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Research article

Existence, uniqueness and controllability results of nonlinear neutral implicit $\mathscr{ABC}$ fractional integro-differential equations with delay and impulses

Sivaranjani Ramasamy[1,*], **Thangavelu Senthilprabu**[1], **Kulandhaivel Karthikeyan**[1], **Palanisamy Geetha**[2], **Saowaluck Chasreechai**[3,4,*] **and Thanin Sitthiwirattham**[4,5]

[1] Department of Mathematics, KPR Institute of Engineering and Technology, Coimbatore 641 407, Tamilnadu, India

[2] Department of Mathematics, KPR College of Arts Science and Research, Coimbatore 641 407, Tamilnadu, India

[3] Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand

[4] Research Group for Fractional Calculus Theory and Applications, Science and Technology Research Institute, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand

[5] Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok 10300, Thailand

* **Correspondence:** Email: sivaranjanirphd@gmail.com, saowaluck.c@sci.kmutnb.ac.th.

Abstract: In this article, the necessary and sufficient conditions for the existence and uniqueness of the mild solutions for nonlinear neutral implicit integro-differential equations of non-integer order $0 < \alpha < 1$ in the sense of $\mathscr{ABC}$ derivative with impulses, delay, and integro initial conditions were established. The existence results were derived using the semi-group theory, measures of non-compactness, and the fixed-point theory in the sense of Arzelà–Ascoli theorem and Schauder's fixed-point theorem. We analyzed the controllability results of the proposed problem by incorporating the ideas of semi-group theory and fixed-point techniques. The Banach contraction principle was used to derive the uniqueness and controllability of the proposed problem. We provide an example to support the theoretical results.

Keywords: fractional derivative; implicit neutral differential equations; controllability; semi-group and fixed point theories
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1. Introduction

Fractional calculus presents numerous applications in modeling complex problems, and analyses using fixed-point techniques are highly effective in fractional integro-differential equations (FIDEs). Models using FIDEs are better for real-world problems compared to those using local derivatives [1–5]. The results obtained from these FIDEs, with various definitions of non-local (fractional) derivatives, demonstrate their unique applications in scientific and non-scientific fields. Among these derivatives, Atangana and Baleanu introduced a non-local derivative with a non-singular kernel based on the Mittag-Leffler function in the sense of Caputo [6–8]. This definition highlights the importance of the Mittag-Leffler function, and together, they present numerous applications in different areas. The $\mathcal{ABC}$ derivative involves the Mittag-Leffler kernel; as such, it is not affected by the singularities of FIDEs compared to other fractional derivatives. Also, it effectively captures the memory effect of the system , performing better than classical derivatives. The ultimate merit of the $\mathcal{ABC}$ derivative is to maintain physical phenomena while evaluating the existence and uniqueness of mild solutions for fractional differential equations.

Numerous problems in the biomedical field involve sudden state changes. Impulsive differential equations of non-integer orderprovide a clear frameworks for addressing such problems in future investigation. Many research works have been conducted in this area [9, 10], and remarkable results have been obtained. In particular, researchers have analyzed impulsive fractional integro-differential equations (IFIDEs) [9–11] using semigroup theory and fixed-point techniques (FPTs). The study of impulsive problems involving non-integer order derivatives is particularly noteworthy due to their distinctiveness.

Several models for physical phenomena completely rely on historical data. In such cases, delay differential equations (DDEs) [7, 12, 13] are used to model scenarios in fields such as control systems, oceanography, and geography. Researchers have investigated the approximate mild solution of the multi-pantograph DDE of second order with singularity [14, 15]. Models involving DDEs account only for past states but not past rates [12–15].

Many researchers are working on coupled delayed fractional systems [16]. For the first time, the fractional adaptive sliding mode control method is being used to study the projective synchronization of uncertain fractional-order reaction-diffusion systems. Adaptive sliding mode control laws are derived by creating a fractional-order integral-type switching function, which makes the fractional-order sliding mode surface reachable in a finite amount of time. In [17], the Lyapunov functional approach and Fillipov's theory were used to derive a novel algebraic necessary condition for the global ML synchronization of fractional-order memristor neural networks (FOMNNs) with leakage delay via a hybrid adaptive controller. In [18], researchers investigated global Mittag-Leffler synchronization by designing a new fractional integral sliding mode surface and its associated control law. In [19, 20], authors examined the well-posedness of systems of incommensurate delay fractional differential equations (DFDEs) of retarded type with non-vanishing constant delay in the space of continuous functions. The behavior of dynamical systems can occasionally vary due to impulses and abrupt process changes. These changes can be modeled [21] using short-memory fractional differential equations

The existence of mild solutions for the given problem and their stability was discussed by

Reunsumrit et al. [11]:

$$
\begin{aligned}
{}^{\mathcal{ABC}}_{0}D^{\alpha}_{t}[s(t) - \mathcal{N}(t, s(t))] &= \mathcal{I}(t, s(t), \mathcal{L}s(t)),\ 0 < \alpha \le 1, t \in [0, \top] = \mathfrak{J}',\\
\Delta(s)\Big|_{t=t_k} &= I_i(s(t_i^-)),\\
s(0) &= \int_0^{\varrho} \frac{(\varrho - \nu)^{\alpha-1}}{\Gamma(\alpha)} \mathfrak{S}(\nu, s(\nu))d\nu,
\end{aligned}
$$

where ${}^{\mathcal{ABC}}_{0}D^{\alpha}_{t}$- is the $\mathcal{ABC}$ fractional derivative of order α and $\mathfrak{U}, \mathfrak{S} : \mathfrak{J}' \times \mathcal{R} \to \mathcal{R}$ and $\mathfrak{V}, g : \mathfrak{J}' \times \mathcal{R}^2 \to \mathcal{R}$ is a continuous function.

Here, $\mathfrak{L}s(t) = \int_0^t g(t, \tau, \phi(\tau))d\tau$, and $I_i : \mathcal{R} \to \mathcal{R},\ i = 1, 2, ...q,\ 0 = t_0 < t_1 < t_2 < ... < t_q = \top$, $\Delta s|_{t=t_i} = s(t_i^+) - s(t_i^-)$, and $s(t_i^+) = \lim_{\mathrm{h}\to 0^+} s(t_i + \mathrm{h})$ and $s(t_i^-) = \lim_{\mathrm{h}\to 0^-} s(t_i - \mathrm{h})$ represents the limits from the left and right sides of $s(t)$ at $t = t_i$.

Benchohra et al. [12] inspected the existence and stability of the mild solution for the below implicit fractional differential equations (FDEs) involving neutrality and impulses

$$
\begin{aligned}
{}^{c}D^{\alpha}_{t_q}[s(t) - \mathcal{N}(t, s_t)] &= \mathcal{I}(t, s_t, {}^{c}D^{\alpha}_{t_q}s(t)),\ \text{for each } t \in (t_q, t_{q+1}],\ q = 0, 1, ...n,\ 0 < \alpha \le 1,\\
\Delta(s)\Big|_{t=t_q} &= I_q(\phi_{t_q^-}),\ q = 1, ...n,\\
s(t) &= \varphi(t),\ t \in [-\nabla, 0],\ \nabla > 0,
\end{aligned}
$$

where ${}^{c}D^{\alpha}_{t_q}$ represents the fractional derivative in Caputo sense, $\mathcal{I} : [0, \top] \times PC([-\nabla, 0], \mathfrak{R}) \times \mathfrak{R} \to \mathfrak{R}$, $\mathcal{N} : [0, \top] \times PC([-\nabla, 0], \mathfrak{R}) \to \mathfrak{R}$ are the given functions with $\mathcal{I}(0, \varphi) = 0$, $I_q : PC([-\nabla, 0], \mathfrak{R}) \to \mathfrak{R}$, $\varphi \in PC([-\nabla, 0], \mathfrak{R})$, $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = \top$. $\Delta s|_{t=t_q} = s(t_q^+) - s(t_q^-)$, where $s(t_q^+) = \lim_{h\to 0^+} s(t_q + h)$ and $s(t_q^-) = \lim_{h\to 0^-} s(t_q - h)$ denote the limit values approaching from the right and left side of s at $t = t_q$, respectively. Here, $s_t(\theta) = s(t + \theta)$.

Gul et al. [1] researched the existence of the mild solution for BVPs, using the $\mathcal{ABC}$ non-integer order derivative

$$
\begin{aligned}
{}^{\mathcal{ABC}}_{0}D^{\alpha}_{t}[s(t) - \mathcal{N}(t, s(t))] &= \mathcal{I}(t, s(t)),\ 0 < \alpha \le 1,\ t \in [0, \top] = \mathcal{J}',\\
s(0) &= \int_0^{\varrho} \frac{(\varrho - \nu)^{\alpha-1}}{\Gamma(\alpha)} \mathcal{U}(\nu, s(\nu))d\nu.
\end{aligned}
$$

Here, ${}^{\mathcal{ABC}}_{0}D^{\alpha}_{t}$- is the $\mathcal{ABC}$ derivative of non-integer order α and $\mathcal{N}, \mathcal{U}, \mathcal{I} : \mathcal{J}' \times \mathfrak{R} \to \mathfrak{R}$.

Karthikeyan et al. [14] studied the existence of the mild solution for implicit FIDEs using $\mathcal{ABC}$ derivatives as mentioned below:

$$
\begin{cases}
{}^{\mathcal{ABC}}_{0}D^{\alpha}_{t}[s(t) - \mathfrak{N}(t, s_{(t)})] = \mathcal{I}(t, s_t, {}^{\mathcal{ABC}}_{0}D^{\alpha}_{t}s(t)),\ t \in [0, \top] = \mathcal{J},\ 0 < \alpha \le 1,\\
\Delta(s)\Big|_{t=t_i} = I_i(s_{t_i^-}),\\
s(t) = \varphi(t),\ t \in [-\mathfrak{r}, 0],\\
s(0) = \displaystyle\int_0^{\top} \frac{(\top - \mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)} \mathfrak{E}(\mathfrak{z}, s_{\mathfrak{z}})d\mathfrak{z},
\end{cases}
$$

where ${}^{\mathcal{ABC}}_{0}D^{\alpha}_{t}$- is the $\mathcal{ABC}$ fractional derivative of order α, $\mathfrak{P}, \mathfrak{E} : \mathcal{J} \times \mathfrak{R} \to \mathfrak{R}$ and $\mathcal{I} : \mathcal{J} \times \mathfrak{R}^2 \to \mathfrak{R}$ are continuous functions. Where $I_i : \mathfrak{R} \to \mathfrak{R},\ \mathfrak{z} = 1, 2, ...\ell, 0 = t_0 < t_1 < t_2 < ... < t_n = \top$, $\Delta s|_t = t_i =$

$s(t_i^+) - s(t_i^-)$, $s(t_i^-) = \lim_{\mathfrak{r}\to 0^-} s(t_i) - \mathfrak{r})$ and $s(t_i^+) = \lim_{\mathfrak{r}\to 0^+} s(t_i + \mathfrak{r})$ represent the limit of $s(t)$ from right and left respectively, at $t = t_i$. For any $t \in \mathcal{J}$, we represent s_t by $s_t(\theta) = s(t+\theta)$ and $-\mathfrak{r} \le s \le 0$.

Nowadays, many researchers are analyzing the exact and approximate controllability of the above mentioned systems involving different non-local derivatives. Aimene et al. [7] verified the controllability for semi-linear FDEs involving the $\mathcal{ABC}$ derivative and delay.

$$\begin{cases} {}_0^{\mathcal{ABC}}D_t^{\alpha}[s(t)] = As(t) + Bc(t) + G(t, s_t, \chi(\Psi(t))),\ t \in [0, \top] = \mathcal{J},\ 0 < \alpha \le 1, \\ \Delta(s)\Big|_{t=t_i} = I_i(s_{t_i^-}), \\ s(t) = \varphi(t),\ t \in [-\mathfrak{r}, 0]. \end{cases}$$

Here, ${}_0^{\mathcal{ABC}}D_t^{\alpha}$- denotes the $\mathcal{ABC}$ non-integer derivative with order α and $A : D(A) \subset \Omega \to \Omega$ is an infinitesimal generator of $\alpha-$ resolvent family T_α & $\mathcal{S}_\alpha$ for $t \ge 0$ forming the solution operator on the Banach space $(\Omega, \|.\|)$. Let $c \in \mathcal{L}^2([0,\top]; \complement)$, here $\complement$ is also a Banach space and B is a bounded linear operator such that $\mathcal{B} = \{g : [-\mathfrak{r}, 0] \mapsto \Omega\}$, g is continuous everywhere except for a finite number of points r at which $g(r^-)$, $g(r^+)$ exists and $g(r^-) = g(r), g \in \complement(\mathcal{J}\ x\ \mathcal{B}\ x\ \Omega,\ \Omega)$, $\chi \in \complement(\mathcal{J},\ \mathcal{J}^+)$ where $\mathcal{J} = \{t_1, ...t_\ell\}$, $\mathcal{J}^+ = [-\mathfrak{r},\ \top]$ and $t - \mathfrak{r} < \chi(t) < t$, $\mathfrak{r} > 0$. $t \in \mathcal{J}$ *and* $t_i < \chi(t) < t$, $t \in (t_i, t_{i+1}]$, $I \in \complement(\Omega,\ \Omega)$, $\Delta(s)\Big|_{t=t_i} = s(t_i^+) - s(t_i^-)$, $0 = t_0 < t_1 < t_2 < ... < t_n = \top$, *for* $n = 1, 2, ..., \ell$, with respect to the right and left side approach, respectively, at $t = t_i$ is $s(t_i^+)$ *and* $s(t_i^-)$. $s_t \in \mathcal{B}$ satisfies $s_t(\theta) = s(t+\theta), \theta \in [-\mathfrak{r}, 0]$. $s_t(.)$ is the history of the state from $t - \mathfrak{r}$ to t.

The controllability of the above defined system has been derived by researchers with the help of k-set contraction mapping.

Inspired by the above mentioned research articles, we aim to investigate nonlinear implicit fractional systems involving impulses and delay in terms of $\mathcal{ABC}$, the non-local derivative of the form,

$$\begin{cases} {}_0^{\mathcal{ABC}}D_t^{\alpha}[s(t) - \mathfrak{N}(t, s(t))] = As(t) + Bc(t) + \mathcal{I}(t, s_t, {}_0^{\mathcal{ABC}}D_t^{\alpha}s(t), c(t)),\ t \in [0, \top] = \mathcal{J},\ 0 < \alpha \le 1, \\ \Delta(s)\Big|_{t=t_i} = I_i(s_{t_i^-}), \\ s(t) = \varphi(t),\ t \in [-\mathfrak{r}, 0], \\ s(0) = \displaystyle\int_0^{\top} \frac{(\top - \mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)} G(\mathfrak{z}, s_{\mathfrak{z}}) d\mathfrak{z}, \end{cases} \tag{1.1}$$

where ${}_0^{\mathcal{ABC}}D_t^{\alpha}$- denotes the $\mathcal{ABC}$ non-integer derivative with order α and $A : D(A) \subset \Omega \to \Omega$ is an infinitesimal generator of $\alpha-$ resolvent family T_α & $\mathcal{S}_\alpha$ for $t \ge 0$ forming the solution operator on the Banach space $(\Omega, \|.\|)$. Let $c \in \mathcal{L}^2([0,\top]; \complement)$, here $\complement$ is also a Banach space and B is a bounded linear operator such that $B : \complement \mapsto \Omega$. The functions $\mathfrak{N}, \mathfrak{G} : \mathcal{J} \times \mathfrak{R} \to \mathfrak{R}$ and $\mathcal{I} : \mathcal{J} \times \mathfrak{R}^3 \to \mathfrak{R}$ are continuous. Also, ${}_0^{\mathcal{ABC}}D_t^{\alpha}$- denotes the $\mathcal{ABC}$ non-integer derivative with order α, and $\mathfrak{N}, \mathfrak{G} : \mathcal{J} \times \mathfrak{R} \to \mathfrak{R}$ are continuous functions. Also, $I_i : \mathfrak{R} \to \mathfrak{R}$, $i = 1, 2, ...n, 0 = t_0 < t_1 < t_2 < ... < t_n = \top$, $\Delta s|_t = t_i = s(t_i^+) - s(t_i^-)$, $s(t_i^-) = \lim_{\mathfrak{r}\to 0^-} s(t_i) - \mathfrak{r})$ and $s(t_i^+) = \lim_{\mathfrak{r}\to 0^+} s(t_i + \mathfrak{r})$ denotes the limit of $s(t)$ with respect to the right and left side approach, respectively, at $t = t_i$. For any $t \in \mathcal{J}$, we represent s_t by $s_t(h) = s(t+h)$ and $-\mathfrak{r} \le s \le 0$.

