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Research article

Distance 2-restricted optimal pebbling in cycles

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Abstract: Let G be a graph and let δ be a distribution of pebbles on G. A pebbling move on the graph G consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. Given a positive integer d, if we can move pebbles to any target vertex v in G only from the vertices in the set $N_d[v] = \{u \in V(G) : d(u, v) \leq d\}$ by pebbling moves, where $d(u, v)$ is the distance between u and v, then such a graph pebbling played on G is said to be distance d-restricted. For each target vertex $v \in V(G)$, we use $m(\delta, d, v)$ to denote the maximum number of pebbles that can be moved to v only from the vertices in the set $N_d[v]$. If $m(\delta, d, v) \geq t$ for each $v \in V(G)$, then we say that δ is (d, t)-solvable. The optimal (d, t)-pebbling number of G, denoted by $\pi^*_{(d,t)}(G)$, is the minimum number of pebbles needed so that there is a (d, t)-solvable distribution of G. In this article, we study distance 2-restricted pebbling in cycles and show that for any n-cycle C_n with $n \geq 6$, $\pi^*_{(2,t)}(C_n) = \pi^*_{(2,t-10)}(C_n) + 4n$ for $t \geq 13$. It follows that if $n \geq 6$, then $\pi^*_{(2,10k+r)}(C_n) = \pi^*_{(2,r)}(C_n) + 4kn$ for $k \geq 1$ and $3 \leq r \leq 12$. Consequently, for $n \geq 6$, the problem of determining the exact value of $\pi^*_{(2,t)}(C_n)$ for all $t \geq 1$ can be reduced to the problem of determining the exact value of $\pi^*_{(2,r)}(C_n)$ for $r \in [1, 12]$. We also consider C_n with $3 \leq n \leq 5$. When $n = 3$, we have $\pi^*_{(2,t)}(C_3) = \pi^*_{(1,t)}(C_3)$, since the diameter of C_3 is one. The exact value of $\pi^*_{(1,t)}(C_3)$ is known. When $n = 4, 5$, we determine the exact value of $\pi^*_{(2,t)}(C_n)$ for $t \geq 1$.

Keywords: graph pebbling; optimal pebbling; capacity-restricted pebbling; distance-restricted pebbling; cycle
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1. Introduction

Let G be a graph. A distribution δ of G is a mapping from $V(G)$ to the set of nonnegative integers. For each vertex $v \in V(G)$, $\delta(v)$ denotes the number of pebbles distributed to v. A pebbling move consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. Let δ be a distribution of G, and let H be an induced subgraph of G. For convenience, we use $|\delta(H)|$

to denote the number of pebbles distributed onto the vertices of H. For each vertex $v \in V(H)$, we use $m(\delta, H, v)$ to denote the maximum number of pebbles that can be moved to v from the vertices of H by using pebbling moves.

Given a positive integer d and a distribution δ of G, if we can move pebbles to any target vertex v in G only from the vertices in the set $N_d[v] = \{u \in V(G) : d(u, v) \leq d\}$ by pebbling moves, where $d(u, v)$ is the distance between u and v, then such a graph pebbling is said to be distance d-restricted. We use $m(\delta, d, v)$ to denote the maximum number of pebbles that can be moved to v only from the vertices in the set $N_d[v]$. That is, $m(\delta, d, v) = m(\delta, H, v)$, where H is a subgraph of G induced by $N_d[v]$. A distribution δ of a graph G is (d, t)-solvable if $m(\delta, d, v) \geq t$ for each vertex v of G. The optimal (d, t)-pebbling number of G, $\pi^*_{(d,t)}(G)$, is the minimum number of pebbles needed so that there is a (d, t)-solvable distribution of G. A distribution δ of G is said to be optimal (d, t)-solvable if it is (d, t)-solvable and $|\delta(G)| = \pi^*_{(d,t)}(G)$. When d is not less than the diameter of G, the subgraph induced by $N_d[v]$ is G itself. In this situation, $\pi^*_{(d,1)}(G) = \pi^*(G)$, which is known as the optimal pebbling number of G, see [2, 5, 7–9, 11] for references. Moreover, $\pi^*_{(d,t)}(G) = \pi^*_t(G)$, which is the optimal t-pebbling number, see [12, 14] for references.

2. Preliminaries

In 2018, the distance-restricted graph pebbling was first proposed by Chen and Shiue [4], and they showed that $\pi^*_{(2,2)}(C_3) = 4$ and $\pi^*_{(2,2)}(C_n) = n$ for all $n \geq 4$. In 2020, Shiue [13] studied distance 1-restricted pebbling in cycles and obtained the following result.

Theorem 2.1. *[13] Let C_n be an n-cycle, $n \geq 3$. Then*

$$\pi^*_{(1,t)}(C_n) = \begin{cases} \frac{nt}{2}, & \text{if } t \text{ is even and } \frac{nt}{2} \text{ is even;} \\ \frac{nt}{2} + 1, & \text{if } t \text{ is even and } \frac{nt}{2} \text{ is odd;} \\ \lceil \frac{2n}{3} \rceil, & \text{if } t = 1; \\ \left\lceil \frac{nt}{2} + \frac{\lceil n/(t+2) \rceil}{2} \right\rceil, & \text{if } t \text{ is odd and } t \geq 3. \end{cases}$$

In [13], Shiue also showed that if G is an r-regular graph of order n with girth at least 5 and t is a multiple of $r^2 + r + 4$, then $\pi^*_{(2,t)}(G) = \frac{4nt}{r^2+r+4}$. Since an n-cycle is 2-regular, we have the following.

Corollary 2.1. *[13] Let C_n be an n-cycle. If $n \geq 5$, then $\pi^*_{(2,10k)}(C_n) = 4nk$ for each positive integer k.*

The conclusion of Corollary 2.1 gives the recurrence relation $\pi^*_{(2,10k)}(C_n) = \pi^*_{(2,10k-10)}(C_n) + 4n = \pi^*_{(2,10k-10)}(C_n) + \pi^*_{(2,10)}(C_n)$ for $k \geq 2$. Hence, we are interested in the following problem.

Problem 1. For $n \geq 5$ and $t \geq 11$, determine the values of t that satisfy the recurrence relation

$$\pi^*_{(2,t)}(C_n) = \pi^*_{(2,t-10)}(C_n) + 4n.$$

It is easy to see that for any graph G and any two positive integers t_1 and t_2, $\pi^*_{(2,t_1+t_2)}(G) \leq \pi^*_{(2,t_1)}(G) + \pi^*_{(2,t_2)}(G)$.

Usually, the equality does not hold. For example, $\pi^*_{(2,2)}(C_6) = 6 < 8 = \pi^*_{(2,1)}(C_6) + \pi^*_{(2,1)}(C_6)$.

In this article, we mainly study the distance 2-restricted pebbling in cycles. Note that the diameter of C_3 is equal to one. This implies that $\pi^*_{(2,t)}(C_3) = \pi^*_t(C_3) = \pi^*_{(1,t)}(C_3)$, see Theorem 2.1. As $C_3 = K_3$, an alternative result about $\pi_t^*(C_3)$ can be seen in [12]. Regarding the research on $\pi_t^*(C_n)$, readers can refer to [14]. Note that the authors in [14] have only given an upper and a lower bound for $\pi_t^*(C_n)$ when $n \geq 4$ and $t \geq 3$. In this article, we show that for $t \geq 13$ and $n \geq 6$, $\pi^*_{(2,t)}(C_n) = \pi^*_{(2,t-10)}(C_n) + 4n$ in Section 2 and completely determine the exact value of $\pi_t^*(C_n)$ for $n = 4, 5$, and $t \geq 1$ in Sections 4 and 5, respectively.

3. Main results

Throughout the rest of this article, let $C_n = v_0, v_1, \cdots, v_{n-1}, v_0$ be an n-cycle, $n \geq 3$, and all subscripts referring to the vertices of the cycle C_n will be interpreted modulo n. We write "a mod b" to denote the remainder when a is divided by b. For convenience, we use "$[\ell_1, \ell_2]$" to denote the set of all integers between ℓ_1 and ℓ_2 and, if δ is a distribution of pebbles on C_n, we use $\langle x_0, x_1, \cdots, x_{n-1}\rangle$ to denote δ, where $x_i = \delta(v_i)$. Thus, $|\delta(C_n)| = \sum_{i=0}^{n-1} x_i$. Note that the diameter of C_n is not more than two for $3 \leq n \leq 5$. It follows that $\pi^*_{(2,t)}(C_n) = \pi^*_t(C_n)$ for $3 \leq n \leq 5$. So we only consider the case when $n \geq 6$ in this section. For $n \geq 6$, and for $i \in [0, n-1]$,

$$m(\delta, 2, v_i) = \left\lfloor \frac{1}{2}(\lfloor \frac{x_{i-2}}{2} \rfloor + x_{i-1}) \right\rfloor + x_i + \left\lfloor \frac{1}{2}(x_{i+1} + \lfloor \frac{x_{i+2}}{2} \rfloor) \right\rfloor. \tag{3.1}$$

The following proposition is an important key for solving Problem 1.

Proposition 3.1. *For $t \geq 11$ and $n \geq 6$, if there exists an optimal $(2, t)$-solvable distribution $\delta = \langle x_0, x_1, \cdots, x_{n-1}\rangle$ of C_n such that $x_i \geq 4$ for $i \in [0, n-1]$, then $\pi^*_{(2,t)}(C_n) = \pi^*_{(2,t-10)}(C_n) + 4n$.*

Proof. Assume that $t \geq 11$ and $n \geq 6$. By Corollary 2.1, we have $\pi^*_{(2,10)}(C_n) = 4n$. It follows that

$$\begin{aligned} \pi^*_{(2,t)}(C_n) &\leq \pi^*_{(2,t-10)}(C_n) + \pi^*_{(2,10)}(C_n) \\ &= \pi^*_{(2,t-10)}(C_n) + 4n. \end{aligned}$$

Thus,

$$\pi^*_{(2,t-10)}(C_n) \geq \pi^*_{(2,t)}(C_n) - 4n.$$

Assume that $\delta = \langle x_0, x_1, \cdots, x_{n-1}\rangle$ is an optimal $(2, t)$-solvable distribution of C_n with $x_i \geq 4$ for all $i \in [0, n-1]$. Then let $\delta' = \langle y_0, y_1, \cdots, y_{n-1}\rangle$ be a distribution of C_n such that $y_i = x_i - 4$ for $i \in [0, n-1]$. Clearly,

$$|\delta'(C_n)| = |\delta(C_n)| - 4n = \pi^*_{(2,t)}(C_n) - 4n.$$

To complete the proof, it is left to show that δ' is $(2, t-10)$-solvable. For $i \in [0, n-1]$, by (3.1), we have

$$\begin{aligned} m(\delta', 2, v_i) &= \left\lfloor \frac{\left\lfloor \frac{y_{i-2}}{2} \right\rfloor + y_{i-1}}{2} \right\rfloor + y_i + \left\lfloor \frac{\left\lfloor \frac{y_{i+2}}{2} \right\rfloor + y_{i+1}}{2} \right\rfloor \\ &= \left\lfloor \frac{\left\lfloor \frac{x_{i-2}-4}{2} \right\rfloor + x_{i-1} - 4}{2} \right\rfloor + x_i - 4 + \left\lfloor \frac{\left\lfloor \frac{x_{i+2}-4}{2} \right\rfloor + x_{i+1} - 4}{2} \right\rfloor \end{aligned}$$

$$= \left\lfloor \frac{\left\lfloor \frac{x_{i-2}}{2} \right\rfloor + x_{i-1}}{2} \right\rfloor + x_i + \left\lfloor \frac{\left\lfloor \frac{x_{i+2}}{2} \right\rfloor + x_{i+1}}{2} \right\rfloor - 10$$

$$= m(\delta, 2, v_i) - 10$$

$$\geq t - 10.$$

Thus, δ' is $(2, t-10)$-solvable, and we have the proof.

Proposition 3.1 gives a sufficient condition for satisfying the recurrence relation in Problem 1. Now, we will find the values of t that satisfy the assumption of the statement in Proposition 3.1.

