omega,\Omega)$.

$$\langle \mathcal{I}'_\lambda(u), v\rangle = \langle \mathcal{L}'(u), v\rangle - \lambda\langle \mathcal{M}'(u), v\rangle = 0.$$

Lemma 2.5. *$\mathcal{L}'$ is a strictly monotone coercive functional that belongs in $(W_0^{1,p(x)}(\omega,\Omega))^*$.*

Proof. For any $u \in W_0^{1,p(x)}(\omega,\Omega)\setminus\{0\}$, by Lemma 2.2, one has

$$\begin{aligned}\mathcal{L}'(u)(u) &= \int_\Omega a(x,u)|\nabla u|^{p(x)-2}\nabla u\nabla u dx + \int_\Omega \frac{b(x)|u|^{q-2}u^2}{|x|^q}dx\\ &\geq a_1\rho_\omega(u)\\ &\geq a_1\cdot\min\{\|u\|^{p^+}, \|u\|^{p^-}\},\end{aligned}$$

thus

$$\lim_{\|u\|\to\infty}\frac{\mathcal{L}'(u)(u)}{\|u\|} \geq a_1\cdot\lim_{\|u\|\to\infty}\frac{\min\{\|u\|^{p^+}, \|u\|^{p^-}\}}{\|u\|} = +\infty,$$

then $\mathcal{L}'$ is coercive in view of $p(x)\in C_+(\overline{\Omega})$.

According to (2.2) of [13], for all $x, y \in \mathbb{R}^N$, there is a positive constant C_p such that

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \geq C_p|x-y|^p,\ \text{if } p\geq 2,$$

and

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \geq \frac{C_p|x-y|^2}{(|x|+|y|)^{2-p}},\ \text{if } 1<p<2,\ \text{and } (x,y)\neq(0,0),$$

where $\langle .,.\rangle$ is the usual inner product in $\mathbb{R}^N$. Thus, for any $u, v \in X$ satisfying $u \neq v$, by standard arguments we can obtain

$$\begin{aligned}\langle \mathcal{L}'(u)-\mathcal{L}'(v), u-v\rangle &= \int_\Omega a(x,u)(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)(\nabla u-\nabla v)dx\\ &\quad + \int_\Omega \frac{b(x)}{|x|^q}(|u|^{q-2}u - |v|^{q-2}v)(u-v))dx\\ &> 0,\end{aligned}$$

hence, one has $\mathcal{L}'$ is strictly monotone in $W_0^{1,p(x)}(\omega,\Omega)$. □

Lemma 2.6. *The functional $\mathcal{L}'$ is a mapping of (S_+)-type, i.e. if $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\omega,\Omega)$, and $\overline{\lim}_{n\to\infty}\langle \mathcal{L}'(u_n) - \mathcal{L}'(u), u_n - u\rangle \le 0$, then $u_n \to u$ in $W_0^{1,p(x)}(\omega,\Omega)$.*

Proof. Let $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\omega,\Omega)$, and $\overline{\lim}_{n\to\infty}\langle \mathcal{L}'(u_n) - \mathcal{L}'(u), u_n - u\rangle \le 0$.

Noting that $\mathcal{L}'$ is strictly monotone in $W_0^{1,p(x)}(\omega,\Omega)$, we have

$$\lim_{n\to\infty}\langle \mathcal{L}'(u_n) - \mathcal{L}'(u), u_n - u\rangle = 0,$$

while

$$\begin{aligned}\langle \mathcal{L}'(u_n) - \mathcal{L}'(u), u_n - u\rangle = &\int_\Omega a(x,u)(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u)dx\\ &+\int_\Omega \Big(\frac{b(x)|u_n|^{q-2}}{|x|^q}u_n(u_n-u) - \frac{b(x)|u|^{q-2}}{|x|^q}u(u_n-u)\Big)dx,\end{aligned}$$

thus we get

$$\overline{\lim}_{n\to\infty}\int_\Omega a(x,u)(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u)dx \le 0.$$

Further, by (1.2) one has

$$\overline{\lim}_{n\to\infty}\int_\Omega \omega(x)(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u)dx \le 0,$$

then $u_n \to u$ in $W_0^{1,p(x)}(\omega,\Omega)$ via Lemma 3.2 in [19]. □

Lemma 2.7. *$\mathcal{L}'$ is an homeomorphism.*

Proof. The strict monotonicity of $\mathcal{L}'$ implies that it is injective. Since $\mathcal{L}'$ is coercive, it is also surjective, and hence $\mathcal{L}'$ has an inverse mapping.

