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Research article

Enhanced decision model for sustainable energy solutions under bipolar hesitant fuzzy soft aggregation information

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Abstract: Energy sustainability is described as an ability to get the energy supplies without diminishing the ability of future generations to provide themselves with any energy. Preceded by the mentioned notion, this paper will be focused on bipolar hesitant fuzzy soft sets (BHFSS) related to the problem of energy sustainability. It actually was a proposed new mathematical method able to conquer ambiguity and uncertainty while determining the different choices in energy-related decisions. In this way, it lead to more informative and better choices to be made, thus leading to the utilization of sustainable energy systems. The paper introduced basic operations and comparison rules for BHFSS. Furthermore, algebraic norms-based aggregation operators were proposed to make the model more robust and flexible so that it was adaptable to a wide range of energy sustainability decisions. Main characteristics of the BHFSS aggregation operators were discussed in detail. Last but not least, this paper also provided a comparison of the BHFSS-based approach with one of the most popular multi-criteria decision-making (MCDM) approaches known as compromise solution (CoCoSo). This comparison confirmed how BHFSS can control for uncertainty and how it can reflect preferences in a mapped way, which afforded it strengths in uses like choosing renewable power and strategy for lowering CO_2 emissions.

Keywords: bipolar hesitant fuzzy soft set; algebraic aggregationoperators; group decision making
Mathematics Subject Classification: 03E72, 90B50

1. Introduction

Energy sustainability needs to meet the present energy requirements without jeopardizing future generations by rendering the economy, society, and environment useless. It encompasses economic growth, minimizing energy waste, and harnessing renewable energy sources. The sustainable use of energy would help combat climate change, reduce our ecological footprint, and create a more efficient energy system. This is quite possible only with collaboration, innovation, and a new commitment by all to put sustainable energy practices into action. All of this will come together through collaborative efforts to make renewable energy drive the future economies to the good of all of us. Frequently in research, results are unforeseen and unpredictable. In my opinion, this is particularly pertinent not only to any energy-related application but in most biological sciences, in many technologies, machine learning, and pattern recognition exercises. Being researchers, it is part of our job to scrutinize and find ways through these uncertainties.

1.1. Review of literature

A multi-criteria decision making (MCDM) technique has been invented by Bhole et al. [1] to rank the alternatives through multiple criteria such as scientific, financial, social, and environmental factors. Comparing with single criteria methods, MCDM deals with more complicated cases with one or more than one objectives and conducts neutral and accurate evaluation. In this direction, MCDM [2–4] plays a crucial role in light of energy sustainability issues, focusing on the use of renewable energy sources and efficiency techniques in terms of financial, environmental, and social purposes. Demir et al. [5] addresses specifically the uncertainties regarding future energy scenarios. MCDM enables decision-makers [6–9] to discover, as well as determine, optimal solutions that promote the progress of long-term sustainability goals.

Although Zadeh presented fuzzy sets (FSs) in 1965 [10], the notion of soft sets (SSs) was first introduced by Molodtsov in 1999 [11]. Subsequently, Maji et al. [12] first applied SSs for solving complex problems when they encountered the methodology in the year 2002. Since then, researchers have explored different properties and applications of SSs in various fields that focus on decision-making and information analysis. The power of SSs lies in their ability to handle uncertainty and errors, thus defining complex relationships among multiple attributes. Since the information considered is uncertain, imprecise, and fuzzy, both SSs and FSs have been found to be quintessential tools for handling such information. Researchers have also developed more sophisticated systems, including combinations like intuitionistic fuzzy soft sets (IFSSs), interval-valued FSSs, hesitant FSSs (HFSSs), and many other variations. In this context, Arora and Garg [13] presented weighted averages and geometric aggregation operations for intuitionistic fuzzy soft numbers (IFSNs). Considering the above discussions, Garg and Arora [14] initiated the weighted averaging and geometric aggregation approach to emphasize further on enhancing the decision-making capability of IFSS frameworks. Further, Agarwal et al. [15] developed some new applications of IFSS and Garg and Arora [16] enhanced the aggregation technique which is useful in the decision-making problems.

Most decisions, including human judgments, are bipolar in nature, containing both positive and negative aspects. In order to express such a duality, Zhang proposed the concept of bipolar FSs (BFSs) in 1998 [17], which is an extension of FSs by allowing membership values within the range [-1, 1] to denote different satisfaction levels under specific conditions [18]. Jana et al. [19, 20] furthered the research work based on BF-structures while dealing with decision-making. In addition, Pandey et al. [21] have proposed bipolar-valued hesitant fuzzy charts that have also been applied to practical decision-making problems. Several measures of uncertainty within fuzzy SSs have also been presented, such as the concept of hesitant FSs (HFSs), first introduced by Torra [22, 23] in 2010 and in 2014, Xia and Xu [24] discussed aggregation of hesitant fuzzy statistics. Later during the same year, Ren and Wei [25] presented the MCDM approach based on dual hesitant fuzzy decision information. Recently, Das et al. [26] developed the weighted hesitant bipolar-valued fuzzy soft set and solved related decision-making problems.

Later on, distance measures between any two elements in HFSSs have been investigated by Beg and Rashid [27], while K. and H. Rezaei [28] presented a new definition of similarity and distance measure for HFSSs. For reducing the uncertainty associated with the membership degree, different data types can be applied. Ullah et al. [29] proposed bipolar-valued hesitant FSs (BVHFSs) and applied aggregation techniques to tackle the decision-making by them. Mandal and Ranadive [30] developed the concepts of hesitant bipolar FSs (BVFSs) and BVHFSs. Wei [31] used aggregation operators for tackling the decision-making induced by bipolar-valued uncertain information. More research by Mahmood et al. on aggregation operators and de-Einstein operational principles related to hesitant fuzzy data with bipolar values also contributes to the difficulties regarding decision-making [32]. Lu and Wei [33] discussed dual hesitant bipolar FSs and decision-making issues associated with it.

Wang et al. [34] introduced the concept of hesitant bipolar FSSs (HBFSSs) and presented the decision-making method based on HBFSS. Akram et al. [35] presented an efficient MCDM approach based on bipolar FSs to tackle real-world problems in decision-making. In confused environments and in knowledge and social gaps, multi-granular hesitant fuzzy linguistic term sets are applied frequently in multi-attribute group decision-making (MAGDM) problems [36]. Some of the recent research was focused on diversified application fields like asymptotic analysis for one-stage stochastic linear complementarity problems [37], demand-side energy management by pricing fluctuation in residential heating and ventilation system [38], and adaptive agent decision models of deep reinforcement learning with autonomous learning [39]. Meng et al. have also proposed a new approach to residential distribution systems [40]. Readers interested in further insights into the results of MCDM across various application areas are advised to study works by Kanan et al. [41], Chakraborty et al. [42], Hussain et al. [43], and Had ikadunic et al. [44].

1.2. Motivation & objectives

This paper introduces the framework of BHFSS, accounting for pressing needs to accommodate complexities in the handling of uncertainty, hesitancy, and multiple objectives in the decision-making scenario. Traditional approaches to decision-making often fail to reflect human judgment correctly, especially in the evaluation of several aspects that are both positive and negative at the same time. BHFSS overcomes this weakness by bringing bipolarity that allows evaluations to reflect both the positive and negative valence while also representing the indecisiveness that brings uncertainty into the decision-making process. The duality then offers a much deeper understanding of mental processes, especially those that then appear in real-life situations characterized by either vagueness or lack of information, thus providing better skill in problem solving.

The subject of the paper is the specific aggregation operations in BHFSS. Although multi-fuzzy bipolar SSs are shown to be efficient in a wide range of decision-making settings, for example, in [45], by now, insufficient investigations have been carried out concerning aggregate methods within the context of FBSS and its extensions. Zeb [46] has clearly emphasized that there is a lot of work needed about aggregation operators for FBSS to fill this gap in the literature. Thus, this paper bridges this divide by presenting aggregation techniques of FBSNs and studying them in the BHFSS framework. In providing these aggregation operations, the paper strengthens the BHFSS theories while at the same time facilitating practical application in situations involving complex decision-making.

Bipolarity facilitates the decision-making process because through it, decision-makers can declare both positive and negative evaluations of alternatives. On the holistic nature of the framework, the complexity of thought in human beings is hence well captured, making it easier to navigate across competing criteria, while the hesitance component gives more richness to the BHFSS as an allowance for uncertainty exists in decision-making given that confidence levels among decision-makers may be varying, especially when there exist incomplete or ambiguous information.

Combining these, BHFSS is an applicable model, suitable for the real-world environment, in many fields such as sustainable energy, healthcare, and environmental management. This forms a robust framework required to face the complex challenges existing in modern decision environments, defined by the combination of bipolarity, hesitancy, and SSs. Thus, it ultimately yields to better quality decision-making processes and outcomes in numerous real-life scenarios, making BHFSS an added value to the existing knowledge. Therefore, practical examples will be presented on how the bipolar hesitant fuzzy soft information in actual situations can influence decision-making in everyday life. For instance, product quality and cost are the major aspects on which most decision-makers face uncertainty during the process of selecting suppliers. Experts may further say that they have different degrees of approvals for product quality, like 0.7, 0.75, and 0.8, with different degrees of rejections, such as 0.2 and 0.3. Similarly, for pricing assessment, acceptance ratings of 0.6, 0.65, and 0.7 would correspond with their corresponding grades in rejection, which would be 0.3 and 0.35; anyway, there will always be such variability in the perception.

In these scenarios, bipolar hesitant fuzzy elements can better represent both the good and bad assessments than is with conventional decision-making techniques. This approach helps the decision-makers solve conflicting criteria very effectively and hence, makes the approach particularly valuable in complex scenarios involving ambiguity and hesitancy, such as in decision-making in medical situations, environmental regulations, and commercial decisions. By including the bipolarity and hesitancy aspects in the framework, it not only enhances the rich complexity of the process of decision but also the precision and reliability of the outcome under a complex, real-world scenario.
The objectives of BHFSS are as follows:

(1) Provide decision-makers with adequate tools and mechanisms in order to make proper and correct decisions when realizing unclear and inconsistent data, in such a manner that complexity will not be a problem anymore.

(2) Develop algorithms capable of working with vague and indistinct data properly, such as those which require powerful algorithms toward data ambiguity and indefiniteness to mainly increase the efficiency of the overall decision-making process.

(3) the robustness of decision-making models can be upgraded with positive and negative feedback, so that assessments could be made holistic for the alternatives and more accurate and rounded decisions can be made in the course.

(4) In the bipolar hesitant fuzzy soft sets, novel aggregation operations are developed and analyzed

to simultaneously capture bipolarity, hesitation, and fuzziness for more robust and flexible decision-making processes. The suggested operations intend to handle complex scenarios of conflicting information under uncertainty and imprecision in decision-making by producing more accurate results and reliability in multi-criteria group decision-making contexts.

(5) To be able to apply bipolar hesitant fuzzy soft sets for energy management in smart grids by integrating sustainable sources of energy and by reducing the overall consumption of energy. This approach proposes efficient techniques for dealing with uncertainty levels while under informed strategic decisions that have been made. Therefore, the integration of renewable energy turns out to improve the overall efficiency and sustainability of smart grids.

In these outcomes, BHFSS will be in the position of offering an integrated framework that equips decision-makers with the ability to address uncertainty and complexity problems in a variety of applications.

1.3. Contributions

This research gives important contributions in the development of the BHFSS framework and applications there of to solve complicated decision-making scenarios. Among these important mathematical techniques applied within BHFSS is the usage of t-norms and t-conorms that play an important role when collecting uncertain data and fusing its sources. This allows representing a certain subtle interaction between variables; ambiguity and uncertainty are efficiently managed, so uncertainty and imprecision in one or several given data are preserved. BHFSS uses t-norm and t-conorm for combining heterogeneous data sources and ensures that decision-making consistency and coherence can prevail throughout the whole process. Then it leads to a strong adaptive framework to handle ambiguous data and arrive at reliable conclusions.

New operational rules of BHFSNs and algebraic norm-based weighted averaging and geometric aggregation operators are put forward. These aggregation operations are used within a new decision-making technique verified with a numerical example. BHFSS and combined compromise solution (CoCoSo) are compared well on the results obtained, their respective strengths, and weaknesses revealed. This in turn enables comparisons to be made so that the best course of action can be identified in relation to specific scenarios so that decisions may be more intelligently made. It is further authenticated through their use in solving sophisticated problems of decision-making. To demonstrate practical applicability of these operators in real-world scenarios, a case study on energy sustainability is made. The results from this paper tend to show that BHFS aggregation operators offer a massive improvement in decision-making related to energy sustainability projects by being able to resolve ambiguity while accommodating multiple criteria. This eventually leads to efficient evaluation as well as overall strategies; hence, improving resource efficiency and facilitating workable sustainable energy solutions.

These implications, therefore, lay on the contributions to academics, but with a good prospect to evolve as a tool supporting energy development through updated decision-making processes and even more accurate policy-making and scheduling within the sector. A list of symbols are given in Table 1 to help read the paper:

Table 1. Symbols list.

Sr.	Full Name	Symbol
1	Fixed Set	$\mathcal{U}$
2	Membership Degree (MD)	$\tilde{\mu}$
3	Positive MD	$\tilde{\mu}^{+}$
4	Negative MD	$\tilde{\mu}^{-}$
5	Hesitant Fuzzy Element	$\tilde{h}$
6	Scoring Function	$\delta(\tilde{h})$
7	Accuracy Function	$a(\tilde{h})$
8	Number of elements	ℓ
9	Weights	w
10	Weights of Experts	ψ

1.4. Structure of the paper

The arrangement of the paper is as follows: Section 2 provides the basic concept; Section 3 BHFSS and its operators; and Section 4 algebraic operations and its norms. Section 5 discusses the aggregation operators of BHFSS; Section 6 covers an approach utilizing suggested operators. Section 7 discusses a case study of incorporating sustainable energies for society. Section 8 discusses the CoCoSo method within BHFSS; and Section 9 offers a comparative analysis while Section 10 provides a brief discussion and Section 11 concludes the article.

2. Preliminaries

This section recapitulates some of the conceptual underpinnings of HFSs and BFS, placing emphasis on why a unification is desirable. This exposition is enlightening enough about unique advantages to be gained from the combination, serving as a more lavish framework in which to frame typical choices given uncertainty and ambiguity.

Definition 2.1. [10] Let U be a fixed set, then the FS A on U is defined as

$$A = \{(x, \tilde{\mu_A}(x))\forall x \in U\}, \tag{2.1}$$

in which membership $\tilde{\mu}_A$ belongs to $\tilde{\mu}_A : U \to [0, 1]$. This function represents the membership degree (MD) of each element x in the universal set U with respect to the fuzzy set A.