Let $\delta = \langle x_0, x_1, \cdots, x_{n-1} \rangle$ be a $(2, t)$-solvable distribution of C_n with $n \geq 6$. Consider any subpath

$$v_{j-5}, v_{j-4}, v_{j-3}, v_{j-2}, v_{j-1}, v_j, v_{j+1}, v_{j+2}, v_{j+3}, v_{j+4}, v_{j+5}$$

in C_n, for some $j \in [0, n-1]$. When $6 \leq n \leq 10$, the subpath is the cycle C_n. We let $n = 10 - k$, where $0 \leq k \leq 4$. Then the vertex $v_{j+5-i} = v_{j-5+k-i}$ for $i \in [0, k]$. For example, if $n = 8$, then $v_{j+5} = v_{j-3}$, $v_{j+4} = v_{j-4}$, and $v_{j+3} = v_{j-5}$. By (3.1), we have

$$m(\delta, 2, v_j) = \lfloor (\lfloor x_{j-2}/2 \rfloor + x_{j-1})/2 \rfloor + x_j + \lfloor (x_{j+1} + \lfloor x_{j+2}/2 \rfloor)/2 \rfloor \geq t.$$

It follows that

$$(\lfloor x_{j-2}/2 \rfloor + x_{j-1}) + (x_{j+1} + \lfloor x_{j+2}/2 \rfloor) \geq 2(t - x_j).$$

Without loss of generality, we can assume that

$$\lfloor x_{j-2}/2 \rfloor + x_{j-1} = t + r - x_j, \tag{3.2}$$

where r is a nonnegative integer. Then we have

$$x_{j+1} + \lfloor x_{j+2}/2 \rfloor \geq t - r - x_j. \tag{3.3}$$

It follows that

$$x_{j-2} \geq 2(t + r - x_j - x_{j-1}) \tag{3.4}$$

and

$$x_{j+2} \geq 2(t - r - x_j - x_{j+1}). \tag{3.5}$$

By (3.1)–(3.5), we have the following four facts:

Fact 1.

$$\begin{aligned} m(\delta, 2, v_{j-1}) &= \lfloor (\lfloor x_{j-3}/2 \rfloor + x_{j-2})/2 \rfloor + x_{j-1} + \lfloor (x_j + \lfloor x_{j+1}/2 \rfloor)/2 \rfloor \\ &\geq \lfloor x_{j-3}/4 \rfloor + \lfloor x_{j-2}/2 \rfloor + x_{j-1} + \lfloor (x_j + \lfloor x_{j+1}/2 \rfloor)/2 \rfloor \\ &= \lfloor x_{j-3}/4 \rfloor + (t + r - x_j) + \lfloor (x_j + \lfloor x_{j+1}/2 \rfloor)/2 \rfloor. \end{aligned}$$

Fact 2.

$$\begin{aligned} m(\delta, 2, v_{j+1}) &= \lfloor (\lfloor x_{j-1}/2 \rfloor + x_j)/2 \rfloor + x_{j+1} + \lfloor (x_{j+2} + \lfloor x_{j+3}/2 \rfloor)/2 \rfloor \\ &\geq \lfloor (\lfloor x_{j-1}/2 \rfloor + x_j)/2 \rfloor + x_{j+1} + \lfloor x_{j+2}/2 \rfloor + \lfloor x_{j+3}/4 \rfloor \end{aligned}$$

$$\geq \lfloor(\lfloor x_{j-1}/2\rfloor + x_j)/2\rfloor + (t - r - x_j) + \lfloor x_{j+3}/4\rfloor.$$

Fact 3.

$$\begin{aligned} m(\delta, 2, v_{j-2}) &= \lfloor(\lfloor x_{j-4}/2\rfloor + x_{j-3})/2\rfloor + x_{j-2} + \lfloor(x_{j-1} + \lfloor x_j/2\rfloor)/2\rfloor \\ &\geq \lfloor(\lfloor x_{j-4}/2\rfloor + x_{j-3})/2\rfloor + 2(t + r - x_j - x_{j-1}) + \lfloor(x_{j-1} + \lfloor x_j/2\rfloor)/2\rfloor. \end{aligned}$$

Fact 4.

$$\begin{aligned} m(\delta, 2, v_{j+2}) &= \lfloor(\lfloor x_j/2\rfloor + x_{j+1})/2\rfloor + x_{j+2} + \lfloor(x_{j+3} + \lfloor x_{j+4}/2\rfloor)/2\rfloor \\ &\geq \lfloor(\lfloor x_j/2\rfloor + x_{j+1})/2\rfloor + 2(t - r - x_j - x_{j+1}) + \lfloor(x_{j+3} + \lfloor x_{j+4}/2\rfloor)/2\rfloor. \end{aligned}$$

Fact 5. If $x_{j-2} \geq 8$ and $\delta' = \langle y_0, y_1, \cdots, y_{n-1}\rangle$ is a distribution of C_n such that $y_j = x_j + 2$, $y_{j-1} = x_{j-1} + 2$, $y_{j-2} = x_{j-2} - 8$, $y_{j-3} = x_{j-3} + 4$, and $y_i = x_i$ for $i \in [0, n-1] \setminus [j-3, j]$, then $|\delta'(C_n)| = |\delta(C_n)|$, $m(\delta', 2, v_{j-2}) \geq m(\delta, 2, v_{j-2}) - 5$ and $m(\delta', 2, v_i) \geq m(\delta, 2, v_i)$ for $i \in [0, n-1] \setminus \{j-2\}$.
Proof. Clearly, $|\delta'(C_n)| = |\delta(C_n)|$. If $n \geq 11$, then the vertices

$$v_{j-5}, v_{j-4}, v_{j-3}, v_{j-2}, v_{j-1}, v_j, v_{j+1}, v_{j+2}, v_{j+3}, v_{j+4}, v_{j+5}$$

are all distinct. Otherwise, $6 \leq n \leq 10$. Let $n = 10 - k$, where $0 \leq k \leq 4$. Then, the vertex $v_{j+5-i} = v_{j-5+k-i}$ for $i \in [0, 4]$. By (3.1), it is easy to see that $m(\delta', 2, v_i) \geq m(\delta, 2, v_i)$ for $i \in [0, n-1] \setminus \{j-2, j-1\}$. So, we only need to check $m(\delta', 2, v_{j-2})$ and $m(\delta', 2, v_{j-1})$. Since $n \geq 6$, $v_i \notin \{v_{j-3}, v_{j-2}, v_{j-1}, v_j\}$ for $i \in \{j-4, j+1\}$. By (3.1), we have

$$\begin{aligned} m(\delta', 2, v_{j-2}) &= \lfloor(\lfloor y_{j-4}/2\rfloor + y_{j-3})/2\rfloor + y_{j-2} + \lfloor(y_{j-1} + \lfloor y_j/2\rfloor)/2\rfloor \\ &= \lfloor(\lfloor x_{j-4}/2\rfloor + x_{j-3} + 4)/2\rfloor + x_{j-2} - 8 + \lfloor(x_{j-1} + 2 + \lfloor(x_j + 2)/2\rfloor)/2\rfloor \\ &\geq \lfloor(\lfloor x_{j-4}/2\rfloor + x_{j-3})/2\rfloor + x_{j-2} + \lfloor(x_{j-1} + \lfloor x_j/2\rfloor)/2\rfloor + 2 - 8 + 1 \\ &= m(\delta, 2, v_{j-2}) - 5, \end{aligned}$$

and

$$\begin{aligned} m(\delta', 2, v_{j-1}) &= \lfloor(\lfloor y_{j-3}/2\rfloor + y_{j-2})/2\rfloor + y_{j-1} + \lfloor(y_j + \lfloor y_{j+1}/2\rfloor)/2\rfloor \\ &= \lfloor(\lfloor(x_{j-3} + 4)/2\rfloor + x_{j-2} - 8)/2\rfloor + x_{j-1} + 2 + \lfloor(x_j + 2 + \lfloor x_{j+1}/2\rfloor)/2\rfloor \\ &= \lfloor(\lfloor x_{j-3}/2\rfloor + x_{j-2})/2\rfloor + x_{j-1} + \lfloor(x_j + \lfloor x_{j+1}/2\rfloor)/2\rfloor - 3 + 2 + 1 \\ &= m(\delta, 2, v_{j-1}). \end{aligned}$$

Fact 6. If $x_{j-2} \geq 4$ and $\delta' = \langle y_0, y_1, \cdots, y_{n-1}\rangle$ is a distribution of C_n such that $y_j = x_j + 1$, $y_{j-1} = x_{j-1} + 1$, $y_{j-2} = x_{j-2} - 4$, $y_{j-3} = x_{j-3} + 2$ and $y_i = x_i$ for $i \in [0, n-1] \setminus [j-3, j]$, then $|\delta'(C_n)| = |\delta(C_n)|$, $m(\delta', 2, v_{j-2}) \geq m(\delta, 2, v_{j-2}) - 3$, $m(\delta', 2, v_{j-1}) \geq m(\delta, 2, v_{j-1}) - 1$ and $m(\delta', 2, v_i) \geq m(\delta, 2, v_i)$ for $i \in [0, n-1] \setminus \{j-1, j-2\}$.
Proof. Clearly, $|\delta'(C_n)| = |\delta(C_n)|$. If $n \geq 11$, then the vertices

$$v_{j-5}, v_{j-4}, v_{j-3}, v_{j-2}, v_{j-1}, v_j, v_{j+1}, v_{j+2}, v_{j+3}, v_{j+4}, v_{j+5}$$

are all distinct. Otherwise, $6 \leq n \leq 10$. Let $n = 10 - k$, where $0 \leq k \leq 4$. Then the vertex $v_{j+5-i} = v_{j-5+k-i}$ for $i \in [0, 4]$. By (3.1), it is easy to see that $m(\delta', 2, v_i) \geq m(\delta, 2, v_i)$ for $i \in [0, n-1] \setminus \{j-2, j-1\}$.

Now, we only need to check $m(\delta', 2, v_{j-2})$ and $m(\delta', 2, v_{j-1})$. Since $n \geq 6$, $v_i \notin \{v_{j-3}, v_{j-2}, v_{j-1}, v_j\}$ for $i \in \{j-4, j+1\}$. By (3.1), we have

$$\begin{aligned} m(\delta', 2, v_{j-2}) &= \lfloor(\lfloor y_{j-4}/2\rfloor + y_{j-3})/2\rfloor + y_{j-2} + \lfloor(y_{j-1} + \lfloor y_j/2\rfloor)/2\rfloor \\ &= \lfloor(\lfloor x_{j-4}/2\rfloor + x_{j-3} + 2)/2\rfloor + x_{j-2} - 4 + \lfloor(x_{j-1} + 1 + \lfloor(x_j + 1)/2\rfloor)/2\rfloor \\ &\geq \lfloor(\lfloor x_{j-4}/2\rfloor + x_{j-3})/2\rfloor + x_{j-2} + \lfloor(x_{j-1} + \lfloor x_j\rfloor)/2\rfloor + 1 - 4 \\ &= m(\delta, 2, v_{j-2}) - 3, \end{aligned}$$

and

$$\begin{aligned} m(\delta', 2, v_{j-1}) &= \lfloor(\lfloor y_{j-3}/2\rfloor + y_{j-2})/2\rfloor + y_{j-1} + \lfloor(y_j + \lfloor y_{j+1}/2\rfloor)/2\rfloor \\ &= \lfloor(\lfloor(x_{j-3} + 2)/2\rfloor + x_{j-2} - 4)/2\rfloor + x_{j-1} + 1 + \lfloor(x_j + 1 + \lfloor x_{j+1}/2\rfloor)/2\rfloor \\ &\geq \lfloor(\lfloor x_{j-3}/2\rfloor + x_{j-2})/2\rfloor + x_{j-1} + \lfloor(x_j + \lfloor x_{j+1}/2\rfloor)/2\rfloor - 2 + 1 \\ &= m(\delta, 2, v_{j-1}) - 1. \end{aligned}$$

By using Eq (3.1) to check, we can obtain Facts 7 and 8.