Next, we show that the inverse mapping $(\mathcal{L}')^{-1}$ is continuous.

Let $\tilde{f}_n, \tilde{f} \in (W_0^{1,p(x)}(\omega,\Omega))^*$ such that $\tilde{f}_n \to \tilde{f}$. We aim to prove that $(\mathcal{L}')^{-1}(\tilde{f}_n) \to (\mathcal{L}')^{-1}(\tilde{f})$.

Indeed, let $(\mathcal{L}')^{-1}(\tilde{f}_n) = u_n$ and $(\mathcal{L}')^{-1}(\tilde{f}) = u$, so that $\mathcal{L}'(u_n) = \tilde{f}_n$ and $\mathcal{L}'(u) = \tilde{f}$. By the coercivity of $\mathcal{L}'$, the sequence u_n is bounded. Without loss of generality, assume $u_n \rightharpoonup u_0$, which implies

$$\lim_{n\to\infty}(\mathcal{L}'(u_n) - \mathcal{L}'(u), u_n - u_0) = \lim_{n\to\infty}\Big(\tilde{f}_n - \tilde{f}, u_n - u_0\Big) = 0.$$

Thus, $u_n \to u_0$ because $\mathcal{L}'$ is of (S_+)-type, which ensures that $\mathcal{L}'(u_n) \to \mathcal{L}'(u_0)$. Combining this with $\mathcal{L}'(u_n) \to \mathcal{L}'(u)$, we deduce that $\mathcal{L}'(u) = \mathcal{L}'(u_0)$. Since $\mathcal{L}'$ is injective, it follows that $u = u_0$, and hence $u_n \to u$. Therefore, we have $(\mathcal{L}')^{-1}(\tilde{f}_n) \to (\mathcal{L}')^{-1}(\tilde{f})$, proving that $(\mathcal{L}')^{-1}$ is continuous.

□

The following critical point theorems constitute the principal tools used to obtain our result.

Theorem 2.1 ([4, Theorem 3.6]). *Let X be a reflexive real Banach space and assume the following*

- *$\mathcal{L} : X \to \mathbb{R}$ be a coercive functional that is continuously Gateaux differentiable and weakly lower semicontinuous in the sequential sense*
- *The Gateaux derivative of $\mathcal{L}$ has a continuous inverse on the dual space X^*.*

- *$\mathcal{M} : X \to \mathbb{R}$ is a continuously Gateaux differentiable functional whith a compact Gateaux derivative.*

Furthermore, suppose that

$$(a_0) \quad \inf_X \mathcal{L} = \mathcal{L}(0) = 0 \text{ and } \mathcal{M}(0) = 0.$$

There exist a positive constant d and a point $\overline{v} \in X$ such that $d < \mathcal{L}(\overline{v})$, and the following conditions are satisfied:

$$(a_1) \quad \frac{\sup_{\mathcal{L}(x)<d} \mathcal{M}(x)}{d} < \frac{\mathcal{M}(\overline{v})}{\mathcal{L}(\overline{v})},$$

$$(a_2) \quad \text{For each } \lambda \in \Lambda_d := \left(\frac{\mathcal{L}(\overline{v})}{\mathcal{M}(\overline{v})}, \frac{d}{\sup_{\mathcal{L}(x)\leq d} \mathcal{M}(x)} \right), \text{ the functional } I_\lambda := \mathcal{L} - \lambda\mathcal{M} \text{ is coercive.}$$

Then, for any $\lambda \in \Lambda_d$, $\mathcal{L} - \lambda\mathcal{M}$ has at least three distinct critical points in X.

3. Main results

In this section, a theorem about the existence of at least three weak solutions to the problem (1.1) is obtained.