Definition 2.2. [17] Let U be a fixed set, then the BFS Q on U is is defined as

$$Q = \{(x, \tilde{\mu}^{+}_{Q}, \tilde{\mu}^{-}_{Q}), x \in U\}, \tag{2.2}$$

where $\tilde{\mu}^{+}_{Q} : U \to [0, 1]$ and $\tilde{\mu}^{-}_{Q} : U \to [-1, 0]$ are positive MD and negative MD, respectively. The pair $(\tilde{\mu}^{+}_{Q}, \tilde{\mu}^{-}_{Q})$ forms what is known as bipolar fuzzy numbers (BFNs). These BFNs are used to handle situations where both positive and negative degrees of membership are needed to fully represent the element s association with the FS.

Definition 2.3. Let $Q = (\tilde{\mu}^{+}_{Q}, \tilde{\mu}^{-}_{Q})$, $Q_{11} = (\tilde{\mu}^{+}_{Q_{11}}, \tilde{\mu}^{-}_{Q_{11}})$, $Q_{12} = (\tilde{\mu}^{+}_{Q_{12}}, \tilde{\mu}^{-}_{Q_{12}})$ be three BFN, then

(1) $Q_{11} \cup Q_{12} = (max(\tilde{\mu}^+_{Q_{11}}, \tilde{\mu}^+_{Q_{12}}), min(\tilde{\mu}^-_{Q_{11}}, \tilde{\mu}^-_{Q_{12}}))$.

(2) $Q_{11} \cap Q_{12} = (min(\tilde{\mu}^+_{Q_{11}}, \tilde{\mu}^+_{Q_{12}}), max(\tilde{\mu}^-_{Q_{11}}, \tilde{\mu}^-_{Q_{12}}))$.

(3) $Q^c = (1 - \tilde{\mu}^+_Q, -1 - \tilde{\mu}^-_Q)$.

Definition 2.4. [22] Let U be a fixed set, then the HFS Y on U is defined as

$$Y = \{(x, \tilde{h}_{Y_i}(x)), x \in U\}, \tag{2.3}$$

where $\tilde{h_{Y_i}}(x)$ has some valued in [0,1].

Definition 2.5. [22] With regard to three hesitant fuzzy sets (HFEs) $\tilde{h} = \bigcup_{\tilde{\mu}\in\tilde{h}}\{\tilde{\mu}\}, \tilde{h}_{11} = \bigcup_{\tilde{\mu}_{11}\in\tilde{h}_{11}}\{\tilde{\mu}_{11}\}, \tilde{h}_{12} = \bigcup_{\tilde{\mu}_{12}\in\tilde{h}_{12}}\{\tilde{\mu}_{12}\}$ and $\lambda > 0$ defined some operators.

(1) $\tilde{h}_{11} \cup \tilde{h}_{12} = \bigcup_{\tilde{\mu}_{11}\in\tilde{h}_{11}, \tilde{\mu}_{12}\in\tilde{h}_{12}} max\{\tilde{\mu}_{11}, \tilde{\mu}_{12}\}$.

(2) $\tilde{h}_{11} \cap \tilde{h}_{12} = \bigcup_{\tilde{\mu}_{11}\in\tilde{h}_{11}, \tilde{\mu}_{12}\in\tilde{h}_{12}} min\{\tilde{\mu}_{11}, \tilde{\mu}_{12}\}$.

(3) $\tilde{h}^c = \bigcup_{\tilde{\mu}\in\tilde{h}}\{1 - \tilde{\mu}\}$.

(4) $\tilde{h}^\lambda = \bigcup_{\tilde{\mu}\in h}\{1 - \tilde{\mu}\}$.

Definition 2.6. [31] Let U be the fixed set. A bipolar hesitant fuzzy set (BHFS) Z is defined as

$$Z = \left\{\left(x, \tilde{h}_Z(x) = \left(\tilde{h}^+_Z(x), \tilde{h}^-_Z(x)\right)\right), x \in U\right\}, \tag{2.4}$$

where $\tilde{h}^+_Z(x)$, also known as the hesitant fuzzy positive element, is a collection of values in the interval $[0, 1]$ that represents the element $x \in U$ fulfillment degree concerning the attribute that corresponds to the set A; the hesitant fuzzy negative element (HFNE), also known as $\tilde{h}^-_Z(x)$, is an accumulation of values in [-1,0] that illustrate the element $x \in U$ fulfillment degree of satisfaction with the implicit opposite feature of the set Z; besides, $\tilde{h}_Z(x) = (\tilde{h}^+_Z(x), \tilde{h}^-_Z(x))$ is a set of values in $[0, 1] \times [-1, 0]$ that are a part of the set Z. It is also known as the bipolar hesitant fuzzy element (BHFE). To keep things simple, we'll refer to the BHFE as $\tilde{h} = (\tilde{h}^+, \tilde{h}^-)$. The formula is $\tilde{h}_Z(x) = (\tilde{h}^+_Z(x), \tilde{h}^-_Z(x))$.

We can yield certain outcomes in situations established on our previous knowledge of BFSS and HFSS. If there is just one value for $\tilde{h}^+_Z(x)$ and $\tilde{h}^-_Z(x)$, then the BHFS is reduced to BVFS. When both $\tilde{h}^-_Z = \phi$ and $\tilde{h}^+_Z = \phi$ are equal, the BHFS becomes an HFS.

Definition 2.7. [31] Let $\tilde{h} = (\tilde{h}^+, \tilde{h}^-)$, $\tilde{h}_{11} = (\tilde{h}^+_{11}, \tilde{h}^-_{11})$, and $\tilde{h}_{12} = (\tilde{h}^+_{12}, \tilde{h}^-_{12})$ be three BHFEs, then

(1) $\tilde{h}_{11} \cup \tilde{h}_{12} = (\bigcup_{\tilde{\mu}_{11}\in\tilde{h}^+_{11}, \tilde{\mu}_{12}\in\tilde{h}^+_{12}} max\{\tilde{\mu}^+_{11}, \tilde{\mu}^+_{12}\}, \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^-_{11}, \tilde{\mu}_{12}\in\tilde{h}^-_{12}} min\{\tilde{\mu}^-_{11}, \tilde{\mu}^-_{12}\})$.

(2) $\tilde{h}_{11} \cap \tilde{h}_{12} = (\bigcup_{\tilde{\mu}_{11}\in\tilde{h}^+_{11}, \tilde{\mu}_{12}\in\tilde{h}^+_{12}} min\{\tilde{\mu}^+_{11}, \tilde{\mu}^+_{12}\}, \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^-_{11}, \tilde{\mu}_{12}\in\tilde{h}^-_{12}} max\{\tilde{\mu}^-_{11}, \tilde{\mu}^-_{12}\})$.

(3) $\tilde{h}^c = (\bigcup_{\tilde{\mu}^+\in\tilde{h}}\{1 - \tilde{\mu}^+\}, \bigcup_{\tilde{\mu}^-\in\tilde{h}}\{-1 - \tilde{\mu}^-\})$.

3. Bipolar hesitant fuzzy soft set and its operations

This section adopts the approach toward providing a novel structure dealing with complexity and uncertainty in the problem domain. Advanced mathematical frameworks and decision techniques of making become integrated in the novel structure to provide more wholesome and flexible solutions. It is designed to enhance the handling of imprecise, conflicting, and incomplete information, and, hence, it is generally associated with settings in which uncertainty arises. This structure also further increases the precision of the decision-making, meaning that it's more efficient, hence, even more apt for solving real-world problems in this area.

Definition 3.1. Let U be fixed set, $\tilde{E}$ is the set of attributes $e \subset \tilde{E}$, then the pair (A, e) called BHFSS over U where

$$A : e \to BHFSS(U). \tag{3.1}$$

⇉ The collection of every BHFSS across U is called BHFSS(U).
⇉ Shown to be $(A, e) = A_e = \{(x, \tilde{h}_e^+(x), \tilde{h}_e^-(x)), x \in U\}$.

To rate practical applications through rankings, we develop the score and accuracy functions.

Definition 3.2. Let $\tilde{h}_e = (\tilde{h}_e^+, \tilde{h}_e^-)$ be a BHFSE, and the score function $S(\tilde{h}_e)$ and the accuracy function $a(\tilde{h}_e)$ is defined as

$$\delta(\tilde{h}_e) = \frac{1}{2}(\frac{1}{\ell_{\tilde{h}_e^+}} \sum_{\tilde{\mu}_1^+ \in \tilde{h}_e^+} \tilde{\mu}^+ - \frac{1}{\ell_{\tilde{h}_e^-}} \sum_{\tilde{\mu}_1^- \in \tilde{h}_e^-} \tilde{\mu}^-), \tag{3.2}$$

$$a(\tilde{h}_e) = \frac{1}{2}(\frac{1}{\ell_{\tilde{h}_e^+}} \sum_{\tilde{\mu}_1^+ \in \tilde{h}_e^+} \tilde{\mu}^+ + \frac{1}{\ell_{\tilde{h}_e^-}} \sum_{\tilde{\mu}_1^- \in \tilde{h}^-} \tilde{\mu}^-), \tag{3.3}$$

where $\ell_{\tilde{h}_e^+}$ and $\ell_{\tilde{h}_e^-}$ are the numbers of the elements in $\tilde{h}_e^+$ and $\tilde{h}_e^-$ correspondingly.

The accuracy function calculates the degree of accuracy of every component in $\tilde{h}_e$. The score function considers the average score for every element in $\tilde{h}_e$. A BHFS, for example, is built from an array of judgments of an option's positive and negative aspects within certain criteria obtained from the opinions of a pair of experts. A typical expert judgment of the alternative's advantages and drawbacks is evaluated using an accurate function, and a score function selects which is preferable.

Definition 3.3. Let $\tilde{h}_{e11} = (\tilde{h}_{e11}^+, \tilde{h}_{e11}^-)$ and $\tilde{h}_{e12} = (\tilde{h}_{e12}^+, \tilde{h}_{e12}^-)$ be two BHFSEs. The following definition applies to the comparison method:

(1) If $\delta(\tilde{h}_{e11}) < \delta(\tilde{h}_{e12})$ then, as $\tilde{h}_{e11}$ is less than $\tilde{h}_{e12}$, it can be demonstrated that $\tilde{h}_{e11} < \tilde{h}_{e12}$.

(2) If $a(\tilde{h}_{e11}) = a(\tilde{h}_{e12})$,then $\tilde{h}_{e11}$ and $\tilde{h}_{e12}$ represent the same information, denoted by $\tilde{h}_{e11}$=$\tilde{h}_{e12}$.

(3) If $a(\tilde{h}_{e11}) < a(\tilde{h}_{e12})$, then $\tilde{h}_{e11}$ is less then $\tilde{h}_{e12}$, demonstrated that $\tilde{h}_{e11} < \tilde{h}_{e12}$.

Theorem 3.1. Let $\tilde{h}_e = (\tilde{h}_e^+, \tilde{h}_e^-)$, $\tilde{h}_{e11} = (\tilde{h}_{e11}^+, \tilde{h}_{e11}^-)$, and $\tilde{h}_{e12} = (\tilde{h}_{e12}^+, \tilde{h}_{e12}^-)$ be three bipolar hesitant fuzzy soft numbers (BHFSNs) and $\lambda, \lambda_1, \lambda_2 > 0$ are real numbers. Then, the data that follows is accurate.

(1) $\tilde{h}_{e11} \bigoplus \tilde{h}_{e12} = \tilde{h}_{e12} \bigoplus \tilde{h}_{e11}$.

(2) $\tilde{h}_{e11} \bigotimes \tilde{h}_{e12} = \tilde{h}_{e12} \bigotimes \tilde{h}_{e11}$.

(3) $\lambda(\tilde{h}_{e11} \bigoplus \tilde{h}_{e12}) = \lambda\tilde{h}_{e12} \bigoplus \lambda\tilde{h}_{e11}$.

(4) $(\tilde{h}_{e11} \bigotimes \tilde{h}_{e12})^{\lambda} = (\tilde{h}_{e12})^{\lambda} \bigotimes \tilde{(}h_{e11})^{\lambda}$.

(5) $\lambda_1\tilde{h}_e \bigoplus \lambda_2\tilde{h}_e = (\lambda_1 + \lambda_2)\tilde{h}_e$.

(6) $(\tilde{h}_{e_{12}})^{\lambda_1} \bigotimes (\tilde{h}_{e_{11}})^{\lambda_2} = (\tilde{h}_{e_{11}})^{\lambda_1+\lambda_2}$.

(7) $(\tilde{h}_e^{\lambda_1})^{\lambda_2} = (\tilde{h}_e^{\lambda_1\lambda_2})$.

Proof. Trivial.

4. Algebraic operations and its norms

This section emphasizes the algebraic norms and operations of BHFSS, which are vital to handling bipolar hesitant fuzzy information.

Definition 4.1. Let b_{11} and b_{12} are two real numbers; formally, the algebric T-norms and S-conorms are described as follows:

$$S_A(b_{11}, b_{12}) = \{b_{11} + b_{12} - b_{11}b_{12}\}.$$

$$T_A(b_{11}, b_{12}) = \{b_{11}b_{12}\}.$$

Definition 4.2. Let $\tilde{h}_e = (\tilde{h}_e^{\ +}, \tilde{h}_e^{\ -})$, $\tilde{h}_{e11} = (\tilde{h}_{e11}^{+}, \tilde{h}_{e11}^{-})$, and $\tilde{h}_{e12} = (\tilde{h}_{e12}^{+}, \tilde{h}_{e12}^{-})$ be three bipolar hesitant fuzzy soft elements (BHFSEs), then

(1) $\tilde{h}_{e11} \cup \tilde{h}_{e12} = (\bigcup_{\tilde{\mu}_{11}\in\tilde{h}_{e11}^{+}, \tilde{\mu}_{12}\in\tilde{h}_{e12}^{+}} max\{\tilde{\mu}_{11}^{+}, \tilde{\mu}_{12}^{+}\}, \bigcup_{\tilde{\mu}_{11}\in\tilde{h}_{e11}^{-}, \tilde{\mu}_{12}\in\tilde{h}_{e12}^{-}} min\{\tilde{\mu}_{11}^{-}, \tilde{\mu}_{12}^{-}\})$.

(2) $\tilde{h}_{e11} \cap \tilde{h}_{e12} = (\bigcup_{\tilde{\mu}_{11}\in\tilde{h}_{e11}^{+}, \tilde{\mu}_{12}\in\tilde{h}_{e12}^{+}} min\{\tilde{\mu}_{11}^{+}, \tilde{\mu}_{12}^{+}\}, \bigcup_{\tilde{\mu}_{11}\in\tilde{h}_{e11}^{-}, \tilde{\mu}_{12}\in\tilde{h}_{e12}^{-}} max\{\tilde{\mu}_{11}^{-}, \tilde{\mu}_{12}^{-}\})$.

(3) $\tilde{h}_e^{\ c} = (\bigcup_{\tilde{\mu}^{+}\in\tilde{h}_e}\{1 - \tilde{\mu}^{+}\}, \bigcup_{\tilde{\mu}^{-}\in\tilde{h}_e}\{-1 - \tilde{\mu}^{-}\})$.