The remaining of this paper is organized as follows: rudimentary concepts, like definitions and lemmas, are given in Section 2. The existence of the mild solution for nonlinear neutral implicit impulsive FIDEs involving $\mathcal{ABC}$ fractional derivative with delay is verified in Section 3. The controllability of the nonlinear neutral implicit impulsive FIDEs involving $\mathcal{ABC}$ with delay is examined in Section 4. An example is provided in Section 5 to demonstrate the applicability of the proposed problem.

2. Preliminaries

Let us define $\grave{\mathcal{P}C}([-\mathfrak{r},0],\mathfrak{R}) = \{s : [-\mathfrak{r},0] \to \mathfrak{R} : s \in \grave{C}((t_i,t_{i+1}],\mathfrak{R}),\ i = 0,1,...\ell,$ and $\exists\ s(t_i^-)$ and $s(t_i^+),\ i = 1....\ell,$ with $s(t_i^-) = s(t_i^+)\}$.

$\grave{\mathcal{P}C}([-\mathfrak{r},0],\mathfrak{R})$ denotes the Banach space, having norm $\|s\|_{\grave{\mathcal{P}C}} = \sup_{t\in[-\mathfrak{r},0]} \|s(t)\|$.

$\grave{\mathcal{P}C}_1([0,\top],\mathfrak{R}) = \{s : [0,\top] \to \mathfrak{R} : s \in \grave{C}((t_i,t_{i+1}],\mathfrak{R}),\ i = 0,1,...\ell,$ and $\exists\ s(t_i^-)$ and $s(t_i^+),\ i = 1,...,\ell,$ with $s(t_i^-) = s(t_i^+)\}$. $\grave{\mathcal{P}C}_1([0,\top],\mathfrak{R})$ represents the Banach space, having norm $\|s\|_{\grave{\mathcal{P}C}_1} = \sup_{t\in[0,\top]} \|s(t)\|$,

$$\Omega = \left\{ s : [-\mathfrak{r},\top] \to \mathfrak{R} : s|_{[-\mathfrak{r},0]} \in \grave{\mathcal{P}C}([-\mathfrak{r},0],\mathfrak{R}) \text{ and } s|_{[0,\top]} \in \grave{\mathcal{P}C}_1([0,\top],\mathfrak{R}) \right\}.$$

Ω holds the properties of Banach space with norm $\|s\|_\Omega = \sup_{t\in[-\mathfrak{r},\top]} \|s(t)\|$.

Remark 1 ([7,8,10,12])**.** *$\grave{\mathcal{P}C}([0,\top],\mathfrak{R})$ is the Banach space, which is a complete normed vector space ($\grave{\mathcal{P}C}, \|.\|$) with the following properties:*

(1) $\|f\| \geq 0$ and $\|f\| = 0$ if and only if $f = 0, \quad \forall\ f \in \grave{\mathcal{P}C}([0,\top],\mathfrak{R})$.

(2) $\|\beta f\| = |\beta| \quad \|f\|$, where β is a scalar, $\forall f \in \grave{\mathcal{P}C}([0,\top],\mathfrak{R})$ and $\beta \in \mathfrak{R}$.

(3) $\|f + g\| \leq \|f\| + \|g\|, \forall\ f$ and $g \in \grave{\mathcal{P}C}([0,\top],\mathfrak{R})$.

Definition 1 ([7,8,10,12])**.** *The non-integer order $\mathcal{ABC}$ derivative of $f(t)$ is*

$${}_0^{\mathcal{ABC}}D_t^\alpha f(t) = \frac{\mathbb{N}(\alpha)}{1-\alpha} \int_0^\top f'(\mathfrak{z})\mathbb{E}_\alpha \left[\frac{-\alpha(t-\mathfrak{z})}{1-\alpha} \right] d\mathfrak{z},$$

where, $\alpha \in (0,1]$ and $\alpha \in \mathbb{E}^1(0,\top)$. $\mathbb{N}(\alpha)$ is the normalization function satisfying $\mathbb{N}(0) = \mathbb{N}(1) = 1$ and $\mathbb{E}_\alpha = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{(\alpha i + 1)}$ is a special function, introduced by Mittag-Leffler.

Definition 2 ([7,8,10,12])**.** *The non-integer order $\mathcal{ABC}$ integral of f is*

$${}_0^{\mathcal{ABC}}\mathbb{I}_t^\alpha f(t) = \frac{1-\alpha}{\mathbb{N}(\alpha)} f(t) + \frac{\alpha}{\mathbb{N}(\alpha)} \int_0^t \frac{(t-\mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)} f(\mathfrak{z}) d\mathfrak{z},$$

where I^α represents the R-L fractional integral.

Remark 2 ([7,8,10,12])**.** *Some important properties of $\mathcal{ABC}$ derivative and the generalized Mittag-Leffler function during the implementation of Laplace transform are as follows:*

(1) $\mathcal{L}[{}^{\mathcal{ABC}}D_{a^+}^\alpha f(t)](s) = \dfrac{\mathbb{N}\alpha}{1-\alpha} \mathcal{L}[\mathbb{E}_\alpha(-\lambda t^\alpha)(s)[s\mathcal{L}(f(t))(s) - f(0)]].$

(2) $\mathcal{L}[t^(\alpha-1)\mathbb{E}_{\alpha,\alpha}(-\lambda t^{\alpha})(s)] = \dfrac{s^{\alpha}-\alpha}{s^{\alpha}+\lambda}$.

(3) $\mathcal{L}[t^(\alpha)](s) = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$.

(4) $\mathcal{L}[f(t) * \Psi(t)](s) = \mathcal{L}[f(t)](s)\mathcal{L}[\Psi(t)](s)$.

Definition 3 ([7, 8, 10, 12]). *The Kurtawoski measure of non-compactness Υ on a bounded set $\mathcal{B} \subset \mathbb{Y}$ is considered as follows:*

$$\Upsilon(\grave{\mathbb{L}}) = inf\Big\{\epsilon > 0 \quad implies \quad \grave{\mathbb{L}} \subset \bigcup_{j=1}^{m} M_j \quad also \quad diam(M_j) \leq \epsilon\Big\},$$

with the following properties:

(1) $\grave{\mathbb{L}}_1 \subset \grave{\mathbb{L}}_2$ *gives* $\Upsilon(\grave{\mathbb{L}}_1) \leq \Upsilon(\grave{\mathbb{L}}_2)$ *where* $\grave{\mathbb{L}}_1, \grave{\mathbb{L}}_2$ *are bounded subsets of* $\mathbb{Y}$.

(2) $\Upsilon(\grave{\mathbb{L}}) = 0$ *iff* $\grave{\mathbb{L}}$ *is relatively compact in* $\mathbb{Y}$.

(3) $\Upsilon(\{z\} \bigcup \grave{\mathbb{L}}) = \Upsilon(\grave{\mathbb{L}})$ *for all* $z \in \mathbb{Y}, \grave{\mathbb{L}} \subseteq \mathbb{Y}$.

(4) $\Upsilon(\grave{\mathbb{L}}_1 \bigcup \grave{\mathbb{L}}_2) \leq max\{\Upsilon(\grave{\mathbb{L}}_1), \Upsilon(\grave{\mathbb{L}}_2)\}$.

(5) $\Upsilon(\grave{\mathbb{L}}_1 + \grave{\mathbb{L}}_2) \leq \Upsilon(\grave{\mathbb{L}}_1) + \Upsilon(\grave{\mathbb{L}}_2)$.

(6) $\Upsilon((\mathfrak{z})\grave{\mathbb{L}}) \leq |(\mathfrak{z})|\Upsilon(\grave{\mathbb{L}})$ *for* $(\mathfrak{z}) \in R$.
Let $\grave{\mathcal{M}} \subset C(I, \mathbb{Y})$ *and* $\grave{\mathcal{M}}((\mathfrak{z})) = \Big\{\upsilon(r) \in \mathbb{Y} | \upsilon \in \grave{\mathcal{M}}\}$. *We define*

$$\int_0^t \grave{\mathcal{M}}((\mathfrak{z}))d\mathfrak{z} = \Big\{\int_0^t \upsilon((\mathfrak{z}))d\mathfrak{z} | \upsilon \in \grave{\mathcal{M}}\Big\}, \quad t \in \grave{\mathbb{J}}.$$

Proposition 1 ([7, 8, 12]). *If* $\grave{\mathbb{M}} \subset C(\grave{\mathbb{J}}, \mathbb{Y})$ *is equi-continuous and bounded, then* $t \to \Upsilon(\grave{\mathbb{M}}(t))$ *is continuous on I, also*

$$\Upsilon(\grave{\mathbb{M}}) = max\Upsilon(\grave{\mathbb{M}}(t)), \ \ \Upsilon\bigg(\int_0^t \upsilon(\mathfrak{z})d\mathfrak{z}\bigg) \leq \int_0^t \Upsilon(\upsilon(\mathfrak{z}))d\mathfrak{z}, \quad for \quad t \in I.$$

Proposition 2 ([7, 8, 12]). *Let the functions* $\{\upsilon_n : \grave{\mathbb{J}} \to \mathbb{Y}, n \in N\}$ *be Bochner integrable. For* $n \in N, \|\upsilon_n\| \leq m(t)$ *a.e* $m \in \mathcal{L}^1(\mathcal{I}, \mathcal{R}^+)$ *and* $\xi(t) = \Upsilon(\{\upsilon_n(t)\}_{n=1}^{\infty}) \in \mathcal{L}^1(\mathcal{I}, \mathcal{R}^+)$, *then it satisfies*

$$\Upsilon(\{\int_0^t \upsilon_n(\mathfrak{z})d\mathfrak{z} : n \in N\}) \leq 2\int_0^t \xi(\mathfrak{z})d\mathfrak{z}.$$

Proposition 3 ([7, 8, 12]). *Let* $\grave{\mathcal{M}}$ *be a bounded set. Then, for each* $\zeta > 0$, *there exists a sequence* $\{\upsilon_n\}_{n=1}^{\infty} \subset \grave{\mathcal{M}}$, *such that*

$$\Upsilon(\grave{\mathcal{M}}) \leq 2\Upsilon\{\upsilon_n\}_{n=1}^{\infty} + \zeta.$$

Definition 4 ([7,8,12]). *Let* $0 < \mu < \pi$ *and* $-1 < \beta < 0$. *We define* $S^0_\mu = \{\grave{\upsilon} \in C \setminus \{0\}$ *that is* $|arg\grave{\upsilon}| < \mu\}$ *and the closure of the form* S_μ, *that is*

$$S_\mu = \{\upsilon \in C \setminus \{0\} | arg\grave{\upsilon}| < \mu\} \bigcup \{0\}.$$

Definition 5 ([7,8,12]). *For* $-1 < \beta < 0, 0 < \omega < \frac{\pi}{2}$, *we define* $\{\odot^\beta_\omega\}$ *as a family of all closed linear operators* $A : D(A) \subset \Omega \to \Omega$; *this implies*

(1) $\sigma(A) \in S_\omega$, *where* $\sigma(A)$ *is the spectrum, which is a complement of the resolvent set.*

(2) For all $\mu \in (\omega, \pi), \exists M_\mu$ *implies* $||\mathcal{R}(z, A)||_{L(X)} \leq M_\mu |z|^\beta$, *where* $\mathcal{R}(z, A) = (zI - A)^{-1}$ *is the resolvent operator and* $A \in \odot^\beta_\omega$ *is said to be an Almost Sectorial operator on* Ω.

Proposition 4 ([7,8,12]). *Let* $A \in \odot^\beta_\omega$ *for* $-1 < \beta < 0$ *and* $0 < \omega < \frac{\pi}{2}$ *and we define* $\{\odot^\beta_\omega\}$ *as a family of all closed linear operators* $A : D(A) \subset \Omega \to \Omega$. *Then, the following properties are fulfilled:*

(1) $\grave{\mho}(t)$ *is analytic and* $\frac{d^n}{dt^n}\grave{\mho}(t) = (-A^n\grave{\mho}(t)(t \in S^0_{\frac{\pi}{2}})$;

(2) $\grave{\mho}(t + s) = \grave{\mho}(t)\grave{\mho}(s) \quad \forall \ t, s \in S^0_{\frac{\pi}{2}}$;

(3) $||\grave{\mho}(t)||_{L(\Omega)} \leq C_0 t^{-\beta-1}(t > 0)$; *where* $C_0 = C_0(\beta) > 0$ *is a constant;*

(4) Let $\sum_{\grave{\mho}} = \{x \in \Omega \ : lim_{t\to 0_+}\grave{\mho}(t)x = x\}$. *Then* $\mathsf{D}(A^\Upsilon) \subset \sum_{\grave{\mho}}$ *if* $\Upsilon > 1 + \beta$;

(5) $\mathcal{R}(z, -A) = \int_0^\infty e^{-zs}\grave{\mho}(s)ds$, $z \in \mathbb{C}$ *with* $Re(z) > 0$;

(6) The range $\mathcal{R}(\grave{\mho}(t))$ *of* $\grave{\mho}(t)$, $t \in S^0_{\frac{\pi}{2}-\omega}$ *is contained in* $\mathsf{D}(\mathsf{A})^\infty$. *Particularly,* $\mathcal{R}(\grave{\mho}(t)$ *is contained in* $\mathsf{D}(\mathsf{A})^\beta$ *for all* $\beta \in C$ *with* $Re\beta > 0$,

$$A^\beta\grave{\mho}(t)x = \frac{1}{2\pi i}\int_{\Gamma\theta} z^\beta e^{-tz}\mathcal{R}(z : A)xdz$$

for all $x \in X$, *and hence there exists a constant* $C' = C'(\varphi, \beta) > 0$, *such that*

$$||A^\beta\grave{\mho}(t)|| \leq C' t^{-\varphi-Re\beta-1}$$

for all $t > 0$.

Remark 3. $\grave{\mho}(t)$ *is a* C_0 *semi-group operator of an infinitesimal generator* A.

Definition 6 ([7,8,12]). *Observe the system represented by the problem given below:*

$$\begin{aligned} {}^{\mathcal{ABC}}_{0}D^\alpha_t s(t) &= f(t), \\ s(0) &= s_0. \end{aligned}$$

The mild solution of the given problem is of the form,

$$s(0) = s_0 + \frac{1-\alpha}{\mathbb{N}(\alpha)}f(t) + \frac{\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_0^t (t - \mathfrak{z})^{\alpha-1} f(\mathfrak{z})d\mathfrak{z}.$$

Proof. From Definition 2, we obtain

$$\begin{aligned} s(t) &= s_0 + {}_0^{\mathcal{ABC}}I_t^{\alpha} f(t) \\ &= s_0 + + \frac{1-\alpha}{\mathbb{N}(\alpha)} f(t) + \frac{\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^t (t-\mathfrak{z})^{\alpha-1} f(\mathfrak{z}) d\mathfrak{z}. \end{aligned}$$

□

Theorem 1 ([7, 8, 12]). *Let (Ω, d) be a complete metric space. Then, a function $\Psi : \Omega \to \Omega$ is said to be a contraction mapping if there is a constant α with $0 \le \alpha < 1$ such that for all $x, y \in \Omega$, $d(\Psi(x), \Psi(y)) \le \alpha d(x, y)$.*

Theorem 2 (Banach contraction principle). *Let $\Psi : \Omega \to \Omega$ represent the completely continuous operator on the Banach space Ω. Consider that the set $\daleth = \{s \in \Omega : s = \lambda \Xi s, \text{for some } \lambda \in (0,1)\}$ is bounded, then Ψ has fixed points.*

Theorem 3 (Arzelà–Ascoli theorem). *Let Ω, d be a compact space. A subset Ω_{μ_b} of $\complement(\Omega)$ is relatively compact if and only if Ω_{μ_b} is uniformly bounded and equi-continuous.*

Theorem 4 (Schauder's fixed-point theorem). *Let Ω, d be a complete metric space. Let Ω_{μ_b} be a closed convex subset of Ω and let $\Psi : \Omega_{\mu_b} \to \Omega_{\mu_b}$ be a mapping such that the set $\{\Psi s : s \in \Omega_{\mu_b}\}$ is relatively compact in Ω, then Ψ has at least one fixed point.*

Lemma 1 ([12]). *Let the real function $\nu(.) : [0, \top] \mapsto (0, \infty)$ and $\rho(t)$ be a non-negative, locally integrable on $[0, \top]$, and assume the constants $c_1 > 0$ & $0 < c_2 \le 1$ such that*

$$\nu(t) \le \rho(t) + c_1 \int_0^t (t-\mathfrak{z})^{-c_2} \nu(\mathfrak{z}) d\mathfrak{z},$$

which implies a constant $\complement = \complement(c_2)$ such that

$$\nu(t) \le \rho(t) + \complement c_1 \int_0^t (t-\mathfrak{z})^{-c_2} \nu(\mathfrak{z}) d\mathfrak{z}, \text{ for every } t \in [0, \top].$$