Recall the Hardy inequality (refer to Lemma 2.1 in [12] for more details), which asserts that for $1 < t < N$, the following inequality holds:

$$\int_\Omega \frac{|u(x)|^t}{|x|^t}\, dx \leq \frac{1}{\mathcal{H}} \int_\Omega |\nabla u|^t\, dx, \quad \forall u \in W_0^{1,t}(\Omega),$$

where the optimal constant $\mathcal{H}$ is given by:

$$\mathcal{H} = \left(\frac{N-t}{t} \right)^t.$$

By combining this with Lemma 2.1 and using the fact that $1 < q < p_h(x) < N$, we deduce the continuous embeddings

$$W_0^{1,p(x)}(\omega, \Omega) \hookrightarrow W_0^{1,p_h(x)}(\Omega) \hookrightarrow W_0^{1,q}(\Omega),$$

which leads to the inequality

$$\int_\Omega \frac{|u(x)|^q}{|x|^q}\, dx \leq \frac{1}{\mathcal{H}} \int_\Omega |\nabla u|^q\, dx, \quad \forall u \in W_0^{1,p(x)}(\omega, \Omega),$$

where $\mathcal{H} = \left(\frac{N-q}{q} \right)^q$.

We are now ready to present our primary result. To this end, we define

$$\tilde{\mathfrak{D}}(x) := \sup \left\{ \tilde{\mathfrak{D}} > 0 \mid B(x, \tilde{\mathfrak{D}}) \subseteq \Omega \right\}$$

for each $x \in \Omega$, here $B(x, \tilde{\mathfrak{D}})$ denotes a ball centered at x with radius $\tilde{\mathfrak{D}}$. It is clear that there exists a point $x^0 \in \Omega$ such that $B(x^0, R) \subseteq \Omega$, where

$$R = \sup_{x \in \Omega} \tilde{\mathfrak{D}}(x).$$

In the remainder, assume that $k(x)$, fulfill this requirement

$$k(x) := \begin{cases} \leq 0, & \text{for } x \in \Omega \setminus B(x^0, R), \\ \geq k_0, & \text{for } x \in B(x^0, \frac{R}{2}), \\ > 0, & \text{for } x \in B(x^0, R) \setminus B(x^0, \frac{R}{2}), \end{cases}$$

where k_0 is a positive constant, the symbol $\tilde{m}$ will represent the constant

$$\tilde{m} = \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma\left(\frac{N}{2}\right)},$$

with Γ denoting the Gamma function.

Theorem 3.1. *Assume that $p^- > s^+$, and, there exist two positive constants d and $\delta > 0$, such that*

$$\frac{1}{p^+}\Big(\frac{2\delta}{R}\Big)^{\check{p}} \|w\|_{L^1(\mathfrak{B})} = d,$$

and

$$A_\delta := \frac{\frac{1}{p^-}\left(\frac{2\delta}{R}\right)^{\hat{p}} \|\omega\|_{L^1(\mathfrak{B})} + \left(\frac{2\delta}{R}\right)^q \frac{\|b\|_\infty}{q\mathcal{H}} \tilde{m}\left(R^N - \left(\frac{R}{2}\right)^N\right)}{\frac{1}{s^+} k_0 \delta^{\check{s}} \tilde{m}\left(\frac{R}{2}\right)^N} < B_d := \frac{d}{\frac{c_{\gamma' s}^{\hat{s}} \|k\|_{\gamma(x)}}{s^-}\big[\left(p^+ d\right)^{\frac{1}{\check{p}}}\big]^{\hat{s}}},$$

then for any $\lambda \in]A_\delta, B_d[$, problem (1.1) has at least three weak solutions.

Proof. It is worth noting that the functional $\mathcal{L}$ and $\mathcal{M}$ associated with problem (1.1) and defined in (2.2) and (2.3), satisfy the regularity assumptions outlined in Theorem 2.1. We will now establish the fulfillment of conditions (a_1) and (a_2). To this end, let's consider

$$\frac{1}{p^+}\Big(\frac{2\delta}{R}\Big)^{\check{p}} \|\omega\|_{L^1(\mathfrak{B})} = d$$

and consider $v_d \in X$ such that

$$v_\delta(x) := \begin{cases} 0 & x \in \Omega \backslash B\left(x^0, R\right) \\ \frac{2\delta}{R}\left(R - \left|x - x^0\right|\right) & x \in \mathfrak{B} := \overline{B}\left(x^0, R\right) \backslash B\left(x^0, \frac{R}{2}\right), \\ \delta & x \in \overline{B}\left(x^0, \frac{R}{2}\right). \end{cases}$$