Fact 7. If $x_{j-1} \geq 4$ and $\delta' = \langle y_0, y_1, \cdots, y_{n-1}\rangle$ is a distribution of C_n such that $y_j = x_j + 2$, $y_{j-1} = x_{j-1} - 4$, $y_{j-2} = x_{j-2} + 2$, and $y_i = x_i$ for $i \in [0, n-1] \setminus [j-2, j]$, then $|\delta'(C_n)| = |\delta(C_n)|$, $m(\delta', 2, v_{j-1}) = m(\delta, 2, v_{j-1}) - 2$ and $m(\delta', 2, v_i) \geq m(\delta, 2, v_i)$ for $i \in [0, n-1] \setminus \{j-1\}$.

Fact 8. If $x_{j-1} \geq 2$ and $\delta' = \langle y_0, y_1, \cdots, y_{n-1}\rangle$ is a distribution of C_n such that $y_j = x_j + 1$, $y_{j-1} = x_{j-1} - 2$, $y_{j-2} = x_{j-2} + 1$, and $y_i = x_i$ for $i \in [0, n-1] \setminus [j-2, j]$, then $|\delta'(C_n)| = |\delta(C_n)|$, $m(\delta', 2, v_{j-1}) \geq m(\delta, 2, v_{j-1}) - 2$ and $m(\delta', 2, v_i) \geq m(\delta, 2, v_i)$ for $i \in [0, n-1] \setminus \{j-1\}$.

Fact 9. If x_{j-1} and x_{j+1} are both odd, and $\delta' = \langle y_0, y_1, \cdots, y_{n-1}\rangle$ is a distribution of C_n such that $y_j = x_j + 2$, $y_{j-1} = x_{j-1} - 1$, $y_{j+1} = x_{j+1} - 1$, and $y_i = x_i$ for $i \in [0, n-1] \setminus [j-1, j, j+1]$, then $|\delta'(C_n)| = |\delta(C_n)|$ and $m(\delta', 2, v_i) \geq m(\delta, 2, v_i)$ for $i \in [0, n-1]$.

Proof. Clearly, $|\delta'(C_n)| = |\delta(C_n)|$. If $n \geq 11$, then the vertices

$$v_{j-5}, v_{j-4}, v_{j-3}, v_{j-2}, v_{j-1}, v_j, v_{j+1}, v_{j+2}, v_{j+3}, v_{j+4}, v_{j+5}$$

are all distinct. Otherwise, $6 \leq n \leq 10$. Let $n = 10 - k$, where $0 \leq k \leq 4$. Then, the vertex $v_{j+5-i} = v_{j-5+k-i}$ for $i \in [0, 4]$. By (3.1), it is easy to see that $m(\delta', 2, v_i) = m(\delta, 2, v_i)$ for $i \in [0, n-1] \setminus \{j-3, j-2, j-1, j, j+1, j+2, j+3\}$. So, we only need to check $m(\delta', 2, v_i)$ for $i \in \{j-3, j-2, j-1, j, j+1, j+2, j+3\}$. Note that x_{j-1} and x_{j+1} are both odd. It follows that $\lfloor(x_{j-1} - 1)/2\rfloor = \lfloor x_{j-1}/2\rfloor$ and $\lfloor(x_{j+1} - 1)/2\rfloor = \lfloor x_{j+1}/2\rfloor$. Since $n \geq 6$, $v_i \notin \{v_{j-1}, v_j, v_{j+1}\}$ for $i \in \{j-4, j-3, j-2, j+2\}$. By using (3.1) to check, we can verify that $m(\delta', 2, v_i) \geq m(\delta, 2, v_i)$ for $i \in \{j, j-1, j-2\}$. For the value of $m(\delta', 2, v_{j-3})$, if $n \geq 7$, then $v_i \notin \{v_{j-1}, v_j, v_{j+1}\}$ for $i \in \{j-5, j-4, j-3, j-2\}$, and

$$\begin{aligned} m(\delta', 2, v_{j-3}) &= \lfloor(\lfloor y_{j-5}/2\rfloor + y_{j-4})/2\rfloor + y_{j-3} + \lfloor(y_{j-2} + \lfloor y_{j-1}/2\rfloor)/2\rfloor \\ &= \lfloor(\lfloor x_{j-5}/2\rfloor + x_{j-4})/2\rfloor + x_{j-3} + \lfloor(x_{j-2} + \lfloor(x_{j-1} - 1)/2\rfloor)/2\rfloor \\ &= \lfloor(\lfloor x_{j-5}/2\rfloor + x_{j-4})/2\rfloor + x_{j-3} + \lfloor(x_{j-2} + \lfloor x_{j-1}/2\rfloor)/2\rfloor \\ &= m(\delta, 2, v_{j-3}). \end{aligned}$$

Otherwise, $n = 6$, and hence $v_{j-5} = v_{j+1}$; then we have

$$\begin{aligned}m(\delta', 2, v_{j-3}) &= \lfloor(\lfloor y_{j-5}/2\rfloor + y_{j-4})/2\rfloor + y_{j-3} + \lfloor(y_{j-2} + \lfloor y_{j-1}/2\rfloor)/2\rfloor\\ &= \lfloor(\lfloor(x_{j-5} - 1)/2\rfloor + x_{j-4})/2\rfloor + x_{j-3} + \lfloor(x_{j-2} + \lfloor(x_{j-1} - 1)/2\rfloor)/2\rfloor\\ &= \lfloor(\lfloor x_{j-5}/2\rfloor + x_{j-4})/2\rfloor + x_{j-3} + \lfloor(x_{j-2} + \lfloor x_{j-1}/2\rfloor)/2\rfloor\\ &= m(\delta, 2, v_{j-3}).\end{aligned}$$

By a similar argument as above, we also have $m(\delta', 2, v_i) \geq m(\delta, 2, v_i)$ for $i \in \{j + 1, j + 2, j + 3\}$.

Fact 10. If x_{j-1} and x_j are both odd, and $\delta' = \langle y_0, y_1, \cdots, y_{n-1}\rangle$ is a distribution of C_n such that $y_j = x_j + 1$, $y_{j-1} = x_{j-1} - 1$, and $y_i = x_i$ for $i \in [0, n-1] \setminus [j-1, j]$, then $|\delta'(C_n)| = |\delta(C_n)|$, $m(\delta', 2, v_{j-1}) \geq m(\delta, 2, v_{j-1}) - 1$ and $m(\delta', 2, v_i) \geq m(\delta, 2, v_i)$ for $i \in [0, n-1] \setminus \{j-1\}$.

Proof. Clearly, $|\delta'(C_n)| = |\delta(C_n)|$. If $n \geq 11$, then, the vertices

$$v_{j-5}, v_{j-4}, v_{j-3}, v_{j-2}, v_{j-1}, v_j, v_{j+1}, v_{j+2}, v_{j+3}, v_{j+4}, v_{j+5}$$

are all distinct. Otherwise, $6 \leq n \leq 10$. Let $n = 10 - k$, where $0 \leq k \leq 4$. Then the vertex $v_{j+5-i} = v_{j-5+k-i}$ for $i \in [0, 4]$. By (3.1), it is easy to see that $m(\delta', 2, v_i) = m(\delta, 2, v_i)$ for $i \in [0, n-1] \setminus \{j-3, j-2, j-1, j, j+1, j+2\}$. So, we only need to check $m(\delta', 2, v_i)$ for $i \in \{j-3, j-2, j-1, j, j+1, j+2\}$. Note that x_{j-1} and x_j are both odd. It follows that $\lfloor(x_{j-1}-1)/2\rfloor = \lfloor x_{j-1}/2\rfloor$ and $\lfloor(x_j+1)/2\rfloor = \lfloor x_j/2\rfloor+1$. Since $n \geq 6$, $v_i \notin \{v_{j-1}, v_j\}$ for $i \in \{j-4, j-3, j-2, j+1, j+2\}$. By using (3.1) to check, we can verify that $m(\delta', 2, v_{j-1}) \geq m(\delta, 2, v_{j-1}) - 1$ and $m(\delta', 2, v_j) \geq m(\delta, 2, v_j)$. For the value of $m(\delta', 2, v_{j-2})$, $v_i \notin \{v_{j-1}, v_j\}$ for $i \in \{j-4, j-3, j-2\}$, and

$$\begin{aligned}m(\delta', 2, v_{j-2}) &= \lfloor(\lfloor y_{j-4}/2\rfloor + y_{j-3})/2\rfloor + y_{j-2} + \lfloor(y_{j-1} + \lfloor y_j/2\rfloor)/2\rfloor\\ &= \lfloor(\lfloor x_{j-4}/2\rfloor + x_{j-4})/2\rfloor + x_{j-2} + \lfloor(x_{j-1} - 1 + \lfloor(x_j + 1)/2\rfloor)/2\rfloor\\ &= \lfloor(\lfloor x_{j-4}/2\rfloor + x_{j-3})/2\rfloor + x_{j-2} + \lfloor(x_{j-1} + \lfloor x_j/2\rfloor)/2\rfloor\\ &= m(\delta, 2, v_{j-2}).\end{aligned}$$

For the value of $m(\delta', 2, v_{j-3})$, $v_i \notin \{v_{j-1}, v_j\}$ for $i \in \{j-5, j-4, j-3, j-2\}$, and

$$\begin{aligned}m(\delta', 2, v_{j-3}) &= \lfloor(\lfloor y_{j-5}/2\rfloor + y_{j-4})/2\rfloor + y_{j-3} + \lfloor(y_{j-2} + \lfloor y_{j-1}/2\rfloor)/2\rfloor\\ &= \lfloor(\lfloor x_{j-5}/2\rfloor + x_{j-4})/2\rfloor + x_{j-3} + \lfloor(x_{j-2} + \lfloor(x_{j-1} - 1)/2\rfloor)/2\rfloor\\ &= \lfloor(\lfloor x_{j-5}/2\rfloor + x_{j-4})/2\rfloor + x_{j-3} + \lfloor(x_{j-2} + \lfloor x_{j-1}/2\rfloor)/2\rfloor\\ &= m(\delta, 2, v_{j-3}).\end{aligned}$$

By a similar argument as above, we also have $m(\delta', 2, v_i) \geq m(\delta, 2, v_i)$ for $i \in \{j + 1, j + 2\}$.

Lemma 3.1. *For $t \geq 13$ and $n \geq 6$, there exists an optimal $(2, t)$-solvable distribution δ of C_n such that $\delta(v) \geq 1$ for each vertex v of C_n.*

Proof. Let $\delta = \langle x_0, x_1, \cdots, x_{n-1}\rangle$ be an optimal $(2, t)$-solvable distribution of C_n. Suppose that there exists a vertex v_j, $j \in [0, n-1]$, with $\delta(v_j) = x_j = 0$. It suffices to show that there exists a $(2, t)$-solvable distribution δ' of C_n such that $\delta'(v_j) \geq 1$ and $\delta'(v_k) \geq 1$ or $\delta'(v_k) \geq x_k$ for $k \in [0, n-1]\setminus\{j\}$ and $|\delta'(C_n)| \leq |\delta(C_n)|$. By (3.1), we have

$$(\lfloor x_{j-2}/2\rfloor + x_{j-1}) + (x_{j+1} + \lfloor x_{j+2}/2\rfloor) \geq 2(t - x_j).$$

Without loss of generality, we can assume that $\lfloor x_{j-2}/2 \rfloor + x_{j-1} = t - x_j + r$, where r is a nonnegative integer. This implies that (3.2)–(3.5) are valid. Now, we will modify the distribution of pebbles on the vertices in $\{v_i | i \in [j-3, j+3]\}$ if necessary. Let $\delta' = \langle y_0, y_1, \cdots, y_{n-1} \rangle$ be a distribution of C_n such that $y_i = x_i$ for $i \in [0, n-1] \setminus [j-3, j+3]$. For $i \in [j-3, j+3]$, y_i will be defined according to the following cases. Initially, we let $y_i = x_i$ for $i \in [j-3, j+3]$.