Lemma 2 ([7, 8, 12]). *Let the BVP with nonlinear integral boundary conditions, if $f \in L(\mathcal{J})$,*

$$\begin{aligned} {}_0^{\mathcal{ABC}}D_t^{\alpha} s(t) &= f(t), \ o < \alpha < 1, \ t \in \mathcal{J}, \\ s(0) &= \int_0^{\top} \frac{(\top-\mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)} \mathbb{E}(\mathfrak{z}, s(\mathfrak{z})) d\mathfrak{z}, \end{aligned}$$

then, the mild solution $s \in \mathcal{AC}(\mathcal{J})$ is,

$$\begin{aligned} s(t) =& PT_{\alpha} \int_0^{\top} \frac{(\top-\mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)} \mathbb{E}(\mathfrak{z}, s(\mathfrak{z})) d\mathfrak{z} + QP \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^t (t-\mathfrak{z})^{\alpha-1} f(\mathfrak{z}) d\mathfrak{z} \\ &+ \frac{\alpha P^2}{\mathbb{N}(\alpha)} \int_0^t \mathcal{S}_{\alpha}(t-\mathfrak{z}) f(\mathfrak{z}) d\mathfrak{z}. \end{aligned} \tag{2.1}$$

Here, P and Q represents the linear operators, $P = \kappa(\kappa I - \mathcal{A})^{-1}$ and $Q = -\eta A(\kappa I - A)^{-1}$, where $\kappa = \frac{\mathbb{N}(\alpha)}{1-\alpha}$,

$$\begin{aligned} T_{\alpha} &= \hat{E}_{\alpha}(-Q(t)^{\alpha}) = \frac{1}{2\pi i} \int_{\Gamma} e^{\nu t} \nu^{\alpha-1} (\nu^{\alpha} I - Q)^{-1} d\nu, \\ \mathcal{S}_{\alpha} &= t^{\alpha-1} \hat{E}_{\alpha,\alpha}(-Q(t)^{\alpha}) = \frac{1}{2\pi i} \int_{\Gamma} e^{\nu t} (\nu^{\alpha} I - Q)^{-1} d\nu. \end{aligned}$$

Proof. We easily prove result (2.1) through Lemma 2 directly by substituting s_0 as the boundary condition. □

Definition 7 ([7, 8, 12]). *Let the BVP with nonlinear integral boundary conditions, if $f \in L(\mathcal{J})$,*

$$\begin{cases} {}^{\mathbb{ABC}}_{0}D^{\alpha}_{t}[s(t)] = As(t) + Bc(t) + G(t, s_t, \chi(\Psi(t))),\ t \in [0, \top] = \mathcal{J},\ 0 < \alpha \leq 1, \\ \Delta(s)\Big|_{t=t_i} = I_i(s_{t_i^-}), \\ s(t) = \varphi(t),\ t \in [-\mathfrak{k}, 0], \end{cases} \tag{2.2}$$

then, the mild solution $s \in \grave{\mathcal{PC}}(\mathcal{J})$ is,

$$s(t) = \begin{cases} \varphi(t),\ t \in [-\mathfrak{k}, 0] \\ PT_{\alpha}s_0 + QP\dfrac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\displaystyle\int_0^t (t-\mathfrak{z})^{\alpha-1} \\ \times[B(c_S(\mathfrak{z})) + G(\mathfrak{z}, s_{\mathfrak{z}}, \chi(\Psi(\mathfrak{z})))]d\mathfrak{z} + \dfrac{\alpha P^2}{\mathbb{N}(\alpha)}\displaystyle\int_0^t \mathcal{S}_{\alpha}(t-\mathfrak{z})[B(c_S(\mathfrak{z})) + G(\mathfrak{z}, s_{\mathfrak{z}}, \chi(\Psi(\mathfrak{z})))]d\mathfrak{z},\ \text{if } t \in [0, t_1], \\ PT_{\alpha}(t-t_j)s(t_j^{-1})) + QP\dfrac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\displaystyle\int_{t_j}^t (t-\mathfrak{z})^{\alpha-1}[B(c_S(\mathfrak{z})) + G(\mathfrak{z}, s_{\mathfrak{z}}, \chi(\Psi(\mathfrak{z})))]d\mathfrak{z} \\ +\frac{\alpha P^2}{\mathbb{N}(\alpha)}\displaystyle\int_{t_j}^t \mathcal{S}_{\alpha}(t-\mathfrak{z})[B(c_S(\mathfrak{z})) + G(\mathfrak{z}, s_{\mathfrak{z}}, \chi(\Psi(\mathfrak{z})))]d\mathfrak{z} + I_j(s(t_j^-)),\ \text{if } t \in (t_j, t_{j+1}],\ j = 1, 2, ...m. \end{cases} \tag{2.3}$$

Here, P and $\mathbb{Q}$ represents the linear operators. $P = \kappa(\kappa I - \mathcal{A})^{-1}$ and $\mathbb{Q} = -\eta A(\kappa I - A)^{-1}$ where $\kappa = \frac{\mathbb{N}(\alpha)}{1-\alpha}$ and

$$T_{\alpha} = \hat{E}_{\alpha}(-\mathbb{Q}(t)^{\alpha}) = \frac{1}{2\pi i}\int_{\Gamma} e^{\nu t}\nu^{\alpha-1}(\nu^{\alpha}I - \mathbb{Q})^{-1}d\nu,$$
$$\mathcal{S}_{\alpha} = t^{\alpha-1}\hat{E}_{\alpha,\alpha}(-\mathbb{Q}(t)^{\alpha}) = \frac{1}{2\pi i}\int_{\Gamma} e^{\nu t}(\nu^{\alpha}I - \mathbb{Q})^{-1}d\nu.$$

Definition 8 ([7, 8, 12]). *The equivalent fractional solution integral for the prescribed system* (1.1) *is*

$$s(t) = \begin{cases} \varphi(t),\ t \in [-\mathfrak{r}, 0], \\ \Re(t, s_t) + PT_{\alpha}\displaystyle\int_0^{\top} \frac{(\top-\mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)}\mathbb{E}(\mathfrak{z}, s(\mathfrak{z}))d\mathfrak{z} + QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_0^t (t-\mathfrak{z})^{\alpha-1} \\ \times[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + \frac{\alpha P^2}{\mathbb{N}(\alpha)}\int_0^t \mathcal{S}_{\alpha}(t-\mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z},\ \text{if } t \in [0, t_1], \\ \\ \Re(t, s_t) + PT_{\alpha}(t_j - t_{j-1})s(t_{j-1}^{-1})) + QP\dfrac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\displaystyle\int_{t_i}^t (t-\mathfrak{z})^{\alpha-1}[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} \\ +\displaystyle\sum_{j=1}^{i} QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_{t_{j-1}}^{t_j} (t_j-\mathfrak{z})^{\alpha-1}[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} \\ +\frac{\alpha P^2}{\mathbb{N}(\alpha)}\displaystyle\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j} \mathcal{S}_{\alpha}(t_j-\mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} \\ +\frac{\alpha P^2}{\mathbb{N}(\alpha)}\displaystyle\int_{t_i}^t \mathcal{S}_{\alpha}(t-\mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + PT_{\alpha}(t)\sum_{j=1}^{i} I_j(s(t_j^-)),\ \text{if } t \in (t_i, t_{i+1}]. \end{cases} \tag{2.4}$$

Definition 9 ([7,8,12]). *Let $\phi \in \Omega$ be an initial function and $s_a \in \grave{\mathcal{P}}C([0, \top], \mathfrak{R}) \subset \Omega$, then there exists a control $c \in L^2(\mathcal{J}, \complement)$, corresponding to the mild solution $s(t)$ of (1.1), that fulfills $s(\top) = s_a$; then the system is controllable on* $[0, \top]$.

Remark 4 ([7, 8, 10, 12]). *The readers may verify the mild solution and the solution operator in [7, 8, 12].*

Remark 5 ([7, 8, 10, 12]). *If $A \in A^{\alpha}(\alpha_0,\beta_0)$, then $\|T_\alpha(t)\| \leq Re^{\beta t}$ and $\|\mathcal{S}_\alpha(t) \leq Qe^{\beta t}(1 + t^{\alpha-1})\|$ for all $t > 0$, $\beta > \beta_0$. Therefore, we get $\hat{R} = \sup_{t\geq 0} \|T_\alpha(t)\|, \hat{R}_1 = \sup_{t\geq 0} Qe^{\beta t}(1 + t^{\alpha-1})$ and so $\|T_\alpha(t)\| \leq \hat{R}; \|\mathcal{S}_\alpha(t)\| \leq t^{\alpha-1}\hat{R}_1$.*

3. Existence and uniqueness results

We examine the existence and uniqueness of the mild solutions of the proposed system by assuming that

(P1) For $\grave{\mathcal{K}}_u > 0$ and for any $s, \rho \in \Omega$

$$|\mathfrak{N}(t, s(t)) - \mathfrak{N}(t, \rho(t))| \leq \grave{\mathcal{K}}_u \|s(t) - \rho(t)\|_{\grave{\mathcal{P}}C}$$

and

$$\|\mathfrak{N}(t, s(t))\| \leq \hat{R}_n.$$

(P2) For $\grave{\mathcal{K}}_v, \grave{\mathcal{L}}_v \& \grave{\mathcal{M}}_v$ and for any $s_1, s_2, s_3, \rho_1, \rho_2, \rho_3(t) \in \Omega$

$$\begin{aligned}&|\mathcal{I}(t, s_1(t), s_2(t), s_3(t)) - \mathcal{I}(t, \rho_1(t), \rho_2(t), \rho_3(t))|\\ \leq&\grave{\mathcal{K}}_v \|s_1(t) - \rho_1(t)\|_{\grave{\mathcal{P}}C} + \grave{\mathcal{L}}_v |s_2(t) - \rho_2(t) + \grave{\mathcal{M}}_v |s_3(t) - \rho_3(t)|\end{aligned}$$

and

$$\|\mathcal{I}(t, s_1(t), s_2(t), s_3(t))\| \leq \hat{R}_q.$$

(P3) For $\grave{\mathcal{K}}_t > 0$ and for any $s, \rho \in \Omega$

$$|I_i s(t) - I_j \rho(t)| \leq \grave{\mathcal{K}}_t \|s_1(t) - \rho_1(t)\|_{\grave{\mathcal{P}}C}$$

and

$$\|I_i s(t) \leq \omega.$$

(P4) For $\grave{\mathcal{K}}_s > 0$ and for any $s, \rho \in \Omega$

$$|G(t, s(t)) - G(t, \rho(t))| \leq \grave{\mathcal{K}}_s \|s(t) - \rho(t)\|_{\grave{\mathcal{P}}C}$$

and

$$\|G(t, s(t))\| \leq \mathcal{K}_g.$$

(P5) There exists $c_1, c_2, c_3, c_4 \in \complement(\mathcal{J}, \mathfrak{R}_+)$ with $c_3^* = \sup_{t\in\mathcal{J}} c_3(t) < 1$ and $c_4^* = \sup_{t\in\mathcal{J}} c_4(t) < 1$ such that

$$|\mathcal{I}(t, s, \rho, \mu)| \leq c_1(t) + c_2(t)\|s\|_{\grave{\mathcal{P}}C} + c_3(t)|\rho| + c_4(t)\|\mu\|_{\grave{\mathcal{P}}C}$$

for $t \in \mathcal{J}$, $s \in \grave{\mathcal{P}C}([-\mathfrak{r}, 0], \mathfrak{R})$ and $\rho, \mu \in \mathfrak{R}$.

(P6) There are constants $\complement_1^*, \complement_2^* > 0$ such that

$$|I_i(s)| \leq \complement_1^* \|s\|_{\grave{\mathcal{P}C}} + \complement_2^*$$

for each $s \in \grave{\mathcal{P}C}([-\mathfrak{r}, 0], \mathfrak{R})$, $i = 1, ..., \ell$.

(P7) Let the completely continuous function be $\mathfrak{N}$, and for each bounded set $B_{\top^*}$ in Ω, the set $t \to \mathfrak{N}(t, s_t) : s \in B_{\top^*}$ is equi-continuous in $\grave{\mathcal{P}C}(\mathcal{J}, \mathfrak{R})$ and hence, there are constants $p_1 > 0$, $p_2 > 0$ with $\ell \complement_1^* + p_1 < 1$ such that

$$|\mathfrak{N}(t, s)| \leq p_1 \|s\|_{\grave{\mathcal{P}C}} + p_2, t \in \mathcal{J}, s \in \grave{\mathcal{P}C}([-\mathfrak{r}, 0], \mathfrak{R}).$$

(P8) The control operator is a bounded linear operator, and for each bounded set $B_{\top^*}$ in Ω,

$$\|c_s(t) - c_\rho(t)\| \leq \hat{R}_c \|s(t) - \rho(t)\|_{\grave{\mathcal{P}C}}.$$

(P9) The linear operator $\mathcal{W} : \grave{\mathcal{P}C}(\mathcal{J}) \mapsto \mathfrak{R}$ is

$$\mathcal{W}(c(.)) = \begin{cases} \frac{QP(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^\top (\top - \mathfrak{z})^{\alpha-1} B(c_S(t)) d\mathfrak{z} + \frac{\alpha P^2}{\mathbb{N}(\alpha)} \int_0^\top \mathcal{S}_\alpha(\top - \mathfrak{z}) B(c_S(t)) d\mathfrak{z}, \text{ if } t \in [0, t_1], \\ \\ \frac{QP(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_{t_j}^\top (\top - \mathfrak{z})^{\alpha-1} B(c_S(t)) d\mathfrak{z} + \frac{\alpha P^2}{\mathbb{N}(\alpha)} \int_{t_j}^\top \mathcal{S}_\alpha(\top - \mathfrak{z}) B(c_S(t)) d\mathfrak{z}, \text{ if } t \in [t_j, t_{j+1}]. \end{cases}$$

Here, we get an invertible operator $\mathcal{W}^{-1} : \mathcal{B} \mapsto L^2((0, \top], C)/ker(\mathcal{W}), \mathcal{W}^{-1}$ is also bounded and hence we have $\|B\| \leq \bar{R}$ and $\|\mathcal{W}^{-1}\| \leq \bar{R}_1$.

(P10) P & Q are linear operators that are bounded on $\mathcal{B}$ and hence $\|P\| \leq \zeta_1$ & $\|Q\| \leq \zeta_2$.

Theorem 5. *Let us consider that hypotheses (P1)–(P8) hold, then the proposed problem* (1.1) *has at least one mild solution.*

Proof. Consider the set,

$$\Omega = \left\{ s : [-\mathfrak{r}, \top] \to \mathfrak{R} : s|_{[-\mathfrak{r},0]} \in \grave{\mathcal{P}C}([-\mathfrak{r}, 0], \mathfrak{R}) \text{ and } s|_{[0,\top]} \in \grave{\mathcal{P}C}_1([0, \top], \mathfrak{R}) \right\}.$$

Ω holds the properties of Banach space with norm

$$\|s\|_\Omega = \sup_{t \in [-\mathfrak{r}, \top]} \|s(t)\|.$$

We define the operator $\Psi_1 : \Omega \to \Omega$ defined by

$$\Psi_1(s(t)) = \begin{cases} \varphi(t); t \in [-\mathfrak{r}, 0] \\ PT_\alpha(t) \int_0^\top \frac{(\top - \mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)} G(\mathfrak{z}, s_\mathfrak{z}) d\mathfrak{z} + QP \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_{t_i}^t (t - \mathfrak{z})^{\alpha-1} [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})] d\mathfrak{z} \\ + \sum_{j=1}^i QP \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_j - \mathfrak{z})^{\alpha-1} [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})] + \frac{\alpha P^2}{\mathbb{N}(\alpha)} \sum_{j=1}^i \int_{t_{j-1}}^{t_j} \mathcal{S}_\alpha(t_j - \mathfrak{z}) \\ \times [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})] d\mathfrak{z} + \frac{\alpha P^2}{\mathbb{N}(\alpha)} \int_{t_i}^t \mathcal{S}_\alpha(t - \mathfrak{z}) [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})] d\mathfrak{z} \\ + PT_\alpha(t) \sum_{j=1}^i I_j(s(t_j^-)), \quad [t_i, t_{i+1}). \end{cases} \tag{3.1}$$

The operator Ψ_1 represented in (3.1) can be formed as $\Psi = \mathfrak{N}(t, s(t)) + \Psi_1$, for all $t \in \mathcal{J}$.

