AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(2): 4473–4491.
DOI: 10.3934/math.2025206
Received: 28 November 2024
Revised: 26 January 2025
Accepted: 17 February 2025
Published: 28 February 2025

Research article

Second-order advanced dynamic equations on time scales: Oscillation analysis via monotonicity properties

Samy E. Affan[1,*], Elmetwally M. Elabbasy[2], Bassant M. El-Matary[3,*], Taher S. Hassan[4,5] and Ahmed M. Hassan[1]

[1] Department of Mathematics, Faculty of Science, Benha University, Benha-Kalubia 13518, Egypt
[2] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
[3] Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia
[4] Department of Mathematics, College of Science, University of Hail, Hail 2440, Saudi Arabia
[5] Jadara University Research Center, Jadara University, Jordan

* **Correspondence:** Email: samy.affan@fsc.bu.edu.eg, b.elmatary@qu.edu.sa; Tel: +201275365648.

Abstract: This paper derives new oscillation criteria for a class of second-order non-canonical advanced dynamic equations of the form

$$\left(\zeta(\ell)\varkappa^{\Delta}(\ell)\right)^{\Delta} + q(\ell)\varkappa(\wp(\ell)) = 0.$$

The derived results are based on establishing dynamic inequalities, which lead to novel monotonicity properties of the solutions. These properties are then used to derive new oscillatory conditions. This approach has been successfully applied to difference and differential equations due to the sharpness of its criteria. However, no analogous studies have adopted a similar methodology for dynamic equations on time scales. Furthermore, this study includes examples to illustrate the importance and sharpness of the main results.

Keywords: Kneser-type; sharp; oscillation; non-canonical; advanced; dynamic equations; differential equations; monotonicity properties
Mathematics Subject Classification: 26E70, 34C10, 34K11, 34K42, 34N05

1. Introduction

The theory of time scales, introduced by Stefan Hilger [24], has gained considerable attention in recent years. Its purpose is to unify continuous and discrete analysis within a single mathematical framework, contributing to the elimination of ambiguities from both. Dynamic equations, which model phenomena by combining continuous and discrete domains have attracted significant research interest, especially with the development of new mathematical analysis based on generalized derivatives and integrals defined on time scales. Recent studies have further highlighted the importance of time scale theory and its effectiveness in various applications such as ecological models, stability analysis of dynamical systems, and fault detection in engineering systems, illustrating its broad applicability, superiority, and practical utility in addressing complex problems; see [10, 30, 31, 36, 37]. For instance, [37] explores a nonlinear periodic Gilpin–Ayala predation ecosystem model with infinite distributed delays on time scales; the study establishes conditions for the existence of periodic solutions and analyzes their global stability using Lyapunov theory.

The main concept involves deriving results for dynamic equations where the domain of the unknown function, defined on a time scale that refers to any arbitrary closed subset of real numbers, includes $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{N}$, and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^\ell : \ell \in \mathbb{N}_0, \text{ for } q > 1\}$. When the time scale is chosen as the real numbers, the outcomes are applicable to ordinary differential equations, whereas using the integers yields results for difference equations. This flexibility enables exploring various time scales beyond these two cases.

To develop a thorough understanding, it is important to revisit some key concepts from time scales theory, which are outlined as follows:

The forward and backward jump operators, σ and $\rho : \mathbb{T} \to \mathbb{T}$, are defined respectively by:

$$\sigma(\ell) := \inf\{\xi \in \mathbb{T} : \xi \geq \ell\},$$

and

$$\rho(t) := \sup\{\xi \in \mathbb{T} : \xi \leq t\}.$$

Using these definitions, a point $\ell \in \mathbb{T}$ is categorized as right-scattered, right-dense, left-scattered, or left-dense depending on whether $\sigma(\ell) > \ell$, $\sigma(\ell) = \ell$, $\rho(\ell) < \ell$, or $\rho(t) = \ell$, respectively. Additionally, the graininess function $\mu : \mathbb{T} \to [0, 1)$ is defined by $\mu(\ell) := \sigma(\ell) - \ell$.

The delta (Hilger) derivative of a function ϕ at ℓ is given by:

$$\phi^\Delta(\ell) = \begin{cases} \lim_{\xi \to \ell} \dfrac{\phi(\ell) - \phi(\xi)}{\ell - \xi}, & \text{if } \ell \text{ is right-dense,} \\ \\ \dfrac{\phi(\sigma(\ell)) - \phi(\ell)}{\mu(\ell)}, & \text{if } \ell \text{ is right-scattered.} \end{cases}$$

The delta derivative of the product and quotient of two differentiable functions ϕ and ϑ is expressed as:

$$(\phi\vartheta)^\Delta(\ell) = \phi^\Delta(\ell)\vartheta(\ell) + \phi(\sigma(\ell))\vartheta^\Delta(\ell) = \phi(\ell)\vartheta^\Delta(\ell) + \phi^\Delta(\ell)\vartheta(\sigma(\ell)),$$

and

$$\left(\frac{\phi}{\vartheta}\right)^\Delta(\ell) = \frac{\phi^\Delta(\ell)\vartheta(\ell) - \phi(\ell)\vartheta^\Delta(\ell)}{\vartheta(\ell)\vartheta(\sigma(\ell))}.$$

For further insights; see [8, 9].

Over the past few decades, extensive research and discussion have been conducted regarding the oscillatory behavior of solutions of various classes of difference, differential, and dynamic equations. Numerous papers have delved into this topic, as shown in references [11, 12, 16, 17, 27–29]. However, there is a noticeable lack of recent results concerning the oscillation of advanced equations, as indicated in [4, 13, 15, 22, 23, 25].

Despite advanced dynamic equations holding significant potential for addressing practical challenges ranging from population dynamics and economics to control theory, this field remains relatively underexplored. In this context, some recent research explores the application of dynamic and differential equations to neural networks, non-Newtonian fluid dynamics, and the turbulent flow of polytropic gas in porous media, highlighting critical aspects of such advanced systems; see [10, 34, 35, 37].

This paper aims to investigate oscillation criteria for a class of second-order advanced dynamic equations

$$\Big(\zeta(\ell)\varkappa^{\Delta}(\ell)\Big)^{\Delta} + q(\ell)\varkappa(\wp(\ell)) = 0, \quad \ell \in [\ell_0, \infty)_{\mathbb{T}}. \tag{1.1}$$

The following conditions are assumed to hold throughout the paper:

(H1) $\zeta, q \in C_{rd}([\ell_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$.

(H2) $\wp \in C_{rd}([\ell_0, \infty)_{\mathbb{T}}, \mathbb{T})$, and $\wp(\ell) \geq \ell$.

(H3) $\Psi(\ell) = \int_{\ell}^{\infty} \frac{1}{\zeta(s)} \Delta s < \infty$.

By a solution of (1.1), we mean a function $\varkappa \in C_{rd}[T_{\varkappa}, \infty)_{\mathbb{T}}$, $T_{\varkappa} \in [\ell_0, \infty)_{\mathbb{T}} = [\ell_0, \infty) \cap \mathbb{T}$, which has the property $[r(\ell)\varkappa^{\Delta}(\ell)] \in C^1_{rd}[T_{\varkappa}, \infty)_{\mathbb{T}}$ and satisfies (1.1) on $[T_{\varkappa}, \infty)_{\mathbb{T}}$, where C_{rd} is the set of right-dense continuous functions. Let $\ell_1 \geq \ell_0$ be a given initial point, ϕ be a given initial rd-continuous function on $[\ell_1, \wp(\ell_1)]$, and α be a given initial constant. An initial value problem of (1.1) with initial conditions $\varkappa(\ell) = \phi(\ell)$ for $\ell \in [\ell_1, \wp(\ell_1)]$ and $\varkappa^{\Delta}(\ell) = \alpha$, has a solution that exists on the whole interval $[\ell_0, \infty)_{\mathbb{T}}$; see [3, 8]. A solution $\varkappa$ of (1.1) is classified as oscillatory if it does not remain strictly positive or strictly negative in the long term. Otherwise, it is referred to as nonoscillatory.

One can deal with canonical or non-canonical cases when studying the oscillation behavior of any class of equations. A notable distinction exists between the structures of non-oscillatory solutions in these cases. Canonical equations are characterized by the first derivative of any positive solution $\varkappa(\ell)$, ultimately having only a positive sign. Conversely, non-canonical equations require considering the possibility of both positive and negative signs. However, relatively few studies have addressed the oscillation behavior in non-canonical dynamic equations, as indicated by references [1,2,19–21,23,33].