By (3.4) and Fact 3, we have

$$x_{j-2} \geq 2(t + r - x_j - x_{j-1}) = 2(13 + r - x_{j-1})$$

and

$$\begin{aligned} m(\delta, 2, v_{j-2}) &\geq \lfloor (\lfloor x_{j-4}/2 \rfloor + x_{j-3})/2 \rfloor + 2(t + r - x_j - x_{j-1}) + \lfloor (x_{j-1} + \lfloor x_j/2 \rfloor)/2 \rfloor \\ &\geq t + 13 + 2r - (2x_{j-1} - \lfloor x_{j-1}/2 \rfloor). \end{aligned}$$

Case 1. $r = 0$.

If $x_{j-1} \leq 5$, then $x_{j-2} \geq 2(13+0-5) = 16$ and $m(\delta, 2, v_{j-2}) \geq t + 13 + 0 - (2 \cdot 5 - \lfloor 5/2 \rfloor) = t + 5$. Let $y_j = x_j + 2 = 2$, $y_{j-1} = x_{j-1} + 2$, $y_{j-2} = x_{j-2} - 8 \geq 8$, and $y_{j-3} = x_{j-3} + 4$. By Fact 5, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2, t)$-solvable.

Similarly, if $x_{j+1} \leq 5$, then, by (3.5) and Fact 4, we have $x_{j+2} \geq 16$ and $m(\delta, 2, v_{j+2}) \geq t + 5$. Let $y_j = x_j + 2 = 2$, $y_{j+1} = x_{j+1} + 2$, $y_{j+2} = x_{j+2} - 8 \geq 8$, $y_{j+3} = x_{j+3} + 4$. Then, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2, t)$-solvable.

Now, we can assume that $x_{j-1} \geq 6$ and $x_{j+1} \geq 6$. Let $y_j = x_j + 1 = 1$, $y_{j-1} = x_{j-1} - 2 \geq 4$, and $y_{j-2} = x_{j-2} + 1$. By Fact 8, we only need to check $m(\delta', 2, v_{j-1})$. By (3.1) and (3.2), we have

$$\begin{aligned} m(\delta', 2, v_{j-1}) &= \lfloor (\lfloor y_{j-3}/2 \rfloor + y_{j-2})/2 \rfloor + y_{j-1} + \lfloor (y_j + \lfloor y_{j+1}/2 \rfloor)/2 \rfloor \\ &\geq \lfloor x_{j-3}/4 \rfloor + \lfloor (x_{j-2} + 1)/2 \rfloor + x_{j-1} - 2 + \lfloor (x_j + 1 + \lfloor x_{j+1}/2 \rfloor)/2 \rfloor \\ &\geq (t - x_j + r) - 2 + \lfloor (1 + \lfloor 6/2 \rfloor)/2 \rfloor \\ &= t. \end{aligned}$$

Case 2. $r = 1$.

If $x_{j-1} \leq 6$, then $x_{j-2} \geq 2(13 + 1 - 6) = 16$ and $m(\delta, 2, v_{j-2}) \geq t + 13 + 2 - (2 \cdot 6 - \lfloor 6/2 \rfloor) = t + 6$. Let $y_j = x_j + 2 = 2$, $y_{j-1} = x_{j-1} + 2$, $y_{j-2} = x_{j-2} - 8 \geq 8$, and $y_{j-3} = x_{j-3} + 4$. Then, by Fact 5, we have $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2, t)$-solvable. Otherwise, $x_{j-1} \geq 7$. If $x_{j+1} \geq 4$, then, by Fact 1, we have $m(\delta, 2, v_{j-1}) \geq (t + r - x_j) + \lfloor (x_j + \lfloor x_{j+1}/2 \rfloor)/2 \rfloor \geq t + 2$. Let $y_j = x_j + 2 = 2$, $y_{j-1} = x_{j-1} - 4 \geq 3$, and $y_{j-2} = x_{j-2} + 2$. Then, by Fact 7, we have $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2, t)$-solvable. If $x_{j+1} \leq 3$, then, by (3.5) and Fact 4, $x_{j+2} \geq 2(t - r - x_j - x_{j+1}) \geq 2(13 - 1 - 0 - 3) = 18$ and $m(\delta, 2, v_{j+2}) \geq \lfloor x_{j+1}/2 \rfloor + 2(t - r - x_j - x_{j+1}) \geq \lfloor 3/2 \rfloor + t + 13 + 2(-1 - 0 - 3) = t + 6$. Let $y_j = x_j + 2 = 2$, $y_{j+1} = x_{j+1} + 2$, $y_{j+2} = x_{j+2} - 8 \geq 10$, and $y_{j+3} = x_{j+3} + 4$. Then, by Fact 5, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2, t)$-solvable.

Case 3. $r \geq 2$.

If $x_{j-1} \leq 8$, then $x_{j-2} \geq 2(13 + 2 - 8) = 14$ and $m(\delta, 2, v_{j-2}) \geq t + 13 + 4 - (2 \cdot 8 - \lfloor 8/2 \rfloor) + 4 = t + 5$. Let $y_j = x_j + 2 = 2$, $y_{j-1} = x_{j-1} + 2$, $y_{j-2} = x_{j-2} - 8 \geq 6$, and $y_{j-3} = x_{j-3} + 4$. Then, by Fact 5, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2, t)$-solvable. Otherwise, $x_{j-1} \geq 9$. By Fact 1, $m(\delta, 2, v_{j-1}) \geq t + r - x_j \geq t + 2$. Let $y_j = x_j + 2 = 2$, $y_{j-1} = x_{j-1} - 4 \geq 5$, $y_{j-2} = x_{j-2} + 2$. Then, by Fact 7, we have $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2, t)$-solvable. □

Lemma 3.2. *For $t \geq 13$ and $n \geq 6$, there exists an optimal $(2,t)$-solvable distribution δ of C_n such that $\delta(v) \geq 2$ for each vertex v of C_n.*

Proof. By Lemma 3.1, we assume that $\delta = \langle x_0, x_1, \cdots, x_{n-1}\rangle$ is an optimal $(2,t)$-solvable distribution of C_n such that $\delta(v) \geq 1$ for each vertex v of C_n. Suppose that $x_j = 1$ for some $j \in [0, n-1]$. Then, let $\delta' = \langle y_0, y_1, \cdots, y_{n-1}\rangle$ be defined as the description in Lemma 3.1. It suffices to show that $\delta'(v_j) \geq 2$ and $\delta'(v_k) \geq 2$ or $\delta'(v_k) \geq x_k$ for $k \in [0, n-1]\backslash\{j\}$ and $|\delta'(C_n)| \leq |\delta(C_n)|$.

By (3.4) and Fact 3, we have $x_{j-2} \geq 2(12+r-x_{j-1})$ and $m(\delta, 2, v_{j-2}) \geq t+11+2r-(2x_{j-1}-\lfloor x_{j-1}/2\rfloor)$.

Case 1. $r = 0$.

If $x_{j-1} \leq 4$, then $x_{j-2} \geq 2(12+0-4) = 16$ and $m(\delta, 2, v_{j-2}) \geq t+11+0-(2\cdot 4-\lfloor 4/2\rfloor) = t+5$. Let $y_j = x_j + 2 = 3$, $y_{j-1} = x_{j-1}+2$, $y_{j-2} = x_{j-2}-8 \geq 8$, and $y_{j-3} = x_{j-3}+4$. By Fact 5, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2,t)$-solvable.

Similarly, if $x_{j+1} \leq 4$, then $x_{j+2} \geq 16$ and $m(\delta, 2, v_{j+2}) \geq t+5$. Let $y_j = x_j+2 = 3$, $y_{j+1} = x_{j+1}+2$, $y_{j+2} = x_{j+2}-8 \geq 8$, $y_{j+3} = x_{j+3}+4$. Then, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2,t)$-solvable.

Now, we assume that $x_{j-1} \geq 5$ and $x_{j+1} \geq 5$. Note that $x_j = 1$ is odd. If x_{j-1} is odd, let $y_j = x_j+1 = 2$, and $y_{j-1} = x_{j-1}-1 \geq 4$. By Fact 10, we only need to check $m(\delta', 2, v_{j-1})$. By using (3.1) and (3.2), we can verify $m(\delta', 2, v_{j-1}) \geq t$.

Similarly, if x_{j+1} is odd, let $y_j = x_j+1 = 2$, and $y_{j+1} = x_{j+1}-1 \geq 4$. Then, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2,t)$-solvable.

Now, we can assume that $x_{j-1} \geq 6$ and $x_{j+1} \geq 6$, and x_{j-1} and x_{j+1} are both even. For the case $x_{j-1} \geq 6$ and $x_{j+1} \geq 8$, let $y_j = x_j+1 = 2$, $y_{j-1} = x_{j-1}-2 \geq 4$, and $y_{j-2} = x_{j-2}+1$. Then, by Fact 8, we only need to check $m(\delta', 2, v_{j-1})$. Also, by using (3.1) and (3.2), it is easy to check that $m(\delta', 2, v_{j-1}) \geq t$.

Similarly, for the case $x_{j-1} \geq 8$ and $x_{j+1} \geq 6$, let $y_j = x_j+1 = 2$, $y_{j+1} = x_{j+1}-2 \geq 4$, and $y_{j+2} = x_{j+2}+1$. Then, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2,t)$-solvable.

For the case $x_{j-1} = x_{j+1} = 6$, let $y_j = x_j+1 = 2$, $y_{j-1} = x_{j-1}-1 = 5$, $y_{j-2} = x_{j-2}-1 \geq 2(t+r-x_j-x_{j-1})-1 \geq 2(13+0-1-6)-1 \geq 11$, and $y_{j-3} = x_{j-3}+1$. Clearly, $|\delta'(C_n)| = |\delta(C_n)|$. If $n \geq 11$, then the vertices

$$v_{j-5}, v_{j-4}, v_{j-3}, v_{j-2}, v_{j-1}, v_j, v_{j+1}, v_{j+2}, v_{j+3}, v_{j+4}, v_{j+5}$$

are all distinct. Otherwise, $6 \leq n \leq 10$. Let $n = 10-k$, where $0 \leq k \leq 4$. Then the vertex $v_{j+5-i} = v_{j-5+k-i}$ for $i \in [0,4]$. By (3.1), it is easy to see that $m(\delta', 2, v_i) = m(\delta, 2, v_i)$ for $i \in [0, n-1] \setminus \{j-5, j-4, j-3, j-2, j-1, j, j+1, j+2\}$. If $n \geq 9$, $v_i \notin \{v_{j-3}, v_{j-2}, v_{j-1}, v_j\}$ for $i \in \{j-7, j-6, j-5, j-4, j+1, j+2, j+3, j+4, j+5\}$. Note that when $n = 9$, $v_{j+5} = v_{j-4}$ and $v_{j+4} = v_{j-5}$, or when $n = 10$, $v_{j+5} = v_{j-5}$. So, for $n \geq 9$, it is easy to see that $m(\delta', 2, v_i) \geq m(\delta, 2, v_i)$ for $i \in \{j-5, j-4, j+1, j+2\}$. By (3.1)–(3.3), we also can verify that $m(\delta', 2, v_i) \geq t$ for $i \in \{j, j-1, j-2, j-3\}$. For the case that $6 \leq n \leq 8$, without loss of generality, let $j = 0$. Then, by (3.1)–(3.3), it is easy to verify that δ' is $(2,t)$-solvable by checking $m(\delta', 2, v_i)$ for each $i \in [0, n-1]$.

Case 2. $r = 1$.