With the help of Schauder's FPT, we derive the existence of a fixed point of Ψ. First, we show that Ψ is completely continuous. Due to the postulate (P7) of $\mathfrak{N}$, it is enough to show that Ψ_1 is completely continuous. □

Step 1: Ψ_1 is continuous. Consider the sequence $\{s_\ell\}$ such that $s_\ell \to s$ in Ω. If $t \in [-\mathfrak{r}, 0]$, then

$$|\Psi_1(s) - \Psi_1(\rho)| = 0.$$

For $t \in \mathcal{J}$, we have

$$\begin{aligned}
|\Psi_1(s) - \Psi_1(\rho)| \leq{}& PT_\alpha(t) \int_0^\top \frac{(\top - \mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)} |G(\mathfrak{z}, s_{\ell\mathfrak{z}}) - G(\mathfrak{z}, s_\mathfrak{z})| d\mathfrak{z} \\
&+ QP \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t - \mathfrak{z})^{\alpha-1} |[B(c_S(\mathfrak{z})) + \mathfrak{q}^*{}_\ell(\mathfrak{z})] - [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]| d\mathfrak{z} \\
&+ \sum_{j=1}^{i} QP \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_j - \mathfrak{z})^{\alpha-1} |[B(c_S(\mathfrak{z})) + \mathfrak{q}^* - \ell(\mathfrak{z})] - [B(c_S(\mathfrak{z})) + \mathfrak{q}^*{}_\ell(\mathfrak{z})]| d(\mathfrak{z}) \\
&+ \frac{\alpha P^2}{\mathbb{N}(\alpha)} \int_{t_i}^{t} \mathcal{S}_\alpha(t - \mathfrak{z}) |[B(c_S(\mathfrak{z})) + \mathfrak{q}^*{}_\ell(\mathfrak{z})] - [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]| d\mathfrak{z} \\
&+ \frac{\alpha P^2}{\mathbb{N}(\alpha)} \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \mathcal{S}_\alpha(t_j - \mathfrak{z}) |[B(c_S(\mathfrak{z})) + \mathfrak{q}^*{}_\ell(\mathfrak{z})] - [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]| d\mathfrak{z} \\
&+ PT_\alpha(t) \sum_{j=1}^{i} |I_j(s(t_j^-)) - I_j(\rho(t_j^-))|,
\end{aligned} \tag{3.2}$$

here, $\mathfrak{q}^*, \mathfrak{q}^*_\ell \in \complement(\mathcal{J}, \mathfrak{R})$ such that

$$\mathfrak{q}^*{}_\ell(t) = \mathcal{I}(t, s_{\ell t}, \mathfrak{q}^*{}_\ell(t), c_\ell(t)),$$

and

$$\mathfrak{q}^*(t) = \mathcal{I}(t, s_t, \mathfrak{q}^*(t), c(t)).$$

By $(P2)$, we have

$$\begin{aligned}
|\mathfrak{q}^*{}_\ell(t) - \mathfrak{q}^*(t)| &= |\mathcal{I}(t, s_{\ell t}), \mathfrak{q}^*{}_\ell(t), c_\ell(t)) - \mathcal{I}(t, s_t, \mathfrak{q}^*(t), ct)| \\
&\leq \grave{\mathcal{K}}_\mathfrak{v} \|s_{\ell t} - s_t\|_{\grave{\mathcal{P}}C} + \grave{\mathcal{L}}_\mathfrak{v} |\mathfrak{q}^*{}_\ell(t) - \mathfrak{q}^*(t)| + \grave{\mathcal{M}}_\mathfrak{w} \|c_\ell(t) - c(t)\|_{\grave{\mathcal{P}}C}, \\
|\mathfrak{q}^*{}_\ell(t) - \mathfrak{q}^*(t)| &\leq \frac{\grave{\mathcal{K}}_\mathfrak{v} \|s_{\ell t} - s_t\|_{\grave{\mathcal{P}}C} + \grave{\mathcal{M}}_\mathfrak{w} \|c_\ell(t) - c(t)\|_{\grave{\mathcal{P}}C}}{1 - \grave{\mathcal{L}}_\mathfrak{v}}.
\end{aligned}$$

Due to the result $s_\ell \to s$, it gives $\mathfrak{q}^*{}_\ell(t) \to \mathfrak{q}^*(t)$ as $\ell \to \infty$ for all $t \in \mathcal{J}$.

Now, consider $\nu > 0$, for each $t \in \mathcal{J}$, we write $|\mathfrak{q}^*{}_\ell(t)| \leq \nu$ and $|\mathfrak{q}^*(t)| \leq \nu$.

Hence, we get

$$\begin{aligned}
(t - \mathfrak{z})^{\beta-1} |\mathfrak{q}^*{}_\ell(\mathfrak{z}) - \mathfrak{q}^*(t)| &\leq (t - \mathfrak{z})^{\beta-1} [|\mathfrak{q}^*{}_\ell(\mathfrak{z})| + |\mathfrak{q}^*(t)|] \\
&\leq 2\nu (t - \mathfrak{z})^{\beta-1},
\end{aligned}$$

and

$$(t_k - \mathfrak{z})^{\beta-1}|\mathfrak{q}^*{}_\ell(\mathfrak{z}) - \mathfrak{q}^*(t)| \leq (t_k - \mathfrak{z})^{\beta-1}[|\mathfrak{q}^*{}_\ell(\mathfrak{z})| + |\mathfrak{q}^*(t)|] \leq 2\nu(t_k - \mathfrak{z})^{\beta-1}.$$

For all $t \in \mathcal{J}$, the maps $\mathfrak{z} \to 2\nu(t-\mathfrak{z})^{\beta-1}$ and $\mathfrak{z} \to 2\nu(t_k-\mathfrak{z})^{\beta-1}$ are integrable on $[0, t]$; hence, by applying the Lebesgue dominated convergence theorem and (3.2), we get

$$|\Psi_1(s_\ell)(t) - \Psi_1(s)(t)| \to 0 \;\text{ as }\; \ell \to \infty,$$

which results in the continuity of Ψ_1.

Step 2: The bounded sets of Ω will be mapped in to bounded sets of Ω by the function Ψ_1. To show this, it is sufficient to prove that for any $\top^* > 0, \exists \wp$ such that for each $s \in B_{\top^*} = \{s \in \Omega : \|s\|_\Omega \leq \top^*\}$, we have $\|\Psi_1(s)|_\Omega \leq \wp$.

For each $t \in \mathcal{J}$, we get

$$\begin{aligned}\Psi_1(s(t)) = {} & PT_\alpha(t)\int_0^\top \frac{(\top-\mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)}G(\mathfrak{z}, s_\mathfrak{z})d\mathfrak{z} + QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_{t_{j-1}}^{t_j}(t-\mathfrak{z})^{\alpha-1}[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} \\ & + \sum_{j=1}^{i} QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\alpha-1}[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})] + \frac{\alpha P^2}{\mathbb{N}(\alpha)}\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}\mathcal{S}_\alpha(t_j-\mathfrak{z}) \\ & \times [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + \frac{\alpha P^2}{\mathbb{N}(\alpha)}\int_{t_i}^{t}\mathcal{S}_\alpha(t-\mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} \\ & + PT_\alpha(t)\sum_{j=1}^{i} I_j(s(t_j^-)), \end{aligned} \tag{3.3}$$

here, $\mathfrak{q}^* \in \complement(\mathcal{J}, \mathfrak{R})$ such that $\mathfrak{q}^*(t) = \mathcal{I}(t, s_t, \mathfrak{q}^*(t), c(t))$. From $(P5)$, for each $t \in \mathcal{J}$, we can write

$$\begin{aligned}|\mathfrak{q}^*(t)| &= |\mathcal{I}(t, s_t, \mathfrak{q}^*(t), ct| \\ &\leq c_1(t) + c_2(t)\|s_t\|_{\hat{\mathcal{P}C}} + c_3(t)|\mathfrak{q}^*(t)| + c_4\|c(t)\|_{\hat{\mathcal{P}C}} \\ &\leq c_1(t) + c_2(t)\|s_t\|_{\hat{\mathcal{P}C}} + c_3(t)|\mathfrak{q}^*(t)| + c_4\|c(t)\|_{\hat{\mathcal{P}C}} \\ &\leq c_1(t) + c_2(t)\top^* + c_3(t)|\mathfrak{q}^*(t)| + c_4\|c(t)\|_{\hat{\mathcal{P}C}} \\ &\leq c_1^* + c_2^*\top^* + c_3^*|\mathfrak{q}^*(t)| + c_4^*\|c(t)\|_{\hat{\mathcal{P}C}},\end{aligned}$$

where $c_1^* = \sup_{t\in\mathcal{J}} c_1(t)$, and $c_2^* = \sup_{t\in\mathcal{J}} c_2(t)$. Then

$$|\mathfrak{q}^*(t)| \leq \frac{c_1^* + c_2^*\top^* + c_4^*\|c(t)\|}{1 - c_3^*} := R.$$

Thus from (3.3),

$$\begin{aligned}|\Psi_1(s)(t)| \leq {} & \Big|PT_\alpha(t)\int_0^\top \frac{(\top-\mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)}G(\mathfrak{z}, s_\mathfrak{z})d\mathfrak{z} + QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_{t_{j-1}}^{t_j}(t-\mathfrak{z})^{\alpha-1}[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} \\ & + \sum_{j=1}^{i} QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\alpha-1}\end{aligned}$$

$$\times [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})] + \frac{\alpha P^2}{\mathbb{N}(\alpha)} \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \mathcal{S}_\alpha(t_j - \mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z}$$

$$+ \frac{\alpha P^2}{\mathbb{N}(\alpha)} \int_{t_i}^{t} \mathcal{S}_\alpha(t - \mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + PT_\alpha(t) \sum_{j=1}^{i} I_j(s(t_j^-))\Big|$$

$$\leq \zeta_1 \hat{R} \frac{\top^\alpha}{\Gamma(\alpha+1)}(f_1\|s\| + f_2) + \zeta_1\zeta_2 \frac{1-\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha)}(\bar{R}M^* + R) + \zeta_1\zeta_2 \frac{1-\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha)}\ell(\bar{R}M^* + R)$$

$$+ \frac{\alpha\zeta_1^2}{\mathbb{N}(\alpha)}(\top - \mathfrak{z})^{\alpha-1}\hat{R}_1\ell(\bar{R}M^* + R) + \frac{\alpha\zeta_1^2}{\mathbb{N}(\alpha)}(\top - \mathfrak{z})^{\alpha-1}\hat{R}_1(\bar{R}M^* + R) + \zeta_1\hat{R}\sum_{i=1}^{\ell}(\mathsf{C}_1^*\|s_{t_k^-}\| + \mathsf{C}_2^*)$$

$$\leq \zeta_1 \hat{R} \frac{\top^\alpha}{\Gamma(\alpha+1)}(f_1 \nu^* + f_2) + \zeta_1\zeta_2 \frac{1-\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha)}(\ell + 1)(\bar{R}M^* + R)$$

$$+ \frac{\alpha\zeta_1^2}{\mathbb{N}(\alpha)}(\top - \mathfrak{z})^{\alpha-1}\hat{R}_1(\ell+1)(\bar{R}M^* + R) + \zeta_1\hat{R}\ell(C_1^*\nu^* + C_2^*) := S.$$

For $t \in [-\mathfrak{r}, 0]$, then

$$|\Psi_1(s)(t)| \leq \|\varphi\|_{\hat{\mathcal{P}C}},$$

Hence,

$$\|\Psi_1(s)\|_\Omega \leq \max\{S, \|\varphi\|_{\hat{\mathcal{P}C}}\} := \wp.$$

Step 3: The function Ψ_1 maps the bounded sets of Ω into equi-continuous sets of Ω.

Let $t_{i-1}, t_i \in (0, \top],\ t_{i-1} < t_i,\ B_{\top^*}$ be a bounded set of Ω as in Step 2, and let $s \in B_{\top^*}$. Then

$$|\Psi_1(s)(t_\ell) - \Psi_1(s)(t_{\ell-1})|$$

$$= \Big| QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_\ell - \mathfrak{z})^{\alpha-1}[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + \sum_{j=1}^{i} QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_\ell - \mathfrak{z})^{\alpha-1}$$

$$\times [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})] + \frac{\alpha P^2}{\mathbb{N}(\alpha)} \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \mathcal{S}_\alpha(t_\ell - \mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + \frac{\alpha P^2}{\mathbb{N}(\alpha)} \int_{t_i}^{t} \mathcal{S}_\alpha(t_\ell - \mathfrak{z})$$

$$\times [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + PT_\alpha(t)\sum_{j=1}^{i} I_j(s(t_\ell^-)) - QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_{\ell-1} - \mathfrak{z})^{\alpha-1}[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z}$$

$$- \sum_{j=1}^{i} QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_{\ell-1} - \mathfrak{z})^{\alpha-1}[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})] - \frac{\alpha P^2}{\mathbb{N}(\alpha)} \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \mathcal{S}_\alpha(t_{\ell-1} - \mathfrak{z})$$

$$\times [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} - \frac{\alpha P^2}{\mathbb{N}(\alpha)} \int_{t_i}^{t} \mathcal{S}_\alpha(t_{\ell-1} - \mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} - PT_\alpha(t)\sum_{j=1}^{i} I_j(s(t_{\ell-1}^-))\Big|.$$

As $t_\ell \to t_{\ell-1}$, the RHS of the above inequality converges to 0. Hence, Ψ_1 is completely continuous.

Step 4: A priori estimates. We show that

$$\daleth = \{s \in \Omega : s = \kappa\Psi_1(s) \text{ for some } \kappa \in (0, 1)\}$$

is bounded. Consider $s \in \daleth$, then $s = \kappa\Psi_1(s)$ for some $\kappa \in (0, 1)$. Now let, for each $t \in \mathcal{J}$,

$$s = \kappa\mathfrak{R}(t, s_t) + \kappa P T_\alpha \int_0^\top \frac{(\top - \mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)} \mathbb{E}(\mathfrak{z}, s(\mathfrak{z}))d\mathfrak{z}$$
$$+ \kappa QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t - \mathfrak{z})^{\alpha-1}[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + \kappa \sum_{j=1}^{i} QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_j - \mathfrak{z})^{\alpha-1}$$
$$\times [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + \kappa\frac{\alpha P^2}{\mathbb{N}(\alpha)} \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \mathcal{S}_\alpha(t_j - \mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + \kappa\frac{\alpha P^2}{\mathbb{N}(\alpha)} \int_{t_i}^{t} \mathcal{S}_\alpha(t - \mathfrak{z})$$
$$\times [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + PT_\alpha(t)\kappa \sum_{j=1}^{i} I_j(s(t_j^-)). \tag{3.4}$$

Hence, for each $t \in \mathcal{J}$ and from $(P5)$, we get,

$$\begin{aligned}
|\mathfrak{q}^*(t)| &= |I(t, s_t, \mathfrak{q}^*(t), c(t))| \\
&\le c_1(t) + c_2(t)\|s_t\|_{\hat{\mathcal{P}C}} + c_3(t)|\mathfrak{q}^*(t)| + c_4(t)|c(t)| \\
&\le c_1(t) + c_2(t)\|s_t\|_{\hat{\mathcal{P}C}} + c_3(t)|\mathfrak{q}^*(t)| + c_4(t)\|c(t)\|_{\hat{\mathcal{P}C}} \\
&\le c_1^* + c_2^*\|s_t\|_{\hat{\mathcal{P}C}} + c_3^*|\mathfrak{q}^*(t)| + c_4^*\|c(t)\|_{\hat{\mathcal{P}C}}, \\
|\mathfrak{q}^*(t)| &\le \frac{1}{1 - c_3^*}(c_1^* + c_2^*\|s_t\|_{\hat{\mathcal{P}C}} + c_4^*\|c(t)\|_{\hat{\mathcal{P}C}}).
\end{aligned}$$