In previous studies, using the Riccati transformation, the comparison theorem, and related techniques to investigate oscillation theory was common. Most recently, Hassan et al. [18] investigated oscillation criteria of (1.1) based on modified Riccati and Hill-type oscillation. However, relatively few studies on oscillation are based on the monotonicity properties of non-oscillatory solutions, as seen in [7, 11–15, 25–29, 32], with no known results analogous to Eq (1.1) on general time scales. For instance, considering the specific time scale $\mathbb{T} = \mathbb{Z}$, Chatzarakis et al. in [14] investigated new oscillation criteria for the second-order non-canonical delay difference equation

$$\Delta\left(\zeta(n)\Delta\varkappa(n)\right) + q(n)\varkappa(\wp(n)) = 0, \quad n \geq n_0 > 0, \tag{1.2}$$

where $\wp(n)$ is an increasing function.

Additionally, the authors in [13] employed this technique to establish novel oscillation criteria for (1.2), where $\wp(\ell)$ is an advanced argument. Gopalakrishnan et al. [15] improved the aforementioned study by deriving new oscillatory criteria for (1.2) in both cases where $\wp(\ell)$ is a delay or an advanced argument.

Indrajith et al. [25] investigated novel oscillation criteria for a class of second-order quasi-linear advanced difference equations

$$\Delta\left(\zeta(n)(\Delta\varkappa(n))^{\alpha}\right) + q(n)\varkappa^{\alpha}(\wp(n)) = 0, \quad n \geq n_0 > 0,$$

where α is a ratio of odd positive integers, and $\sum_{n=n_0}^{\infty} \frac{1}{\zeta^{1/\alpha}(s)} < \infty$.

For $\mathbb{T} = \mathbb{R}$, Baculíková [5] introduced oscillatory criteria for the second-order non-canonical differential equation

$$\left(\zeta(\ell)\varkappa'(\ell)\right)' + q(\ell)\varkappa(\wp(\ell)) = 0, \quad \ell \geq \ell_0 > 0,$$

with delay/advanced argument $\wp(\ell)$ by establishing monotonicity properties. Additionally, the classical Kneser oscillation theorem was extended by Jadlovská [26] to cover a broader category of second-order half-linear advanced differential equations

$$\left(\zeta(\ell)\left(\varkappa'(\ell)\right)^{\alpha}\right)' + q(\ell)\varkappa^{\alpha}(\wp(\ell)) = 0, \quad \ell \geq \ell_0 > 0, \tag{1.3}$$

where $\Psi(\ell_0) := \int_{\ell_0}^{\infty} \frac{ds}{\zeta^{1/\alpha}(s)} < \infty$.

Baculíková and Džurina [6] used the same technique to derive new oscillatory criteria by linearizing the half-linear second-order canonical differential equations (1.3), where $\wp(\ell) \leq \ell$ and $\lim_{\ell\to\infty} \wp(\ell) = \infty$.

Taking inspiration from this observation, our paper aims to enhance the underdeveloped oscillation theory concerning second-order non-canonical advanced dynamic equations (1.1). The derived oscillation criteria are based on establishing novel dynamic inequalities that yield new monotonicity properties for the non-oscillatory solutions of (1.1). Moreover, the results in this paper enhance and supplement the findings of previous studies, as referenced in [13, 15].

2. Preliminaries

Define

$$\varsigma_* = \liminf_{\ell\to\infty} \zeta(\ell)\Psi(\wp(\ell))\Psi(\sigma(\ell))q(\ell), \tag{2.1}$$

and

$$\gamma_* = \liminf_{\ell\to\infty} \frac{\Psi(\ell)}{\Psi(\wp(\ell))}. \tag{2.2}$$

According to (2.1) and (2.2), for sufficiently large $\ell_1 \geq \ell_0$, we can define arbitrary fixed constants $\varsigma_0 \in (0, \varsigma_*)$ and $\gamma_0 \in [1, \gamma_*)$ for $\ell \geq \ell_1$

$$\zeta(\ell)\Psi(\wp(\ell))\Psi(\sigma(\ell))q(\ell) \geq \varsigma_o, \tag{2.3}$$

and

$$\frac{\Psi(\ell)}{\Psi(\wp(\ell))} \geq \gamma_o. \tag{2.4}$$

The following auxiliary results are essential for proving our main results.

Theorem 2.1. *[8] Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $y : \tilde{\mathbb{T}} \to \mathbb{R}$. If $y^{\tilde{\Delta}}(\nu(\ell))$ and $\nu^{\Delta}(\ell)$ exist for $\ell \in \mathbb{T}^{\kappa}$, then*

$$(y \circ \nu)^{\Delta}(\ell) = y^{\tilde{\Delta}}(\nu(\ell))\nu^{\Delta}(\ell).$$

Where $\tilde{\Delta}$ refers to the derivative on $\tilde{\mathbb{T}}$.

If $\varkappa(\ell)$ represents a non-oscillatory solution of (1.1), and assuming without loss of generality that $\varkappa(\ell)$ is eventually positive; it follows that $\left(\zeta(\ell)\varkappa^{\Delta}(\ell)\right)^{\Delta} < 0$. Consequently, we introduce the following lemma.

Lemma 2.1. *If $\varkappa(\ell)$ is an eventually positive solution of* (1.1)*, then there are two possibilities:*

(I) $\varkappa^{\Delta}(\ell) > 0$ *and* $\left(\zeta(\ell)\varkappa^{\Delta}(\ell)\right)^{\Delta} < 0$ *eventually;*

or

(II) $\varkappa^{\Delta}(\ell) < 0$ *and* $\left(\zeta(\ell)\varkappa^{\Delta}(\ell)\right)^{\Delta} < 0$ *eventually.*

Proof. Since the solution $\varkappa(\ell)$ is eventually positive, it follows that $\varkappa(\wp(\ell)) > 0$ is also eventually positive. Therefore, from Eq (1.1), the function $\zeta(\ell)\varkappa^{\Delta}(\ell)$ is decreasing, which implies that $\zeta(\ell)\varkappa^{\Delta}(\ell)$ eventually has one sign. Due to $\zeta(\ell) > 0$, Lemma 2.1 has been proved. □

Lemma 2.2. *Assume that*

$$\int_{\ell_0}^{\infty} \Psi(s)q(s)\Delta s = \infty, \tag{2.5}$$

holds. Then every eventually positive solution $\varkappa(\ell)$ of (1.1) *satisfies (II), moreover*

(i) $\lim_{\ell\to\infty} \varkappa(\ell) = 0$*;*

(ii) $\varphi(\ell) := \zeta(\ell)\Psi(\ell)\varkappa^{\Delta}(\ell) + \varkappa(\ell)$ *is eventually positive decreasing function;*

(iii) $\dfrac{\varkappa(\ell)}{\Psi(\ell)}$ *is eventually increasing.*

Proof. On the contrary, assume that $\varkappa(\ell)$ is an eventually positive solution of (1.1) satisfies (I) for $\ell \in [\ell_1, \infty)_{\mathbb{T}}$, where $\ell_1 \in [\ell_0, \infty)_{\mathbb{T}}$. Integrating (1.1) from ℓ_1 to ∞, leads to

$$\zeta(\ell_1)\varkappa^{\Delta}(\ell_1) \geq \int_{\ell_1}^{\infty} q(s)\varkappa(\wp(s))\Delta s.$$

Since $\varkappa^{\Delta} > 0$ and $\Psi^{\Delta} < 0$, then there exists a constant $k > 0$ such that $\varkappa(\wp(\ell)) \geq k \geq \Psi(\ell)$, for $\ell \in [\ell_1, \infty)_{\mathbb{T}}$. Combining this with the last inequality, we obtain

$$\zeta(\ell_1)\varkappa^{\Delta}(\ell_1) \geq k\int_{\ell_1}^{\infty} q(s)\Delta s \geq \int_{\ell_1}^{\infty} \Psi(s)q(s)\Delta s.$$

This contradicts (2.5); hence, we conclude that $\varkappa(\ell)$ satisfies (II).

Now, we claim that $\lim_{\ell\to\infty} \varkappa(\ell) = 0$. If not, then there exists a positive constant $\lambda > 0$ such that $\varkappa(\ell) \geq \lambda$ and $\varkappa(\varrho(\ell)) \geq \lambda$ for $\ell \in [\ell_1, \infty)_{\mathbb{T}}$. Integrating (1.1) from ℓ_1 to ℓ, we get

$$-\zeta(\ell)\varkappa^{\Delta}(\ell) \geq \lambda \int_{\ell_1}^{\ell} q(s)\Delta s.$$

Integrating this result from ℓ_1 to ∞ yields

$$\begin{aligned}\varkappa(\ell_1) \geq & \lambda \int_{\ell_1}^{\infty} \frac{1}{\zeta(\xi)} \int_{\ell_1}^{\xi} q(s)\Delta s\Delta\xi \\ = & \lambda \int_{\ell_1}^{\infty} \Psi(s)q(s)\Delta s \to \infty.\end{aligned}$$

This is a contradiction; then we conclude that $\lim_{\ell\to\infty} \varkappa(\ell) = 0$.