Definition 4.3. Let $\tilde{h}_e = (\tilde{h}_e^{\ +}, \tilde{h}_e^{\ -})$, $\tilde{h}_{e11} = (\tilde{h}_{e11}^{+}, \tilde{h}_{e11}^{-})$, and $\tilde{h}_{e12} = (\tilde{h}_{e12}^{+}, \tilde{h}_{e12}^{-})$ be three BHFSEs, then

(1) $\tilde{h}_{e11} \bigoplus \tilde{h}_{e12} = (\bigcup_{\tilde{\mu}_{11}\in\tilde{h}_{e11}^{+}, \tilde{\mu}_{12}\in\tilde{h}_{e12}^{+}}\{\tilde{\mu}_{11}^{+} + \tilde{\mu}_{12}^{+} - \tilde{\mu}_{11}^{+}\tilde{\mu}_{12}^{+}\}, \bigcup_{\tilde{\mu}_{11}\in\tilde{h}_{e11}^{-}, \tilde{\mu}_{12}\in\tilde{h}_{e12}^{-}}\{-\tilde{\mu}_{11}^{-}\tilde{\mu}_{12}^{-}\})$.

(2) $\tilde{h}_{e11} \bigotimes \tilde{h}_{e12} = (\bigcup_{\tilde{\mu}_{11}\in\tilde{h}_{e11}^{+}, \tilde{\mu}_{12}\in\tilde{h}_{e12}^{+}}\{\tilde{\mu}_{11}^{+}\tilde{\mu}_{12}^{+}\}, \bigcup_{\tilde{\mu}_{11}\in\tilde{h}_{e11}^{-}, \tilde{\mu}_{12}\in\tilde{h}_{e12}^{-}}\{\tilde{\mu}_{11}^{-} + \tilde{\mu}_{12}^{-} - \tilde{\mu}_{11}^{-}\tilde{\mu}_{12}^{-}\})$.

(3) $\lambda\tilde{h}_e = (\bigcup_{\tilde{\mu}^{+}\in\tilde{h}_e^{+}}\{(1 - (1 - \tilde{\mu}^{+})^{\lambda}\}, \bigcup_{\tilde{\mu}^{-}\in\tilde{h}_e^{-}}\{-|\tilde{\mu}^{-}|^{\lambda}\})$.

(4) $\tilde{h}_e^{\ \lambda} = (\bigcup_{\tilde{\mu}^{+}\in\tilde{h}_e^{+}}\{(\tilde{\mu}^{+})^{\lambda}\}, \bigcup_{\tilde{\mu}^{-}\in\tilde{h}_e^{-}}\{-1 + (1 + \tilde{\mu}^{-})^{\lambda}\})$.

5. Bipolar hesistant fuzzy soft aggregation operators

This section introduces and examines the aggregation operations for BHFSS, having emphasis on both averaging and geometric aggregation techniques. These operators are essential for transforming a large number of hesitant fuzzy soft values into one accurate value required for the decision-making process. To build a solid conceptual foundation for these operators, we shall describe their characteristics and present instructive examples and proofs.

Definition 5.1. Suppose $\tilde{h}_{e_{\beth\shortparallel}} = \left(\tilde{h}^{+}_{e_{\beth\shortparallel}}, \tilde{h}^{-}_{e_{\beth\shortparallel}}\right)$ $\beth$={1,2,.....,n} $\shortparallel$={1,2,...,m} is BHFSNs, then a function is represented by the bipolar hesitant fuzzy soft weighted average (BHFSWA) operator $\tilde{h}_e^{\,n} \to \tilde{h}_e$ such that

$$BHFSWA(\tilde{h_{e_{11}}}, \tilde{h_{e_{12}}}, ..., \tilde{h_{e_{nm}}}) = \bigoplus_{\shortparallel=1}^{m} \psi_{\shortparallel} \left(\bigoplus_{\beth=1}^{n} w_{\beth} \tilde{h}_{e_{\beth\shortparallel}} \right), \tag{5.1}$$

where $\tilde{h}_e$ represents the collection of BHFSNs, the weight vector $w_{\beth}(\beth = 1, 2, ..., n)$ and $\psi_{\shortparallel}(\shortparallel = 1, 2, ..., m)$ describe the experts and parameters, respectively, and $\beth, \shortparallel > 0$ so that $\sum_{\shortparallel=1}^{m} \psi_{\shortparallel} = 1$, $\sum_{\beth=1}^{n} w_{\beth} = 1$.

Theorem 5.1. Suppose $\tilde{h}_{e_{\beth\shortparallel}} = (\tilde{h}^{+}_{e_{\beth\shortparallel}}, \tilde{h}^{-}_{e_{\beth\shortparallel}})$ $\beth$={1,2,...,n} $\shortparallel$={1,2,...,m} is BHFSNs, then a function is represented by the BHFSWA operator $\tilde{h}_e^{\,n} \to \tilde{h}_e$ such that

$$BHFSWA(\tilde{h_{e_{11}}}, \tilde{h_{e_{12}}}, ..., \tilde{h_{e_{nm}}}) = \bigoplus_{\shortparallel=1}^{m} \psi_{\shortparallel} \left(\bigoplus_{\beth=1}^{n} w_{\beth} \tilde{h}_{e_{\beth\shortparallel}} \right) \tag{5.2}$$

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{+}_{e_{11}}, ..., \tilde{\mu}_{nm} \in \tilde{h}^{+}_{e_{nm}}} 1 - \prod_{\shortparallel=1}^{m} (\prod_{\beth=1}^{n} (1 - \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}})^{w_{\beth}})^{\psi_{\shortparallel}}, \\ \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{-}_{e_{11}}, ..., \tilde{\mu}_{nm} \in \tilde{h}^{-}_{e_{nm}}} -(\prod_{\shortparallel=1}^{m} (\prod_{\beth=1}^{n} (|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|)^{w_{\beth}})^{\psi_{\shortparallel}}) \end{array} \right\},$$

where $\tilde{h}_e$ represents the collection of BHFSNs, the weight vector $w_{\beth}(\beth = 1, 2, ..., n)$ and $\psi_{\shortparallel}(\shortparallel = 1, 2, ..., m)$ describe the experts and parameters, respectively, and $\beth, \shortparallel > 0$ so that $\sum_{\shortparallel=1}^{m} \psi_{\shortparallel} = 1$, $\sum_{\beth=1}^{n} w_{\beth} = 1$.

Proof. See Appendix A. □

Example 1. The BHFS information is provided in Table 2. Using this information find the value of the operator BHFSWA by the weighted vectors below.

Table 2. BHFSWA.

A1	e_1	e_2	e_3	e_4
x_1	$\{0.4, 0.2\}, \{-0.7\}$	$\{0.2\}, \{-0.1\}$	$\{0.2\}, \{-0.1, -0.3\}$	$\{0.1\}, \{-0.2\}$
x_2	$\{0.3\}, \{-0.4\}$	$\{0.5\}, \{-0.3\}$	$\{0.5\}, \{-0.7\}$	$\{0.2, 0.6\}, \{-0.5, -0.3\}$
x_3	$\{0.5\}, \{-0.5, -0.3\}$	$\{0.7\}, \{-0.6\}$	$\{0.3, 0.1\}, \{-0.8\}$	$\{0.4\}, \{-0.8\}$

Suppose $\psi = (0.1, 0.4, 0.2, 0.3)$ and $w = (0.3, 0.5, 0.2)$ are weighted vectors for the parameters and the experts, correspondingly,

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^{+}_{e11},...,\tilde{\mu}_{34}\in\tilde{h}^{+}_{e34}} 1 - \prod_{\shortparallel=1}^{4}(\prod_{\beth=1}^{3}(1-\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}})^{w_{\beth}})^{\psi_{\shortparallel}}, \\ \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^{-}_{e11},...,\tilde{\mu}_{34}\in\tilde{h}^{-}_{e34}} -(\prod_{\shortparallel=1}^{4}(\prod_{\beth=1}^{3}(|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|)^{w_i})^{\psi_{\shortparallel}}) \end{array} \right\}.$$

$$= \left\{ \begin{array}{c} (1-\{(1-0.4)^{0.3}(1-0.3)^{0.5}(1-0.5)^{0.2}\}^{0.1} \\ \{(1-0.2)^{0.3}(1-0.5)^{0.5}(1-0.7)^{0.2}\}^{0.4} \\ \{(1-0.2)^{0.3}(1-0.5)^{0.5}(1-0.3)^{0.2}\}^{0.2} \\ \{(1-0.1)^{0.3}(1-0.2)^{0.5}(1-0.4)^{0.2}\}^{0.3}), \\ -(\{|-0.7|^{0.3}|-0.4|^{0.5}|-0.5|^{0.2}\}^{0.1} \\ \{|-0.1|^{0.3}|-0.3|^{0.5}|-0.6|^{0.2}\}^{0.4} \\ \{|-0.1|^{0.3}|-0.7|^{0.5}|-0.8|^{0.2}\}^{0.2} \\ \{|-0.2|^{0.3}|-0.5|^{0.5}|-0.8|^{0.2}\}^{0.3}) \end{array} \right\}.$$

$$= \left\{ \begin{array}{c} 0.3808, 0.3754, 0.3745, 0.3691, 0.4419, 0.4371, 0.4363, 0.4314, \\ -0.3419, -0.3384, -0.3652, -0.3615, -0.3167, -0.3134, -0.3382, -0.3348 \end{array} \right\}.$$

Properties of (BHFSWA) operator

Idempotency:

If $\tilde{h}_{e\beth\shortparallel} = \tilde{h}_e = (\tilde{h}^{+}_{e\beth\shortparallel}, \tilde{h}^{-}_{e\beth\shortparallel})\ \forall\ \beth, \shortparallel$ then $BHFSWA(\tilde{h}_{e_{11}}, \tilde{h}_{e_{12}}, ..., \tilde{h}_{e_{nm}}) = \tilde{h}_e$.

Proof. See Appendix B. □

Homogeneity:

We derive for each real number $\lambda > 0$ then

$$BHFSWA(\lambda\tilde{h}_{e_{11}}, \lambda\tilde{h}_{e_{12}}, ..., \lambda\tilde{h}_{e_{nm}}) = \lambda\{BHFSWA(\tilde{h}_{e_{11}}, \tilde{h}_{e_{12}}, ..., \tilde{h}_{e_{nm}})\}.$$

Proof. See Appendix C. □

Boundedness:

Suppose $\tilde{h}_{e\beth\shortparallel} = (\tilde{h}^{+}_{e\beth\shortparallel}, \tilde{h}^{-}_{e\beth\shortparallel})$, then $\tilde{h}^{-}_{e\beth\shortparallel} = (\min_{\shortparallel}\min_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\}, \max_{\shortparallel}\max_{\beth}\{\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}\})$ and $\tilde{h}^{+}_{e\beth\shortparallel} = (\max_{\shortparallel}\max_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\}, \min_{\shortparallel}\min_{\beth}\{\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}\})$.

Let, $\gamma_{min} = (\min_{\shortparallel}\min_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\}|\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}} \epsilon \tilde{h}^{-}_{e\beth\shortparallel}), \zeta_{max} = (\max_{\shortparallel}\max_{\beth}\{\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}\}|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}} \epsilon \tilde{h}^{-}_{e\beth\shortparallel})$.

$\zeta_{min} = (\min_{\shortparallel}\min_{\beth}\{\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}\}|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}} \epsilon \tilde{h}^{+}_{e\beth\shortparallel}), \gamma_{max} = (\max_{\shortparallel}\max_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\}|\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}} \epsilon \tilde{h}^{+}_{e\beth\shortparallel})$

$$\tilde{h}^{-}_{e\beth\shortparallel} \leq BHFSWA(\tilde{h}_{e_{11}}, \tilde{h}_{e_{12}}, ..., \tilde{h}_{e_{nm}}) \leq \tilde{h}^{+}_{e\beth\shortparallel}, \tag{5.3}$$

where

$$\tilde{h}^{-}_{e\beth\shortparallel} = (\{\gamma_{min}\}, \{\zeta_{max}\}) = \bigcup_{\substack{\gamma^{-}\epsilon\gamma_{min} \\ \zeta^{+}\epsilon\zeta_{max}}} (\{\gamma^{-}\}, \{\zeta^{+}\}),$$

$$\tilde{h}^{+}_{e\beth\shortparallel} = (\{\gamma_{max}\}, \{\zeta_{min}\}) = \bigcup_{\substack{\gamma^{+}\epsilon\gamma_{max} \\ \zeta^{-}\epsilon\zeta_{min}}} (\{\gamma^{+}\}, \{\zeta^{-}\}).$$

Proof. See Appendix D. □

Definition 5.2. Suppose $\tilde{h}_{e\sigma(\beth\shortparallel)} = (\tilde{h}^{+}_{e_{\sigma(\beth\shortparallel)}}, \tilde{h}^{-}_{e_{\sigma(\beth\shortparallel)}})$ $\beth$={1,2,...,n} $\shortparallel$={1,2,...,m} is a bipolar hesitant fuzzy soft ordered numbers (BHFSONs), then a function is represented by the bipolar hesitant fuzzy soft ordered weighted average (BHFSOWA) operator $\tilde{h_e}^n \rightarrow \tilde{h}_e$ such that

$$BHFSOWA(\tilde{h}_{e\sigma_{(11)}}, \tilde{h}_{e\sigma_{(12)}}, ..., \tilde{h}_{e\sigma_{(nm)}}) = \bigoplus_{\shortparallel=1}^{m} \psi_{\shortparallel} \left(\bigoplus_{\beth=1}^{n} w_{\beth} \tilde{h}_{e\sigma_{(\beth\shortparallel)}} \right), \tag{5.4}$$

where $\tilde{h}_e$ represents the collection of BHFSNs, the weight vector $w_{\beth}(\beth = 1, 2, ..., n)$ and $\psi_{\shortparallel}(\shortparallel = 1, 2, ..., m)$ describe the experts and parameters, respectively, and $\beth, \shortparallel > 0$ so that $\sum_{\shortparallel=1}^{m} \psi_{\shortparallel} = 1, \sum_{\beth=1}^{n} w_{\beth} = 1.$

Theorem 5.2. Suppose $\tilde{h}_{e\sigma(\beth\shortparallel)} = (\tilde{h}^{+}_{e_{\sigma(\beth\shortparallel)}}, \tilde{h}^{-}_{e_{\sigma(\beth\shortparallel)}})$ $\beth$={1,2,...,n} $\shortparallel$={1,2,...,m} is BHFSONs, then a function is represented by the BHFSOWA operator $\tilde{h_e}^n \rightarrow \tilde{h}_e$ such that

$$BHFSOWA(\tilde{h}_{e\sigma_{(11)}}, \tilde{h}_{e\sigma_{(12)}}, ..., \tilde{h}_{e\sigma_{(nm)}}) = \bigoplus_{\shortparallel=1}^{m} \psi_{\shortparallel} \left(\bigoplus_{\beth=1}^{n} w_{\beth} \tilde{h}_{e\sigma(\beth\shortparallel)} \right)$$