For each $t \in \mathcal{J}$ and by (3.4), (P6), and (P7), we have

$$|s| \le p_1\|s_t\|_{\hat{\mathcal{P}C}} + p_2 + \zeta_1\hat{R}\int_0^\top \frac{(\top - \mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)}(f_1\|s_t\|_{\hat{\mathcal{P}C}} + f_2)d\mathfrak{z} + \zeta_1\zeta_2\frac{1-\alpha}{(1-c_3^*)(\mathbb{N}(\alpha))\Gamma\alpha}(c_1^* + c_2^*\|s\|_{\hat{\mathcal{P}C}}$$
$$+ c_4^*\|c(t)\|_{\hat{\mathcal{P}C}}) + \frac{\ell\zeta_1\zeta_2(1-\alpha)}{(1-c_3^*)(\mathbb{N}(\alpha))\Gamma\alpha}(c_1^* + c_2^*\|s\|_{\hat{\mathcal{P}C}} + c_4^*\|c(t)\|_{\hat{\mathcal{P}C}})$$
$$+ \ell\frac{\alpha\hat{R}^2}{(1-c_3^*)\mathbb{N}(\alpha)} \sum_{i=1}^{i} \int_{t_{i-1}}^{t_i} (t_i - \mathfrak{z})^{\alpha-1}(c_1^* + c_2^*\|s\|_{\hat{\mathcal{P}C}} + c_4^*\|c(t)\|_{\hat{\mathcal{P}C}})d\mathfrak{z}$$
$$+ \frac{\alpha\hat{R}^2}{(1-c_3^*)\mathbb{N}(\alpha)} \int_{t_i}^{t} (t - \mathfrak{z})^{\alpha-1}(c_1^* + c_2^*\|s\|_{\hat{\mathcal{P}C}} + c_4^*\|c(t)\|_{\hat{\mathcal{P}C}})d\mathfrak{z} + \zeta_1\hat{R}\ell(\mathsf{C}_1^*\|s_{t_i^-}\|_{\hat{\mathcal{P}C}} + \mathsf{C}_2^*).$$

Define ν by

$$\nu(t) = \sup\{|s(\mathfrak{z})| : \mathfrak{z} \in [-\mathfrak{r}, t]\},\ t \in [0, \top], c(t) = \sup\{|c(\mathfrak{z})| : \mathfrak{z} \in [-\mathfrak{r}, t]\},\ t \in [0, \top].$$

Then there exists $t^* \in [-\mathfrak{r}, \top]$ such that $\nu(t) = |s(t^*)|$. If $t \in [0, \top]$, then by the previous inequality, we have for $t \in \mathcal{J}$

$$\nu(t) \le p_1\nu(t) + p_2 + \zeta_1\hat{R}\int_0^\top \frac{(\top - \mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)}(f_1\nu(\mathfrak{z}) + f_2)d\mathfrak{z} + \frac{\zeta_1\zeta_2(\ell+1)(1-\alpha)}{(1-c_3^*)(\mathbb{N}(\alpha))\Gamma(\alpha)}[c_1^* + c_2^*\nu(t) + c_4^*c(t)]$$
$$+ \frac{\alpha\hat{R}^2}{(1-c_3^*)\mathbb{N}(\alpha)} \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} S_\alpha(t - \mathfrak{z})^{\alpha-1}[c_1^* + c_2^*\nu(\mathfrak{z}) + c_4^*c(\mathfrak{z})]d\mathfrak{z}$$

$$
\begin{aligned}
&+\frac{\alpha\hat{R}^2}{(1-c_3^*)\mathbb{N}(\alpha)}\int_{t_i}^{t} S_\alpha(t-\mathfrak{z})^{\alpha-1}[c_1^*+c_2^*\nu(\mathfrak{z})+c_4^*c(\mathfrak{z})]d\mathfrak{z}+\zeta_1\hat{R}\ell[\complement_1^*\nu(t)+\complement_2^*]\\
&\leq\left(p_1+\frac{\zeta_1\zeta_2(\ell+1)(1-\alpha)}{(1-c_3^*)\mathbb{N}(\alpha)\Gamma(\alpha)}c_2^*+\zeta_1\hat{R}\ell\complement_1^*\right)\nu(t)+\left(p_2+\frac{\zeta_1\zeta_2(\ell+1)(1-\alpha)}{(1-c_3^*)\mathbb{N}(\alpha)\Gamma(\alpha)}c_1^*+\zeta_1\hat{R}\ell\complement_2^*\right)\\
&+\frac{\zeta_1\zeta_2(\ell+1)(1-\alpha)}{(1-c_3^*)\mathbb{N}(\alpha)\Gamma(\alpha)}c_4^*c(t)+\frac{\alpha\hat{R}^2\ell c_1^*}{\mathbb{N}(\alpha)(1-c_3^*)}\top^{(\alpha-1)}\hat{R}_1+\frac{\alpha\hat{R}^2}{(1-c_3^*)\mathbb{N}(\alpha)}\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j} S_\alpha(t-\mathfrak{z})^{\alpha-1}\\
&\times[c_2^*\nu(\mathfrak{z})+c_4^*c(\mathfrak{z})]d\mathfrak{z}+\frac{\alpha\hat{R}^2c_1^*}{\mathbb{N}(\alpha)(1-c_3^*)}\top^{(\alpha-1)}\hat{R}_1+\frac{\alpha\hat{R}^2}{(1-c_3^*)\mathbb{N}(\alpha)}\int_{t_i}^{t} S_\alpha(t-\mathfrak{z})^{\alpha-1}[c_2^*\nu(\mathfrak{z})+c_4^*c(\mathfrak{z})]d\mathfrak{z}\\
&\leq\frac{1}{\left(1-[p_1+\frac{\zeta_1\zeta_2(\ell+1)(1-\alpha)}{(1-c_3^*)\mathbb{N}(\alpha)\Gamma(\alpha)}c_2^*+\zeta_1\hat{R}\ell\complement_1^*]\right)}\left(p_2+\frac{\zeta_1\zeta_2(\ell+1)(1-\alpha)}{(1-c_3^*)\mathbb{N}(\alpha)\Gamma(\alpha)}c_1^*+\zeta_1\hat{R}\ell\complement_2^*+\frac{\alpha\hat{R}^2(\ell+1)c_1^*}{\mathbb{N}(\alpha)(1-c_3^*)}\top^{(\alpha-1)}\hat{R}_1\right)\\
&+\frac{1}{\left(1-[p_1+\frac{\zeta_1\zeta_2(\ell+1)(1-\alpha)}{(1-c_3^*)\mathbb{N}(\alpha)\Gamma(\alpha)}c_2^*+\zeta_1\hat{R}\ell\complement_1^*]\right)}\frac{\zeta_1\zeta_2(\ell+1)(1-\alpha)}{(1-c_3^*)\mathbb{N}(\alpha)\Gamma(\alpha)}c_4^*c(t)\\
&+\frac{1}{\left(1-[p_1+\frac{\zeta_1\zeta_2(\ell+1)(1-\alpha)}{(1-c_3^*)\mathbb{N}(\alpha)\Gamma(\alpha)}c_2^*+\zeta_1\hat{R}\ell\complement_1^*]\right)}\left(\frac{\alpha(\ell+1)c_2^*\hat{R}_1^2}{(1-c_3^*)\mathbb{N}(\alpha)}\int_{t_i}^{t} S_\alpha(t-\mathfrak{z})^{\alpha-1}\nu(\mathfrak{z})d\mathfrak{z}\right)\\
&+\frac{1}{\left(1-[p_1+\frac{\zeta_1\zeta_2(\ell+1)(1-\alpha)}{(1-c_3^*)\mathbb{N}(\alpha)\Gamma(\alpha)}c_2^*+\ell\zeta_1\hat{R}\complement_1^*]\right)}\left(\frac{\alpha(\ell+1)c_4^*\hat{R}_1^2}{(1-c_3^*)\mathbb{N}(\alpha)}\int_{t_i}^{t} S_\alpha(t-\mathfrak{z})^{\alpha-1}c(\mathfrak{z})d\mathfrak{z}\right).
\end{aligned}
$$

Applying Lemma 1, we get

$$
\begin{aligned}
\nu(t)\leq&\frac{1}{1-[p_1+\frac{\zeta_1\zeta_2(\ell+1)(1-\alpha)}{(1-c_3^*)\mathbb{N}(\alpha)\Gamma(\alpha)}c_2^*+\zeta_1\hat{R}\ell\complement_1^*]}\\
&\times\left[\left(p_2+\frac{\zeta_1\zeta_2(\ell+1)(1-\alpha)}{(1-c_3^*)\mathbb{N}(\alpha)\Gamma(\alpha)}c_1^*+\zeta_1\hat{R}\ell\complement_2^*\right)+\left(\frac{\alpha\hat{R}^2(\ell+1)c_1^*}{\mathbb{N}(\alpha)(1-c_3^*)}\top^{(\alpha-1)}\hat{R}_1\right)+\left(\frac{\zeta_1\zeta_2(\ell+1)(1-\alpha)}{(1-c_3^*)\mathbb{N}(\alpha)\Gamma(\alpha)}c_4^*\bar{R}\right)\right]\\
&+\frac{1}{1-[p_1+\frac{\zeta_1\zeta_2(\ell+1)(1-\alpha)}{(1-c_3^*)\mathbb{N}(\alpha)\Gamma(\alpha)}c_2^*+\zeta_1\hat{R}\ell\complement_1^*]}\left[\left(\frac{\alpha(\ell+1)c_2^*\hat{R}_1^2}{(1-c_3^*)\mathbb{N}(\alpha)}\complement(c_2)\right)+\left(\frac{\alpha(\ell+1)c_4^*\hat{R}_1^2}{(1-c_3^*)\mathbb{N}(\alpha)}\bar{R}\right)\right],
\end{aligned}
$$

where $\complement(c_2)$ is a constant. If $t^*\in[-\mathrm{r},0]$, then $\nu(t)=\|\phi\|_{\grave{\mathcal{P}C}}$, thus for any $t\in\mathcal{J}$, $\|s\|_\Omega\leq\nu(t)$, we get

$$\|s\|_\Omega\leq\max\{\|\mathfrak{z}\|_{\grave{\mathcal{P}C}},A\}.$$

Hence the set $\Finv$ is bounded. By Theorems 3 and 4, Ψ has at least one fixed point in Ω which is a mild solution of the problem (1.1).

Theorem 6. *Under hypotheses (P1)–(P8), the considered problem* (1.1) *has a unique mild solution if*

$$
\begin{aligned}
\Theta_a=\grave{\mathcal{K}}_u+\frac{\zeta_1\hat{R}\top^\alpha}{\Gamma(\alpha+1)}\grave{\mathcal{K}}_s+\frac{\zeta_1\zeta_2(1-\alpha)\top^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}(\ell+1)\left(\bar{R}\hat{R}_c+\frac{\grave{\mathcal{K}}_\nu+\grave{\mathcal{M}}_\nu\hat{R}_c}{1-\grave{\mathcal{L}}_\nu}\right)&\\
+\frac{\alpha\zeta_1^2}{\mathbb{N}(\alpha)}\hat{R}_1(\ell+1)\left(\bar{R}\hat{R}_c+\frac{\grave{\mathcal{K}}_\nu+\grave{\mathcal{M}}_\nu\hat{R}_c}{1-\grave{\mathcal{L}}_\nu}\right)+\zeta_1\hat{R}\ell\grave{\mathcal{K}}_t<1.&
\end{aligned}
$$

Proof. Define a set,

$$\Omega=\left\{s:[-\mathrm{r},\top]\to\mathfrak{R}:s|_{[-\mathrm{r},0]}\in\grave{\mathcal{P}C}([-\mathrm{r},0],\mathfrak{R})\text{ and }s|_{[0,\top]}\in\grave{\mathcal{P}C}_1([0,\top],\mathfrak{R})\right\}.$$

Ω holds the properties of Banach space with the norm

$$\|s\|_{\Omega} = \sup_{t\in[-\mathfrak{r},\top]} \|s(t)\|.$$

Consider the operator $\Psi_1 : \Omega \to \Omega$ by

$$\Psi_1(s(t)) = \begin{cases} \varphi(t); t \in [-\mathfrak{r}, 0] \\ \mathfrak{N}(t, s_t) + PT_\alpha(t)\displaystyle\int_0^\top \frac{(\top - \mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)} G(\mathfrak{z}, s_\mathfrak{z})d\mathfrak{z} + QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_{t_i}^t (t-\mathfrak{z})^{\alpha-1} \\ \times[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + \displaystyle\sum_{j=1}^i QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\alpha-1}[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} \\ +\frac{\alpha P^2}{\mathbb{N}(\alpha)}\displaystyle\sum_{j=1}^i\int_{t_{j-1}}^{t_j} \mathcal{S}_\alpha(t_j-\mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + \frac{\alpha P^2}{\mathbb{N}(\alpha)}\int_{t_i}^t \mathcal{S}_\alpha(t-\mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} \\ +PT_\alpha(t)\displaystyle\sum_{j=1}^i I_j(s(t_j^-)) \end{cases} \tag{3.5}$$

where $\mathfrak{q}^*(t) \in \mathfrak{C}(\mathcal{J}, \mathfrak{R})$ and

$$\mathfrak{q}^*(t) = \mathcal{I}(t, s_t, {}_0^{\mathcal{ABC}}D_t^\alpha, c(t)).$$

If $s, \rho \in \Omega$, for $t \in [-\mathfrak{r}, 0]$, which implies

$$\|\Psi(s) - \Psi(\rho)\| = 0.$$

For $t \in \mathcal{J}$ and from (3.5), we have

$$\begin{aligned} \|\Psi(s) - \Psi(\rho)\|_\Omega &= \max_{t\in\mathcal{J}} \left|\Psi s(t) - \Psi\rho(t)\right| \\ &\le \max_{t\in\mathcal{J}} \Bigg|\mathfrak{N}(t, s_t) + PT_\alpha(t)\int_0^\top \frac{(\top-\mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)} G(\mathfrak{z}, s_\mathfrak{z})d\mathfrak{z} + QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_{t_{j-1}}^{t_j}(t-\mathfrak{z})^{\alpha-1} \\ &\times [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + \sum_{j=1}^i QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\alpha-1}[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} \\ &+ \frac{\alpha P^2}{\mathbb{N}(\alpha)}\sum_{j=1}^i\int_{t_{j-1}}^{t_j}\mathcal{S}_\alpha(t_j-\mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + \frac{\alpha P^2}{\mathbb{N}(\alpha)}\int_{t_i}^t \mathcal{S}_\alpha(t-\mathfrak{z}) \\ &\times [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + PT_\alpha(t)\sum_{j=1}^i I_j(s(t_j^-)) - \Bigg\{\mathfrak{N}(t, \rho_t) + PT_\alpha(t)\int_0^\top \frac{(\top-\mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)} \\ &\times G(\mathfrak{z}, \rho_\mathfrak{z})d\mathfrak{z} + QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_{t_{j-1}}^{t_j}(t-\mathfrak{z})^{\alpha-1}[B(c_\rho(\mathfrak{z})) + \bar{\mathfrak{q}}^*(\mathfrak{z})]d\mathfrak{z} + \sum_{j=1}^i QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \\ &\times \int_{t_{j-1}}^{t_j}(t_j-\mathfrak{z})^{\alpha-1}[B(c_\rho(\mathfrak{z})) + \bar{\mathfrak{q}}^*(\mathfrak{z})]d\mathfrak{z} + \frac{\alpha P^2}{\mathbb{N}(\alpha)}\sum_{j=1}^i\int_{t_{j-1}}^{t_j}\mathcal{S}_\alpha(t_j-\mathfrak{z})[B(c_\rho(\mathfrak{z})) + \bar{\mathfrak{q}}^*(\mathfrak{z})]d\mathfrak{z} \end{aligned}$$

$$+\frac{\alpha P^2}{\mathbb{N}(\alpha)}\int_{t_i}^{t}\mathcal{S}_\alpha(t-\mathfrak{z})[B(c_\rho(\mathfrak{z}))+\bar{\mathfrak{q}}^*(\mathfrak{z})]d\mathfrak{z}+PT_\alpha(t)\sum_{j=1}^{i}I_j(\rho(t_j^-))\Big\}\Big|$$

$$\leq \max_{t\in\mathcal{J}}|\mathfrak{N}(t,s(t))-\mathfrak{N}(t,\rho(t))|+PT_\alpha(t)\int_0^{\top}\frac{(\top-\mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)}|G(\mathfrak{z},s_\mathfrak{z})-G(\mathfrak{z},\rho_\mathfrak{z})|d\mathfrak{z}$$

$$+QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\|[B(c_S(t))+\mathfrak{q}^*(t)]-[B(c_\rho(t))+\bar{\mathfrak{q}}^*(t)]\|+\sum_{j=1}^{i}QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}$$

$$\times\|[B(c_S(t))+\mathfrak{q}^*(t_j)]-[B(c_\rho(t))+\bar{\mathfrak{q}}^*(t_j))]\|+\frac{\alpha P^2}{\mathbb{N}(\alpha)}\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}\mathcal{S}_\alpha(t_j-\mathfrak{z})^{\alpha-1}$$

$$\times\|[B(c_S(\mathfrak{z}))+\mathfrak{q}^*(\mathfrak{z})]-[B(c_\rho(\mathfrak{z}))+\bar{\mathfrak{q}}^*(\mathfrak{z})]\|d\mathfrak{z}+\frac{\alpha P^2}{\mathbb{N}(\alpha)}\int_{t_i}^{t}\mathcal{S}_\alpha(t-\mathfrak{z})^{\alpha-1}$$

$$\times\|[B(c_S(\mathfrak{z}))+\mathfrak{q}^*(\mathfrak{z})]-[B(c_\rho(\mathfrak{z}))+\bar{\mathfrak{q}}^*(\mathfrak{z})]\|d\mathfrak{z}+PT_\alpha(t)\sum_{j=1}^{i}|I_j(s(t_j^-))-I_j(\rho(t_j^-))|,$$

here, $\mathfrak{q}^*,\bar{\mathfrak{q}}^*\in\mathfrak{C}(\mathcal{J},\mathfrak{R})$ is

$$\mathfrak{q}^*(t)=\mathcal{I}(t,s_t,\mathfrak{q}^*(t),c_S(t)),$$

and

$$\bar{\mathfrak{q}}^*(t)=\mathcal{I}(t,\rho_t,\bar{\mathfrak{q}}^*(t),c_\rho(t)).$$