Next, we prove (*ii*). Since $\zeta(\ell)\varkappa^{\Delta}(\ell)$ is a decreasing function, it follows that

$$\varkappa(\ell) > -\int_{\ell}^{\infty} \frac{\zeta(s)\varkappa^{\Delta}(s)}{\zeta(s)}\Delta s \geq -\zeta(\ell)\varkappa^{\Delta}(\ell)\int_{\ell}^{\infty} \frac{1}{\zeta(s)}\Delta s = -\zeta(\ell)\Psi(\ell)\varkappa^{\Delta}(\ell).$$

Then $\varphi(\ell) = \zeta(\ell)\Psi(\ell)\varkappa^{\Delta}(\ell) + \varkappa(\ell) > 0$. Using the last inequality with (1.1), it follows that

$$\varphi^{\Delta}(\ell) = \Psi(\sigma(\ell))\left(\zeta(\ell)\varkappa^{\Delta}(\ell)\right)^{\Delta} = -\Psi(\sigma(\ell))q(\ell)\varkappa(\wp(\ell)) < 0. \tag{2.6}$$

Finally, the proof of (*iii*) comes directly by using (*ii*) as follows:

$$\left(\frac{\varkappa(\ell)}{\Psi(\ell)}\right)^{\Delta} = \frac{\Psi(\ell)\varkappa^{\Delta}(\ell) + \frac{\varkappa(\ell)}{\zeta(\ell)}}{\Psi(\ell)\Psi(\sigma(\ell))} > 0,$$

which implies that $\frac{\varkappa(\ell)}{\Psi(\ell)}$ is eventually increasing for $\ell \geqslant \ell_1$. This completes the proof. □

Lemma 2.3. *Assume* (2.5) *holds. If* $\varkappa(\ell)$ *is an eventually positive solution of* (1.1)*, then for any fixed* $\varsigma_0 \in (0, \varsigma_*)$ *the following are satisfied:*

(i) $\zeta(\ell)\Psi(\ell)\varkappa^{\Delta}(\ell) + \varsigma_o\varkappa(\ell) \leq 0$*;*

(ii) $\zeta(\ell)\Psi(\ell)\varkappa^{\Delta}(\ell) + (1 - \varsigma_o)\varkappa(\ell) \geq 0$*;*

(iii) The function $\dfrac{\varkappa(\ell)}{\Psi^{1-\varsigma_o}(\ell)}$ *is eventually increasing.*

Proof. Let $\varkappa(\ell) > 0$ and $\varkappa(\wp(\ell)) > 0$ for all $\ell \in [\ell_1, \infty)_{\mathbb{T}}$, where $\ell_1 \in [\ell_0, \infty)_{\mathbb{T}}$. First, by integrating (1.1) from ℓ_1 to ℓ, we obtain

$$-\zeta(\ell)\varkappa^{\Delta}(\ell) = -\zeta(\ell_1)\varkappa^{\Delta}(\ell_1) + \int_{\ell_1}^{\ell} q(s)\varkappa(\wp(s))\Delta s. \tag{2.7}$$

Using the monotonic properties of $\varkappa(\ell)$ and $\frac{\varkappa(\ell)}{\Psi(\ell)}$, we conclude, for $s \leq \ell$, that

$$\varkappa(\wp(s)) \geq \varkappa(s)\frac{\Psi(\wp(s))}{\Psi(s)} \geq \varkappa(\ell)\frac{\Psi(\wp(s))}{\Psi(s)}. \tag{2.8}$$

From (2.7), (2.8), and using (2.3), leads to

$$-\zeta(\ell)\varkappa^{\Delta}(\ell) \geq -\zeta(\ell_1)\varkappa^{\Delta}(\ell_1) + \varkappa(\ell)\int_{\ell_1}^{\ell} \frac{q(s)\Psi(\wp(s))}{\Psi(s)}\Delta s$$

$$
\begin{aligned}
&\geq -\zeta(\ell_1)\varkappa^{\Delta}(\ell_1)+\varsigma_o\varkappa(\ell)\int_{\ell_1}^{\ell}\frac{1}{\Psi(s)\Psi(\sigma(s))\zeta(s)}\Delta s\\
&= -\zeta(\ell_1)\varkappa^{\Delta}(\ell_1)+\varsigma_o\varkappa(\ell)\int_{\ell_1}^{\ell}\left(\frac{1}{\Psi(s)}\right)^{\Delta}\Delta s\\
&= -\zeta(\ell_1)\varkappa^{\Delta}(\ell_1)+\varsigma_o\varkappa(\ell)\left(\frac{1}{\Psi(\ell)}-\frac{1}{\Psi(\ell_1)}\right)\\
&= -\zeta(\ell_1)\varkappa^{\Delta}(\ell_1)+\varsigma_o\frac{\varkappa(\ell)}{\Psi(\ell)}-\varsigma_o\frac{\varkappa(\ell)}{\Psi(\ell_1)}.
\end{aligned}
$$

Using Lemma 2.2 (i) in the last inequality leads to

$$
-\zeta(\ell_1)\varkappa^{\Delta}(\ell_1)-\varsigma_o\frac{\varkappa(\ell)}{\Psi(\ell_1)}\geq 0,\ \text{for sufficiently large } \ell\geq\ell_1,
$$

which together with the last inequality, implies (i) holds.

Next, to prove (ii) by a simple calculation, one can show Eq (1.1) is equivalent to

$$
\varphi^{\Delta}(\ell)+\Psi(\sigma(\ell))q(\ell)\varkappa(\wp(\ell))=0, \tag{2.9}
$$

where $\varphi(\ell)=\zeta(\ell)\Psi(\ell)\varkappa^{\Delta}(\ell)+\varkappa(\ell)$.

Integrating (2.9) from ℓ to ∞, this with using (2.3), the decreasing fact of $\varphi(\ell)$, and the eventually increasing fact of $\frac{\varkappa(\ell)}{\Psi(\ell)}$, we obtain

$$
\begin{aligned}
\zeta(\ell)\Psi(\ell)\varkappa^{\Delta}(\ell)+\varkappa(\ell)\geq&\int_{\ell}^{\infty}\Psi(\sigma(s))q(s)\varkappa(\wp(s))\Delta s\\
\geq&\varsigma_o\int_{\ell}^{\infty}\frac{\varkappa(\wp(s))}{\zeta(s)\Psi(\wp(s))}\Delta s\\
\geq&\varsigma_o\frac{\varkappa(\ell)}{\Psi(\ell)}\int_{\ell}^{\infty}\frac{1}{\zeta(s)}\Delta s\\
\geq&\varsigma_o\varkappa(\ell),
\end{aligned}
$$

which implies

$$
\zeta(\ell)\Psi(\ell)\varkappa^{\Delta}(\ell)+(1-\varsigma_o)\varkappa(\ell)\geq 0. \tag{2.10}
$$

Finally, by using the Pötzsche chain rule

$$
-\left(\Psi^{1-\varsigma_o}(\ell)\right)^{\Delta}\geq(1-\varsigma_o)\frac{\Psi^{-\varsigma_o}(\ell)}{\zeta(\ell)}. \tag{2.11}
$$

Now, applying Δ-Derivative to the function $\dfrac{\varkappa(\ell)}{\Psi^{1-\varsigma_o}(\ell)}$ in view of (2.10) and (2.11), we obtain

$$
\begin{aligned}
\left(\frac{\varkappa(\ell)}{\Psi^{1-\varsigma_o}(\ell)}\right)^{\Delta}=&\frac{\varkappa^{\Delta}(\ell)\Psi^{1-\varsigma_o}(\ell)-\varkappa(\ell)\left(\Psi^{1-\varsigma_o}(\ell)\right)^{\Delta}}{\Psi^{1-\varsigma_o}(\ell)\Psi^{1-\varsigma_o}(\sigma(\ell))}\\
\geq&\frac{\varkappa^{\Delta}(\ell)\Psi^{1-\varsigma_o}(\ell)+(1-\varsigma_o)\varkappa(\ell)\frac{\Psi^{-\varsigma_o}(\ell)}{\zeta(\ell)}}{\Psi^{1-\varsigma_o}(\ell)\Psi^{1-\varsigma_o}(\sigma(\ell))}\\
=&\frac{\zeta(\ell)\Psi(\ell)\varkappa^{\Delta}(\ell)+(1-\varsigma_o)\varkappa(\ell)}{\zeta(\ell)\Psi(\ell)\Psi^{1-\varsigma_o}(\sigma(\ell))}\geq 0.
\end{aligned}
$$