If $x_{j-1} \leq 5$, then $x_{j-2} \geq 2(12+1-5) = 16$ and $m(\delta, 2, v_{j-2}) \geq t+11+2-(2\cdot 5-\lfloor 5/2\rfloor) = t+5$. Let $y_j = x_j+2 = 3$, $y_{j-1} = x_{j-1}+2$, $y_{j-2} = x_{j-2}-8 \geq 8$, and $y_{j-3} = x_{j-3}+4$. By Fact 5, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2,t)$-solvable. If $x_{j+1} \leq 2$, then, by (3.5) and Fact 4, $x_{j+2} \geq 2(t-r-x_j-x_{j+1}) \geq 2(13-1-1-2) = 18$ and $m(\delta, 2, v_{j+2}) \geq \lfloor x_{j+1}/2\rfloor + 2(t-r-x_j-x_{j+1}) \geq \lfloor 2/2\rfloor + t+13-2(1+1+2) = t+6$. Let $y_j = x_j+2 = 3$,

$y_{j+1} = x_{j+1} + 2$, $y_{j+2} = x_{j+2} - 8 \geq 10$, and $y_{j+3} = x_{j+3} + 4$. Then, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2,t)$-solvable. Now, we can assume that $x_{j-1} \geq 6$ and $x_{j+1} \geq 3$. If $x_{j+1} \geq 4$, let $y_j = x_j + 1 = 2$, $y_{j-1} = x_{j-1} - 2 \geq 4$, and $y_{j-2} = x_{j-2} + 1$. By Fact 8, we only need to check $m(\delta', 2, v_{j-1})$. By using (3.1) and (3.2), we can verify that $m(\delta', 2, v_{j-1}) \geq t$. For the case $x_{j-1} \geq 6$ and $x_{j+1} = 3$, let $y_j = x_j + 1 = 2$, $y_{j-1} = x_{j-1} - 2 \geq 4$, $y_{j-2} = x_{j-2} + 1$, $y_{j+1} = x_{j+1} + 1$, $y_{j+2} = x_{j+2} - 2 \geq 2(13 - 1 - 1 - 3) - 2 \geq 14$ and $y_{j+3} = x_{j+3} + 1$. By Fact 8, we only need to check $m(\delta', 2, v_{j-1})$ and $m(\delta', 2, v_{j+2})$. By using (3.1), (3.2) and (3.5), we can verify that $m(\delta', 2, v_{j-1}) \geq t$ and $m(\delta', 2, v_{j+2}) \geq t + 4$.

Case 3. $r \geq 2$.

If $x_{j-1} \leq 6$, then $x_{j-2} \geq 2(12 + 2 - 6) = 16$ and $m(\delta, 2, v_{j-2}) \geq t + 11 + 4 - (2 \cdot 6 - \lfloor 6/2 \rfloor) = t + 6$. Let $y_j = x_j + 2 = 3$, $y_{j-1} = x_{j-1} + 2$, $y_{j-2} = x_{j-2} - 8 \geq 8$, and $y_{j-3} = x_{j-3} + 4$. Then, by Fact 5, we have $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2,t)$-solvable. Otherwise, $x_{j-1} \geq 7$. Let $y_j = x_j + 1 = 2$, $y_{j-1} = x_{j-1} - 2 \geq 5$, and $y_{j-2} = x_{j-2} + 1$. Then, by Fact 8, we only need to check $m(\delta', 2, v_{j-1})$. By using (3.1) and (3.2), we can verify that $m(\delta', 2, v_{j-1}) \geq t$.

Lemma 3.3. *For $t \geq 13$ and $n \geq 6$, there exists an optimal $(2,t)$-solvable distribution δ of C_n such that $\delta(v) \geq 3$ for each vertex v of C_n.*

Proof. By Lemma 3.2, we assume that $\delta = \langle x_0, x_1, \cdots, x_{n-1} \rangle$ is an optimal $(2,t)$-solvable distribution of C_n such that $\delta(v) \geq 2$ for each vertex v of C_n. Suppose that $x_j = 2$ for some $j \in [0, n-1]$. Then, let $\delta' = \langle y_0, y_1, \cdots, y_{n-1} \rangle$ be defined as the description in Lemma 3.1. It suffices to show that $\delta'(v_j) \geq 3$ and $\delta'(v_k) \geq 3$ or $\delta'(v_k) \geq x_k$ for $k \in [0, n-1] \backslash \{j\}$ and $|\delta'(C_n)| \leq |\delta(C_n)|$.

By (3.4) and Fact 3, we have $x_{j-2} \geq 2(11 + r - x_{j-1})$ and $m(\delta, 2, v_{j-2}) \geq t + 10 + 2r - (2x_{j-1} - \lfloor (x_{j-1} + 1)/2 \rfloor)$.

Case 1. $r = 0$.

If $x_{j-1} \leq 3$, then $x_{j-2} \geq 2(11 + 0 - 3) = 16$ and $m(\delta, 2, v_{j-2}) \geq t + 10 + 0 - (2 \cdot 3 - \lfloor (3+1)/2 \rfloor) = t + 6$. Let $y_j = x_j + 2 = 4$, $y_{j-1} = x_{j-1} + 2$, $y_{j-2} = x_{j-2} - 8 \geq 8$, and $y_{j-3} = x_{j-3} + 4$. Similarly, if $x_{j+1} \leq 3$, then $x_{j+2} \geq 16$ and $m(\delta, 2, v_{j+2}) \geq t + 6$. Let $y_j = x_j + 2 = 4$, $y_{j+1} = x_{j+1} + 2$, $y_{j+2} = x_{j+2} - 8 \geq 8$, and $y_{j+3} = x_{j+3} + 4$. By Fact 5, we have $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2,t)$-solvable.

Now, we can assume that $x_{j-1} \geq 4$ and $x_{j+1} \geq 4$. For the case $x_{j-1} = 4$ and $x_{j+1} \geq 4$, we have $x_{j-2} \geq 2(11 + 0 - 4) = 14$ and $m(\delta, 2, v_{j-2}) \geq t + 10 + 0 - (2 \cdot 4 - \lfloor (4+1)/2 \rfloor) = t + 4$. Let $y_j = x_j + 1 = 3$, $y_{j-1} = x_{j-1} + 1$, $y_{j-2} = x_{j-2} - 4 \geq 10$, and $y_{j-3} = x_{j-3} + 2$. Then, by Fact 6, we only need to check $m(\delta', 2, v_{j-1})$. By (3.1) and (3.2), we can verify $m(\delta', 2, v_{j-1}) \geq t$.

Similarly, for the case $x_{j-1} \geq 4$ and $x_{j+1} = 4$, let $y_j = x_j + 1 = 3$, $y_{j+1} = x_{j+1} + 1$, $y_{j+2} = x_{j+2} - 4 \geq 10$, and $y_{j+3} = x_{j+3} + 2$. Then, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2,t)$-solvable. Now, we can assume that $x_{j-1} \geq 5$ and $x_{j+1} \geq 5$. For the case $x_{j-1} = x_{j+1} = 5$, let $y_j = x_j + 2 = 4$, $y_{j-1} = x_{j-1} - 1 = 4$, and $y_{j+1} = x_{j+1} - 1 = 4$. Note that x_{j-1} and x_{j+1} are both odd. By Fact 9, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2,t)$-solvable. For the case $x_{j-1} \geq 5$ and $x_{j+1} \geq 6$, let $y_j = x_j + 1 = 3$, $y_{j-1} = x_{j-1} - 2 \geq 3$, and $y_{j-2} = x_{j-2} + 1$. Then, by Fact 8, we only need to check $m(\delta', 2, v_{j-1})$. By using (3.1) and (3.2), we can verify $m(\delta', 2, v_{j-1}) \geq t$.

Similarly, for the case $x_{j-1} \geq 6$ and $x_{j+1} \geq 5$, let $y_j = x_j + 1 = 3$, $y_{j+1} = x_{j+1} - 2 \geq 3$, and $y_{j+2} = x_{j+2} + 1$. Then, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2,t)$-solvable.

Case 2. $r = 1$.

If $x_{j-1} \leq 4$, then $x_{j-2} \geq 2(11 + 1 - 4) = 16$ and $m(\delta, 2, v_{j-2}) \geq t + 10 + 2 - (2 \cdot 4 - \lfloor 4/2 \rfloor) = t + 6$. Let $y_j = x_j + 2 = 4$, $y_{j-1} = x_{j-1} + 2$, $y_{j-2} = x_{j-2} - 8 \geq 8$, and $y_{j-3} = x_{j-3} + 4$. Then, by Fact 5, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2,t)$-solvable. Otherwise, $x_{j-1} \geq 5$, let $y_j = x_j + 1 = 3$, $y_{j-1} = x_{j-1} - 2 \geq 3$, and $y_{j-2} = x_{j-2} + 1$. By Fact 8, we only need to check $m(\delta', 2, v_{j-1})$. By (3.1) and (3.2), we can verify $m(\delta', 2, v_{j-1}) \geq t$.

Case 3. $r \geq 2$.

If $x_{j-1} \leq 6$, then $x_{j-2} \geq 2(11 + 2 - 6) = 14$ and $m(\delta, 2, v_{j-2}) \geq t + 10 + 4 - (2 \cdot 6 - \lfloor 6/2 \rfloor) = t + 5$. Let $y_j = x_j + 2 = 4$, $y_{j-1} = x_{j-1} + 2$, $y_{j-2} = x_{j-2} - 8 \geq 6$, and $y_{j-3} = x_{j-3} + 4$. Then, by Fact 5, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2, t)$-solvable. Otherwise, $x_{j-1} \geq 7$, let $y_j = x_j + 1 = 3$, $y_{j-1} = x_{j-1} - 2 \geq 5$, and $y_{j-2} = x_{j-2} + 1$. By Fact 8, we only need to check $m(\delta', 2, v_{j-1})$. By (3.1) and (3.2), we can verify $m(\delta', 2, v_{j-1}) \geq t + 1$. □

Lemma 3.4. *For $t \geq 13$ and $n \geq 6$, there exists an optimal $(2, t)$-solvable distribution δ of C_n such that $\delta(v) \geq 3$ for each vertex v of C_n and $x_k = 3$ implies $x_{k+1} \geq 4$ for any $k \in [0, n-1]$.*

Proof. By Lemma 3.3, we assume that $\delta = \langle x_0, x_1, \cdots, x_{n-1} \rangle$ is an optimal $(2, t)$-solvable distribution of C_n such that $\delta(v) \geq 3$ for each vertex v of C_n. Suppose that $x_{j-1} = 3$ and $x_j = 3$ or $x_j = 3$ and $x_{j+1} = 3$ for some $j \in [0, n-1]$. Then, let $\delta' = \langle y_0, y_1, \cdots, y_{n-1} \rangle$ be defined as the description in Lemma 3.1. For the case that $x_{j-1} = 3$ and $x_j = 3$, it suffices to show that $\delta'(v_k) \geq 4$ for $k = j$ or $j - 1$ and $\delta'(v_i) \geq 4$ or $\delta'(v_i) = x_i$ for $i \in [0, n-1] \setminus \{k\}$ and $|\delta'(C_n)| \leq |\delta(C_n)|$. For the case that $x_j = 3$ and $x_{j+1} = 3$, it suffices to show that $\delta'(v_k) \geq 4$ for $k = j$ or $j + 1$ and $\delta'(v_i) \geq 4$ or $\delta'(v_i) = x_i$ for $i \in [0, n-1] \setminus \{k\}$ and $|\delta'(C_n)| \leq |\delta(C_n)|$.

Case 1. $x_{j-1} = 3$ and $x_j = 3$.

By (3.4) and Fact 3, we have

$$\begin{aligned} x_{j-2} &\geq 2(t + r - x_j - x_{j-1}) \\ &= 2(13 + r - 3 - 3) \\ &\geq 14 \end{aligned}$$

and

$$\begin{aligned} m(\delta, 2, v_{j-2}) &\geq \lfloor (\lfloor x_{j-4}/2 \rfloor + x_{j-3})/2 \rfloor + 2(t + r - x_j - x_{j-1}) + \lfloor (x_{j-1} + \lfloor x_j/2 \rfloor)/2 \rfloor \\ &\geq \lfloor (\lfloor 3/2 \rfloor + 3)/2 \rfloor + t + 13 + 2r - 6 - 6 + \lfloor (3 + \lfloor 3/2 \rfloor)/2 \rfloor \\ &\geq t + 5. \end{aligned}$$

Let $y_j = x_j + 2 = 5$, $y_{j-1} = x_{j-1} + 2 = 5$, $y_{j-2} = x_{j-2} - 8 \geq 6$, and $y_{j-3} = x_{j-3} + 4$. Then, by Fact 5, δ' is a desired distribution.