By $(P2)$, we prove

$$\begin{aligned}|\mathfrak{q}^*(t)-\bar{\mathfrak{q}}^*(t)|&=|\mathcal{I}(t,s_t,\mathfrak{q}^*(t),c_S(t))-\mathcal{I}(t,\rho_t,\bar{\mathfrak{q}}^*(t),c_\rho(t))|\\&\leq\grave{\mathfrak{K}}_v\|s_t-\rho_t\|_{\hat{\mathcal{P}C}}+\grave{\mathfrak{L}}_v|\mathfrak{q}^*(t)-\bar{\mathfrak{q}}^*(t)|+\grave{\mathfrak{M}}_v\|c_S(t))-c_\rho(t)\|,\\|\mathfrak{q}^*(t)-\bar{\mathfrak{q}}^*(t)|&\leq\frac{\grave{\mathfrak{K}}_v}{1-\grave{\mathfrak{L}}_v}\|s_t-\rho_t\|_{\hat{\mathcal{P}C}}+\frac{\grave{\mathfrak{M}}_v}{1-\grave{\mathfrak{L}}_v}\|c_S(t))-c_\rho(t)\|\\&\leq\frac{\grave{\mathfrak{K}}_v}{1-\grave{\mathfrak{L}}_v}\|s_t-\rho_t\|_{\hat{\mathcal{P}C}}+\frac{\grave{\mathfrak{M}}_v\hat{R}_c}{1-\grave{\mathfrak{L}}_v}\|s_t-\rho_t\|_{\hat{\mathcal{P}C}}\\&\leq\left[\frac{\grave{\mathfrak{K}}_v+\grave{\mathfrak{M}}_v\hat{R}_c}{1-\grave{\mathfrak{L}}_v}\right]\|s_t-\rho_t\|_{\hat{\mathcal{P}C}},\end{aligned}$$

$$\begin{aligned}\|\Psi(s)-\Psi(\rho)\|_\Omega&\leq\grave{\mathcal{K}}_u\|s_t-\rho_t\|_{\hat{\mathcal{P}C}}+\frac{\zeta_1\hat{R}\top^\alpha}{\Gamma(\alpha+1)}\grave{\mathcal{K}}_s\|s_t-\rho_t\|_{\hat{\mathcal{P}C}}+\frac{\zeta_1\zeta_2(1-\alpha)\top^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}\bar{R}\hat{R}_c\|s_t-\rho_t\|_{\hat{\mathcal{P}C}}\\&+\frac{\zeta_1\zeta_2(1-\alpha)\top^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}\left[\frac{\grave{\mathfrak{K}}_v+\grave{\mathfrak{M}}_v\hat{R}_c}{1-\grave{\mathfrak{L}}_v}\right]\|s_t-\rho_t\|_{\hat{\mathcal{P}C}}+\frac{\zeta_1\zeta_2(1-\alpha)\top^\alpha\ell}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}\bar{R}\hat{R}_c\|s_t-\rho_t\|_{\hat{\mathcal{P}C}}\\&+\frac{\zeta_1\zeta_2(1-\alpha)\top^\alpha\ell}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}\left[\frac{\grave{\mathfrak{K}}_v+\grave{\mathfrak{M}}_v\hat{R}_c}{1-\grave{\mathfrak{L}}_v}\right]\|s_t-\rho_t\|_{\hat{\mathcal{P}C}}+\frac{\alpha\zeta_1^2\ell}{\mathbb{N}(\alpha)}\hat{R}_1\bar{R}\hat{R}_c\|s_t-\rho_t\|_{\hat{\mathcal{P}C}}\\&+\frac{\alpha\zeta_1^2}{\mathbb{N}(\alpha)}\hat{R}_1\bar{R}\hat{R}_c\|s_t-\rho_t\|_{\hat{\mathcal{P}C}}+\frac{\alpha\zeta_1^2\ell}{\mathbb{N}(\alpha)}\hat{R}_1\left[\frac{\grave{\mathfrak{K}}_v+\grave{\mathfrak{M}}_v\hat{R}_c}{1-\grave{\mathfrak{L}}_v}\right]\|s_t-\rho_t\|_{\hat{\mathcal{P}C}}\\&+\frac{\alpha\zeta_1^2}{\mathbb{N}(\alpha)}\hat{R}_1\left[\frac{\grave{\mathfrak{K}}_v+\grave{\mathfrak{M}}_v\hat{R}_c}{1-\grave{\mathfrak{L}}_v}\right]\|s_t-\rho_t\|_{\hat{\mathcal{P}C}}+\zeta_1\hat{R}\ell\grave{\mathcal{K}}_t\|s_t-\rho_t\|_{\hat{\mathcal{P}C}}\end{aligned}$$

$$\leq \left[\frac{\zeta_1\hat{R}\top^{\alpha}}{\Gamma(\alpha+1)}\grave{\mathcal{K}}_s + \frac{\zeta_1\zeta_2(1-\alpha)\top^{\alpha}}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}(\ell+1)\left(\bar{R}\hat{R}_c + \frac{\grave{\mathcal{K}}_v + \grave{\mathcal{M}}_v\hat{R}_c}{1-\grave{\mathcal{L}}_v}\right)\right]\|s_t-\rho_t\|_{\hat{\mathcal{P}C}}$$
$$+\left[\frac{\alpha\zeta_1^2}{\mathbb{N}(\alpha)}\hat{R}_1(\ell+1)\left(\bar{R}\hat{R}_c + \frac{\grave{\mathcal{K}}_v + \grave{\mathcal{M}}_v\hat{R}_c}{1-\grave{\mathcal{L}}_v}\right) + \grave{\mathcal{K}}_u + \zeta_1\hat{R}\ell\grave{\mathcal{K}}_t\right]\|s_t-\rho_t\|_{\hat{\mathcal{P}C}}.$$

Hence, we obtain

$$\|\Psi(s)-\Psi(\rho)\|_{\Omega} \leq \Theta_a\|s-\rho\|_{\Omega}. \tag{3.5}$$

Therefore, Ψ is a contraction and (1.1) has a unique mild solution by Theorem 2 in Ω. □

4. Controllability

Theorem 7. *Let us consider that hypotheses (P1)–(P10) hold, then the proposed problem* (1.1) *is controllable if*

$$\frac{\zeta_1\hat{R}\top^{\alpha}\mathcal{K}_g}{\Gamma(\alpha+1)} + \zeta_1\hat{R}\|s_j^-\| + \frac{\zeta_2\zeta_1(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}(\hat{R}_q+\hat{R}_n)(\top^{\alpha}+\eta)$$
$$+\frac{\alpha\zeta^2\hat{R}_1\top^{\alpha}}{\mathbb{N}(\alpha)}(\hat{R}_q+\hat{R}_n)(\top^{\alpha}+\eta) + \complement_1^*\|s\| + \complement_2^* < 1. \tag{4.1}$$

Proof. Let us define the set

$$\Omega_{\mu_b} = \{s\in\Omega; \|s\|_{\Omega} \leq \mu_b\} \in C : \|c\|] \leq \mu_b,$$

where

$$\mu_b \geq \frac{\left[1-2\zeta_1\hat{R}\kappa\eta\bar{R}\bar{R}_1(1+\zeta_1\hat{R})\right]\left[\hat{R}_n + \frac{\zeta_1\hat{R}\mathcal{K}_g\top^{\alpha}}{\Gamma(\alpha+1)} + \kappa(\bar{R}\bar{R}_1\kappa_a + \hat{R}_q)\right]\left[(1-\kappa\bar{R}\bar{R}_1)\left(\hat{R}_n + 2\kappa\eta(\bar{R}\bar{R}_1\kappa_b + \hat{R}_q)\right)\right]}{2(1-\kappa\bar{R}\bar{R}_1)\left[1-2\zeta_1\hat{R}\kappa\eta\bar{R}\bar{R}_1(1+\zeta_1\hat{R})\right]},$$

$$\kappa = \frac{\zeta_1\zeta_2(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha+1)} + \frac{\hat{R}_1\zeta_1^2}{\mathbb{N}(\alpha)}.$$

$\Omega_{\mu_b} \subset \Omega$ is closed, bounded, and convex. We observe that the fixed points of the operator Ψ_1 are the mild solutions of the formulated problem (1.1) with $\Psi_1(s)(\top) = s_1$. This implies that the system is controllable. Now, we derive the postulates of Theorem 2.

We define the operator $\Psi_1 : \Omega \to \Omega$ defined by,

$$\Psi_1(s(t)) = \begin{cases} \varphi(t); t\in[-\mathfrak{r},0] \\ PT_{\alpha}(t)\displaystyle\int_0^t \frac{(\top-\mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)}G(\mathfrak{z},s_{\mathfrak{z}})d\mathfrak{z} + QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\displaystyle\int_0^t (t-\mathfrak{z})^{\alpha-1}[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} \\ +\frac{\alpha P^2}{\mathbb{N}(\alpha)}\displaystyle\int_0^t \mathcal{S}_{\alpha}(t-\mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z}, \quad t\in(0,t_1] \\ PT_{\alpha}(t_j-t_{j-1})s(t_{j-1}^{-1})) + QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\displaystyle\int_{t_{j-1}}^{t_j}(t-\mathfrak{z})^{\alpha-1}[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} \\ +\frac{\alpha P^2}{\mathbb{N}(\alpha)}\displaystyle\int_{t_{j-1}}^{t_j}\mathcal{S}_{\alpha}(t-\mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + PT_{\alpha}(t)\sum_{j=1}^{i} I_{j-1}(s(t_{j-1}^-)), \quad (t_{j-1},t_j]. \end{cases} \tag{4.2}$$

By $(P2)$, we define the control, $c_s(t)$

$$c_s(t) = \mathcal{W}^{-1}\begin{cases} s_\top - PT_\alpha(t)\int_0^\top \frac{(\top-\mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)}G(\mathfrak{z}, s_\mathfrak{z})d\mathfrak{z} - QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_0^\top (\top-\mathfrak{z})^{\alpha-1}\mathfrak{q}^*(\mathfrak{z})d\mathfrak{z} \\ -\frac{\alpha P^2}{\mathbb{N}(\alpha)}\int_0^\top \mathcal{S}_\alpha(\top-\mathfrak{z})(\mathfrak{q}^*(\mathfrak{z}))d\mathfrak{z} - QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_0^\top (\top-\mathfrak{z})^{\alpha-1}\mathfrak{N}((\mathfrak{z}), s(\mathfrak{z}))d\mathfrak{z} \\ -\frac{\alpha P^2}{\mathbb{N}(\alpha)}\int_0^\top \mathcal{S}_\alpha(\top-\mathfrak{z})\mathfrak{N}((\mathfrak{z}), s(\mathfrak{z}))d\mathfrak{z}, \quad t\in(0,t_1] \\ s_\top - PT_\alpha(\top-t_j)s(t_j^{-1}) - QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_{t_j}^\top (\top-\mathfrak{z})^{\alpha-1}\mathfrak{q}^*(\mathfrak{z})d\mathfrak{z} \\ -\frac{\alpha P^2}{\mathbb{N}(\alpha)}\int_{t_i}^t \mathcal{S}_\alpha(t-\mathfrak{z})(\mathfrak{q}^*(\mathfrak{z}))d\mathfrak{z} - QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_{t_j}^\top (\top-\mathfrak{z})^{\alpha-1}\mathfrak{N}((\mathfrak{z}), s(\mathfrak{z}))d\mathfrak{z} \\ -\frac{\alpha P^2}{\mathbb{N}(\alpha)}\int_{t_i}^t \mathcal{S}_\alpha(t-\mathfrak{z})\mathfrak{N}((\mathfrak{z}), s(\mathfrak{z}))d\mathfrak{z} + PT_\alpha(t)\sum_{j=1}^{i} I_j(s(t_j^-)) \ \ (t_i, t_{i+1}]. \end{cases} \tag{4.3}$$

Step 1: Ψ_1 is continuous.

$$\begin{aligned} \|\Psi_1(s_r) - \Psi_1(s)\| \le \|Q\| \ \|P\|\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \\ \times \int_0^t (t-\mathfrak{z})^{\alpha-1}[\|B\| \ \|c_{s_r}(\mathfrak{z}) - c_s(\mathfrak{z})\| + \|\mathfrak{q}^*(\mathfrak{z}) - \bar{\mathfrak{q}}^*(\mathfrak{z})\|]d\mathfrak{z} \\ + \frac{\alpha\|P^2\|}{\mathbb{N}(\alpha)}\int_0^t \|\mathcal{S}_\alpha(t-\mathfrak{z})\|[\|B\| \ \|c_{s_r}(\mathfrak{z}) - c_s(\mathfrak{z})\| + \|\mathfrak{q}^*(\mathfrak{z} - \bar{\mathfrak{q}}^*(\mathfrak{z})\|]d\mathfrak{z} \\ \le \frac{\zeta_1\zeta_2(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\bar{R}\int_0^t (t-\mathfrak{z})^{\alpha-1}\Big\{\bar{R}\Big\{\frac{\zeta_1\zeta_2(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_0^\top (\top-\varphi)^{\alpha-1} \\ \times \|\mathfrak{q}^*(\varphi) - \mathfrak{q}^*(\varphi)\|d\varphi + \frac{\alpha\zeta_1^2}{\mathbb{N}(\alpha)}\hat{R}_1\int_0^\top (\top-\varphi)^{\alpha-1}\|\mathfrak{q}^*(\varphi) - \mathfrak{q}^*(\varphi)\|d\varphi\Big\} \\ + \|\mathfrak{q}^*(\mathfrak{z}) - \bar{\mathfrak{q}}^*(\mathfrak{z})\|\Big\}d\mathfrak{z} + \frac{\alpha\zeta_1^2}{\mathbb{N}(\alpha)}\hat{R}_1\int_0^t (t-\mathfrak{z})^{\alpha-1} \\ \times \Big\{\bar{R}\Big\{\frac{\zeta_1\zeta_2(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_0^\top (\top-\varphi)^{\alpha-1}\|\mathfrak{q}^*(\varphi) - \mathfrak{q}^*(\varphi)\|d\varphi + \frac{\alpha\zeta_1^2}{\mathbb{N}(\alpha)}\hat{R}_1 \\ \times \int_0^\top (\top-\varphi)^{\alpha-1}\|\mathfrak{q}^*(\varphi) - \mathfrak{q}^*(\varphi)\|d\varphi\Big\} + \|\mathfrak{q}^*(\mathfrak{z}) - \bar{\mathfrak{q}}^*(\mathfrak{z})\|\Big\}d\mathfrak{z}. \end{aligned}$$