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{\sigma(11)} \in \tilde{h}^{+}_{e_{\sigma(11)}}, ..., \tilde{\mu}_{\sigma(nm)} \in \tilde{h}^{+}_{e_{\sigma(nm)}}} 1 - \prod_{\shortparallel=1}^{m} (\prod_{\beth=1}^{n} (1 - \tilde{\mu}^{+}_{\tilde{h}_{e\sigma(\beth\shortparallel)}})^{w_{\beth}})^{\psi_{\shortparallel}}, \\ \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{-}_{e_{\sigma(11)}}, ..., \tilde{\mu}_{\sigma(nm)} \in \tilde{h}^{-}_{e_{\sigma(nm)}}} -(\prod_{\shortparallel=1}^{m} (\prod_{\beth=1}^{n} (|\tilde{\mu}^{-}_{\tilde{h}_{e\sigma(\beth\shortparallel)}}|)^{w_{\beth}})^{\psi_{\shortparallel}}) \end{array} \right\},$$

where $\tilde{h}_e$ represents the collection of BHFSNs, the weight vector $w_{\beth}(\beth = 1, 2, ..., n)$ and $\psi_{\shortparallel}(\shortparallel = 1, 2, ..., m)$ describe the experts and parameters, respectively, and $\beth, \shortparallel > 0$ so that $\sum_{\shortparallel=1}^{m} \psi_{\shortparallel} = 1, \sum_{\beth=1}^{n} w_{\beth} = 1.$

Definition 5.3. Suppose $\tilde{h}'_{e_{\sigma_{(\beth\shortparallel)}}} = (\tilde{h}'^{+}_{e_{\sigma_{(\beth\shortparallel)}}}, \tilde{h}'^{-}_{e_{\sigma_{(\beth\shortparallel)}}})$ $\beth$={1,2,...,n} $\shortparallel$={1,2,...,m} is a bipolar hesitant fuzzy soft hybrid numbers (BHFSHNs), then a function is represented by the bipolar hesitant fuzzy soft hybrid weighted average (BHFSHWA) operator $\tilde{h_e}^n \rightarrow \tilde{h}_e$ such that

$$BHFSHWA(\tilde{h'}_{e\sigma_{(11)}}, \tilde{h'}_{e\sigma_{(12)}}, ..., \tilde{h'}_{e\sigma_{(nm)}}) = \bigoplus_{\shortparallel=1}^{m} \psi_{\shortparallel} \left(\bigoplus_{\beth=1}^{n} w_{\beth} \tilde{h}'_{e\sigma_{(\beth\shortparallel)}} \right), \tag{5.5}$$

where $\tilde{h}_e$ represents the collection of BHFSNs, the weight vector $w_{\beth}(\beth = 1, 2, ..., n)$ and $\psi_{\shortparallel}(\shortparallel = 1, 2, ..., m)$ describe the experts and parameters, respectively, and $\beth, \shortparallel > 0$ so that $\sum_{\shortparallel=1}^{m} \psi_{\shortparallel} = 1, \sum_{\beth=1}^{n} w_{\beth} = 1.$

Theorem 5.3. Suppose $\tilde{h}'_{e_{\sigma_{(\beth\shortparallel)}}} = (\tilde{h}'^{+}_{e_{\sigma(\beth\shortparallel)}}, \tilde{h}'^{-}_{e_{\sigma_{(\beth\shortparallel)}}})$ $\beth$={1,2,...,n} $\shortparallel$={1,2,...,m} is BHFSHNs, then a function is represented by the BHFSHWA operator $\tilde{h_e}^n \rightarrow \tilde{h}_e$ such that

$$BHFSHWA(\tilde{h'}_{e\sigma_{(11)}}, \tilde{h'}_{e\sigma_{(12)}}, ..., \tilde{h'}_{e\sigma_{(nm)}}) = \bigoplus_{\shortparallel=1}^{m} \psi_{\shortparallel} \left(\bigoplus_{\beth=1}^{n} w_{\beth} \tilde{h}'_{e\sigma_{(\beth\shortparallel)}} \right)$$

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{\sigma(11)} \in \tilde{h}'^{+}_{e_{\sigma(11)}}, ..., \tilde{\mu}_{\sigma(nm)} \in \tilde{h}'^{+}_{e_{\sigma(nm)}}} 1 - \prod_{\shortparallel=1}^{m} (\prod_{\beth=1}^{n} (1 - \tilde{\mu}^{+}_{\tilde{h}'_{e\sigma(\beth\shortparallel)}})^{w_{\beth}})^{\psi_{\shortparallel}}, \\ \bigcup_{\tilde{\mu}_{11} \in \tilde{h}'^{-}_{e_{\sigma(11)}}, ..., \tilde{\mu}_{\sigma(nm)} \in \tilde{h}'^{-}_{e_{\sigma(nm)}}} -(\prod_{\shortparallel=1}^{m} (\prod_{\beth=1}^{n} (|\tilde{\mu}^{-}_{\tilde{h}'_{e\sigma(\beth\shortparallel)}}|)^{w_{\beth}})^{\psi_{\shortparallel}}) \end{array} \right\},$$

where $\tilde{h}_e$ represents the collection of BHFSNs, the weight vector $w_\beth(\beth = 1, 2, ..., n)$ and $\psi_\parallel(\parallel = 1, 2, ..., m)$ describe the experts and parameters, respectively, and $\beth, \parallel > 0$ so that $\sum_{\parallel=1}^{m} \psi_\parallel = 1$, $\sum_{\beth=1}^{n} w_\beth = 1$.

Definition 5.4. Suppose $\tilde{h}_{e_{\beth\parallel}} = (\tilde{h}^+_{e_{\beth\parallel}}, \tilde{h}^-_{e_{\beth\parallel}})$ $\beth$={1,2,...,n} $\parallel$={1,2,...,m} is BHFSNs, then a function is represented by the bipolar hesitant fuzzy soft geometric (BHFSWG) operator $\tilde{h}_e^{\,n} \to \tilde{h}_e$ such that

$$BHFSWG(\tilde{h_{e_{11}}}, \tilde{h_{e_{12}}}, ..., \tilde{h_{e_{nm}}}) = \bigotimes_{\parallel=1}^{m} \left(\bigotimes_{\beth=1}^{n} (\tilde{h}_{e_{\beth\parallel}})^{w_\beth} \right)^{\psi_\parallel}, \tag{5.6}$$

where $\tilde{h}_e$ represents the collection of BHFSNs, the weight vector $w_\beth(\beth = 1, 2, ..., n)$ and $\psi_\parallel(\parallel = 1, 2, ..., m)$ describe the experts and parameters, respectively, and $\beth, \parallel > 0$ so that $\sum_{\parallel=1}^{m} \psi_\parallel = 1$, $\sum_{\beth=1}^{n} w_\beth = 1$.

Theorem 5.4. Suppose $\tilde{h}_{e_{\beth\parallel}} = (\tilde{h}^+_{e_{\beth\parallel}}, \tilde{h}^-_{e_{\beth\parallel}})$ $\beth$={1,2,...,n} $\parallel$={1,2,...,m} is BHFSNs, then a function is represented by the BHFSWG operator $\tilde{h}_e^{\,n} \to \tilde{h}_e$ such that

$$BHFSWG(\tilde{h_{e_{11}}}, \tilde{h_{e_{12}}}, ..., \tilde{h_{e_{nm}}}) = \bigotimes_{\parallel=1}^{m} \left(\bigotimes_{\beth=1}^{n} (\tilde{h}_{e_{\beth\parallel}})^{w_\beth} \right)^{\psi_\parallel}$$

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^+_{e_{11}}, ..., \tilde{\mu}_{nm} \in \tilde{h}^+_{e_{nm}}} \prod_{\parallel=1}^{m} (\prod_{\beth=1}^{n} (\tilde{\mu}^+_{\tilde{h}_{e(\beth\parallel)}})^{w_\beth})^{\psi_\parallel}, \\ \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^-_{e_{11}}, ..., \tilde{\mu}_{nm} \in \tilde{h}^-_{e_{nm}}} -1 + \prod_{\parallel=1}^{m} (\prod_{\beth=1}^{n} (1 + \tilde{\mu}^-_{\tilde{h}_{e(\beth\parallel)}})^{w_\beth})^{\psi_\parallel} \rangle \end{array} \right\},$$

where $\tilde{h}_e$ represents the collection of BHFSNs, the weight vector $w_\beth(\beth = 1, 2, ..., n)$ and $\psi_\parallel(\parallel = 1, 2, ..., m)$ describe the experts and parameters, respectively, and $\beth, \parallel > 0$ so that $\sum_{\parallel=1}^{m} \psi_\parallel = 1$, $\sum_{\beth=1}^{n} w_\beth = 1$.

Proof. Similarly, as this study follows directly from Theorem 3.2, we will not restate it here. □

Example 2. The BHFS information is provided in Table 3. Using this information find the value of the operator BHFSWG by the weighted vectors below.

Table 3. BHFSWG.

A1	e_1	e_2	e_3	e_4
x_1	$\{0.1, 0.5\}, \{-0.3\}$	$\{0.1\}, \{-0.1\}$	$\{0.2\}, \{-0.1, -0.2\}$	$\{0.1\}, \{-0.2\}$
x_2	$\{0.2\}, \{-0.4\}$	$\{0.6\}, \{-0.4\}$	$\{0.5\}, \{-0.7\}$	$\{0.2, 0.5\}, \{-0.6, -0.4\}$
x_3	$\{0.6\}, \{-0.8, -0.2\}$	$\{0.7\}, \{-0.5\}$	$\{0.4, 0.1\}, \{-0.4\}$	$\{0.3\}, \{-0.8\}$

Suppose $\psi = (0.1, 0.4, 0.2, 0.3)$ and $w = (0.3, 0.5, 0.2)$ are weighted vectors for the parameters and the experts, correspondingly,

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^+_{e_{11}}, ..., \tilde{\mu}_{nm} \in \tilde{h}^+_{e_{nm}}} \prod_{\parallel=1}^{m} (\prod_{\beth=1}^{n} (\tilde{\mu}^+_{\tilde{h}_{e(\beth\parallel)}})^{w_\beth})^{\psi_\parallel}, \\ \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^-_{e_{11}}, ..., \tilde{\mu}_{nm} \in \tilde{h}^-_{e_{nm}}} -1 + \prod_{\parallel=1}^{m} (\prod_{\beth=1}^{n} (1 + \tilde{\mu}^-_{\tilde{h}_{e(\beth\parallel)}})^{w_\beth})^{\psi_\parallel} \end{array} \right\}.$$

$$= \left\{ \begin{array}{c} (\{(0.1)^{0.3}(0.2)^{0.5}(0.6)^{0.2}\}^{0.1}\{(0.1)^{0.3}(0.6)^{0.5}(0.7)^{0.2}\}^{0.4} \\ \{(0.2)^{0.3}(0.5)^{0.5}(0.4)^{0.2}\}^{0.2}\{(0.1)^{0.3}(0.2)^{0.5}(0.3)^{0.2}\}^{0.3}), \\ (-1 + \{(1 + (-0.3))^{0.3}(1 + (-0.4))^{0.5}(1 + (-0.8))^{0.2}\}^{0.1} \\ \{(1 + (-0.1))^{0.3}(1 + (-0.4))^{0.5}(1 + (-0.5))^{0.2}\}^{0.4} \\ \{(1 + (-0.1))^{0.3}(1 + (-0.7))^{0.5}(1 + (-0.4))^{0.2}\}^{0.2} \\ \{(1 + (-0.2))^{0.3}(1 + (-0.6))^{0.5}(1 + (-0.8))^{0.2}\}^{0.3}) \end{array} \right\}.$$

$$= \left\{ \begin{array}{c} 0.2752, 0.2888, 0.2604, 0.2733, 0.3158, 0.3314, 0.2987, 0.3135, -0.4727, \\ -0.4578, -0.4769, -0.4617, -0.4396, -0.4239, -0.4436, -0.4279 \end{array} \right\}.$$

Properties of (BHFSWG) operator.

Idempotency:

If $\tilde{h}_{e_{\beth\daleth}} = \tilde{h}_e = (\tilde{h}^{+}_{e_{\beth\daleth}}, \tilde{h}^{-}_{e_{\beth\daleth}})\ \forall\ \beth, \daleth$ then $BHFSWG(\tilde{h_{e_{11}}}, \tilde{h_{e_{12}}}, ..., \tilde{h_{e_{nm}}}) = \tilde{h}_e$.

Proof. See Appendix E. □

Homogeneity:

We derive for each real number $\lambda > 0$, then

$$BHFSWG(\tilde{h}^{\lambda}_{e_{11}}, \tilde{h}^{\lambda}_{e_{12}}, ..., \tilde{h}^{\lambda}_{e_{nm}}) = \{BHFSWG(\tilde{h}_{e_{11}}, \tilde{h}_{e_{12}}, ..., \tilde{h}_{e_{nm}})\}^{\lambda}.$$

Proof. See Appendix F. □

Boundedness:

Suppose $\tilde{h}_{e_{\beth\daleth}} = (\tilde{h}^{+}_{e_{\beth\daleth}}, \tilde{h}^{-}_{e_{\beth\daleth}})$.

Then $\tilde{h}^{-}_{e_{\beth\daleth}} = (\min_{\daleth}\min_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\daleth)}}\}, \max_{\daleth}\max_{\beth}\{\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\daleth)}}\})$ and $\tilde{h}^{+}_{e_{\beth\daleth}} = (\max_{\daleth}\max_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\daleth)}}\}, \min_{\daleth}\min_{\beth}\{\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\daleth)}}\})$.