This completes the proof. □

3. Main results

Theorem 3.1. *Assume* (2.5) *holds. If*

$$\lim_{\ell\to\infty}\frac{\Psi(\ell)}{\Psi(\wp(\ell))}=\infty,$$

then (1.1) *is oscillatory.*

Proof. Let $\varkappa(\ell)$ be a nonoscillatory solution of (1.1), such that $\varkappa(\ell)>0$ and $\varkappa(\wp(\ell))>0$ for all $\ell\in[\ell_1,\infty)_{\mathbb{T}}$, where $\ell_1\in[\ell_0,\infty)_{\mathbb{T}}$. From (1.1) and (2.3), this, with the eventually increasing of $\frac{\varkappa(\ell)}{\Psi^{1-\varsigma_o}(\ell)}$, and the decreasing of $\Psi(\ell)$, we obtain

$$\begin{aligned}-\left(\zeta(\ell)\varkappa^{\Delta}(\ell)\right)^{\Delta}=&q(\ell)\varkappa(\wp(\ell))\\ \geq&\varsigma_o\frac{\varkappa(\wp(\ell))}{\zeta(\ell)\Psi(\wp(\ell))\Psi(\sigma(\ell))}\\ \geq&\varsigma_o\frac{\varkappa(\ell)}{\zeta(\ell)\Psi^{1-\varsigma_o}(\ell)\Psi^{\varsigma_o}(\wp(\ell))\Psi(\sigma(\ell))}\\ \geq&\varsigma_o\gamma_o\frac{\varkappa(\ell)}{\zeta(\ell)\Psi(\ell)\Psi(\sigma(\ell))}.\end{aligned}\tag{3.1}$$

Integrating (3.1) from $\ell_2\geq\ell_1$ to ℓ with considering the fact that $\varkappa(\ell)$ is decreasing, we obtain

$$\begin{aligned}-\zeta(\ell)\varkappa^{\Delta}(\ell)>&\varsigma_o\gamma_o\int_{\ell 2}^{\ell}\frac{\varkappa(s)}{\zeta(s)\Psi(s)\Psi(\sigma(s))}\Delta s\\ \geq&\varsigma_o\gamma_o\varkappa(\ell)\int_{\ell 2}^{\ell}\frac{1}{\zeta(s)\Psi(s)\Psi(\sigma(s))}\Delta s\\ \geq&\varsigma_o\gamma_o\varkappa(\ell)\left(\frac{1}{\Psi(\ell)}-\frac{1}{\Psi(\ell_2)}\right)\\ \geq&\iota\varsigma_o\gamma_o\frac{\varkappa(\ell)}{\Psi(\ell)},\end{aligned}$$

where $0<\iota<1$ for sufficiently large ℓ_2, by choosing arbitrarily large γ_o such that $\gamma_o\geq\frac{1}{\iota\varsigma_o}$, leads to

$$\varphi(\ell)=\zeta(\ell)\Psi(\ell)\varkappa^{\Delta}(\ell)+\varkappa(\ell)<0,$$

which contradicts Lemma 2.2 (*ii*). □

Theorem 3.2. *Assume* (2.5) *holds. If*

$$\varsigma_o>\frac{1}{2},$$

then (1.1) *is oscillatory.*

Proof. Let $\varkappa(\ell)$ be a nonoscillatory solution of (1.1), such that $\varkappa(\ell)>0$ and $\varkappa(\wp(\ell))>0$ for all $\ell\in[\ell_1,\infty)_{\mathbb{T}}$, where $\ell_1\in[\ell_0,\infty)_{\mathbb{T}}$. The proof comes directly from Lemma 2.3 (*i*) and (*ii*), which implies $\varsigma_o\leq\frac{1}{2}$. This leads to a contradiction. □

Theorem 3.3. *Assume* (2.5) *holds. If*

$$\limsup_{\ell\to\infty}\left[\Psi(\sigma(\ell))\int_{\ell 1}^{\ell} q(s)\frac{\Psi(\wp(s))}{\Psi(s)}\Delta s\right] > \frac{1-\varsigma_o}{\gamma_o^{\varsigma_o}}, \tag{3.2}$$

then (1.1) *is oscillatory.*

Proof. Let $\varkappa(\ell)$ be a nonoscillatory solution of (1.1), such that $\varkappa(\ell) > 0$ and $\varkappa(\varrho(\ell)) > 0$ for all $\ell \in [\ell_1, \infty)_{\mathbb{T}}$, where $\ell_1 \in [\ell_0, \infty)_{\mathbb{T}}$. Integrating (1.1) from ℓ_1 to ℓ, we obtain

$$-\zeta(\ell)\varkappa^{\Delta}(\ell) \geq \int_{\ell 1}^{\ell} q(s)\varkappa(\wp(s))\Delta s. \tag{3.3}$$

Since $\dfrac{\varkappa(\ell)}{\Psi^{1-\varsigma_o}(\ell)}$ is eventually increasing and $\varkappa(\ell)$ is decreasing, then for $\wp(s) \geq s$ and $\ell \geq s$, we obtain

$$\frac{\varkappa(\wp(s))}{\Psi^{1-\varsigma_o}(\wp(s))} \geq \frac{\varkappa(s)}{\Psi^{1-\varsigma_o}(s)} \geq \frac{\varkappa(\ell)}{\Psi^{1-\varsigma_o}(s)}.$$

This, along with (2.4), leads to

$$\varkappa(\wp(s)) \geq \varkappa(\ell)\frac{\Psi(\wp(s))}{\Psi(s)}\left(\frac{\Psi(s)}{\Psi(\wp(s))}\right)^{\varsigma_o} \geq \gamma_o^{\varsigma_o}\varkappa(\ell)\frac{\Psi(\wp(s))}{\Psi(s)}.$$

Combining the last inequality and Lemma 2.3 (*ii*) with (3.3) leads to

$$\frac{(1-\varsigma_o)\varkappa(\ell)}{\Psi(\ell)} \geq -\zeta(\ell)\varkappa^{\Delta}(\ell) \geq \gamma_o^{\varsigma_o}\varkappa(\ell)\int_{\ell 1}^{\ell} q(s)\frac{\Psi(\wp(s))}{\Psi(s)}\Delta s. \tag{3.4}$$

Using the decreasing fact of $\Psi(\ell)$ implies

$$\frac{(1-\varsigma_o)}{\gamma_o^{\varsigma_o}} \geq \Psi(\sigma(\ell))\int_{\ell 1}^{\ell} q(s)\frac{\Psi(\wp(s))}{\Psi(s)}\Delta s. \tag{3.5}$$

This contradicts (3.2). □

In case $\varsigma_o \leq \frac{1}{2}$, we can improve the last results by introducing the constant $\varsigma_1 \geq \varsigma_o$ such that

$$\varsigma_1 = \varsigma_o\frac{\gamma_o^{\varsigma_o}}{1-\varsigma_o}.$$

Now, using the last procedure, we can generalize this improvement in the case where $\varsigma_o + \varsigma_i \leq 1$ for $i = 1, 2, 3, \ldots, \kappa - 1$. We introduce the constant $\varsigma_\kappa \geq \varsigma_{\kappa-1}$ such that

$$\varsigma_\kappa = \varsigma_o\frac{\gamma_o^{\varsigma_{\kappa-1}}}{1-\varsigma_{\kappa-1}}. \tag{3.6}$$

This procedure is valid as long as $\varsigma_{\kappa-1} < 1$ and directly provides the following results.

Lemma 3.1. *Assume* (2.5) *holds. If $\varkappa(\ell)$ is an eventually positive solution of* (1.1)*, then for ς_κ the following are satisfied:*

(i) $\zeta(\ell)\Psi(\ell)\varkappa^{\Delta}(\ell) + (1-\varsigma_\kappa)\varkappa(\ell) \geq 0$*;*

(ii) The function $\dfrac{\varkappa(\ell)}{\Psi^{1-\varsigma_\kappa}(\ell)}$ *is eventually increasing.*

Proof. Let $\varkappa(\ell) > 0$ and $\varkappa(\wp(\ell)) > 0$ for all $\ell \in [\ell_1,\infty)_{\mathbb{T}}$, where $\ell_1 \in [\ell_0,\infty)_{\mathbb{T}}$. In order to prove (*i*), it is sufficient to show that

$$\zeta(\ell)\Psi(\ell)\varkappa^{\Delta}(\ell) + (1-\varepsilon_\kappa\varsigma_\kappa)\varkappa(\ell) \geq 0; \lim_{\varsigma_o\to\varsigma_*,\gamma_o\to\gamma_*} \varepsilon_\kappa = 1, \tag{3.7}$$

holds for sufficient large ℓ, where $\varepsilon_o = \dfrac{\varsigma_o}{\varsigma_*} \in (0,1)$ and $\varepsilon_{\kappa+1} = \dfrac{(1-\varsigma_\kappa)\varsigma_o\gamma_o^{\varepsilon_\kappa\varsigma_\kappa}}{(1-\varepsilon_\kappa\varsigma_\kappa)\varsigma_*\gamma_*^{\varsigma_\kappa}} \in (0,1)$.