Then, by the definition of $\mathcal{L}$, we have

$$\begin{aligned} &\frac{1}{p^+}\Big(\frac{2\delta}{R}\Big)^{\check{p}} \|\omega\|_{L^1(\mathfrak{B})} \\ &\quad < \mathcal{L}(v_\delta) \\ &\quad \leq \frac{1}{p^-}\Big(\frac{2\delta}{R}\Big)^{\hat{p}} \|\omega\|_{L^1(\mathfrak{B})} + \Big(\frac{2\delta}{R}\Big)^q \frac{\|b\|_\infty}{q\mathcal{H}} \tilde{m}\left(R^N - \left(\frac{R}{2}\right)^N\right) \end{aligned}$$

Therefore, $\mathcal{L}(v_\delta) > d$. However, it is important to consider the following

$$\mathcal{M}(v_\delta) \geq \int_{B\left(x_0, \frac{R}{2}\right)} \frac{k(x)}{s(x)} |v_\delta|^{\gamma(x)}\, dx \geq \frac{1}{s^+} k_0 \delta^{\check{s}} \tilde{m} \left(\frac{R}{2}\right)^N \tag{3.1}$$

In addition, for each $u \in \mathcal{L}^{-1}(]-\infty, d])$, we have

$$\frac{1}{p^+}\|u\|^{\check{p}} \leq d. \tag{3.2}$$

therefore,

$$\|u\| \leq \left(p^+\mathcal{L}(u)\right)^{\frac{1}{\check{p}}} < \left(p^+ d\right)^{\frac{1}{\check{p}}}.$$

Furthermore, we can deduce using Lemmas 2.3, 2.4 and Remark 2.1 the following

$$\mathcal{M}(u) \leq \frac{1}{s^-}\|k\|_{\gamma(x)} \||u|^{s(x)}\|_{\gamma'(x)} \leq \frac{1}{s^-}\|k\|_{s(x)} (c_{\gamma' s}\|u\|)^{\hat{s}}, \tag{3.3}$$

where $c_{\gamma' s}$ is the constant from the continuous embedding of $W_0^{1,p(x)}(\omega, \Omega)$ into $W^{1,\gamma'(x)s(x)}(\Omega)$.

This leads to the following result

$$\sup_{\mathcal{L}(u)<d} \mathcal{M}(u) \quad \leq \quad \frac{c_{\gamma' s}^{\hat{s}}\|k\|_{\gamma(x)}}{s^-}\left[\left(p^+ d\right)^{\frac{1}{\check{p}}}\right]^{\hat{s}},$$

and

$$\frac{1}{d}\sup_{\mathcal{L}(u)<d} \mathcal{M}(u) < \frac{1}{\lambda}.$$

Furthermore, we can establish the coerciveness of $\mathcal{I}_\lambda$ for any positive value of λ by employing inequality (3.3) once more. This yields the following result

$$\mathcal{M}(u) \leq \frac{c_{\gamma' s}^{\hat{s}}\|k\|_{\gamma(x)}}{s^-}\|u\|^{\hat{s}}.$$

When $\|u\|$ is great enough, the following can be inferred

$$\mathcal{L}(u) - \lambda\mathcal{M}(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \lambda\frac{c_{\gamma' s}^{\hat{s}}\|k\|_{\gamma(x)}}{s^-}\|u\|^{\hat{s}}.$$

By considering the fact that $p^- > s^+$, we can reach the desired conclusion. In conclusion, considering the aforementioned fact that

$$\bar{\Lambda}_d := (A_\delta, B_d) \subseteq \left(\frac{\mathcal{L}(v_\delta)}{\mathcal{M}(v_\delta)}, \frac{d}{\sup_{\mathcal{L}(u)<d}\mathcal{M}(u)}\right),$$

since all assumptions of Theorem 2.1 are fulfilled, it can be deduced that for any $\lambda \in \bar{\Lambda}_d$, the function $\mathcal{L} - \lambda\mathcal{M}$ possesses at least three critical points that belong in $X := W_0^{1,p}(\omega, \Omega)$. Consequently these critical points are exactly weak solutions of problem (1.1). □

Author contributions

Khaled Kefi: Conceptualization, Methodology, Writing–original draft, Supervision; Nasser S. Albalawi: Conceptualization, Methodology, Writing–original draft, Supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, KSA for funding this research work through the project number NBU-FPEJ-2025-1706-01.

Conflict of interest

The authors declare that they have no conflicts of interest.

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