Case 2. $x_j = 3$ and $x_{j+1} = 3$.

If $r = 0$, then by (3.5) and Fact 4, we have $x_{j+2} \geq 14$ and $m(\delta, 2, v_{j+2}) \geq t + 5$. Let $y_j = x_j + 2 = 5$, $y_{j+1} = x_{j+1} + 2 = 5$, $y_{j+2} = x_{j+2} - 8 \geq 6$, and $y_{j+3} = x_{j+3} + 4 \geq 7$. Then, by Fact 5, δ' is a desired distribution. Otherwise, $r \geq 1$. Suppose that $r = 1$. Then, by (3.5) and Fact 4, we have $x_{j+2} \geq 12$ and $m(\delta, 2, v_{j+2}) \geq t + 3$. Let $y_{j+1} = x_{j+1} + 2 = 5$, $y_{j+2} = x_{j+2} - 4 \geq 8$, and $y_{j+3} = x_{j+3} + 2 \geq 5$. Then, by Fact 7, δ' is a desired distribution. Suppose that $r \geq 2$ and $x_{j-1} \leq 5$. Then, by (3.4) and Fact 3, we have $x_{j-2} \geq 14$ and $m(\delta, 2, v_{j-2}) \geq t + 5$. Let $y_j = x_j + 2 = 5$, $y_{j-1} = x_{j-1} + 2 \geq 5$, $y_{j-2} = x_{j-2} - 8 \geq 6$, and $y_{j-3} = x_{j-3} + 4 \geq 7$. Then, by Fact 5, δ' is a desired distribution. Suppose that $r \geq 2$ and $x_{j-1} \geq 6$. Then, by Fact 1, we have $m(\delta, 2, v_{j-1}) \geq t + 1$. If x_{j-1} is odd, let $y_j = x_j + 1 = 4$, $y_{j-1} = x_{j-1} - 1 \geq 5$. We only need to check $m(\delta', 2, v_{j-2})$ and $m(\delta', 2, v_{j-3})$. By using (3.1), it is easy to see that $m(\delta', 2, v_{j-2}) = m(\delta, 2, v_{j-2})$ and $m(\delta', 2, v_{j-3}) = m(\delta, 2, v_{j-3})$. Otherwise, x_{j-1} is even, let

$y_j = x_j + 1 = 4$, $y_{j-1} = x_{j-1} - 2 \geq 4$, and $y_{j-2} = x_{j-2} + 1 \geq 4$. We only need to check $m(\delta', 2, v_{j-1})$. By (3.1) and (3.2), we have

$$\begin{aligned} m(\delta', 2, v_{j-1}) &= \lfloor (\lfloor y_{j-3}/2 \rfloor + y_{j-2})/2 \rfloor + y_{j-1} + \lfloor (y_j + \lfloor y_{j+1}/2 \rfloor)/2 \rfloor \\ &= \lfloor (\lfloor x_{j-3}/2 \rfloor + x_{j-2} + 1)/2 \rfloor + x_{j-1} - 2 + \lfloor (x_j + 1 + \lfloor x_{j+1}/2 \rfloor)/2 \rfloor \\ &\geq \lfloor (\lfloor 3/2 \rfloor + 1)/2 \rfloor + \lfloor x_{j-2}/2 \rfloor + x_{j-1} - 2 + \lfloor (4 + \lfloor x_{j+1}/2 \rfloor)/2 \rfloor \\ &\geq 1 + (t + r - x_j) - 2 + 2 \\ &= t. \end{aligned}$$

This completes the proof. □

We demonstrate the techniques of Lemmas 3.1–3.4 with the following example.

Let $\delta = \langle 4, 8, 12, 0, 3, 14 \rangle$ be a distribution of C_6 such that $0 = x_j$. Note that δ is $(2, 13)$-solvable. By Eq (3.2), $r = 3$. Since $x_{j-1} \geq 9$, by Case 3 in Lemma 3.1, δ is modified to $\langle 4, 10, 8, 2, 3, 14 \rangle$ with $x_j = 2$. Note that at this stage, the new δ is still $(2, 13)$-solvable, $\delta(v) \geq 2$ for all $v \in V(C_6)$ and $r = 2$. Since $x_{j-1} \geq 7$, by Case 3 of Lemma 3.3, δ is modified to $\langle 4, 11, 6, 3, 3, 14 \rangle$. Note that at this stage the new δ is still $(2, 13)$-solvable, $\delta(v) \geq 3$ for all $v \in V(C_6)$, $x_j = x_{j+1} = 3$, and $r = 1$. By Case 2 of Lemma 3.4, δ is modified to $\langle 6, 11, 6, 3, 5, 10 \rangle$. Note that $x_{j+1} \geq 4$ and δ is still $(2, 13)$-solvable. Now, we are in a position to make a $(2, 13)$-solvable δ such that $\delta(v) \geq 4$ for each vertex v of the cycle with the following proposition.

Proposition 3.2. *If $t \geq 13$ and $n \geq 6$, then there exists an optimal $(2, t)$-solvable distribution δ of C_n such that $\delta(v) \geq 4$ for each vertex v of C_n.*

Proof. By Lemma 3.4, we assume that $\delta = \langle x_0, x_1, \cdots, x_{n-1} \rangle$ is an optimal $(2, t)$-solvable distribution of C_n such that $x_i \geq 3$ for all $i \in [0, n-1]$ and $x_k = 3$ implies $x_{k+1} \geq 4$ for any $k \in [0, n-1]$. Suppose that $x_j = 3$ for some $j \in [0, n-1]$. Then, let $\delta' = \langle y_0, y_1, \cdots, y_{n-1} \rangle$ be defined as the description in Lemma 3.1. It suffices to show that $\delta'(v_j) \geq 4$ and $\delta'(v_k) \geq 4$ or $\delta'(v_k) = x_k$ for $k \in [0, n-1] \backslash \{j\}$ and $|\delta'(C_n)| \leq |\delta(C_n)|$.

By (3.4) and Fact 3, we have $x_{j-2} \geq 2(10 + r - x_{j-1})$ and $m(\delta, 2, v_{j-2}) \geq t + 9 + 2r - (2x_{j-1} - \lfloor (x_{j-1} + 1)/2 \rfloor)$.

Case 1. $r = 0$.

If $4 \leq x_{j-1} \leq 6$ and $x_{j+1} \geq 4$, let $y_j = x_j + 1 = 4$, $y_{j-2} = x_{j-2} - 2 \geq 2(t + r - x_j - x_{j-1}) - 2 \geq 2(13 + 0 - 3 - 6) - 2 \geq 6$, and $y_{j-3} = x_{j-3} + 1$. Clearly, $|\delta'(C_n)| = |\delta(C_n)|$. By (3.1), it is easy to see that $m(\delta', 2, v_i) = m(\delta, 2, v_i)$ for $i \in [0, n-1] \setminus \{j-5, j-4, j-3, j-2, j-1, j, j+1, j+2\}$. For $n \geq 9$, $v_i \notin \{v_{j-3}, v_{j-2}, v_j\}$ for $i \in \{j-7, j-6, j-5, j-4, j+1, j+2, j+3, j+4, j+5\}$. By (3.1), it is easy to check that $m(\delta', 2, v_i) \geq m(\delta, 2, v_i)$ for $i \in \{j-5, j-4, j-3, j, j+1, j+2\}$. Hence, we only need to check $m(\delta', 2, v_{j-1})$ and $m(\delta', 2, v_{j-2})$. Note that $x_{j-3} \geq 4$ or $x_{j-4} \geq 4$. It follows that $\lfloor x_{j-4}/2 \rfloor + x_{j-3} \geq 5$. By (3.1)–(3.3), we have

$$\begin{aligned} m(\delta', 2, v_{j-1}) &= \lfloor (x_{j-3} + 1)/4 \rfloor + \lfloor (x_{j-2} - 2)/2 \rfloor + x_{j-1} + \lfloor (x_j + 1 + \lfloor x_{j+1}/2 \rfloor)/2 \rfloor \\ &\geq \lfloor (3 + 1)/4 \rfloor + (t + r - x_j) - 1 + \lfloor (4 + \lfloor 4/2 \rfloor)/2 \rfloor \\ &= t, \end{aligned}$$

and

$$m(\delta', 2, v_{j-2}) = \lfloor (\lfloor x_{j-4}/2 \rfloor + x_{j-3} + 1)/2 \rfloor + x_{j-2} - 2 + \lfloor (x_{j-1} + \lfloor (x_j + 1)/2 \rfloor)/2 \rfloor$$

$$\geq \lfloor(5+1)/2\rfloor + 2(t+r-x_j-x_{j-1}) - 2 + \lfloor(x_{j-1}+\lfloor 4/2\rfloor)/2\rfloor$$
$$\geq t.$$

For the case that $6 \leq n \leq 8$, without loss of generality, let $j = 0$. Then by (3.1)–(3.3), it is easy to verify that δ' is $(2,t)$-solvable by checking $m(\delta', 2, v_i)$ for each $i \in [0, n-1]$.

Similarly, if $x_{j-1} \geq 4$ and $4 \leq x_{j+1} \leq 6$, let $y_j = x_j + 1 = 4$, $y_{j+2} = x_{j+2} - 2 \geq 6$, and $y_{j+3} = x_{j+3} + 1$. Then, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2,t)$-solvable.

Now, we can assume that $x_{j-1} \geq 7$ and $x_{j+1} \geq 7$. If $x_{j-1} = x_{j+1} = 7$, then x_{j-1} and x_{j+1} are both odd. Let $y_j = x_j + 2 = 5$, $y_{j-1} = x_{j-1} - 1 = 6$ and $y_{j+1} = x_{j+1} - 1 = 6$. Then, by Fact 9, we have $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2,t)$-solvable.

For the case $x_{j-1} \geq 7$ and $x_{j+1} \geq 8$, let $y_j = x_j + 1 = 4$, $y_{j-1} = x_{j-1} - 2 \geq 5$, and $y_{j-2} = x_{j-2} + 1$. Then, by Fact 8, we only need to check $m(\delta', 2, v_{j-1})$. By (3.1) and (3.2), we can verify $m(\delta', 2, v_{j-1}) \geq t$.

Similarly, for the case $x_{j-1} \geq 8$ and $x_{j+1} \geq 7$, let $y_j = x_j + 1 = 4$, $y_{j+1} = x_{j+1} - 2 \geq 5$, and $y_{j+2} = x_{j+2} + 1$. Then, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2,t)$-solvable.

Case 2. $r = 1$.

If $x_{j-1} = 4$, then $x_{j-2} \geq 2(10+1-4) = 14$ and $m(\delta, 2, v_{j-2}) \geq t + 9 + 2 - (2 \cdot 4 - \lfloor 4/2 \rfloor) = t + 5$. Let $y_j = x_j + 2 = 5$, $y_{j+1} = x_{j+1} + 2$, $y_{j-2} = x_{j-2} - 8 \geq 6$, and $y_{j-3} = x_{j-3} + 4$. By Fact 5, $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2,t)$-solvable. Otherwise, $x_{j-1} \geq 5$. If $x_{j-1} = 5$, then x_j and x_{j-1} are both odd. Let $y_j = x_j + 1 = 4$ and $y_{j-1} = x_{j-1} - 1 = 4$. By Fact 10, we only need to check $m(\delta', 2, v_{j-1})$. By (3.1) and (3.2), we can verify $m(\delta', 2, v_{j-1}) \geq t$.

Now, we can assume that $x_{j-1} \geq 6$. Let $y_j = x_j + 1 = 4$, $y_{j-1} = x_{j-1} - 2 \geq 4$, and $y_{j-2} = x_{j-2} + 1$. By Fact 8, we only need to check $m(\delta', 2, v_{j-1})$. By using (3.1) and (3.2), we can verify $m(\delta', 2, v_{j-1}) \geq t$.

Case 3. $r \geq 2$.