Let, $\gamma_{min} = (\min_{\daleth}\min_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\daleth)}}\}|\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\daleth)}} \epsilon \tilde{h}^{-}_{e_{\beth\daleth}}), \zeta_{max} = (\max_{\daleth}\max_{\beth}\{\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\daleth)}}\}|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\daleth)}} \epsilon \tilde{h}^{-}_{e_{\beth\daleth}})$.

$$\zeta_{min} = (\min_{\daleth}\min_{\beth}\{\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\daleth)}}\}|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\daleth)}} \epsilon \tilde{h}^{+}_{e_{\beth\daleth}}), \gamma_{max} = (\max_{\daleth}\max_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\daleth)}}\}|\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\daleth)}} \epsilon \tilde{h}^{+}_{e_{\beth\daleth}}).$$

$$\tilde{h}^{-}_{e_{\beth\daleth}} \leq BHFSWG(\tilde{h_{e_{11}}}, \tilde{h_{e_{12}}}, ..., \tilde{h_{e_{nm}}}) \leq \tilde{h}^{+}_{e_{\beth\daleth}}. \tag{5.7}$$

where

$$\tilde{h}^{-}_{e_{\beth\daleth}} = (\{\gamma_{min}\}, \{\zeta_{max}\}) = \bigcup_{\substack{\gamma^{-}\epsilon\gamma_{min} \\ \zeta^{+}\epsilon\zeta_{max}}} (\{\gamma^{-}\}, \{\zeta^{+}\}).$$

$$\tilde{h}^{+}_{e_{\beth\daleth}} = (\{\gamma_{max}\}, \{\zeta_{min}\}) = \bigcup_{\substack{\gamma^{+}\epsilon\gamma_{max} \\ \zeta^{-}\epsilon\zeta_{min}}} (\{\gamma^{+}\}, \{\zeta^{-}\}).$$

Proof. According to the property 3, the proof is simple and can be easily derived, so we omit it for conciseness. □

Definition 5.5. Suppose $\tilde{h}_{e\sigma(\beth\daleth)} = (\tilde{h}^{+}_{e_{\sigma(\beth\daleth)}}, \tilde{h}^{-}_{e_{\sigma(\beth\daleth)}})$ $\beth$={1,2,...,n} $\daleth$={1,2,...,m} is BHFSONs, then a function is represented by the bipolar hesitant fuzzy soft ordered weighted geometric (BHFSOWG) operator $\tilde{h}_e^{\,n} \rightarrow \tilde{h}_e$ such that

$$BHFSOWG(\tilde{h_{e_{\sigma(11)}}}, \tilde{h_{e_{\sigma(12)}}}, ..., \tilde{h_{e_{\sigma(nm)}}}) = \bigotimes_{\daleth=1}^{m} \left(\bigotimes_{\beth=1}^{n} (\tilde{h}_{e_{\sigma(\beth\daleth)}})^{w_{\beth}} \right)^{\psi_{\daleth}}, \tag{5.8}$$

where $\tilde{h}_e$ represents the collection of BHFSNs, the weight vector $w_{\beth}(\beth = 1, 2, ..., n)$ and $\psi_{\daleth}(\daleth = 1, 2, ..., m)$ describe the experts and parameters, respectively, and $\beth, \daleth > 0$ so that $\sum_{\daleth=1}^{m} \psi_{\daleth} = 1$, $\sum_{\beth=1}^{n} w_{\beth} = 1$.

Theorem 5.5. Suppose $\tilde{h}_{e\sigma(\beth\daleth)} = (\tilde{h}^{+}_{e\sigma(\beth\daleth)}, \tilde{h}^{-}_{e\sigma(\beth\daleth)})$ $\beth$={1,2,...,n} $\daleth$={1,2,...,m} is BHFSONs, then a function is represented by the BHFSOWG operator $\tilde{h}_e^{\,n} \to \tilde{h}_e$ such that

$$BHFSOWG(\tilde{h_{e_{\sigma(11)}}}, \tilde{h_{e_{\sigma(12)}}}, ..., \tilde{h_{e_{\sigma(nm)}}}) = \bigotimes_{\daleth=1}^{m} \left(\bigotimes_{\beth=1}^{n} (\tilde{h}_{e_{\sigma(\beth\daleth)}})^{w_{\beth}} \right)^{\psi_{\daleth}}$$

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{\sigma(11)} \in \tilde{h}^{+}_{e_{\sigma(11)}}, ..., \tilde{\mu}_{\sigma(nm)} \in \tilde{h}^{+}_{e_{\sigma(nm)}}} \prod_{\daleth=1}^{m} (\prod_{\beth=1}^{n} (\tilde{\mu}^{+}_{\tilde{h}_{e\sigma(\beth\daleth)}})^{w_{\beth}})^{\psi_{\daleth}}, \\ \bigcup_{\tilde{\mu}_{\sigma(11)} \in \tilde{h}^{-}_{e_{\sigma(11)}}, ..., \tilde{\mu}_{\sigma(nm)} \in \tilde{h}^{-}_{e_{\sigma(nm)}}} -1 + \prod_{\daleth=1}^{m} (\prod_{\beth=1}^{n} (1 + \tilde{\mu}^{-}_{\tilde{h}_{e\sigma(\beth\daleth)}})^{w_{\beth}})^{\psi_{\daleth}} \end{array} \right\},$$

where $\tilde{h}_e$ represents the collection of BHFSNs, the weight vector $w_{\beth}(\beth = 1, 2, ..., n)$ and $\psi_{\daleth}(\daleth = 1, 2, ..., m)$ describe the experts and parameters, respectively, and $\beth, \daleth > 0$ so that $\sum_{\daleth=1}^{m} \psi_{\daleth} = 1$, $\sum_{\beth=1}^{n} w_{\beth} = 1$.

Definition 5.6. Suppose $\tilde{h}'_{e\sigma(\beth\daleth)} = (\tilde{h}'^{+}_{e\sigma(\beth\daleth)}, \tilde{h}'^{-}_{e\sigma(\beth\daleth)})$ $\beth$={1,2,...,n} $\daleth$={1,2,...,m} is BHFSHNs, then a function is represented by the bipolar hesitant fuzzy soft hybrid weighted geometric (BHFSHWG) operator $\tilde{h}_e^{\,n} \to \tilde{h}_e$ such that

$$BHFSHWG(\tilde{h'_{e_{\sigma(11)}}}, \tilde{h'_{e_{\sigma(12)}}}, ..., \tilde{h'_{e_{\sigma(nm)}}}) = \bigotimes_{\daleth=1}^{m} \left(\bigotimes_{\beth=1}^{n} (\tilde{h}'_{e_{\sigma(\beth\daleth)}})^{w_{\beth}} \right)^{\psi_{\daleth}}, \tag{5.9}$$

where $\tilde{h}_e$ represents the collection of BHFSNs, the weight vector $w_{\beth}(\beth = 1, 2, ..., n)$ and $\psi_{\daleth}(\daleth = 1, 2, ..., m)$ describe the experts and parameters, respectively, and $\beth, \daleth > 0$ so that $\sum_{\daleth=1}^{m} \psi_{\daleth} = 1$, $\sum_{\beth=1}^{n} w_{\beth} = 1$.

Theorem 5.6. Suppose $\tilde{h}'_{e\sigma(\beth\daleth)} = (\tilde{h}'^{+}_{e\sigma(\beth\daleth)}, \tilde{h}'^{-}_{e\sigma(\beth\daleth)})$ $\beth$={1,2,...,n} $\daleth$={1,2,...,m} is BHFSHNs, then a function is represented by the BHFSHWG operator $\tilde{h}_e'^{\,n} \to \tilde{h}'_e$ such that

$$BHFSHWG(\tilde{h}'_{e\sigma(11)}, \tilde{h}'_{e\sigma(12)}, ..., \tilde{h}'_{e\sigma(nm)}) = \bigotimes_{\daleth=1}^{m} \left(\bigotimes_{\beth=1}^{n} (\tilde{h}'_{e\sigma(\beth\daleth)})^{w_{\beth}} \right)^{\psi_{\daleth}}$$

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{\sigma(11)} \in \tilde{h}'^{+}_{e\sigma(11)}, ..., \tilde{\mu}_{\sigma(nm)} \in \tilde{h}'^{+}_{e\sigma(nm)}} \prod_{\daleth=1}^{m} (\prod_{\beth=1}^{n} (\tilde{\mu}^{+}_{\tilde{h}'_{e\sigma(\beth\daleth)}})^{w_{\beth}})^{\psi_{\daleth}}, \\ \bigcup_{\tilde{\mu}_{\sigma(11)} \in \tilde{h}'^{-}_{e\sigma(11)}, ..., \tilde{\mu}_{\sigma(nm)} \in \tilde{h}'^{-}_{e\sigma(nm)}} -1 + \prod_{\daleth=1}^{m} (\prod_{\beth=1}^{n} (1 + \tilde{\mu}^{-}_{\tilde{h}'_{e\sigma(\beth\daleth)}})^{w_{\beth}})^{\psi_{\daleth}} \end{array} \right\},$$

where $\tilde{h}_e$ represents the collection of BHFSNs, the weight vector $w_{\beth}(\beth = 1, 2, ..., n)$ and $\psi_{\daleth}(\daleth = 1, 2, ..., m)$ describe the experts and parameters, respectively, and $\beth, \daleth > 0$ so that $\sum_{\daleth=1}^{m} \psi_{\daleth} = 1$, $\sum_{\beth=1}^{n} w_{\beth} = 1$.

6. A new approach utilizing on proposed operators

Let $C = \{C_1, C_2, ..., C_f\}$ represent the collection of f distinct alternatives, which are going to be evaluated by s experts $x_1, x_2, ..., x_s$ under the constraints of r parameters, $E = \{e_1, e_2, ..., e_r\}$. Suppose

$w = (w_1, w_2, ..., w_n)^T$ and $\psi = (\psi_1, \psi_2, ..., \psi_m)^T$ represent weighted vectors of experts and parameters correspondingly, $w_{\sqsupset} \epsilon (0, 1]$, $\sum_{\sqsupset=1}^{n} w_{\sqsupset} = 1$ and $\psi_{\shortparallel} \epsilon (0, 1]$ $\sum_{\shortparallel=1}^{m} \psi_{\shortparallel} = 1$. Those who make decisions will discuss alternatives to BHFSNs $\tilde{h}_{e_{\sqsupset\shortparallel}} = (\tilde{h}^{+}_{e_{\sqsupset\shortparallel}}, \tilde{h}^{-}_{e_{\sqsupset\shortparallel}})$. Then the information is gathered into a decision matrix $D = (\tilde{h}_{e_{\sqsupset\shortparallel}})_{n\times m}$. The BHFSWA or BHFSWG operators are used to aggregate the BFSNs. Lastly, the options are ranked using the score function of the combined FBSN. We are presenting the already mentioned approach eventually.

Step 1. Compile an all-inclusive information of all the alternatives for various conditions under one hesitant bipolar fuzzy soft matrix systematically. Such a matrix allows the manifestation of the degrees of uncertainty and hesitation associated with each alternative. Organization in such a structured form enables decision-makers to evaluate and compare choices more effectively within the nuances of the decision environment,

$$D_{n\times m} = D_{n\times m} = (\tilde{h}^{+}_{e_{\sqsupset\shortparallel}}, \tilde{h}^{-}_{e_{\sqsupset\shortparallel}}).$$

$$\begin{bmatrix} \{\tilde{h}^{+}_{e_{11}}, \tilde{h}^{-}_{e_{11}}\} & \{\tilde{h}^{+}_{e_{12}}, \tilde{h}^{-}_{e_{12}}\} & ... & \{\tilde{h}^{+}_{e_{1m}}, \tilde{h}^{-}_{e_{1m}}\} \\ \{\tilde{h}^{+}_{e_{21}}, \tilde{h}^{-}_{e_{21}}\} & \{\tilde{h}^{+}_{e_{22}}, \tilde{h}^{-}_{e_{22}}\} & ... & \{\tilde{h}^{+}_{e_{2m}}, \tilde{h}^{-}_{e_{2m}}\} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ \{\tilde{h}^{+}_{e_{n1}}, \tilde{h}^{-}_{e_{n1}}\} & \{\tilde{h}^{+}_{e_{n2}}, \tilde{h}^{-}_{e_{n2}}\} & ... & \{\tilde{h}^{+}_{e_{nm}}, \tilde{h}^{-}_{e_{nm}}\} \end{bmatrix}.$$

Step 2. To normalize the entire decision matrix, a normalization technique would be used to assign values to the cost-type variables, so that lower values of these variables are going to increase the value of better options, and to account for the benefit-type variables as well, where higher values of those variables represent more favorable outcomes. In other words, it seeks to transform the original data onto a standard scale to enhance comparability across criteria. With normalized decision matrixes, decision-makers can assist in better assessments of alternatives that allow more clarity in relative merits and drawbacks.

$$\Re_{\sqsupset\shortparallel} = \begin{cases} \tilde{h}^{c}_{e_{\sqsupset\shortparallel}}; \text{about cost type variables} \\ \tilde{h}_{e_{\sqsupset\shortparallel}}; \text{about benefit type variables} \end{cases}.$$

Step 3. Aggregating the BHFSNs $\tilde{h}_{e_{\sqsupset\shortparallel}}$ ($\sqsupset = 1, 2, ..., n; \shortparallel = \{1, 2, ..., m\}$) for each alternative $C_k(k = 1, 2, ..., f)$ using any of the aggregation operators presented herein. This will lead to an overall decision-matrix that aggregates all of the assessments made by the BHFSNs in the process of evaluating each prospect. The composite value aggregated will allow decision-making by combining the various judgments and uncertainties associated with an alternative.

Step 4. Determine all possible alternative scores $\tilde{h}_{e_{\sqsupset\shortparallel}}$ for all $C_k(k = 1, 2, ..., f)$ using the determined equations. It means applying all mathematical operations defined in the steps to compute the appropriate scores that represent the appraisals of each alternative. In this way, decision-makers can determine quantifiable representations about the performance of each alternative, which lends clear comparison and analytical assessments on options in a decision-making framework.

$$\delta(\tilde{h}_{e_{\beth''}}) = \frac{1}{2}(\frac{1}{\ell_{\tilde{h}^{+}_{e_{\beth''}}}} \sum_{\tilde{\mu}^{+}_{1} \in \tilde{h}^{+}_{e_{\beth''}}} \tilde{\mu}^{+} - \frac{1}{\ell_{\tilde{h}^{-}_{e_{\beth''}}}} \sum_{\tilde{\mu}^{-}_{1} \in \tilde{h}^{-}_{e_{\beth''}}} \tilde{\mu}^{-}). \tag{6.1}$$

Then calculate the actual values of the alternatives as in: This is the step in which score values calculated in relation to the established decision-making criteria are used to get an overall value of the alternatives. Therefore, by finding the above precise values, decision makers will be able to make an effective ranking of the alternatives with the further decision-making process being based on the given complex situation.

$$a(\tilde{h}_{e_{\beth''}}) = \frac{1}{2}(\frac{1}{\ell_{\tilde{h}^{+}_{e_{\beth''}}}} \sum_{\tilde{\mu}^{+}_{1} \in \tilde{h}^{+}_{e_{\beth''}}} \tilde{\mu}^{+} + \frac{1}{\ell_{\tilde{h}^{-}_{e_{\beth''}}}} \sum_{\tilde{\mu}^{-}_{1} \in \tilde{h}^{-}_{e_{\beth''}}} \tilde{\mu}^{-}), \tag{6.2}$$

where $\ell_{\tilde{h}^{+}}$ and $\ell_{\tilde{h}^{-}}$ are the numbers of the elements in $\tilde{h}^{+}$ and $\tilde{h}^{-}$, correspondingly.

Step 5. Organize the options $C_k (k = 1, 2, ..., f)$ systematically such that based on rating evaluation, you determine which one is best ranked and which one is the worst. The ranking will determine how good or bad some calculation values are for a given option versus predefined criteria. Then, from all these choices, select an appropriate option as the final choice that fits decision-making objectives and has the highest potential for finding desired results.

Step 6. Conclude.

The flowchart of the proposed methodology is given in Figure 1.