For, $t \in (t_{j-1}, t_j]$, we get

$$\begin{aligned} \|\Psi_1(s_r) - \Psi_1(s)\| \le \|P\| \ \|T_\alpha(t_{j-1} - t_j)\| \ \|(s_r(t_j^-) - (s(t_j^-))\| + Q\| \ \|P\|\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \\ \int_{t_j}^\top (t-\mathfrak{z})^{\alpha-1}[\|B\| \ \|c_{s_r}(\mathfrak{z}) - c_s(\mathfrak{z})\| + \|\mathfrak{q}^*(\mathfrak{z}) - \bar{\mathfrak{q}}^*(\mathfrak{z})\|]d\mathfrak{z} \\ + \frac{\alpha\|P^2\|}{\mathbb{N}(\alpha)}\int_{t_j}^\top \|\mathcal{S}_\alpha(t-\mathfrak{z})\|[\|B\| \ \|c_{s_r}(\mathfrak{z}) - c_s(\mathfrak{z})\| + \|\mathfrak{q}^*(\mathfrak{z} - \bar{\mathfrak{q}}^*(\mathfrak{z})\|]d\mathfrak{z} \\ + PT_\alpha(t)\|\sum_{j=1}^{i} I_j(s_r(t_j^-)) - I_j(s(t_j^-))\| \end{aligned}$$

$$
\begin{aligned}
&\le \zeta_1\hat{R}\|(s_r(t_j^-) - (s(t_j^-))\| + \frac{\zeta_1\zeta_2(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\bar{R}\int_{t_j}^{\top}(t-\mathfrak{z})^{\alpha-1}\\
&\times\Bigg\{\bar{R}\Big\{\frac{\zeta_1\zeta_2(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_{t_j}^{\top}(\top-\varphi)^{\alpha-1}\|\mathfrak{q}^*(\varphi)-\mathfrak{q}^*(\varphi)\|d\varphi\\
&+\frac{\alpha\zeta_1^2}{\mathbb{N}(\alpha)}\hat{R}_1\int_{t_j}^{\top}(\top-\varphi)^{\alpha-1}\|\mathfrak{q}^*(\varphi)-\mathfrak{q}^*(\varphi)\|d\varphi\Big\}+\|\mathfrak{q}^*(\mathfrak{z})-\bar{\mathfrak{q}}^*(\mathfrak{z})\|\Bigg\}d\mathfrak{z}\\
&+\frac{\alpha\zeta_1^2}{\mathbb{N}(\alpha)}\hat{R}_1\int_0^t(t-\mathfrak{z})^{\alpha-1}\Bigg\{\bar{R}\Big\{\frac{\zeta_1\zeta_2(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_{t_j}^{\top}(\top-\varphi)^{\alpha-1}\|\mathfrak{q}^*(\varphi)-\mathfrak{q}^*(\varphi)\|d\varphi\\
&+\frac{\alpha\zeta_1^2}{\mathbb{N}(\alpha)}\hat{R}_1\int_{t_j}^{\top}(\top-\varphi)^{\alpha-1}\|\mathfrak{q}^*(\varphi)-\mathfrak{q}^*(\varphi)\|d\varphi\Big\}+\|\mathfrak{q}^*(\mathfrak{z})-\bar{\mathfrak{q}}^*(\mathfrak{z})\|\Bigg\}d\mathfrak{z}\\
&+\zeta_{1\hat{R}}\sum_{j=1}^{i}\|I_j(s_r(t_j^-))-I_j(s(t_j^-))\|.
\end{aligned}
$$

We easily observe that $\Psi_1(s_r)\mapsto\Psi_1(s)$ in Ω_{μ_b} due to the continuity of the functions $\mathfrak{q}^*$ *and* I. This implies the proof of continuity of Ψ_1.

Step 2: Ψ_1 maps the bounded sets into bounded sets.

$$
\|c_s(t)\| = \|\mathcal{W}^{-1}\|\begin{cases}
\|s_\top\| + \|P\|\ \|T_\alpha(t)\|\ s_0 + \|Q\|\ \|P\|\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_0^{\top}(t-\mathfrak{z})^{\alpha-1}\|\mathfrak{q}^*(\mathfrak{z})\|d\mathfrak{z}\\
+\frac{\alpha\|P^2\|}{\mathbb{N}(\alpha)}\int_0^t\|\mathcal{S}_\alpha(t-\mathfrak{z})\|\ \|\mathfrak{q}^*(\mathfrak{z})\|d\mathfrak{z}+\|Q\|\ \|P\|\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\\
\times\int_0^{\top}(\top-\mathfrak{z})^{\alpha-1}\|\mathfrak{R}((\mathfrak{z}),s(\mathfrak{z}))\|d\mathfrak{z}+\frac{\alpha\|P^2\|}{\mathbb{N}(\alpha)}\int_0^{\top}\|\mathcal{S}_\alpha(\top-\mathfrak{z})\|\ \|\mathfrak{R}((\mathfrak{z}),s(\mathfrak{z}))\|d\mathfrak{z}\\
\|s_\top\| + \|P\|\ \|T_\alpha(\top-t_j)\|\ \|s_j^-\|+\|Q\|\ \|P\|\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\\
\times\int_{t_j}^{\top}(t-\mathfrak{z})^{\alpha-1}\|\mathfrak{q}^*(\mathfrak{z})\|d\mathfrak{z}+\frac{\alpha\|P^2\|}{\mathbb{N}(\alpha)}\int_{t_j}^t\|\mathcal{S}_\alpha(t-\mathfrak{z})\|\ \|\mathfrak{q}^*(\mathfrak{z})\|d\mathfrak{z}+\|I_j(s(t_j^-))\|\\
+\|Q\|\ \|P\|\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_{t_j}^{\top}(\top-\mathfrak{z})^{\alpha-1}\|\mathfrak{R}((\mathfrak{z}),s(\mathfrak{z}))\|d\mathfrak{z}\\
+\frac{\alpha\|P^2\|}{\mathbb{N}(\alpha)}\int_{t_j}^{\top}\|\mathcal{S}_\alpha(\top-\mathfrak{z})\|\ \|\mathfrak{R}((\mathfrak{z}),s(\mathfrak{z}))\|d\mathfrak{z}+\|P\|\ \|T_\alpha(t)\|\ \|\sum_{j=1}^{i}I_j(s(t_j^-))\|.
\end{cases}
$$

By the postulates (P1)–(P10),

$$
\|c_s(t)\| \le \bar{R}_1\begin{cases}
s_\top+\frac{\zeta_1\hat{R}\top^\alpha\mathcal{K}_g}{\Gamma(\alpha+1)}+\frac{\zeta_2\zeta_1(1-\alpha)\hat{R}_q\top^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}+\frac{\alpha\zeta^2\hat{R}_1\hat{R}_q\top^\alpha}{\mathbb{N}(\alpha)}+\frac{\zeta_2\zeta_1(1-\alpha)\hat{R}_n\top^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}+\frac{\alpha\zeta^2\hat{R}_1\hat{R}_n\top^\alpha}{\mathbb{N}(\alpha)}\\
\|s_\top\|+\zeta_1\hat{R}\|s_j^-\|+\frac{\zeta_2\zeta_1(1-\alpha)(\top-t_j)^\alpha\hat{R}_q}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}+\frac{\zeta_1\hat{R}(\top-t_j)^\alpha}{\Gamma\alpha+1}+\frac{\zeta_2\zeta_1(1-\alpha)\hat{R}_n(\top-t_j)^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}\\
+\frac{\alpha\zeta^2\hat{R}_1\hat{R}_n(\top-t_j)^\alpha}{\mathbb{N}(\alpha)}+\zeta_1\hat{R}\omega,
\end{cases}
$$

$$\|c_s(t)\| \leq \bar{R}_1 \begin{cases} s_\top + \dfrac{\zeta_1 \hat{R} \top^\alpha \mathcal{K}_g}{\Gamma(\alpha+1)} + \dfrac{\zeta_2\zeta_1(1-\alpha)\top^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}(\hat{R}_q + \hat{R}_n) + \dfrac{\alpha\zeta^2\hat{R}_1\top^\alpha}{\mathbb{N}(\alpha)}(\hat{R}_q + \hat{R}_n) \\ s_\top + \zeta_1\hat{R}\|s_j^-\| + \dfrac{\zeta_2\zeta_1(1-\alpha)(\top - t_j)^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}(\hat{R}_q + \hat{R}_n) + \dfrac{\alpha\zeta^2\hat{R}_1(\top - t_j)^\alpha}{\mathbb{N}(\alpha)}(\hat{R}_q + \hat{R}_n) \\ +\zeta_1\hat{R}\omega. \end{cases}$$

Let the two constants be

$$\begin{aligned} \|c_s(t)\| &\leq \bar{R}_1 \left[s_\top + \frac{\zeta_1 \hat{R} \top^\alpha \mathcal{K}_g}{\Gamma(\alpha+1)} + \frac{\zeta_2\zeta_1(1-\alpha)\top^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}(\hat{R}_q + \hat{R}_n) + \frac{\alpha\zeta^2\hat{R}_1\top^\alpha}{\mathbb{N}(\alpha)}(\hat{R}_q + \hat{R}_n) \right] \\ &\leq \bar{R}_1 s_\top + \bar{R}_1 \kappa_a, \end{aligned}$$

for all $t \in (0, t_1]$, where

$$\kappa_a = \frac{\zeta_1 \hat{R} \top^\alpha \mathcal{K}_g}{\Gamma(\alpha+1)} + \frac{\zeta_2\zeta_1(1-\alpha)\top^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}(\hat{R}_q + \hat{R}_n) + \frac{\alpha\zeta^2\hat{R}_1\top^\alpha}{\mathbb{N}(\alpha)}(\hat{R}_q + \hat{R}_n),$$

$$\begin{aligned} \|c_s(t)\| &\leq \bar{R}_1 \left[s_\top + \zeta_1\hat{R}\|s_j^-\| + \frac{\zeta_2\zeta_1(1-\alpha)\eta}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}(\hat{R}_q + \hat{R}_n) + \frac{\alpha\zeta^2\hat{R}_1\eta}{\mathbb{N}(\alpha)}(\hat{R}_q + \hat{R}_n) + \zeta_1\hat{R}\omega \right] \\ &\leq \bar{R}_1 s_\top + \bar{R}_1\zeta_1\hat{R} s_\top + \bar{R}_1\kappa_b, \end{aligned}$$

where

$$\kappa_b = \frac{\zeta_2\zeta_1(1-\alpha)\eta}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}(\hat{R}_q + \hat{R}_n) + \frac{\alpha\zeta^2\hat{R}_1\eta}{\mathbb{N}(\alpha)}(\hat{R}_q + \hat{R}_n) + \zeta_1\hat{R}\omega$$

for all $t \in (t_j, t_{j-1}]$, where $\eta = max(\top - t_j)^\alpha$.

And therefore, for $t \in (0, t_1]$

$$\Psi_1(c_s(t)) \leq \frac{\zeta_1 \hat{R} \top^\alpha \mathcal{K}_g}{\Gamma(\alpha+1)} + \frac{\zeta_2\zeta_1(1-\alpha)\top^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}(\hat{R}_q + \hat{R}_n) + \frac{\alpha\zeta^2\hat{R}_1\top^\alpha}{\mathbb{N}(\alpha)}(\hat{R}_q + \hat{R}_n)$$

for $t \in (t_j, t_{j-1}]$,

$$\Psi_1(c_s(t)) \leq \zeta_1\hat{R}\|s_j^-\| + \frac{\zeta_2\zeta_1(1-\alpha)\eta}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}(\hat{R}_q + \hat{R}_n) + \frac{\alpha\zeta^2\hat{R}_1\eta}{\mathbb{N}(\alpha)}(\hat{R}_q + \hat{R}_n) + \zeta_1\hat{R}\omega.$$

Hence we have,

$$\begin{aligned} \Psi_1(c_s(t)) \leq & \frac{\zeta_1 \hat{R} \top^\alpha \mathcal{K}_g}{\Gamma(\alpha+1)} + \zeta_1\hat{R}\|s_j^-\| + \frac{\zeta_2\zeta_1(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}(\hat{R}_q + \hat{R}_n)(\top^\alpha + \eta) \\ & + \frac{\alpha\zeta^2\hat{R}_1\top^\alpha}{\mathbb{N}(\alpha)}(\hat{R}_q + \hat{R}_n)(\top^\alpha + \eta) + \zeta_1\hat{R}\omega \\ \leq & \mu_b. \end{aligned}$$

This implies that, $\|\Psi_1(s)\| \leq \mu_b$. So, $\Psi_1(\Omega_{\mu_b}) \subset \Omega_{\mu_b}$.

Step 3: Verify the equi-continuity of Ψ_1. Consider $s \in \Omega_{\mu_b}$ and $\rho_1, \rho_2 \in (t_{j-1}, t_j]$. Here, $\zeta_1 < \zeta_2, j = 1, 2, ...\ell$,

$$
\begin{aligned}
&\|\Psi_1(s)(\rho_2) - \Psi_1(s)(\rho_1)\| \\
=&\|PT_\alpha(\rho_2)\int_0^{\rho_1}\frac{(\top - \mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)}G(\mathfrak{z}, s_\mathfrak{z})d\mathfrak{z} + QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_0^{\rho_1}(\rho_2 - \mathfrak{z})^{\alpha-1} \\
&\times [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + \frac{\alpha P^2}{\mathbb{N}(\alpha)}\int_0^{\rho_1}\mathcal{S}_\alpha(\rho_2 - \mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} \\
&- PT_\alpha(\rho_1)\int_0^{\rho_1}\frac{(\top - \mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)}G(\mathfrak{z}, s_\mathfrak{z})d\mathfrak{z} - QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_0^{\rho_1}(\rho_1 - \mathfrak{z})^{\alpha-1} \\
&\times [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + \frac{\alpha P^2}{\mathbb{N}(\alpha)}\int_0^{\rho_1}\mathcal{S}_\alpha(\rho_1 - \mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z}\| \\
\leq&\zeta_1\frac{[(\top - \rho_1)^\alpha - \top^\alpha\mathcal{K}_g]}{\Gamma(\alpha+1)}\|T_\alpha(\rho_2) - T_\alpha(\rho_1) + \frac{\zeta_2\zeta_1(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}(\bar{R}g_1 + \hat{R}_q) \\
&\times \int_0^{\rho_1}(\rho_2 - \mathfrak{z})^{\alpha-1} - (\rho_1 - \mathfrak{z})^{\alpha-1}d\mathfrak{z} + \frac{\alpha\zeta_1^2}{\mathbb{N}(\alpha)}(\bar{R}g_1 + \hat{R}_q) \\
&\times \int_0^{\rho_1}\|\mathcal{S}_\alpha(\rho_2 - \mathfrak{z}) - \mathcal{S}_\alpha(\rho_1 - \mathfrak{z})\|d\mathfrak{z} + \frac{\zeta_2\zeta_1(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}(\bar{R}g_1 + \hat{R}_q) \\
&\times \int_{\rho_1}^{\rho_2}(\rho_2 - \mathfrak{z})^{\alpha-1}d\mathfrak{z} + \frac{\alpha\zeta_1^2}{\mathbb{N}(\alpha)}(\bar{R}g_1 + \hat{R}_q)\int_{\rho_1}^{\rho_2}\|\mathcal{S}_\alpha(\rho_2 - \mathfrak{z})\|d\mathfrak{z}.\rho_1, \rho_2 \in (0, t_1].
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\|\Psi_1(s)(\rho_2) - \Psi_1(s)(\rho_1)\| \\
=&\|PT_\alpha(\rho_2 - t_{j-1})s(t_{j-1}) + QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_0^{\rho_1}(\rho_2 - \mathfrak{z})^{\alpha-1} \\
&\times [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + \frac{\alpha P^2}{\mathbb{N}(\alpha)}\int_0^{\rho_1}\mathcal{S}_\alpha(\rho_2 - \mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} \\
&- PT_\alpha(\rho_1)\int_0^{\rho_1}\frac{(\top - \mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)}G(\mathfrak{z}, s_\mathfrak{z})d\mathfrak{z} - QP\frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}\int_0^{\rho_1}(\rho_1 - \mathfrak{z})^{\alpha-1} \\
&\times [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} + \frac{\alpha P^2}{\mathbb{N}(\alpha)}\int_0^{\rho_1}\mathcal{S}_\alpha(\rho_1 - \mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})]d\mathfrak{z} \\
&+ P(T_\alpha(\rho_2 - t_{j-1}) - T_\alpha(\rho_1 - t_{j-1}))I_{j-1}(s(t_{j-1}^-))\| \\
\leq&\zeta_1\|T_\alpha(\rho_2 - t_{j-1}) - T_\alpha(\rho_1 - t_{j-1})\| + \frac{\zeta_2\zeta_1(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}(\bar{R}g_1) + \hat{R}_q) \\
&\times \int_0^{\rho_1}(\rho_2 - \mathfrak{z})^{\alpha-1} - (\rho_1 - \mathfrak{z})^{\alpha-1}d\mathfrak{z} + \frac{\alpha\zeta_1^2}{\mathbb{N}(\alpha)}(\bar{R}g_1) + \hat{R}_q) \\
&\times \int_0^{\rho_1}\|\mathcal{S}_\alpha(\rho_2 - \mathfrak{z}) - \mathcal{S}_\alpha(\rho_1 - \mathfrak{z})\|d\mathfrak{z} + \frac{\zeta_2\zeta_1(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)}(\bar{R}g_1 + \hat{R}_q) \\
&\times \int_{\rho_1}^{\rho_2}(\rho_2 - \mathfrak{z})^{\alpha-1}d\mathfrak{z} + \frac{\alpha\zeta_1^2}{\mathbb{N}(\alpha)}(\bar{R}g_1 + \hat{R}_q)
\end{aligned}
$$

$$\times \int_{\rho_1}^{\rho_2} \|\mathcal{S}_\alpha(\rho_2 - \mathfrak{z})\| d\mathfrak{z} + \zeta_1 \hat{R}(\rho_2 - \rho_1)\|I_{j-1}(s(t_{j-1}^-))\| \ \rho_1, \rho_2 \in (t_{j-1}, t_j],\ j = 1, 2...\ell.$$

This result converges to 0 during ρ_1 tending to ρ_2. By the compactness and the strong continuity of the operators $T_\alpha(t)$ and $\mathcal{S}_\alpha(t)$, we easily get that Ψ_1 is continuous in uniform operator topology. Hence, $\Psi(\Omega_{\mu_b})$ satisfies the condition of equi-continuous.