Certainly, for $\kappa = o$, the inequality (3.7) holds directly from Lemma 2.3 (*ii*). Second, we assume that (3.7) holds for $\kappa = m$, i.e.,

$$\zeta(\ell)\Psi(\ell)\varkappa^{\Delta}(\ell) + (1-\varepsilon_m\varsigma_m)\varkappa(\ell) \geq 0; \lim_{\varsigma_o\to\varsigma_*,\gamma_o\to\gamma_*} \varepsilon_m = 1. \tag{3.8}$$

Finally, for $\kappa = m+1$, we need to prove (3.7) holds. Substituting from (2.3) into (2.6), we get

$$\varphi^{\Delta}(\ell) \leq -\varsigma_o \frac{\varkappa(\wp(\ell))}{\zeta(\ell)\Psi(\wp(\ell))} = -\varsigma_o \frac{\varkappa(\wp(\ell))}{\zeta(\ell)\Psi^{1-\varepsilon_m\varsigma_m}(\wp(\ell))\Psi^{\varepsilon_m\varsigma_m}(\wp(\ell))}.$$

Combining this with (2.4) leads to

$$\varphi^{\Delta}(\ell) \leq -\varsigma_o\gamma_o^{\varepsilon_m\varsigma_m} \frac{\varkappa(\wp(\ell))}{\zeta(\ell)\Psi^{1-\varepsilon_m\varsigma_m}(\wp(\ell))\Psi^{\varepsilon_m\varsigma_m}(\ell)}. \tag{3.9}$$

Similarly, as in the proof of Lemma 2.3 (*iii*), inequality (3.8) directly implies the eventual increase of $\dfrac{\varkappa(\ell)}{\Psi^{1-\varepsilon_m\varsigma_m}(\ell)}$, and this, together with (3.9) and (3.8), leads to

$$\varphi^{\Delta}(\ell) \leq -\varsigma_o\gamma_o^{\varepsilon_m\varsigma_m} \frac{\varkappa(\ell)}{\zeta(\ell)\Psi(\ell)} \leq \frac{\varsigma_o\gamma_o^{\varepsilon_m\varsigma_m}}{(1-\varepsilon_m\varsigma_m)}\varkappa^{\Delta}(\ell) \leq \varepsilon_{m+1}\varsigma_{m+1}\varkappa^{\Delta}(\ell).$$

Integrating from ℓ to ∞ and using the definition of $\varphi(\ell)$, we obtain

$$\zeta(\ell)\Psi(\ell)\varkappa^{\Delta}(\ell) + (1-\varepsilon_{m+1}\varsigma_{m+1})\varkappa(\ell) \geq 0, \text{ where } \lim_{\varsigma_o\to\varsigma_*,\gamma_o\to\gamma_*} \varepsilon_{m+1} = 1.$$

Now, the proof of (*ii*) comes directly from (*i*) as in the proof of Lemma 2.3 (*iii*). □

Theorem 3.4. *Assume* (2.5) *holds. If there exists an integer* $\kappa \in \mathbb{N}$ *such that*

$$\varsigma_o + \varsigma_\kappa > 1,$$

then (1.1) *is oscillatory.*

Proof. Let $\varkappa(\ell)$ be a nonoscillatory solution of (1.1), such that $\varkappa(\ell) > 0$ and $\varkappa(\wp(\ell)) > 0$ for all $\ell \in [\ell_1,\infty)_{\mathbb{T}}$, where $\ell_1 \in [\ell_0,\infty)_{\mathbb{T}}$. The proof comes directly from Lemma 2.3 (*i*) and Lemma 3.1 (*i*), which implies $\varsigma_o + \varsigma_\kappa \leq 1$. This leads to a contradiction. □

Theorem 3.5. *Assume* (2.5) *holds. If there exists an integer* $\kappa \in \mathbb{N}$ *such that*

$$\limsup_{\ell\to\infty}\left[\Psi(\sigma(\ell))\int_{\ell 1}^{\ell} q(s)\frac{\Psi(\wp(s))}{\Psi(s)}\Delta s\right] > \frac{1-\varsigma_\kappa}{\gamma_o^{\varsigma_\kappa}}, \tag{3.10}$$

then (1.1) *is oscillatory.*

Proof. Let $\varkappa(\ell)$ be a nonoscillatory solution of (1.1), such that $\varkappa(\ell) > 0$ and $\varkappa(\wp(\ell)) > 0$ for all $\ell \in [\ell_1,\infty)_{\mathbb{T}}$, where $\ell_1 \in [\ell_0,\infty)_{\mathbb{T}}$. The proof follows the same manner as the proof of Theorem 3.3, with the key difference being the use of Lemma 3.1 (*i*) instead of Lemma 2.3 (*ii*). □

Theorem 3.6. *Assume* (2.5) *holds and* $\gamma_* < \infty$. *If the equation*

$$\beta(1-\beta) = \varsigma_*\gamma_*^{\beta}, \tag{3.11}$$

has no positive solution on $\beta \in (0,1)$, *then* (1.1) *is oscillatory.*

Proof. Let $\varkappa(\ell)$ be a nonoscillatory solution of (1.1), such that $\varkappa(\ell) > 0$ and $\varkappa(\wp(\ell)) > 0$ for all $\ell \in [\ell_1,\infty)_{\mathbb{T}}$, where $\ell_1 \in [\ell_0,\infty)_{\mathbb{T}}$. Since $\varkappa(\ell)$ is decreasing and $\frac{\varkappa(\ell)}{\Psi^{1-\varsigma_\kappa}(\ell)}$ is eventually increasing, it follows that $\varsigma_\kappa < 1$ for any $\kappa \in \mathbb{N}_0$. Hence, the sequence $\{\varsigma_\kappa\}$ defined by (3.6) is increasing and bounded from above; thus, it is convergent, such that $\lim_{\kappa\to\infty}\varsigma_\kappa = y$, where y is the smallest positive root of the equation

$$y(1-y) = \varsigma_*\gamma_*^{y}.$$

This contradiction completes the proof. □

Corollary 3.1. *Assume* (2.5) *holds and* $\gamma_* < \infty$. *If*

$$\varsigma_* > \max\{\omega(\beta) := \beta(1-\beta)\gamma_*^{-\beta} : 0 < \beta < 1\},$$

then (1.1) *is oscillatory.*

By applying straightforward calculations, we obtain the following expression:

$$\max\{\omega(\beta) := \beta(1-\beta)\gamma_*^{-\beta} : 0 < \beta < 1\} = \omega(\beta_{\max}),$$

where

$$\beta_{\max} = \begin{cases} \dfrac{1}{2}, & \text{for} \quad \gamma_* = 1 \\ \dfrac{-\sqrt{r^2+4}+r+2}{2r}, \text{ where } r = \ln\gamma_*, & \text{for} \quad \gamma_* \neq 1. \end{cases}$$

This result leads to the following corollary.

Corollary 3.2. *Assume* (2.5) *holds and* $\gamma_* < \infty$. *If*

$$\varsigma_* > \omega(\beta_{\max}),$$

then (1.1) *is oscillatory.*

4. Illustrative examples

This section presents several examples to illustrate the novelty and sharpness of our derived oscillation criteria. These examples cover special cases of time scales, including advanced dynamic equations on specific time scales. We also compare our results with existing findings in the literature to highlight their sharper and more generalized nature. Additionally, remarks are included to provide insights into the superiority of our results.