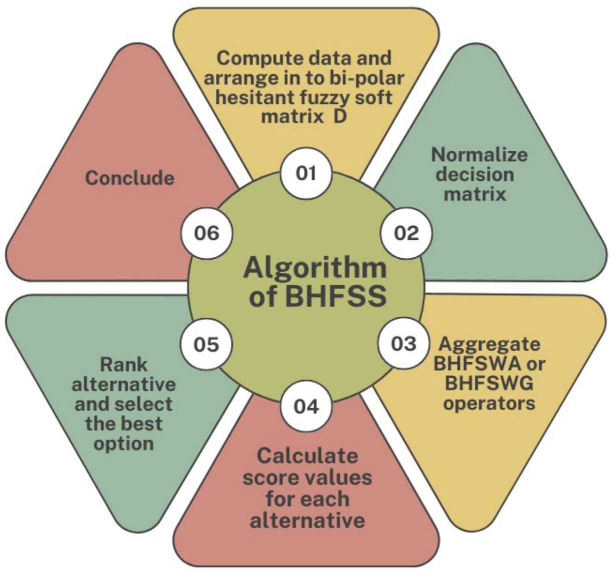

Figure 1. Step by step illustration of algorithm.

7. Sustainable energy solutions for society

Sustainable energy solutions are essential for the well-being of society and are the foundation for a cleaner, greener future. We must move to renewable energy sources that include solar, hydroelectricity, wind, and thermal energy to minimize our dependency on fossil fuels and improve the impacts of climate change. Investing in low-carbon solutions and forming energy cooperation can promote scalability and innovation. Efficient power sources, such as solar panels and modern windmills, can ensure a steady energy supply. Community-based work that addresses an array of issues, such as education, health care, prosperity, and protecting the environment, is also essential. Improving equipment and adding energy storage technologies such as capacitors can help provide continuous and reliable electricity supplies. Support rules and regulations are needed to encourage the use of sustainable energy sources. Sustainable energy solutions have several benefits. Reduced emission of greenhouse gases can benefit people's health by decreasing air quality, mitigating climate change, and encouraging regional growth and industries. Moreover, sustainable energy solutions may help with safety, energy independence, and a decrease in power deprivation [47, 48]. In sustainable energy case studies, a suitable energy source or technique may be picked and examined to generate the BHFSS. BHFSS helps decision-makers to pick alternatives, define criteria, and get expert views to determine a wide range of perspectives and various kinds of ambiguity. This helps them to compare the positive aspects and drawbacks of numerous solutions. We must select the most suitable choice for sustainable energy options for society.

Solar Energy (C_1)**:** It is possible to produce heat or power from the sun by consuming solar energy. This sustainable and eco-friendly approach to energy production reduces the dependence on fossil fuels and mitigates their impact on global warming. Photovoltaic panels use sunlight to create electricity, whereas solar thermal systems heat water or air. The benefits of solar energy include reduced costs, energy independence, and a lower carbon footprint. It is critical to renewable energy's future.

Wind energy (C_2)**:** Wind energy to harness wind power allows for the production of electricity. Due to the adaptable wind speeds, its dependability is mediocre. Wind conditions affect consistency and reliability. Technological advancements boost efficiency and dependability. For a sustainable energy future, wind energy is still essential.

Hydro energy (C_3)**:** Hydro energy uses the power of flowing water to generate electricity. It is a renewable energy source that is trustworthy and has good energy conversion efficiency. Facilities that use hydro energy are adaptable because their making may be changed to meet demand. As a good alternative, hydro energy is abundant in water resources. It is a clean, sustainable way to supply energy that utilizes less fossil fuel.

The mentioned alternatives provide different approaches to changing societies by sustainable energy, and emphasize the significance of community engagement, urban resilience, and power development for getting an equitable and sustainable energy system. Three professionals were selected to share their knowledge to decide the most effective strategy for improving society through the use of renewable energy. Because they all give different ideas and abilities in a discussion, it is essential to make significant changes.

Energy Researcher (x_1)**:** The energy researcher accurately estimated numerous energy alternatives. They utilized an accurate mathematical framework for examination. Their awareness of energy systems and regulations affected their rating. They examined a range of components, including productivity and impact. The best energy source was picked with their opinions in mind.

Environmentalist (x_2)**:** An environmentalist is a professional who examines the environmental impact of different forms of energy, taking into consideration pollution, carbon dioxide emissions, and land usage. They give an extensive perspective on how environmentally friendly certain energy alternatives are, helping to select the best alternatives.

Economist (x_3)**:** Economists examine energy sources through their financial viability, including affordability, costs, and future development capacity into consideration. They examine the economic effects of each alternative, giving a vital perspective on which are the most cost-effective and environmentally friendly energy sources. Their results help decision-makers understand the economic decisions relating to renewable energy sources.

When placed together, these experts represent a multidisciplinary group able to successfully handle the obstacles faced by renewable energy projects to achieve significant progress in a green and equitable society. In the present situation, a team of individuals hold the project, deciding within the specified restrictions, but determining the best option to take demands as a thorough examination of all the essential components. E = $\{e_1$ = Reliability ,e_2= Efficiency,e_3 =Cost, e_4 = Sustainability$\}$ $\psi = (0.3, 0.4, 0.1, 0.2)$, and $w = (0.2, 0.5, 0.3)$, respectively, represent the weighted vectors for the parameters and the expert, correspondingly.

The steps for implementing it are given below.

Table 4. BHFS matrix.

C_1	e_1	e_2	e_3	e_4
x_1	$\{0.1, 0.2\}, \{-0.3\}$	$\{0.5, 0.4\}, \{-0.1\}$	$\{0.2\}, \{-0.4, -0.6\}$	$\{0.1\}, \{-0.2\}$
x_2	$\{0.4\}, \{-0.5\}$	$\{0.7\}, \{-0.3, -0.2\}$	$\{0.6\}, \{-0.2\}$	$\{0.2, 0.3\}, \{-0.5\}$
x_3	$\{0.5, 0.4\}, \{-0.1\}$	$\{0.4\}, \{-0.6, -0.4\}$	$\{0.7, 0.3\}, \{-0.1\}$	$\{0.6\}, \{-0.7\}$
C_2	e_1	e_2	e_3	e_4
x_1	$\{0.1\}, \{-0.3\}$	$\{0.5\}, \{-0.1\}$	$\{0.2\}, \{-0.6, -0.4\}$	$\{0.1\}, \{-0.2\}$
x_2	$\{0.4\}, \{-0.5\}$	$\{0.7, 0.3\}, \{-0.3\}$	$\{0.6\}, \{-0.5\}$	$\{0.2, 0.8\}, \{-0.5\}$
x_3	$\{0.5\}, \{-0.1\}$	$\{0.4\}, \{-0.6, -0.4\}$	$\{0.7\}, \{-0.1\}$	$\{0.6\}, \{-0.7\}$
C_3	e_1	e_2	e_3	e_4
x_1	$\{0.5\}, \{-0.8\}$	$\{0.6\}, \{-0.1\}$	$\{0.2\}, \{-0.6, -0.4\}$	$\{0.1\}, \{-0.2\}$
x_2	$\{0.4\}, \{-0.6\}$	$\{0.7, 0.3\}, \{-0.3\}$	$\{0.6\}, \{-0.5\}$	$\{0.2, 0.8\}, \{-0.5, -0.3\}$
x_3	$\{0.1, 0.2\}, \{-0.5\}$	$\{0.4\}, \{-0.6, -0.4\}$	$\{0.7\}, \{-0.1\}$	$\{0.4\}, \{-0.1\}$

Step 1. Three experts, $x_s(s = 1, 2, 3)$, examine each alternative, $C_k(k = 1, 2, 3)$, considering parameters $e_r(r = 1, 2, 3, 4)$, and offer their rating values under the BHFSS. Table 4 summarizes the combined ratings of alternatives given by expert preferences, using BHFSNs $(\tilde{h}^{+}_{e_{\beth}}, \tilde{h}^{-}_{e_{\beth}})$.

Step 2. No normalization is needed as all parameters are of the same type, so the dataset is homogeneous.

Step 3. Total score values for each of the alternative $C_k(k = 1, 2, 3)$ are calculated through operators proposed and gives a summarization in terms of performance based on criterion defined beforehand. This scoring of alternatives helps make an effective comparison among the alternatives, which brings the way to making adequate decisions. The results are mentioned in Tables 5–10.

Table 5. Results proposed by operators ($BHFSWA$).

	$BHFSWA$
C_1	{0.4887 ,0.4802 ,0.4923 ,0.4839 ,0.4812 ,0.4726 ,0.4848 ,0.4763, 0.4755,0.4668,0.4792,0.4706,0.4678,0.4590,0.4716,0.4628, 0.4955, 0.4871, 0.4990, 0.4907, 0.4881, 0.4796, 0.4917, 0.4833, 0.4825, 0.4739, 0.4861, 0.4776, 0.4749, 0.4663, 0.4786, 0.4699}, {−0.3034, −0.2889, −0.2798, −0.2665, −0.3059, −0.2913, −0.2820, −0.2686}
C_2	{0.4887 , 0.3943 ,0.5549 , 0.4727} {−0.3202,−0.3049 , −0.3176, −0.3025 }
C_3	{0.4762 , 0.4817 ,0.3796 , 0.3860,0.5440,0.5488,0.4598,0.4655} {−0.3589,−0.3419 , −0.3560, −0.3392,−0.3411,−0.3249,−0.3383,−0.3222 }

Table 6. Results proposed by operators ($BHFSWG$).

	$BHFSWG$
C_1	{0.3958 ,0.3879 ,0.4126 ,0.4044 ,0.3888 ,0.3810 ,0.4053 ,0.3972, 0.3859 ,0.3782 ,0.4023 ,0.3943 ,0.3790 ,0.3715 ,0.3951 ,0.3872 , 0.4122 ,0.4039 ,0.4297 ,0.4211 ,0.4049 ,0.3968 ,0.4220 ,0.4137, 0.4018 ,0.3938 ,0.4189 ,0.4106 ,0.395 ,0.3869 ,0.4115 ,0.4033}, {−0.3932 ,−0.3629 ,−0.3768 ,−0.3457 ,−0.3981 ,−0.3681 ,−0.3818 ,−0.3510}
C_2	{0.3958 ,0.3341 ,0.4547 ,0.3838} {−0.4120 ,−0.3828 ,−0.4073 ,−0.3778}
C_3	{0.3735 ,0.3975 ,0.3153 ,0.3356 ,0.4290 ,0.4566 ,0.3621 ,0.3855} {−0.4657 ,−0.4391 ,−0.4614 ,−0.4345 ,−0.4474 ,−0.4199 ,−0.4429 ,−0.4152 }

The performance is measured for this scoring system, which means it would make it easy and effective to compare the alternatives between themselves, and decision-makers may determine which one of them is the best.

Table 7. Results proposed by operators ($BHFSOWA$).

	$BHFSOWA$
C_1	$\{0.4269, 0.4109, 0.4329, 0.4172, 0.4142, 0.3979, 0.4204, 0.4043, 0.4121, 0.3958, 0.4183, 0.4022, 0.3991, 0.3825, 0.4055, 0.3889, 0.4345, 0.4188, 0.4404, 0.4249, 0.4219, 0.4059, 0.4281, 0.4122, 0.4199, 0.4038, 0.4260, 0.4101, 0.4070, 0.3906, 0.4133, 0.3970\}$, $\{-0.2733, -0.2519, -0.2645, -0.2439, -0.2788, -0.2571, -0.2699, -0.2489\}$
C_2	$\{0.4829, 0.3874, 0.5498, 0.4667\}$ $\{-0.2521, -0.2441, -0.2491, -0.2411\}$
C_3	$\{0.4887, 0.4941, 0.3943, 0.4007, 0.5163, 0.5214, 0.4269, 0.4330\}$ $\{-0.2984, -0.2889, -0.2948, -0.2854, -0.2924, -0.2831, -0.2889, -0.2796\}$

Table 8. Results proposed by operators ($BHFSOWG$).

	$BHFSOWG$
C_1	$\{0.3389, 0.3277, 0.3607, 0.3488, 0.3299, 0.3191, 0.3512, 0.3396, 0.3304, 0.3195, 0.3517, 0.3401, 0.3217, 0.3111, 0.3424, 0.33109, 0.3529, 0.3413, 0.3756, 0.3633, 0.3436, 0.3323, 0.3657, 0.3537, 0.3441, 0.3327, 0.3662, 0.3542, 0.3349, 0.3239, 0.3565, 0.3448\}$, $\{-0.3799, -0.3276, -0.3733, -0.3204, -0.3924, -0.3411, -0.3859, -0.3339)\}$
C_2	$\{0.3716, 0.3137, 0.4269, 0.3603\}$ $\{-0.3457, -0.3241, -0.3377, -0.3159\}$
C_3	$\{0.3819, 0.6019, 0.5080, 0.5080, 0.6362, 0.6362, 0.5370, 0.5370\}$ $\{-0.4242, -0.4052, -0.4171, -0.3979, -0.4164, -0.3971, -0.4092, -0.3897\}$

The performance is also measured for this scoring system, which means it would make it easy and effective to compare the alternatives between themselves, and decision-makers may determine which one of them is the best.

Table 9. Results proposed by operators ($BHFSHWA$).

	$BHFSHWA$
C_1	$\{0.3923, 0.3961, 0.3883, 0.3921, 0.3679, 0.3719, 0.3637, 0.3677, 0.3773,$ $0.3812, 0.3732, 0.3771, 0.3523, 0.3564, 0.3479, 0.3521, 0.3981,$ $0.4019, 0.3941, 0.3979, 0.3739, 0.3779, 0.3698, 0.3738, 0.3832,$ $0.3871, 0.3792, 0.3831, 0.3585, 0.3625, 0.3542, 0.3583\},$ $\{-0.2917, -0.2752, -0.2817, -0.2658, -0.3062, -0.2889, -0.2958, -0.2790)\}$
C_2	$\{0.4633, 0.3555, 0.5143, 0.4168\}$ $\{-0.2606, -0.2545, -0.2531, -0.2472\}$
C_3	$\{0.5078, 0.5091, 0.4507, 0.4521, 0.5271, 0.5283, 0.4722, 0.4735\}$ $\{-0.2278, -0.2226, -0.2213, -0.2162, -0.2245, -0.2193, -0.2180, -0.2129\}$

Table 10. Results proposed by operators ($BHFSHWG$).

	$BHFSHWG$
C_1	$\{0.2559, 0.2729, 0.2452, 0.2615, 0.2507, 0.2674, 0.2403, 0.2563,$ $0.2494, 0.2659, 0.2389, 0.2549, 0.2444, 0.2606, 0.2342, 0.2497,$ $0.2669, 0.2847, 0.2558, 0.2728, 0.2616, 0.2789, 0.2507, 0.2673,$ $0.2602, 0.2775, 0.2493, 0.2659, 0.2549, 0.2718, 0.2443, 0.2605\},$ $\{-0.5181, -0.4727, -0.5133, -0.4675, -0.5236, -0.4787, -0.5188, -0.4735)\}$
C_2	$\{0.2866, 0.3485, 0.4107, 0.3485\}$ $\{-0.4711, -0.4516, -0.4674, -0.4478\}$
C_3	$\{0.2871, 0.3068, 0.2602, 0.2779, 0.3053, 0.3262, 0.2766, 0.2955\}$ $\{-0.5547, -0.5384, -0.5516, -0.5352, -0.5477, -0.5311, -0.5446, -0.5279\}$

Step 4. The score values $\delta(\tilde{h}_{e_{\beth_{ll}}})$ for alternatives $C_k (k = 1, 2, 3)$ are obtained by applying the scoring equation already developed, which assigns a numerical score to each alternative under the set criteria. This systematic approach enables one to sum up several different factors into just one score that reflects the sum total performance of each alternative. Therefore, these score values can prove a useful tool for meaningful comparison among the alternatives available while assisting a decision-maker to choose the most appropriate choice. The results are mentioned in Table 9.