Step 4: Ψ_1 is a contraction on Ω_{μ_b}. For $t \in (0, t_1]$,

$$\begin{aligned}
\|\Psi_1(s) - \Psi_1(\rho)\| =& \|PT_\alpha(t) \int_0^t \frac{(\top - \mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)} G(\mathfrak{z}, s_\mathfrak{z}) d\mathfrak{z} + QP \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^t (t - \mathfrak{z})^{\alpha-1} \\
& \times [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})] d\mathfrak{z} + \frac{\alpha P^2}{\mathbb{N}(\alpha)} \int_0^t \mathcal{S}_\alpha(t - \mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})] d\mathfrak{z} \\
& - PT_\alpha(t) \int_0^t \frac{(\top - \mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)} G(\mathfrak{z}, \rho_\mathfrak{z}) d\mathfrak{z} - QP \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_0^t (t - \mathfrak{z})^{\alpha-1} \\
& \times [B(c_\rho(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})] d\mathfrak{z} - \frac{\alpha P^2}{\mathbb{N}(\alpha)} \int_0^t \mathcal{S}_\alpha(t - \mathfrak{z})[B(c_\rho(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})] d\mathfrak{z}\|, t \in (0, t_1] \\
& \leq \frac{\zeta_1 \hat{R} \top^\alpha}{\Gamma(\alpha+1)} \grave{\mathcal{K}}_s \|s_t - \rho_t\|_{\grave{\mathcal{P}C}} + \frac{\zeta_2\zeta_1(1-\alpha)\top^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)} \left[\frac{\grave{\mathfrak{K}}_v + \grave{\mathfrak{M}}_v \hat{R}_c}{1 - \grave{\mathfrak{L}}_v}\right] \|s_t - \rho_t\|_{\grave{\mathcal{P}C}} \\
& + \frac{\alpha\zeta_1^2 \hat{R}_1}{\mathbb{N}(\alpha)} \left[\frac{\grave{\mathfrak{K}}_v + \grave{\mathfrak{M}}_v \hat{R}_c}{1 - \grave{\mathfrak{L}}_v}\right] \|s_t - \rho_t\|_{\grave{\mathcal{P}C}} + \frac{\zeta_2\zeta_1(1-\alpha)\bar{R}\top^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)} \hat{R}_c \|s_t - \rho_t\|_{\grave{\mathcal{P}C}} \\
& + \frac{\alpha\zeta_1^2 \hat{R}_1 \bar{R}}{\mathbb{N}(\alpha)} \hat{R}_c \|s_t - \rho_t\|_{\grave{\mathcal{P}C}} \\
& \leq \frac{\zeta_1 \hat{R} \top^\alpha}{\Gamma(\alpha+1)} \grave{\mathcal{K}}_s \|s_t - \rho_t\|_{\grave{\mathcal{P}C}} + \frac{\zeta_2\zeta_1(1-\alpha)\top^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)} \left[\frac{\grave{\mathfrak{K}}_v + \grave{\mathfrak{M}}_v \hat{R}_c + (1 - \grave{\mathfrak{L}}_v)\bar{R}\hat{R}_c}{1 - \grave{\mathfrak{L}}_v}\right] \\
& \times \|s_t - \rho_t\|_{\grave{\mathcal{P}C}} + \frac{\alpha\zeta_1^2 \hat{R}_1}{\mathbb{N}(\alpha)} \left[\frac{\grave{\mathfrak{K}}_v + \grave{\mathfrak{M}}_v \hat{R}_c + (1 - \grave{\mathfrak{L}}_v)\bar{R}\hat{R}_c}{1 - \grave{\mathfrak{L}}_v}\right] \|s_t - \rho_t\|_{\grave{\mathcal{P}C}} \\
& \leq \left(\frac{\zeta_1 \hat{R} \top^\alpha}{\Gamma(\alpha+1)} \grave{\mathcal{K}}_s + \frac{\zeta_2\zeta_1(1-\alpha)\top^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)} \left[\frac{\grave{\mathfrak{K}}_v + \grave{\mathfrak{M}}_v \hat{R}_c + (1 - \grave{\mathfrak{L}}_v)\bar{R}\hat{R}_c}{1 - \grave{\mathfrak{L}}_v}\right]\right) \|s_t - \rho_t\|_{\grave{\mathcal{P}C}} \\
& + \left(\frac{\alpha\zeta_1^2 \hat{R}_1}{\mathbb{N}(\alpha)} \left[\frac{\grave{\mathfrak{K}}_v + \grave{\mathfrak{M}}_v \hat{R}_c + (1 - \grave{\mathfrak{L}}_v)\bar{R}\hat{R}_c}{1 - \grave{\mathfrak{L}}_v}\right]\right) \|s_t - \rho_t\|_{\grave{\mathcal{P}C}}.
\end{aligned}$$

For $t \in (t_{j-1}, t_j]$,

$$\begin{aligned}
\|\Psi_1(s) - \Psi_1(\rho)\| =& \|PT_\alpha(t_j - t_{j-1}) s(t_{j-1}^{-1})) + QP \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_j - \mathfrak{z})^{\alpha-1} [B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})] d\mathfrak{z} \\
& + \frac{\alpha P^2}{\mathbb{N}(\alpha)} \int_{t_{j-1}}^{t_j} \mathcal{S}_\alpha(t_j - \mathfrak{z})[B(c_S(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})] d\mathfrak{z} + P(T_\alpha \sum_{j=1}^{i} I_{j-1}(s(t_{j-1}^-)) \\
& - PT_\alpha(t_j - t_{j-1}) \rho(t_{j-1}^{-1})) - QP \frac{(1-\alpha)}{\mathbb{N}(\alpha)\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_j - \mathfrak{z})^{\alpha-1} [B(c_\rho(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})] d\mathfrak{z} \\
& - \frac{\alpha P^2}{\mathbb{N}(\alpha)} \int_{t_{j-1}}^{t_j} \mathcal{S}_\alpha(t_j - \mathfrak{z})[B(c_\rho(\mathfrak{z})) + \mathfrak{q}^*(\mathfrak{z})] d\mathfrak{z} - P(T_\alpha \sum_{j=1}^{i} I_{j-1}(s(t_{j-1}^-))\|, t \in (t_{j-1}, t_j]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\zeta_2\zeta_1(1-\alpha)(t_j-t_{j-1})^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}\left[\frac{\grave{\mathfrak{K}}_v+\grave{\mathfrak{M}}_v\hat{R}_c}{1-\grave{\mathfrak{L}}_v}\right]\|s_t-\rho_t\|_{\grave{\mathcal{P}C}}\\
&+\frac{\alpha\zeta^2\hat{R}_1}{\mathbb{N}(\alpha)}\left[\frac{\grave{\mathfrak{K}}_v+\grave{\mathfrak{M}}_v\hat{R}_c}{1-\grave{\mathfrak{L}}_v}\right]\|s_t-\rho_t\|_{\grave{\mathcal{P}C}}+\frac{\zeta_2\zeta_1(1-\alpha)\bar{R}(t_j-t_{j-1})^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}\hat{R}_c\|s_t-\rho_t\|_{\grave{\mathcal{P}C}}\\
&+\frac{\alpha\zeta^2\hat{R}_1\bar{R}}{\mathbb{N}(\alpha)}\hat{R}_c\|s_t-\rho_t\|_{\grave{\mathcal{P}C}}\\
&\leq \frac{\zeta_2\zeta_1(1-\alpha)(t_j-t_{j-1})^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}\left[\frac{\grave{\mathfrak{K}}_v+\grave{\mathfrak{M}}_v\hat{R}_c+(1-\grave{\mathfrak{L}}_v)\bar{R}\hat{R}_c}{1-\grave{\mathfrak{L}}_v}\right]\|s_t-\rho_t\|_{\grave{\mathcal{P}C}}\\
&+\frac{\alpha\zeta^2\hat{R}_1}{\mathbb{N}(\alpha)}\left[\frac{\grave{\mathfrak{K}}_v+\grave{\mathfrak{M}}_v\hat{R}_c+(1-\grave{\mathfrak{L}}_v)\bar{R}\hat{R}_c}{1-\grave{\mathfrak{L}}_v}\right]\|s_t-\rho_t\|_{\grave{\mathcal{P}C}}\\
&\leq \left(\frac{\zeta_2\zeta_1(1-\alpha)(t_j-t_{j-1})^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}\left[\frac{\grave{\mathfrak{K}}_v+\grave{\mathfrak{M}}_v\hat{R}_c+(1-\grave{\mathfrak{L}}_v)\bar{R}\hat{R}_c}{1-\grave{\mathfrak{L}}_v}\right]\right)\|s_t-\rho_t\|_{\grave{\mathcal{P}C}}\\
&+\left(\frac{\alpha\zeta^2\hat{R}_1}{\mathbb{N}(\alpha)}\left[\frac{\grave{\mathfrak{K}}_v+\grave{\mathfrak{M}}_v\hat{R}_c+(1-\grave{\mathfrak{L}}_v)\bar{R}\hat{R}_c}{1-\grave{\mathfrak{L}}_v}\right]\right)\|s_t-\rho_t\|_{\grave{\mathcal{P}C}}.
\end{aligned}
$$

This implies that Ψ_1 is a contraction on Ω_{μ_b} for

$$
\left(\frac{\zeta_1\hat{R}\top^\alpha}{\Gamma(\alpha+1)}\grave{\mathcal{K}}_s+\frac{\zeta_2\zeta_1(1-\alpha)\top^\alpha}{\mathbb{N}(\alpha)\Gamma(\alpha+1)}\left[\frac{\grave{\mathfrak{K}}_v+\grave{\mathfrak{M}}_v\hat{R}_c+(1-\grave{\mathfrak{L}}_v)\bar{R}\hat{R}_c}{1-\grave{\mathfrak{L}}_v}\right]+\frac{\alpha\zeta^2\hat{R}_1}{\mathbb{N}(\alpha)}\left[\frac{\grave{\mathfrak{K}}_v+\grave{\mathfrak{M}}_v\hat{R}_c+(1-\grave{\mathfrak{L}}_v)\bar{R}\hat{R}_c}{1-\grave{\mathfrak{L}}_v}\right]\right)<1.
$$

Hence, Ψ_1 possesses a fixed point and so it is a mild solution of the proposed system (1.1) based on the defined control function $c_s(t)$ given in (4.3) by Theorem 2. By the definition of controllability (Definition 9), the proposed problem is controllable. □

5. Implementation

The following application is provided for evidencing the theoretical results:

$$
\begin{cases}
{}^{\mathcal{ABC}}_{0}D_t^{\frac{1}{10}}\left[s(t,\kappa)-\dfrac{\cos|s(t,\kappa)|}{45}\right]=\frac{\partial^2 s(t,\kappa)}{\partial\kappa^2}+c(t,\kappa)+\frac{t+\sin|s(t,\kappa)|}{45}\\
+\frac{e^t}{11+e^t}\frac{|{}^{\mathcal{ABC}}_{0}D_t^{\frac{1}{10}}s(t,\kappa)|}{1+|{}^{\mathcal{ABC}}_{0}D_t^{\frac{1}{10}}s(t,\kappa)|}+c(t,\kappa), t\in[0,1], t\neq\frac{1}{10},\\
\Delta s(t,\kappa)=\dfrac{s(\frac{1}{10}^-,\kappa)}{28+s(\frac{1}{10}^-,\kappa)},\\
s(t,\kappa)=\varphi(t,\kappa),\ t\in[-\mathfrak{r},0],\kappa\in[0,\pi]\ \mathfrak{r}>0,\\
s(t,0)=\displaystyle\int_0^1\frac{(1-\mathfrak{z})^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{25}\exp(-s(t,\mathfrak{z}))d\mathfrak{z}, s(t,\pi)=0.
\end{cases}
\tag{5.1}
$$

Here, $A:D(A)\subset\Omega\mapsto\Omega$ is an infinitesimal generator $A\chi=\chi^\ell$ where, $\Omega=\mathcal{L}^2[0,\pi]$ and the domain is defined by $D(A) = \{\chi \in \Omega : \chi$ and χ' are absolutely continuous, $\chi'' \in \Omega, \chi(0) = 1 = \chi(1)\}$,

$$A\chi = \sum_{\ell=1}^{\infty} \ell^2 < \chi, \chi \in D(A).$$

Whence the eigenvectors that are orthogonal are

$$\chi_\ell(\wp) = \sqrt{\frac{2}{\pi}} sin(\ell\wp), \ell \in \mathcal{N}.$$

Hence, the corresponding analytic semi-group $\mathcal{S}(t)$ related to A in Ω is $\mathcal{S}(t)\chi = \sum_{\ell=1}^{\infty} e^{-}\ell^2 < \chi, \chi \in \Omega$ and $\|(\mathcal{S}(t)\| \leq 1$. The resolvent operator $Q(\grave{\mu}, A) = (\grave{\mu}\mathcal{I} - A)^{-1}$ where $\grave{\mu} \in \rho(A)$. So the proposed system (5.1) will take the form of (1.1) by replacing

$$s(t,\kappa) = s(t), \quad c(t,\kappa) = c(t),$$

$$\Re(t, s(t)) = \frac{\cos|s(t)|}{45},$$

$$\mathcal{I}(t, s, \rho, c) = \frac{t + \sin|s(t)|}{45} + \frac{e^t}{11 + e^t}\frac{|\rho|}{1 + |\rho|} + |c(t)|,$$

where

$$\rho = {}_0^{\mathcal{ABC}} D_t^{\frac{1}{10}} s(t), \quad \mathbb{E}(t, s(t)) = \frac{1}{25}\exp(-s(t)).$$

We can easily verify that (5.1) fulfills the postulates (P1)–(P10) and so the proposed system is controllable by (4.1) on $[0, \pi]$.

6. Discussion

This research article gathers the results of existence, uniqueness , and controllability. In previous studies, authors either developed the results of existence and uniqueness or controllability. However, this article verifies the controllability results, being sufficient to verify the existence of a mild solution for the proposed system. Additionally, we have shown the uniqueness results using the Banach contraction principle to some extent. Due to this uniqueness, a single trajectory can be obtained for a unique control input. Also, researchers can design control strategies according to the system due to the uniqueness of the mild solution of the problem. We can ensure the well-defined controls, making the study of controllability results more straightforward. A new researcher can improve the system or include some delays in state space or control, obtaining new results. Highlighting the stability results of the problem is a key focus in current research scenarios. The comparative analysis of numerical solutions and theoretical results are gaining significant attention among researchers.

7. Conclusions

This work has successfully investigated the existence results for the nonlinear neutral fractional implicit impulsive differential equation with impulses, delay, and integro initial conditions by means of semi-group theory and fixed-point techniques. These types of problems have numerous applications, namely to the mathematical modeling of human diseases and complex problems. Based on Arzelà Ascoli theorem and Schauder's fixed-point theorem, we established the adequate results for at least one mild solution. Banach contraction principle helped to derive the uniqueness and controllability results of the defined system. The derived results were justified by providing a suitable illustration. Researchers can establish the stability results of the given problem as a future work. Also, changing the initial condition and including state delay, control delay, or both will obtain innovative results. Future work may be extended to non-instantaneous impulses and comparative analysis with numerical techniques.

Author contributions

Sivaranjani Ramasamy: Writing original draft, Conceptualization and Methodology, and Validation; Thangavelu Senthilprabu: Writing original draft, Validation and Resources; Kulandhaivel Kathikeyan: Investigation and Validation; Palanisamy Geetha: Investigation and Validation; Saowaluck Chasreechai: Writing original draft, Conceptualization and Methodology, and Investigation; Thanin Sitthiwirattham: Investigation and Validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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