Example 4.1. *For any time scale $\mathbb{T}$ that satisfies $\wp(\ell) \geq \ell\sigma(\ell)$ for all $\ell \in \mathbb{T}$. Consider the second-order advanced dynamic equation*

$$\left(\ell\sigma(\ell)\varkappa^{\Delta}(\ell)\right)^{\Delta} + q_o\frac{\wp(\ell)}{\ell}\varkappa(\wp(\ell)) = 0, \quad q_o > 0, \quad \ell \geq 1. \tag{4.1}$$

Here $\zeta(\ell) = \ell\sigma(\ell)$, $q(\ell) = q_o\frac{\wp(\ell)}{\ell}$, $\wp(\ell) \geq \ell\sigma(\ell)$, and $\Psi(\ell) = \frac{1}{\ell}$. Also,

$$\lim_{\ell\to\infty}\frac{\Psi(\ell)}{\Psi(\wp(\ell))} = \lim_{\ell\to\infty}\frac{\wp(\ell)}{\ell} \geq \lim_{\ell\to\infty}\sigma(\ell) = \infty,$$

then by Theorem 3.1, Eq (4.1) *is oscillatory.*

To illustrate the sharpness of our oscillation results for Eq (4.1), *Figure 1 presents some numerical solutions of Eq* (4.1) *for particular time scales $\mathbb{T} = q^{\mathbb{N}_0}$.*

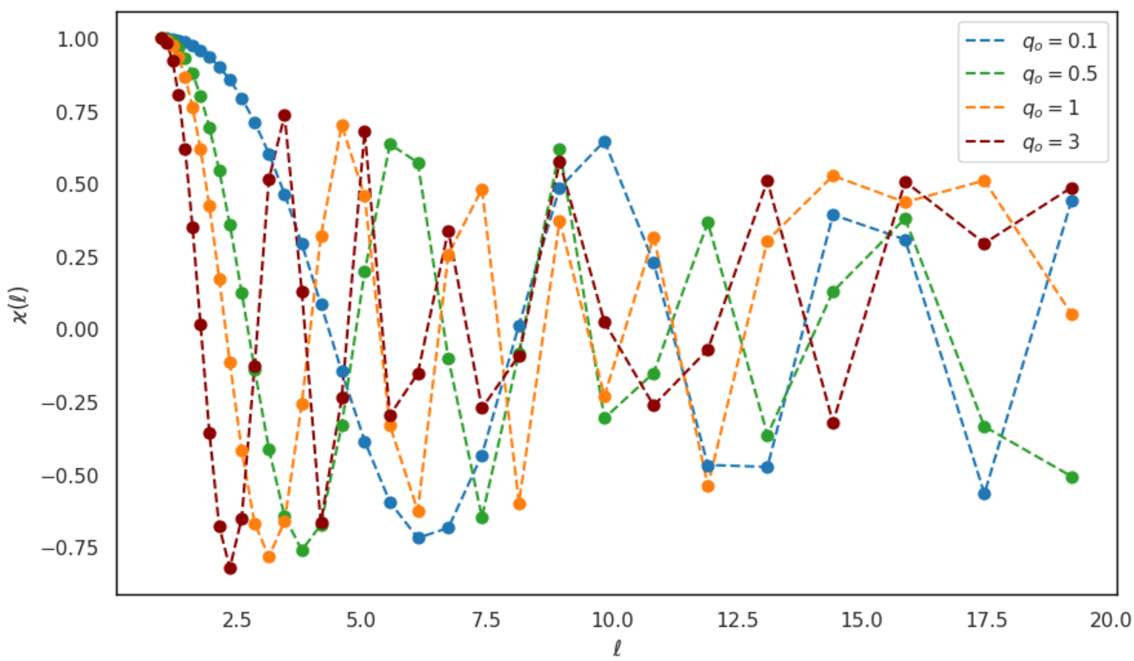

Figure 1. Some numerical solutions of Eq (4.1) for particular time scales $\mathbb{T} = q^{\mathbb{N}_0}$, where $q = 1.1$ and $\wp(\ell) = \ell\sigma(\ell) = q\ell^2$.

Remark 4.1. *Applying [18, Theorem 4] demonstrates that Eq* (4.1) *is oscillatory for $q_o > \frac{1}{4}$. Therefore, our criterion provides a sharper tool for testing oscillation.*

Example 4.2. *For any time scale $\mathbb{T}$ that satisfies $\wp(\ell) \geq \ell\sigma(\ell)$ for all $\ell \in \mathbb{T}$. Consider the second-order advanced dynamic equation*

$$\left(\ell\sigma(\ell)\varkappa^{\Delta}(\ell)\right)^{\Delta} + \ell\varkappa(\wp(\ell)) = 0, \quad \ell \geq 1. \tag{4.2}$$

Here $\zeta(\ell) = \ell\sigma(\ell)$, $q(\ell) = \ell$, $\wp(\ell) \geq \ell\sigma(\ell)$, and $\Psi(\ell) = \frac{1}{\ell}$. Also,

$$\lim_{\ell\to\infty} \frac{\Psi(\ell)}{\Psi(\wp(\ell))} = \lim_{\ell\to\infty} \frac{\wp(\ell)}{\ell} \geq \lim_{\ell\to\infty} \sigma(\ell) = \infty,$$

then, by Theorem 3.1, we conclude that Eq (4.2) *is oscillatory.*

Example 4.3. *For any time scale $\mathbb{T}$ that satisfies $b\ell \in \mathbb{T}$ for all $\ell \in \mathbb{T}$, where b is a constant. Consider the second-order advanced dynamic equation*

$$\left(\ell\sigma(\ell)\varkappa^{\Delta}(\ell)\right)^{\Delta} + q_o\varkappa(b\ell) = 0, \quad \text{where } b \geq 1, \quad \ell \geq 0. \tag{4.3}$$

Here $\zeta(\ell) = \ell\sigma(\ell)$, $q(\ell) = q_o > 0$, $\wp(\ell) = b\ell$, and $\Psi(\ell) = \frac{1}{\ell}$. Also, $\varsigma_* = \frac{q_o}{b}$ and $\gamma_* = b$. For $b = 1$, by Corollary 3.2, the second-order advanced dynamic equation

$$\left(\ell\sigma(\ell)\varkappa^{\Delta}(\ell)\right)^{\Delta} + q_o\varkappa(\ell) = 0, \quad \text{where } b \geq 1, \quad \ell \geq 0,$$

is oscillatory if $q_o > \frac{1}{4}$. Otherwise, in case $b > 1$, by Corollary 3.2, Eq (4.3) *is oscillatory if*

$$q_o > \frac{\sqrt{r^2+4}-2}{r^2} e^{\left(\sqrt{r^2+4}-2+r\right)/2}, \quad \text{where } r = \ln(b).$$

Example 4.4. *A special case of* (4.3) *occurs when $b = 2$. Consider the second-order advanced dynamic equation:*

$$\left(\ell\sigma(\ell)\varkappa^{\Delta}(\ell)\right)^{\Delta} + q_o\varkappa(2\ell) = 0, \quad \text{where } q_o > 0, \quad \ell \geq 0. \tag{4.4}$$

According to Theorem 3.2, Eq (4.4) *is oscillatory if $q_o > 1$. Otherwise, in the case where $q_o \leq 1$, by a simple calculation for* $0.364174 \leq q_o \leq 1$, *we can determine*

$$\varsigma_\kappa = q_o \frac{2^{\varsigma_{\kappa-1}-1}}{1-\varsigma_{\kappa-1}}$$

such that $\varsigma_o + \varsigma_\kappa > 1$. Therefore, by Theorem 3.4, Eq (4.4) *is oscillatory if $q_o \geq 0.364174$, as shown in Table 1. Furthermore, by Corollary 3.2, Eq* (4.4) *is oscillatory if $q_o > 0.364173$.*

Table 1. Numerical verification of oscillation of Eq (4.4) via Theorem 3.4.

q_o	κ	ς_κ	$\varsigma_o + \varsigma_\kappa$
1	1	1.41421	1.91421
0.8	1	0.879672	1.27967
0.6	2	0.915536	1.21554
0.4	9	1.33204	1.53204
0.38	14	0.841967	1.03197
0.37	26	1.61925	1.80425
0.365	75	1.21863	1.40113
0.3645	118	1.4952	1.67745
0.364178	975	0.988949	1.17104
0.364174	2197	0.849027	1.03111

Remark 4.2. *Applying [18, Theorem 4] shows that Eq* (4.4) *is oscillatory for* $q_o \geq 0.421$. *Thus, our result is stronger for testing oscillation.*

Remark 4.3. *For the special case of* (4.4) *when* $\mathbb{T} = \mathbb{Z}$, *applying [13, Theorem 2.6] implies that the advanced difference equation*

$$\Delta\left(\ell(\ell+1)\Delta\varkappa_\ell\right) + q_o\varkappa_{2\ell} = 0, \quad \ell \geq 0, \tag{4.5}$$

is oscillatory for $q_o \geq \frac{3}{4}$. *Furthermore, by [25, Theorem 2.4], Eq* (4.5) *is oscillatory if* $q_o > \frac{1}{\sqrt{2}}$. *Consequently, our criteria provide more precise and stringent oscillation results compared to the previously established bounds.*