If $x_{j-1} \leq 5$, then $x_{j-2} \geq 2(10+2-5) = 14$ and $m(\delta, 2, v_{j-2}) \geq t + 9 + 4 - (2 \cdot 5 - \lfloor 5/2 \rfloor) = t + 5$. Let $y_j = x_j + 2 = 5$, $y_{j-1} = x_{j-1} + 2$, $y_{j-2} = x_{j-2} - 8 \geq 6$, and $y_{j-3} = x_{j-3} + 4$. By Fact 5, we have $|\delta'(C_n)| = |\delta(C_n)|$ and δ' is $(2,t)$-solvable. Otherwise, $x_{j-1} \geq 6$. Let $y_j = x_j + 1 = 4$, $y_{j-1} = x_{j-1} - 2 \geq 4$, and $y_{j-2} = x_{j-2} + 1$. By Fact 8, we only need to check $m(\delta', 2, v_{j-1})$. By using (3.1) and (3.2), we can verify $m(\delta', 2, v_{j-1}) \geq t + 1$. □

As a demonstration of the techniques used in the proof of Proposition 3.2, we give one example for each of the three cases.

For $t = 13, n = 6$, we let $\delta = \langle x_{j-3}, x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2} \rangle$.

If $\delta = \langle 4, 12, 4, 3, 4, 12 \rangle$, then, δ is a $(2, 13)$-solvable distribution of C_6 with $x_j = 3$. By Eq (3.2), $r = 0$. Since $x_{j-1} = x_{j+1} = 4$, $\delta' = \langle 5, 10, 4, 4, 4, 12 \rangle$.

If $\delta = \langle 6, 11, 6, 3, 5, 10 \rangle$, then δ is $(2, 13)$-solvable and $r = 1$. Since $x_{j-1} \geq 6$, $\delta' = \langle 6, 12, 4, 4, 5, 10 \rangle$. Note that this is a continuation of the example above Proposition 3.2.

If $\delta = \langle 4, 12, 6, 3, 4, 12 \rangle$, then δ is $(2, 13)$-solvable and $r = 2$. Since $x_{j-1} \geq 6$, $\delta' = \langle 4, 13, 4, 4, 4, 12 \rangle$.

Note that in all examples above $\delta'(v) \geq 4$ for each $v \in V(C_6)$ and δ' is still $(2, 13)$-solvable.

By combining Propositions 3.1 and 3.2, we have the following.

Theorem 3.1. *For $t \geq 13$ and $n \geq 6$, $\pi^*_{(2,t)}(C_n) = \pi^*_{(2,t-10)}(C_n) + 4n$.*

By using Theorem 3.1 repeatedly (if necessary), we have the following.

Corollary 3.1. *For $n \geq 6$, $\pi^*_{(2,10k+r)}(C_n) = \pi^*_{(2,r)}(C_n) + 4kn$, where $k \geq 1$ and $3 \leq r \leq 12$.*

4. The optimal $(2,t)$-pebbling number of C_4

Let $\delta = \langle x_0, x_1, x_2, x_3\rangle$ be a $(2,t)$-solvable distribution of C_4. Note that $v_{i-2} = v_{i+2}$ for all $i \in [0,3]$. Hence, Eq (3.1) is not true for C_4; it must be modified into the following:

$$m(\delta, 2, v_i) = \max\{m_i | y_{i-2} + y_{i+2} = x_{i+2}\}, \tag{4.1}$$

where $m_i = \lfloor(\lfloor y_{i-2}/2\rfloor + x_{i-1})/2\rfloor + x_i + \lfloor(x_{i+1} + \lfloor y_{i+2}/2\rfloor)/2\rfloor$.

This implies that if $x_{i+2} \le 1$, then

$$m(\delta, 2, v_i) = \left\lfloor \frac{x_{i-1}}{2} \right\rfloor + x_i + \left\lfloor \frac{x_{i+1}}{2} \right\rfloor + \lfloor \frac{x_{i+2}}{4} \rfloor.$$

Otherwise,

$$m(\delta, 2, v_i) = x_i + \left\lfloor \frac{1}{2}(x_{i-1} + x_{i+1} + \lfloor \frac{x_{i+2}}{2} \rfloor) \right\rfloor.$$

It follows that $\frac{9}{4}(x_0 + x_1 + x_2 + x_3) \ge 4t$ or $x_0 + x_1 + x_2 + x_3 \ge \frac{16t}{9}$. This implies

$$\pi^*_{(2,t)}(C_4) \ge \left\lceil \frac{16t}{9} \right\rceil. \tag{4.2}$$

It is not difficult to see $\langle 4k, 4k, 4k, 4k\rangle$ is an optimal $(2,9k)$-solvable distribution of C_4 for $k \ge 1$. Thus, we have the following.

Proposition 4.1. *For $k \ge 1$, $\pi^*_{(2,9k)}(C_4) = 16k$.*

Clearly, $\pi^*_{(2,9k+r)}(C_4) \le \pi^*_{(2,9k)}(C_4) + \pi^*_{(2,r)}(C_4)$ for $r \ge 1$. Also, by Proposition 4.1, for $k \ge 1$, we have

$$\pi^*_{(2,9k+r)}(C_4) \le 16k + \pi^*_{(2,r)}(C_4). \tag{4.3}$$

It is known that $\pi^*(C_4) = \lceil \frac{2\times 4}{3} \rceil = 3$, see [2]. So, we have the following.

Proposition 4.2. $\pi^*_{(2,1)}(C_4) = 3$.

Proposition 4.3. *If $t = 9k + 1$ and $k \ge 1$, then $\pi^*_{(2,t)}(C_4) = \lceil 16t/9 \rceil$.*

Proof. For $k = 1$, by (4.2), we have $\pi^*_{(2,10)}(C_4) \ge \lceil 160/9 \rceil = 18$. By (4.1), it is easy to check that $\langle 4, 5, 4, 5\rangle$ is a $(2,10)$-solvable distribution of C_4. Hence, we have $\pi^*_{(2,10)}(C_4) = 18$. For $k \ge 2$, by (4.2), we have $\pi^*_{(2,9k+1)}(C_4) \ge \lceil 16(9k+1)/9 \rceil$. Also, by (4.3), we have $\pi^*_{(2,9k+1)}(C_4) = \pi^*_{(2,9(k-1)+10)}(C_4) \le 16(k-1) + \pi^*_{(2,10)}(C_4) = 16(k-1) + 18 = \lceil 16(9k+1)/9 \rceil$. This completes the proof. □

Proposition 4.4. *For $r \in \{2,3,4,6,7,8\}$, if $t = 9k + r$ and $k \ge 0$, then $\pi^*_{(2,t)}(C_4) = \lceil 16t/9 \rceil$.*

Proof. For $k \ge 0$ and $r \ge 0$, by (4.2), we have $\pi^*_{(2,9k+r)}(C_4) \ge \lceil 16(9k+r)/9 \rceil = 16k + \lceil \frac{16r}{9} \rceil$. Also, by (4.3), we have $\pi^*_{(2,9k+r)}(C_4) \le 16k + \pi^*_{(2,r)}(C_4)$. So, it suffices to prove that $\pi^*_{(2,r)}(C_4) \le \lceil \frac{16r}{9} \rceil$ for $r \in \{2,3,4,6,7,8\}$. We will prove it by constructing a $(2,r)$-solvable distribution with $\lceil \frac{16r}{9} \rceil$ pebbles for $r \in \{2,3,4,6,7,8\}$. Let

$$\delta_2 = \langle 2,0,2,0\rangle,\ \delta_3 = \langle 2,0,4,0\rangle,\ \delta_4 = \langle 2,2,2,2\rangle,$$

$$\delta_6 = \langle 4,2,3,2\rangle,\ \delta_7 = \langle 4,3,3,3\rangle,\ \delta_8 = \langle 4,4,3,4\rangle.$$

It is easy to see $|\delta_r(C_5)| = \lceil \frac{16r}{9} \rceil$ and by using (4.1), it is not difficult to verify that δ_r is a $(2,r)$-solvable distribution of C_4 for each $r \in \{2,3,4,6,7,8\}$. □

Proposition 4.5. *If $t = 9k+5$ and $k \geq 0$, then $\pi^*_{(2,t)}(C_4) = \lceil 16t/9 \rceil + 1$.*

Proof. For $k = 0$, by (4.1), it is easy to check that $\langle 4,2,2,2\rangle$ is a $(2,5)$-solvable distribution of C_4. Hence, we have $\pi^*_{(2,5)}(C_4) \leq 10$. For $k \geq 1$, by $(**2)$, we have $\pi^*_{(2,9k+5)}(C_4) \leq 16k + \pi^*_{(2,5)}(C_4) \leq 16k+10$. Assume that $\delta = \langle x_0, x_1, x_2, x_3\rangle$ is a $(2,9k+5)$-solvable distribution of C_4 with $x_0+x_1+x_2+x_3 = 16k+9$, where $k \geq 0$. Let $x_i = 4k + r_i$, $i = 0,1,2,3$. Then, $r_0 + r_1 + r_2 + r_3 = 9$. Without loss of generality, let $r_0 = \min\{r_i | i = 0,1,2,3\}$. By (4.1), we have $x_0 + (x_1 + x_3)/2 + x_2/4 = 9k + r_0 + (r_1 + r_3)/2 + r_2/4 \geq m(\delta, 2, v_0) \geq 9k+5$. This implies that $r_0+(r_1+r_3)/2+r_2/4 \geq 5$, equivalently, $r_0+r_1+r_2+r_3 \geq 10-r_0+r_2/2$. Thus, $9 \geq 10 - r_0 + r_2/2$, and we have $r_0 \geq 1 + r_2/2 \geq 1 + r_0/2$. Hence, $r_0 \geq 2$. This implies that δ is one of the following distributions:

$$\langle 4k+2, 4k+3, 4k+2, 4k+2\rangle,$$
$$\langle 4k+2, 4k+2, 4k+3, 4k+2\rangle,$$
$$\langle 4k+2, 4k+2, 4k+2, 4k+3\rangle.$$

It is easy to check that δ is not $(2,9k+5)$-solvable, which is a contradiction. Therefore, $\pi^*_{(2,9k+5)}(C_4) \geq 16k + 10 = \lceil 16(9k+5)/9 \rceil + 1$, and we have the proof. □

By Propositions 4.1–4.5, we conclude the following.

Theorem 4.1. *Let t be a positive integer. If $t = 1$ or $t \mod 9 = 5$, then $\pi^*_{(2,t)}(C_4) = \lceil 16t/9 \rceil + 1$. Otherwise, $\pi^*_{(2,t)}(C_4) = \lceil 16t/9 \rceil$.*

5. The optimal $(2,t)$-pebbling number of C_5

Let $\delta = \langle x_0, x_1, x_2, x_3, x_4\rangle$ be a $(2,t)$-solvable distribution of C_5. Then we have

$$x_i + (x_{i+1} + x_{i-1})/2 + (x_{i+2} + x_{i-2})/4 \geq m(\delta, 2, v_i) \geq t. \tag{5.1}$$

for each $i \in [0,4]$. This implies that $\frac{5}{2}(x_0 + x_1 + x_2 + x_3 + x_4) \geq 5t$. Thus,

$$\pi^*_{(2,t)}(C_5) \geq 2t. \tag{5.2}$$

Note that Eq (3.1) is not always true for C_5. For example, let $\delta = \langle 0,1,1,2,0\rangle$. Then, we can move one pebble to v_0 by applying three pebbling moves. First, we can move one pebble to v_2 from v_3, and then move one pebble to v_1 from v_2. Finally, we can move one pebble to v_0 from v_1. But, when we use (3.1), we have $\left\lfloor \frac{1}{2}(\lfloor \frac{x_3}{2} \rfloor + x_4) \right\rfloor + x_0 + \left\lfloor \frac{1}{2}(x_1 + \lfloor \frac{x_2}{2} \rfloor) \right\rfloor = 0$. So, Eq (3.1) should be modified into the following:

$$m(\delta, 2, v_i) = \max\{m_{i,j} | j = 1,2,3\}, \tag{5.3}$$

where

$$m_{i,1} = \lfloor (\lfloor x_{i-2}/2 \rfloor + x_{i-1})/2 \rfloor + x_i + \lfloor (x_{i+1} + \lfloor x_{i+2}/2 \rfloor)/2 \rfloor,$$

$$m_{i,2} = \begin{cases} m_{i,1}, & \text{if } x_{i-2} \le 1; \\ \left\lfloor \frac{\lfloor (x_{i-2}-2)/2 \rfloor + x_{i-1}}{2} \right\rfloor + x_i + \left\lfloor \frac{x_{i+1} + \lfloor (x_{i+2}+1)/2 \rfloor}{2} \right\rfloor, & \text{otherwise,} \end{cases}$$

and

$$m_{i,2} = \begin{cases} m_{i,1}, & \text{if } x_{i+2} \le 1; \\ \left\lfloor \frac{\lfloor (x_{i-2}+1)/2 \rfloor + x_{i-1}}{2} \right\rfloor + x_i + \left\lfloor \frac{x_{i+1} + \lfloor (x_{i+2}-2)/2 \rfloor}{2} \right\rfloor, & \text{otherwise.} \end{cases}$$

By Corollary 2.1, we have the following.