Step 5. A tabular presentation in Table 11 ranks the options according to their calculated score values. The ranking clearly permits a comparison of the performance of the different options. Ranking helps in choosing the best-performing alternative as it guides decision makers through the selection process.

All scores reflect an aggregate alternative assessment based upon defined criteria.

Table 11. Scoring and ranking.

Operators	C_1	C_2	C_3	Ranking
$BHFSWA$	0.3826	0.3945	0.4040	$C_3 > C_2 > C_1$
$BHFSWG$	0.3859	0.3935	0.4113	$C_3 > C_2 > C_1$
$BHFSOWA$	0.3365	0.3591	0.3742	$C_3 > C_2 > C_1$
$BHFSOWG$	0.3495	0.3495	0.4752	$C_3 > C_2 > C_1$
$BHFSHWA$	0.3304	0.3457	0.3552	$C_3 > C_2 > C_1$
$BHFSHWG$	0.3925	0.4040	0.4167	$C_3 > C_2 > C_1$

Table 11 clearly depicts how C_3 is the only alternative that stands out to be the most advisable, demonstrated through superior performance for all the criteria. Further validation is hence drawn from the graphical representation as presented in Figure 2, with a visual illustration that enhances C_3 as a better alternative as against others and, therefore, the selection of the best alternative in this particular analysis.

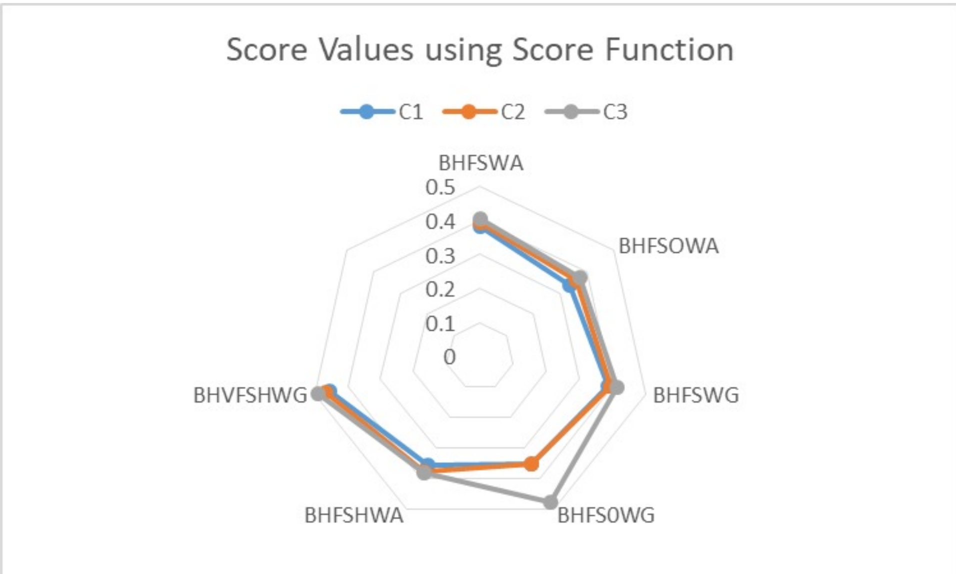

Figure 2. Graphical representation of score values using score function.

8. Extended combining compromised solution technique for bipolar hesitant fuzzy soft information

The suggested method is based on the combined basic addition and progressively weighted product(WP) models. It may represent a collection of applicable choices.

The advanced MADM technique, CoCoSo, combines compromise and aggregation strategies to evaluate and rank alternatives. In this method, two aggregation methods are involved-the power average and geometric mean-that are used to ascertain the performance scores in the evaluation of the alternatives. CoCoSo provides a balanced and adaptable framework for decision-making, particularly in situations where criteria have different levels of importance and where there is a need for compromise toward identifying the optimal alternative. This makes CoCoSo highly effective in complex decision environments. To resolve a CoCoSo decision choice, one has to first regulate the options and the associated criterion, and the subsequent actions are verified [49, 50].

Step 1. As mentioned below, the fundamental decision-making matrix is developed.
$Dn \times m = (\tilde{h}^{+}_{e_{\beth\parallel}}, \tilde{h}^{-}_{e_{\beth\parallel}})$

$$\begin{bmatrix} \{\tilde{h}^{+}_{e_{11}}, \tilde{h}^{-}_{e_{11}}\} & \{\tilde{h}^{+}_{e_{12}}, \tilde{h}^{-}_{e_{12}}\} & \dots & \{\tilde{h}^{+}_{e_{1m}}, \tilde{h}^{-}_{e_{1m}}\} \\ \{\tilde{h}^{+}_{e_{21}}, \tilde{h}^{-}_{e_{21}}\} & \{\tilde{h}^{+}_{e_{22}}, \tilde{h}^{-}_{e_{22}}\} & \dots & \{\tilde{h}^{+}_{e_{2m}}, \tilde{h}^{-}_{e_{2m}}\} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ \{\tilde{h}^{+}_{e_{n1}}, \tilde{h}^{-}_{e_{n1}}\} & \{\tilde{h}^{+}_{e_{n2}}, \tilde{h}^{-}_{e_{n2}}\} & \dots & \{\tilde{h}^{+}_{e_{nm}}, \tilde{h}^{-}_{e_{nm}}\} \end{bmatrix}.$$

Step 2. To normalize the entire decision matrix, apply the normalizing technique to allocate numbers as cost-type variables and, if necessary, benefit-type variables.

$$\Re_{\beth\parallel} = \begin{cases} \tilde{h}^{c}_{e_{\beth\parallel}}; \text{about cost type variables} \\ \tilde{h}_{e_{\beth\parallel}}; \text{about benefit type variables} \end{cases}.$$

Step 3. Aggregate BHFSNs $\tilde{h}_{e_{\beth\parallel}}$ ($\beth = 1, 2, ..., n; \parallel = \{1, 2, ..., m\}$) for every possibility $C_k(k = 1, 2, ..., f)$that are entered by anyone of the suggested operators into the overall decision-matrix. Determine the measurement of the weighted sum(WS). The measurement of a WS using the BHFSWA operator may be used to find $Y_{\beth}$, thus,

$$Y_{\beth} = BHFSWA(\tilde{h}_{e_{11}}, \tilde{h}_{e_{12}}, ..., \tilde{h}_{e_{nm}}) = \bigoplus_{\parallel=1}^{m} \psi_{\parallel} \left(\bigoplus_{\beth=1}^{n} w_{\beth} \tilde{h}_{e_{\beth\parallel}} \right)$$

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{+}_{e11}, ..., \tilde{\mu}_{nm} \in \tilde{h}^{+}_{enm}} 1 - \prod_{\parallel=1}^{m} (\prod_{\beth=1}^{n} (1 - \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\parallel)}})^{w_{\beth}})^{\psi_{\parallel}}, \\ \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{-}_{e11}, ..., \tilde{\mu}_{nm} \in \tilde{h}^{-}_{enm}} -(\prod_{\parallel=1}^{m} (\prod_{\beth=1}^{n} (|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\parallel)}}|)^{w_{\beth}})^{\psi_{\parallel}})\rangle \end{array} \right\}.$$

Determine the measurement of the WP. The measurement of a WP using the BHFSWG operator may be used to find $Z_{\beth}$, thus,

$$Z_{\beth} = BHFSWG(\tilde{h}_{e_{11}}, \tilde{h}_{e_{12}}, ..., \tilde{h}_{e_{nm}}) = \bigotimes_{\parallel=1}^{m} \left(\bigotimes_{\beth=1}^{n} (\tilde{h}_{e_{\beth\parallel}})^{w_{\beth}} \right)^{\psi_{\parallel}}$$

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{+}_{e11}, ..., \tilde{\mu}_{nm} \in \tilde{h}^{+}_{enm}} \prod_{\parallel=1}^{m} (\prod_{\beth=1}^{n} (\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\parallel)}})^{w_{\beth}})^{\psi_{\parallel}}, \\ \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{-}_{e11}, ..., \tilde{\mu}_{nm} \in \tilde{h}^{-}_{enm}} -1 + \prod_{\parallel=1}^{m} (\prod_{\beth=1}^{n} (1 + \tilde{\mu}^{-}_{\tilde{h}_{e(\beth\parallel)}})^{w_{\beth}})^{\psi_{\parallel}} \end{array} \right\}.$$

Step 4. Determine the scores values for both the WS and the WP. $\delta(\tilde{h}_{e_{\beth_{\shortparallel}}})$ of $\tilde{h}_{e_{\beth_{\shortparallel}}}$ for all possible alternatives $C_k(k = 1, 2, ..., f)$ from the equation

$$\delta(\tilde{h}_{e_{\beth_{\shortparallel}}}) = \frac{1}{2}(\frac{1}{\ell_{\tilde{h}^{+}_{e_{\beth_{\shortparallel}}}}} \sum_{\tilde{\mu}_1^{+} \in \tilde{h}^{+}_{e_{\beth_{\shortparallel}}}} \tilde{\mu}^{+} - \frac{1}{\ell_{\tilde{h}^{-}}} \sum_{\tilde{\mu}_1^{-} \in \tilde{h}^{-}_{e_{\beth_{\shortparallel}}}} \tilde{\mu}^{-}), \tag{8.1}$$

where $\ell_{\tilde{h}^+}$ and $\ell_{\tilde{h}^-}$ are the numbers of the elements in $\tilde{h}^+$ and $\tilde{h}^-$, correspondingly.

Step 5. Three methods are used in this step to obtain the relative appraisal scores that correspond to the other alternatives' weights, which are calculated using formulas. The arithmetic mean of the combined WS measure and WP measure scores is displayed in

$$K^1_{\beth} = \frac{\mathbb{Y}_{\beth} + \mathbb{Z}_{\beth}}{\sum_{\beth=1}^{n}(\mathbb{Y}_{\beth} + \mathbb{Z}_{\beth})}$$
$$K^2_{\beth} = \frac{\mathbb{Y}_{\beth}}{\min_{\beth} \mathbb{Y}_{\beth}} + \frac{\mathbb{Z}_{\beth}}{\min_{\beth} \mathbb{Z}_{\beth}},$$

symbolizing the total of the respective scores for the WS measure and WP measure with regard to the finest

$$K^3_{\beth} = \frac{\kappa\mathbb{Y}_{\beth} + (1-\kappa)\mathbb{Z}_{\beth}}{\kappa \max_{\beth} \mathbb{Y}_{\beth} + (1-\kappa)\max_{\beth} \mathbb{Z}_{\beth}},$$

the balancing compromised score findings for the WS measure and WP measure models. In the equation previously, decision-makers select $\kappa(often\ \kappa = 0.5)$. However, the recommended CoCoSo's toughness and flexibility might vary based on a number of factors.

Step 6. The following shows how the overall ranking function is assembled together.

$$K_{\beth} = (K^1_{\beth} K^2_{\beth} K^3_{\beth})^{\frac{1}{3}} + \frac{K^1_{\beth} + K^2_{\beth} + K^3_{\beth}}{3}$$

Step 7. The solutions are presented in decreasing order of the $K_{\beth}$ values.

The explicit and sequential nature of an application of CoCoSo to decision-making in real life can easily be facilitated because of the graphical presentation of these steps in the Figure 3.

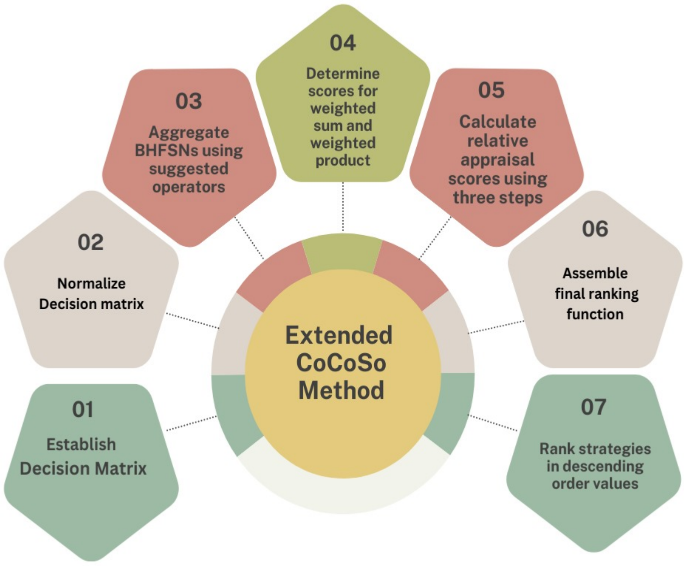

Figure 3. Extended CoCoSo method flowchart.

9. Comparative analysis

In this section, we evaluate the COCOSO technique with BHFSS to figure out its effectiveness in complex decision-making scenarios. The comparison helps to identify the most effective solution by examining the strengths and weaknesses of each technique, emphasizing how BHFSS can give more flexible and adaptive approaches in difficult situations.

Step 1. We will apply this strategy for the BHFS matrix in Table 4 suggested by experts.

Step 2. When the information provided is cost-related, normalization is necessary, but here, normalizing the data is not needed because all parameters are of benefit type.

Step 3. Table 6 gives the WS measures derived from Theorem 5.1, whereas Table 6 gives the WP measures derived from Theorem 5.4.

Step 4. Table 11 presents scoring for WS and WP measures.

Steps 5 and 6. In this step, we use three distinctive methods to calculate the relative appraisal scores based on the WS and WP measures by taking $\kappa = 0.5$. In the subsequent step, the final appraisal scores are obtained by using these three solutions. The final answers for both phases are shown in Table 11.

Step 7. The final ranking is obtained in Table 12.

Table 12. Extended CoCoSo with BHFSSs.

	$K^1_{\beth}$	$K^2_{\beth}$	$K^3_{\beth}$	$K_{\beth}$	Ranking
C_1	0.3240	2	1.3523	2.1824	3
C_2	0.3322	2.0506	1.3766	2.2322	2
C_3	0.3437	2.1216	1.4257	2.3101	1

In a comparison of CoCoSo and aggregation operators, C_3 was found to be the best method for handling HFSs. Although CoCoSo effectively balances multiple criteria, C_3 improves by accurately addressing all MD and hesitation levels utilizing algebraic norms.

The graphical representation of results are shown in Figure 4.