Example 4.5. *For the discrete time scale* $\mathbb{T} = \mathbb{Z}$. *Consider the second-order advanced difference equation*

$$\Delta\Big(2^\ell\Delta\varkappa_\ell\Big) + \frac{2^\ell}{3}\varkappa_{\ell+1} = 0, \quad \ell \geq 0. \tag{4.6}$$

Here $\zeta(\ell) = 2^\ell$, $q(\ell) = \dfrac{2^\ell}{3}$, $\wp(\ell) = \ell + 1$, $\Psi(\ell) = 2^{1-\ell}$, $\varsigma_* = \dfrac{1}{3}$, *and* $\gamma_* = 2$. *Now, using the iterative formula*

$$\varsigma_\kappa = \varsigma_o\frac{2^{\varsigma_{\kappa-1}}}{1-\varsigma_{\kappa-1}},$$

we can determine $\varsigma_o + \varsigma_2 = 1.72735 > 1$. *Therefore, by Theorem 3.4, Eq* (4.6) *is oscillatory.*

Example 4.6. *For the real time scale* $\mathbb{T} = \mathbb{R}$. *Consider the second-order advanced differential equation*

$$\Big(\ell^2\varkappa'(\ell)\Big)' + \frac{q_o\ell + \ln(\ell)}{\ell}\varkappa(5\ell) = 0, \quad \ell \geq 1. \tag{4.7}$$

Here, $\zeta(\ell) = \ell^2$, $q(\ell) = \dfrac{q_o\ell + \ln(\ell)}{\ell}$, $\wp(\ell) = 5\ell$, *and* $\Psi(\ell) = \dfrac{1}{\ell}$. *Also,* $\varsigma_* = \dfrac{q_o}{5}$ *and* $\gamma_* = 5$. *By Theorem 3.2, Eq* (4.7) *is oscillatory if* $q_o > \dfrac{5}{2}$. *Otherwise, in case* $q_0 \leq \dfrac{5}{2}$ *by a simple calculation for*

$0.6501235 \le q_o \le \frac{5}{2}$, *we can determine* $\varsigma_\kappa = q_o \frac{5^{\varsigma_{\kappa-1}-1}}{1-\varsigma_{\kappa-1}}$. *Therefore,* $\varsigma_o + \varsigma_\kappa > 1$, *then by Theorem 3.4, Eq* (4.7) *is oscillatory if* $q_o \ge 0.6501235$. *Moreover, condition* (3.10) *takes the form*

$$\limsup_{\ell\to\infty}\left[\frac{1}{5\ell}\int_1^\ell\left(q_o + \frac{\ln(s)}{s}\right)ds\right] > \frac{1-\varsigma_\kappa}{5^{\varsigma_\kappa}},$$

which holds. Consequently, by Theorem 3.5, Eq (4.7) *is oscillatory if* $q_o \ge 0.6501235$. *Furthermore, by Corollary 3.2, Eq* (4.7) *is oscillatory if* $q_o > 0.650123$.

The oscillatory behavior of Eq (4.7) *is numerically verified using Theorems 3.4 and 3.5, as presented in Table 2. Additionally, Figure 2 illustrates some numerical solutions of Eq* (4.7)*, highlighting the accuracy of our oscillation results.*

Table 2. Numerical verification of oscillation of Eq (4.7) via Theorems 3.4 and 3.5.

q_o	Theorem 3.4 criterion Holds for			Theorem 3.5 criterion Holds for	
	κ	ς_κ	$\varsigma_o + \varsigma_\kappa$	κ	ς_κ
2.5	1	2.23607	2.73607	1	2.23607
1.7	1	0.890399	1.2304	1	0.890399
1.3	2	1.31735	1.57735	1	0.533917
0.9	4	1.12082	1.30082	3	0.586952
0.7	12	1.88673	2.02673	11	0.751349
0.66	30	2.73882	2.87082	29	0.819713
0.6505	163	0.947643	1.07774	163	0.947643
0.6502	365	1.13734	1.26738	364	0.666022
0.65015	619	2.20295	2.33298	618	0.789636
0.650124	3221	1.74472	1.87474	3220	0.750579
0.6501235	4546	1.07187	1.2019	4545	0.65301

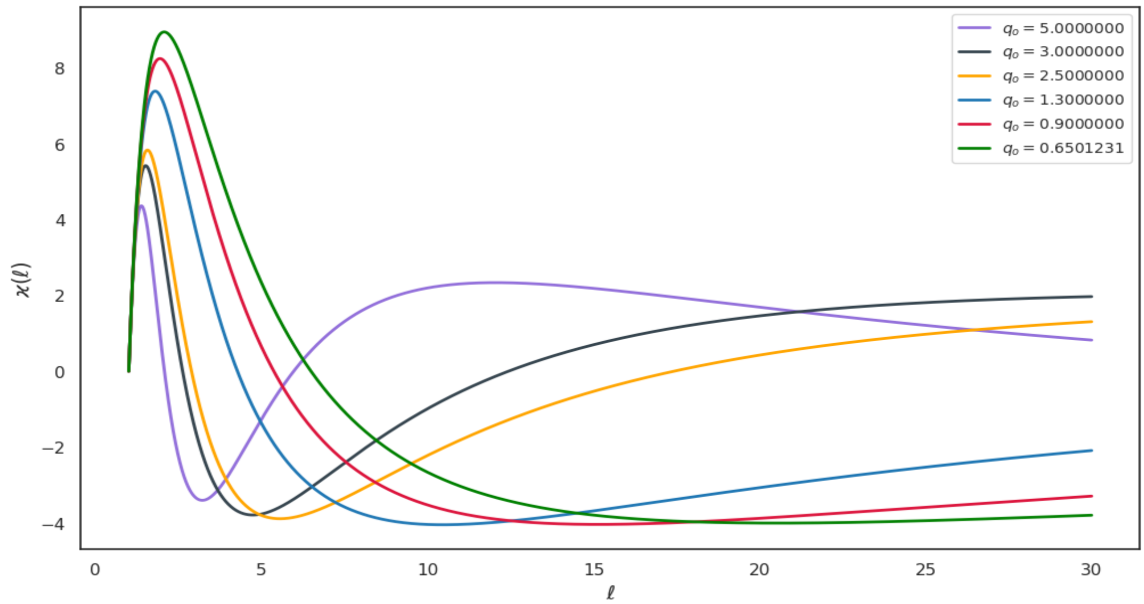

Figure 2. Some numerical solutions of Eq (4.7).

Remark 4.4. *Applying [5, Theorem 4.3] shows that Eq* (4.7) *is oscillatory for* $q_o \geq 0.6501235$. *Thus, our results provide stronger assurance for oscillation.*

5. Conclusions

This paper investigated Kneser-type oscillation for a class of second-order noncanonical dynamic equations with an advanced argument, a topic that has received limited attention in the time scales literature. The derived oscillation criteria are sharp and refined; some previous studies have been extended. For example, our findings generalize and improve the results presented in [13, 15] and extend the special case where $\alpha = 1$, as discussed in [25, 26]. Moreover, when $\mathbb{T} = \mathbb{R}$, our results differ from those in [6], where $\alpha = 1$, since we focused on the case where $\wp(\ell)$ represents an advanced argument. Additionally, our results are sharper than those in [18], which are based on modified Riccati and Hill-type oscillation. Furthermore, our findings are valid for all time scales, such as $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = q^{\mathbb{N}_0}$ where $q > 1$, etc. An interesting extension of this approach would be to consider the second-order half-linear dynamic equation with an advanced argument

$$\left(\zeta(\ell)\left(\varkappa^{\Delta}(\ell)\right)^{\alpha}\right)^{\Delta} + q(\ell)\varkappa^{\alpha}(\wp(\ell)) = 0, \quad \ell \in [\ell_0, \infty)_{\mathbb{T}},$$

where α is a ratio of odd positive integers.

Author contributions

Samy E. Affan: Writing-original draft, Conceptualization, Formal analysis, Making major revisions; Elmetwally M. Elabbasy: Supervision, Writing-original draft, Methodology, Visualization; Bassant M. El-Matary: Writing-original draft, Validation, Data curation; Taher S. Hassan: Supervision, Writing-original draft, Formal analysis, Writing-review and editing; Ahmed M. Hassan: Supervision, Writing-original draft, Methodology, Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2025).