Proposition 5.1. $\pi^*_{(2,10k)}(C_5) = 20k$ *for* $k \ge 1$.

By Proposition 5.1, we have the following.

Lemma 5.1. $\pi^*_{(2,10k+r)}(C_5) \le 20k + \pi^*_{(2,r)}(C_5)$ *for* $k \ge 1$ *and* $r \ge 1$.

Now, we will give a lower bound for $\pi^*_{(2,t)}(C_5)$ when $t \mod 10 \ne 0$.

Lemma 5.2. $\pi^*_{(2,t)}(C_5) \ge 2t + 1$ *when* $t \mod 10 \ne 0$.

Proof. Let $\delta = \langle x_0, x_1, x_2, x_3, x_4 \rangle$ be an optimal $(2,t)$-solvable distribution of C_5. By (5.2), we have $\pi^*_{(2,t)}(C_5) \ge 2t$. By (5.1), we can write $x_i + (x_{i+1} + x_{i-1})/2 + (x_{i+2} + x_{i-2})/4 = t + s_i$, where s_i is a nonnegative real number for each $i \in [0,4]$. If $\pi^*_{(2,t)}(C_5) = 2t$, then $s_i = 0$ for all $i \in [0,4]$, and this system of linear equations can be represented as a matrix.

$$\begin{bmatrix} 1 & 1/2 & 1/4 & 1/4 & 1/2 & t \\ 1/2 & 1 & 1/2 & 1/4 & 1/4 & t \\ 1/4 & 1/2 & 1 & 1/2 & 1/4 & t \\ 1/4 & 1/4 & 1/2 & 1 & 1/2 & t \\ 1/2 & 1/4 & 1/4 & 1/2 & 1 & t \end{bmatrix}.$$

After Gaussian elimination, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2t/5 \\ 0 & 1 & 0 & 0 & 0 & 2t/5 \\ 0 & 0 & 1 & 0 & 0 & 2t/5 \\ 0 & 0 & 0 & 1 & 0 & 2t/5 \\ 0 & 0 & 0 & 0 & 1 & 2t/5 \end{bmatrix}.$$

Thus, the system of equations has a unique solution $x_0 = x_1 = x_2 = x_3 = x_4 = \frac{2t}{5}$. If $t \mod 5 \ne 0$, then the solution is not integral, hence, there exists at least one $i \in [0,4]$ such that $s_i > 0$. This implies $\pi^*_{(2,t)}(C_5) \ge 2t + 1$ because $\pi^*_{(2,t)}(C_5)$ is an integer. If $t \mod 10 = 5$ and $\pi^*_{(2,t)}(C_5) = 2t$, let $t = 10k + 5$, where k is a nonnegative integer, then $\delta = \langle 4k+2, 4k+2, 4k+2, 4k+2, 4k+2 \rangle$. It is easy to check that δ is not $(2,t)$-solvable. It leads to a contradiction and hence $\pi^*_{(2,t)}(C_5) \ge 2t + 1$. This completes the proof. □

It is known that $\pi^*(C_5) = \lceil \frac{2 \times 5}{3} \rceil = 4$, see [2]. So, we have the following.

Proposition 5.2. $\pi^*_{(2,1)}(C_5) = 4$.

In 2018, Chen and Shiue [4] showed that $\pi^*_{(2,2)}(C_3) = 4$ and $\pi^*_{(2,2)}(C_n) = n$ for $n \geq 4$. So, we have the following.

Proposition 5.3. $\pi^*_{(2,2)}(C_5) = 5$.

Lemma 5.3. $\pi^*_{(2,r)}(C_5) = 2r + 1$ *for* $r \in [2, 9] \cup \{11\}$.

Proof. By Proposition 5.3, it is true for $r = 2$. By Lemma 5.2, we have $\pi^*_{(2,r)}(C_5) \geq 2r + 1$ for $r \in [3, 9] \cup \{11\}$. We will prove that the lower bound is also an upper bound by constructing a $(2, r)$-solvable distribution with $2r + 1$ pebbles. Let

$$\delta_3 = \langle 1, 2, 1, 2, 1\rangle,\ \delta_4 = \langle 1, 3, 1, 2, 2\rangle,\ \delta_5 = \langle 3, 2, 2, 2, 2\rangle,$$
$$\delta_6 = \langle 4, 2, 2, 3, 2\rangle,\ \delta_7 = \langle 3, 3, 3, 3, 3\rangle,\ \delta_8 = \langle 3, 4, 3, 3, 4\rangle,$$
$$\delta_9 = \langle 3, 4, 4, 4, 4\rangle,\ \textit{and}\ \delta_{11} = \langle 5, 4, 5, 5, 4\rangle.$$

Clearly, $|\delta_r(C_5)| = 2r + 1$. Then, by using (5.3), it is not difficult to verify that δ_r is a $(2, r)$-solvable distribution of C_5 for each $r \in [3, 9] \cup \{11\}$. □

Proposition 5.4. $\pi^*_{(2,10k+r)}(C_5) = 20k + 2r + 1$ *for* $k \geq 0$ *and* $r \in [2, 9] \cup \{11\}$.

Proof. By Lemma 5.3, it is true for $k = 0$. By combining Proposition 5.1 and Lemma 5.3, we have $\pi^*_{(2,10k+r)}(C_5) \leq 20k + \pi^*_{(2,r)}(C_5) = 20k + 2r + 1$ for $k \geq 1$. By Lemma 5.2, we can see that the upper bound is also a lower bound, hence, we have the proof. □

By combining Propositions 5.1, 5.2, and 5.4, we have the main result of this section.

Theorem 5.1. *Let t be a positive integer. Then,*

$$\pi^*_{(2,t)}(C_5) = \begin{cases} 4, & \textit{if } t = 1; \\ 2t, & \textit{if } t \mod 10 = 0; \\ 2t + 1, & \textit{if } t \geq 2 \textit{ and } t \mod 10 \neq 0. \end{cases}$$

6. Conclusions

In 2018, the distance-restricted pebbling was first proposed by Chen and Shiue [4], and they showed that $\pi^*_{(2,2)}(C_3) = 4$ and $\pi^*_{(2,2)}(C_n) = n$ for $n \geq 4$. In 2020, Shiue [13] studied distance 1-restricted pebbling in cycles, and he determined the exact value of $\pi^*_{(1,t)}(C_n)$ for all $t \geq 1$ and $n \geq 3$; see Theorem 2.1. For $t = 1$, $\pi^*_{(1,1)}(C_n) = \pi^*(C_n) = \left\lceil \frac{2n}{3} \right\rceil$; see [2]. This implies that $\pi^*_{(d,1)}(C_n) = \left\lceil \frac{2n}{3} \right\rceil$ for all $d \geq 1$. Thus, we have $\pi^*_{(2,1)}(C_n) = \left\lceil \frac{2n}{3} \right\rceil$.

"(d, t)-pebbling" can be seen as a generalization of optimal pebbling. In addition to this, it is our belief that (d, t)-pebbling is more applicable than other versions of pebbling. For example, in a transportation and resource allocation system, the resource must be delivered to a target in a set amount of time, thus, in practicality, we need to restrict the distance to the target. Ideally, in such a resource allocation scheme, the storehouses for the resources should have a capacity restriction since the space for the storehouse is limited. Studies on capacity-restricted optimal pebbling can be found in [3, 10]. We discussed the optimal capacity and distance-restricted t-fold pebbling, also known as the optimal (c, d, t)-pebbling number, in [15]. In some sense, $\pi^*_{(d,t)}(G)$ can also be viewed as a distance-based parameter, which is a topic that has been studied before, for example, in [1].

In this article, we give a further study into "(d, t)-pebbling". Our result makes progress towards understanding (d, t)-pebbling for cycles. Specifically, we study $(2, t)$-pebbling as a continuation from the study of $(1, t)$-pebbling done in [13]. We were able to show that the optimal $(2, t)$-pebbling number of cycles can be determined by considering only a finite number of cases, namely, $1 \le t \le 12$. We were also able to completely determine the optimal $(2, t)$-pebbling number for C_4 and C_5. For cycles of length greater than 5, our result relies on repeated application of the tools developed in Facts 1–10. For cycles of length not greater than 5, the key to our results is obtained by modifying Eq (3.1). The process of obtaining a lower bound for $\pi^*_{(2,t)}(C_5)$ involves the solution of a system of linear equations whose matrix is positive semi-definite. For arbitrarily large distance d, the matrix can be very large. Thus, in order to obtain a lower bound for $\pi^*_{(d,t)}(C_n)$ when $d > 2$, one may consider the use of the conjugate gradient method; see [6].

Finally, we will give a summary of progress towards the determination of $\pi^*_{(2,t)}(C_n)$ for all $n \ge 3$ and $t \ge 1$. Corollary 3.1 implies that the problem of determining the exact value of $\pi^*_{(2,t)}(C_n)$ for $n \ge 6$ and $t \ge 1$ can be reduced to the problem of determining the exact value of $\pi^*_{(2,r)}(C_n)$ for $n \ge 6$ and $r \in [1, 12]$. By the discussion above, we know the exact value of $\pi^*_{(2,t)}(C_n)$ for $n \ge 3$ and $t = 1, 2$. Furthermore, note that the diameter of C_3 is equal to one. This implies that $\pi^*_{(2,t)}(C_3) = \pi^*_{(1,t)}(C_3)$ for $t \ge 1$, see Theorem 2.1. Theorems 4.1 and 5.1 give the exact value of $\pi^*_{(2,t)}(C_n)$ for $n = 4, 5$ and $t \ge 1$. So, to completely determine the exact value of $\pi^*_{(2,t)}(C_n)$ for $n \ge 3$ and $t \ge 1$, it is left to solve the following problem.

Problem 2. Determine the exact value of $\pi^*_{(2,t)}(C_n)$ for $n \ge 6$ and $3 \le t \le 12$.

Recently, we have shown that $\pi^*_{(2,3)}(C_n) = \lceil \frac{4n}{3} \rceil$ for $n \ge 4$, see [15]. By combining Corollary 3.1, we have $\pi^*_{(2,10k+3)}(C_n) = \lceil \frac{4n}{3} \rceil + 4kn$ for $n \ge 6$ and $k \ge 0$. For example, by using the two facts that $\langle 4, 0, 0, 4, 0, 0 \rangle$ is an optimal $(2, 3)$-solvable distribution of C_6 (see [15]) and that $\langle 4, 4, 4, 4, 4, 4 \rangle$ is an optimal $(2, 10)$-solvable distribution of C_6 (see [13]), we can conclude that $\langle 8, 4, 4, 8, 4, 4 \rangle$ is an optimal $(2, 13)$-solvable distribution of C_6.

Author contributions

Chin-Lin Shiue: Conceptualization, Supervision, Writing–original draft; Tzu-Hsien Kwong: Conceptualization, Validation, Writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest in this paper.

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