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

References

1. S. Abbas, S. Grace, J. Graef, S. Negi, Oscillation of second-order non-canonical non-linear dynamic equations with a sub-linear neutral term, *Differ. Equ. Dyn. Syst.*, **32** (2024), 819–829. http://dx.doi.org/10.1007/s12591-022-00592-0
2. S. Affan, T. Hassan, E. Elabbasy, E. Saied, A. Hassan, Oscillatory behavior of second-order nonlinear delay dynamic equations with multiple sublinear neutral terms utilizing canonical transformation, *Math. Meth. Appl. Sci.*, **48** (2025), 1589–1600. http://dx.doi.org/10.1002/mma.10397
3. R. Agarwal, S. Grace, D. O'Regan, *Oscillation theory for second order dynamic equations*, CRC Press, 2003. https://doi.org/10.4324/9780203222898
4. B. Baculíková, Oscillation of second-order nonlinear noncanonical differential equations with deviating argument, *Appl. Math. Lett.*, **91** (2019), 68–75. http://dx.doi.org/10.1016/j.aml.2018.11.021
5. B. Baculíková, Oscillatory behavior of the second order noncanonical differential equations, *Electron. J. Qual. Theory Differ. Equ.*, **89** (2019), 1–11. http://dx.doi.org/10.14232/ejqtde.2019.1.89
6. B. Baculíková, J. Dzurina, Oscillatory criteria via linearization of half-linear second order delay differential equations, *Opuscula Math.*, **40** (2020), 523–536. https://doi.org/10.7494/OpMath.2020.40.5.523
7. M. Bohner, S. Grace, I. Jadlovská, Sharp oscillation criteria for second-order neutral delay differential equations, *Math. Meth. Appl. Sci.*, **43** (2020), 10041–10053. http://dx.doi.org/10.1002/mma.6677
8. M. Bohner, A. Peterson, *Dynamic equations on time scales: An introduction with applications*, Springer Science & Business Media, 2001. https://doi.org/10.1007/978-1-4612-0201-1
9. M. Bohner, A. Peterson, *Advances in dynamic equations on time scales*, Springer Science & Business Media, 2002. https://doi.org/10.1007/978-0-8176-8230-9
10. Z. Cai, L. Huang, Z. Wang, X. Pan, S. Liu, Periodicity and multi-periodicity generated by impulses control in delayed cohen–grossberg-type neural networks with discontinuous activations, *Neural Netw.*, **143** (2021), 230–245. http://dx.doi.org/10.1016/j.neunet.2021.06.013
11. G. Chatzarakis, S. Grace, I. Jadlovská, On the sharp oscillation criteria for half-linear second-order differential equations with several delay arguments, *Appl. Math. Comput.*, **397** (2021), 125915. http://dx.doi.org/10.1016/j.amc.2020.125915
12. G. Chatzarakis, S. Grace, I. Jadlovská, A sharp oscillation criterion for second-order half-linear advanced differential equations, *Acta Math. Hungar.*, **163** (2021), 552–562. http://dx.doi.org/10.1007/s10474-020-01110-w
13. G. Chatzarakis, N. Indrajith, S. Panetsos, E. Thandapani, Improved oscillation criteria of second-order advanced non-canonical difference equations, *Aust. J. Math. Anal. Appl.*, **19** (2022), 5.
14. G. Chatzarakis, N. Indrajith, E. Thandapani, K. Vidhyaa, Oscillatory behavior of second-order non-canonical retarded difference equations, *Aust. J. Math. Anal. Appl.*, **18** (2021), 20.

15. P. Gopalakrishnan, A. Murugesan, C. Jayakumar, Oscillation conditions of the second order noncanonical difference equations, *J. Math. Comput. SCI-JM.*, **25** (2022), 351–360. http://dx.doi.org/10.22436/jmcs.025.04.05
16. A. Hassan, S. Affan, New oscillation criteria and some refinements for second-order neutral delay dynamic equations on time scales, *J. Math. Comput. SCI-JM.*, **28** (2023), 192–202. http://dx.doi.org/10.22436/jmcs.028.02.07
17. A. Hassan, S. Affan, Oscillation criteria for second-order delay dynamic equations with a sub-linear neutral term on time scales, *Filomat*, **37** (2023), 7445–7454. http://dx.doi.org/10.2298/FIL2322445H
18. A. Hassan, O. Moaaz, S. Askar, A. Alshamrani, S. Affan, Enhanced oscillation criteria for non-canonical second-order advanced dynamic equations on time scales, *Symmetry*, **16** (2024), 1457. http://dx.doi.org/10.3390/sym16111457
19. A. Hassan, I. Odinaev, T. Hassan, Oscillatory behavior of noncanonical quasilinear second-order dynamic equations on time scales, *J. Math.*, **2023** (2023), 5585174. http://dx.doi.org/10.1155/2023/5585174
20. A. Hassan, H. Ramos, O. Moaaz, Second-order dynamic equations with noncanonical operator: Oscillatory behavior, *Fractal Fract.*, **7** (2023), 134. http://dx.doi.org/10.3390/fractalfract7020134
21. T. Hassan, M. Bohner, I. Florentina, A. Abdel Menaem, M. Mesmouli, New criteria of oscillation for linear sturm–liouville delay noncanonical dynamic equations, *Mathematics*, **11** (2023), 4850. http://dx.doi.org/10.3390/math11234850
22. T. Hassan, C. Cesarano, M. Mesmouli, H. Zaidi, I. Odinaev, Iterative hille-type oscillation criteria of half-linear advanced dynamic equations of second order, *Math. Meth. Appl. Sci.*, **74** (2024), 5651–5663. http://dx.doi.org/10.1002/mma.9883
23. T. Hassan, R. El-Nabulsi, N. Iqbal, A. Abdel Menaem, New criteria for oscillation of advanced noncanonical nonlinear dynamic equations, *Mathematics*, **12** (2024), 824. http://dx.doi.org/10.3390/math12060824
24. S. Hilger, Analysis on measure chains — a unified approach to continuous and discrete calculus, *Results Math.*, **18** (1990), 18–56. https://doi.org/10.1007/BF03323153
25. N. Indrajith, J. Graef, E. Thandapani, Kneser-type oscillation criteria for second-order half-linear advanced difference equations, *Opuscula Math.*, **42** (2022), 55–64. http://dx.doi.org/10.7494/OpMath.2022.42.1.55
26. I. Jadlovská, Oscillation criteria of kneser-type for second-order half-linear advanced differential equations, *Appl. Math. Lett.*, **106** (2020), 106354. http://dx.doi.org/10.1016/j.aml.2020.106354
27. I. Jadlovská, New criteria for sharp oscillation of second-order neutral delay differential equations, *Mathematics*, **9** (2021), 2089. http://dx.doi.org/10.3390/math9172089
28. C. Jayakumar, A. Murugesan, Oscillation result for half-linear delay difference equations of second-order, *Malaya J. Mat.*, **9** (2021), 1153–1159.
29. C. Jayakumar, S. Santra, D. Baleanu, R. Edwan, V. Govindan, A. Murugesan, et al., Oscillation result for half-linear delay difference equations of second-order, *Math. Biosci. Eng.*, **19** (2022), 3879–3891. http://dx.doi.org/10.3934/mbe.2022178

30. M. Liu, H. Shi, Exponential stability of dynamical systems on time scales with application to multi-agent systems, *Axioms*, **13** (2024), 100. http://dx.doi.org/10.3390/axioms13020100
31. J. Shi, S. Gu, S. Xing, C. Chen, Dynamic event-triggered fault detection for multi time scale systems: Application to grid connected converters, *J. Frank. Inst.*, **361** (2024), 106738. http://dx.doi.org/10.1016/j.jfranklin.2024.106738
32. S. Shi, Z. Han, A new approach to the oscillation for the difference equations with several variable advanced arguments, *J. Appl. Math. Comput.*, **68** (2022), 2083–2096. http://dx.doi.org/10.1007/s12190-021-01605-x
33. A. Soliman, A. Hassan, S. Affan, Oscillatory behavior of second order delay dynamic equa-tions with a sub-linear neutral term on time scales, *J. Math. Comput. SCI-JM.*, **24** (2022), 97–109. http://dx.doi.org/10.22436/jmcs.024.02.01
34. Y. Tian, X. Su, C. Shen, X. Ma, Exponentially extended dissipativity-based filtering of switched neural networks, *Automatica*, **161** (2024), 111465. http://dx.doi.org/10.1016/j.automatica.2023.111465
35. H. Zhao, J. Zhang, J. Li, Decay estimates of solution to the two-dimensional fractional quasi-geostrophic equation, *Math. Meth. Appl. Sci.*, **47** (2024), 4043–4057. http://dx.doi.org/10.1002/mma.9802
36. K. Zhao, Existence and stability of a nonlinear distributed delayed periodic ag-ecosystem with competition on time scales, *Axioms*, **12** (2023), 315. https://doi.org/10.3390/axioms12030315
37. K. Zhao, Asymptotic stability of a periodic ga-predation system with infinite distributed lags on time scales, *Int. J. Control*, **97** (2024), 1542–1552. http://dx.doi.org/10.1080/00207179.2023.2214251

AIMS Press