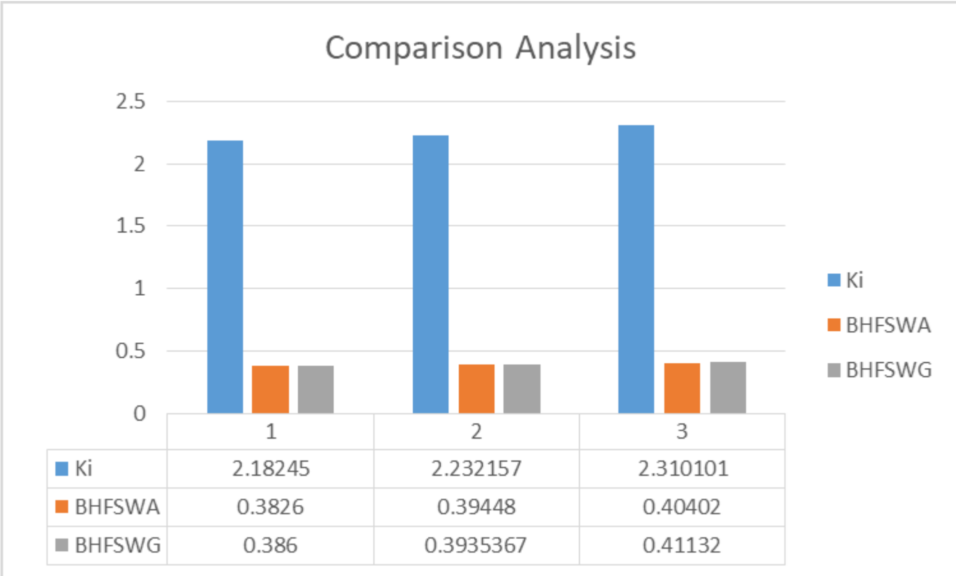

Figure 4. Graphical representation of comparative analysis.

10. Discussion

This section presents the comparable results from our investigation, with an emphasis on the BHFSS architecture and the CoCoSo approach. Our comparison research is divided into two parts. First, we demonstrate how to compare our methodology by addressing a mathematical instance utilizing the BHFSS structure, which acts as a foundation for evaluating alignments. In addition, we contain information gathered from the most recent research into our BHFSS structure to compare our method's ranks to those obtained using existing CoCoSo aggregation methods.

The first section compared our BHFSS-based approach to the CoCoSo technique, and the findings show remarkable consistency. This alignment illustrates the efficacy and resilience of our method within the BHFSS architecture. The reliable findings support the credibility of our procedure, demonstrating that it delivers outcomes equivalent to what was obtained through the CoCoSo technique. This concurrence emphasizes the usefulness and applicability of our BHFSS approach, showing its capacity to deal with complicated choices in a manner comparable to the recognized CoCoSo methodology.

11. Conclusions

In this study, the article examined the MCDM issues using hesitant bipolar fuzzy soft information. We have worked on BHFSSs, introducing aggregation operators by using geometric and arithmetic operations. These operators' prominent characteristics are examined. The BHFS multiple attribute decision-making issues have then been approached in different ways by using these operators. With the help of a case study on energy sustainability in society, the BHFSS's practical applicability is presented.

Furthermore, a comparison study with the CoCoSo method is executed. To manage vagueness and imprecision in a diversity of fields and provide better informed and efficient decision-making, BHFSS will be an essential tool in the future. It will be used in many altered industries, such as energy, transportation, medical care, and smart cities. However, the computational difficulty of BHFSS and its handling of insufficient reluctance causes problems. Future studies should focus on developing enhanced techniques and examining larger applications to improve decision-making ability.

Despite all the advantages it presents, BHFSS is not without its limitations. Its computationally intensive nature makes it difficult to apply to more industrial-scale applications, and in certain modeling scenarios, the models can be very cautious, which tends to reduce the outcome accuracy. As has usually been the case with fuzzy logic theories with such bipolar connotations, there are certain concepts that may be a little difficult to follow for one who does not understand bipolar fuzzy logic, thereby limiting the applicability of this model. However, the sensitivity of input data may be a challenge to the framework as inaccuracies in the data can have a significant impact on the outcome. Moreover, several restrictions on its application may limit BHFSS to only a few problems in practice, thereby weakening the effectiveness of the model within broader decision-making contexts.

Author Contributions

Zaheer Ahmad: Conceptualization; Validation, Analysis, Investigation, Review and editing; Shahzaib Ashraf: Conceptualization, Methodology, Writing-original draft; Shawana khan: Investigation, Review and editing and Supervision; Mehdi Tlija: Analysis, Investigation, Review and editing; Chiranjibe Jana: Conceptualization, Methodology, Supervision, Review and editing; Dragan Pamucar: Investigation, Review and editing, and Supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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Appendix.

List of symbols are given in Table 13.

Table 13. Symbols list.

Sr.	Full Name	Symbol
1	Fixed Set	$\mathcal{U}$
2	Membership Degree (MD)	$\tilde{\mu}$
3	Positive MD	$\tilde{\mu}^{+}$
4	Negative MD	$\tilde{\mu}^{-}$
5	Hesitant Fuzzy Element	$\tilde{h}$
6	Scoring Function	$\delta(\tilde{h})$
7	Accuracy Function	$a(\tilde{h})$
8	Number of elements	ℓ
9	Weights	w
10	Weights of Experts	ψ

Appendix A.

Proof. We use mathematical induction in order to obtain the required result.

For $n = 1$ $\sum\limits_{\beth=1}^{n} w_{\beth} = 1$. By Definition 3.1

$$BHFSWA(\tilde{h_{e_{11}}}, \tilde{h_{e_{12}}}, ..., \tilde{h_{e_{1m}}}) = \bigoplus_{\shortparallel=1}^{m} \psi_{\shortparallel} \tilde{h}_{e_{1\shortparallel}}$$

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{+}_{e_{11}}, ..., \tilde{\mu}_{1m} \in \tilde{h}^{+}_{e_{1m}}} 1 - \prod\limits_{\shortparallel=1}^{m} (\prod\limits_{\beth=1}^{1} (1 - \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}})^{w_{\beth}})^{\psi_{\shortparallel}}, \\ \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{-}_{e_{11}}, ..., \tilde{\mu}_{1m} \in \tilde{h}^{-}_{e_{1m}}} -(\prod\limits_{\shortparallel=1}^{m} (\prod\limits_{\beth=1}^{1} (|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|)^{w_{\beth}})^{\psi_{\shortparallel}}) \end{array} \right\}.$$

Likewise, for $m = 1$, we have $\sum\limits_{\shortparallel=1}^{m} \psi_{\shortparallel} = 1$ so that,

$$BHFSWA(\tilde{h_{e_{11}}}, \tilde{h_{e_{12}}}, ..., \tilde{h_{e_{n1}}}) = \bigoplus_{\beth=1}^{n} w_{\beth} \tilde{h}_{e_{\beth 1}}$$

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{+}_{e_{11}}, \ldots, \tilde{\mu}_{n1} \in \tilde{h}^{+}_{e_{n1}}} 1 - \prod_{\shortparallel=1}^{1} (\prod_{\beth=1}^{n} (1 - \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}})^{w_{\beth}})^{\psi_{\shortparallel}}, \\ \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{-}_{e_{11}}, \ldots, \tilde{\mu}_{n1} \in \tilde{h}^{-}_{e_{n1}}} -(\prod_{\shortparallel=1}^{1} (\prod_{\beth=1}^{n} (|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|)^{w_{\beth}})^{\psi_{\shortparallel}}) \end{array} \right\}.$$

Thus, the conclusion is true for both $n = 1$ and $m = 1$. Now, suppose the result is true for $m = c_1 + 1$, $n = c_2$, and $m = c_1$, $n = c_2 + 1$, *i.e.*,

$$= \bigoplus_{\shortparallel=1}^{c_1+1} \psi_{\shortparallel} (\bigoplus_{\beth=1}^{c_2} w_{\beth} \tilde{h}_{e_{\beth\shortparallel}})$$

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{+}_{e_{11}}, \ldots, \tilde{\mu}_{(c_2)(c_1+1)} \in \tilde{h}^{+}_{e_{(c_2)(c_1+1)}}} 1 - \prod_{\shortparallel=1}^{c_1+1} (\prod_{\beth=1}^{c_2} (1 - \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}})^{w_{\beth}})^{\psi_{\shortparallel}}, \\ \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{-}_{e_{11}}, \ldots, \tilde{\mu}_{(c_2)(c_1+1)} \in \tilde{h}^{-}_{e_{(c_2)(c_1+1)}}} -(\prod_{\shortparallel=1}^{c_1+1} (\prod_{\beth=1}^{c_2} (|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|)^{w_{\beth}})^{\psi_{\shortparallel}}) \end{array} \right\}.$$

and

$$= \bigoplus_{\shortparallel=1}^{c_1} \psi_{\shortparallel} (\bigoplus_{\beth=1}^{c_2+1} w_i \tilde{h}_{e_{\beth\shortparallel}})$$

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{+}_{e_{11}}, \ldots, \tilde{\mu}_{(c_2+1)(c_1)} \in \tilde{h}^{+}_{e_{(c_2+1)(c_1)}}} 1 - \prod_{\shortparallel=1}^{c_1} (\prod_{\beth=1}^{c_2+1} (1 - \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}})^{w_{\beth}})^{\psi_{\shortparallel}}, \\ \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{-}_{e_{11}}, \ldots, \tilde{\mu}_{(c_2+1)(c_1)} \in \tilde{h}^{-}_{e_{(c_2+1)(c_1)}}} -(\prod_{\shortparallel=1}^{c_1} (\prod_{\beth=1}^{c_2+1} (|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|)^{w_{\beth}})^{\psi_{\shortparallel}}) \end{array} \right\}.$$

Now, for $m = c_1 + 1$, $n = c_2 + 1$, we have,

$$= \bigoplus_{\shortparallel=1}^{c_1+1} \psi_{\shortparallel} (\bigoplus_{\beth=1}^{c_2+1} w_{\beth} \tilde{h}_{e_{\beth\shortparallel}}) = \bigoplus_{\shortparallel=1}^{c_1+1} \psi_{\shortparallel} (\bigoplus_{\beth=1}^{c_2} w_{\beth} \tilde{h}_{e_{\beth\shortparallel}} \oplus w_{c_2+1} \tilde{h}_{e_{(c_2+1)\shortparallel}}) = \bigoplus_{\shortparallel=1}^{c_1+1} \bigoplus_{\beth=1}^{c_2} \psi_{\shortparallel} w_{\beth} \tilde{h}_{e_{\beth\shortparallel}} \bigoplus_{\shortparallel=1}^{c_1+1} \psi_{\shortparallel} w_{c_2+1} \tilde{h}_{e_{(c_2+1)\shortparallel}}$$

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{+}_{e_{11}}, \ldots, \tilde{\mu}_{(c_2)(c_1+1)} \in \tilde{h}^{+}_{e_{(c_2)(c_1+1)}}} 1 - \prod_{\shortparallel=1}^{c_1+1} (\prod_{\beth=1}^{c_2} (1 - \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}})^{w_{\beth}})^{\psi_{\shortparallel}} \oplus \\ \bigcup_{\tilde{\mu}_{1} \in \tilde{h}^{+}_{e_{1}}, \ldots, \tilde{\mu}_{(c_1+1)} \in \tilde{h}^{+}_{e_{(c_1+1)}}} 1 - \prod_{\shortparallel=1}^{c_1+1} ((1 - \tilde{\mu}^{+}_{\tilde{h}_{e((c_2+1)\shortparallel)}})^{w_{c_2+1}})^{\psi_{\shortparallel}}, \\ \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{-}_{e_{11}}, \ldots, \tilde{\mu}_{c_2)(c_1+1)} \in \tilde{h}^{-}_{e_{(c_2)(c_1+1)}}} -(\prod_{\shortparallel=1}^{c_1+1} (\prod_{\beth=1}^{c_2} (|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|)^{w_{\beth}})^{\psi_{\shortparallel}} \oplus \\ \bigcup_{\tilde{\mu}_{1} \in \tilde{h}^{-}_{e_{1}}, \ldots, \tilde{\mu}_{(c_1+1)} \in \tilde{h}^{-}_{e_{(c_1+1)}}} - \prod_{\shortparallel=1}^{c_1+1} ((|\tilde{\mu}^{-}_{\tilde{h}_{e((c_2+1)\shortparallel)}}|)^{w_{c_2+1}})^{\psi_{\shortparallel}} \end{array} \right\},$$

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{+}_{e_{11}}, \ldots, \tilde{\mu}_{(c_2+1)(c_1+1)} \in \tilde{h}^{+}_{e_{(c_2+1)(c1+1)}}} 1 - \prod_{\shortparallel=1}^{c_1+1} (\prod_{\beth=1}^{c_2+1} (1 - \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}})^{w_{\beth}})^{\psi_{\shortparallel}}, \\ \bigcup_{\tilde{\mu}_{11} \in \tilde{h}^{-}_{e_{11}}, \ldots, \tilde{\mu}_{(c_2+1)(c_1+1)} \in \tilde{h}^{-}_{e_{c_2+1)(c_1+1)}}} -(\prod_{\shortparallel=1}^{c_1+1} (\prod_{\beth=1}^{c_2+1} (|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|)^{w_{\beth}})^{\psi_{\shortparallel}}) \end{array} \right\}.$$

Hence, true for $m = c_1 + 1, n = c_2 + 1$. Thus, the result is true for n, m ≥ 1. Also, $0 \leq \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}} \leq 1$

$\Leftrightarrow 0 \leq \bigcup_{\tilde{\mu} \in \tilde{h}_{e}^{+}} 1 - \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}} \leq 1$

$\Leftrightarrow 0 \leq \bigcup_{\tilde{\mu}_1 \in \tilde{h}^{+}_{e_1}, \ldots, \tilde{\mu}_n \in \tilde{h}^{+}_{e_n}} \prod_{\beth=1}^{n} (1 - \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}})^{w_{\beth}} \leq 1$

$\Leftrightarrow 0 \leq \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^{+}_{e11},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^{+}_{enm}} \prod_{\gimel=1}^{m} (\prod_{\beth=1}^{n} (1-\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\gimel)}})^{w_\beth})^{\psi_\gimel} \leq 1$

$\Leftrightarrow 0 \leq \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^{+}_{e11},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^{+}_{enm}} 1-\prod_{\gimel=1}^{m} (\prod_{\beth=1}^{n} (1-\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\gimel)}})^{w_\beth})^{\psi_\gimel} \leq 1.$

And $-1 \leq \bigcup_{\tilde{\mu}\in\tilde{h}^{-}_{e}} -|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\gimel)}}| \leq 0$

$\Leftrightarrow -1 \leq \bigcup_{\tilde{\mu}_{1}\in\tilde{h}^{-}_{e1},\ldots,\tilde{\mu}_{n}\in\tilde{h}^{-}_{en}} - \prod_{\beth=1}^{n} (|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\gimel)}}|)^{w_\beth} \leq 0$

$\Leftrightarrow -1 \leq \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^{-}_{e11},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^{-}_{enm}} - \prod_{\gimel=1}^{m} (\prod_{p=1}^{n} (|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\gimel)}}|)^{w_\beth})^{\psi_\gimel} \leq 0.$

Finally,

$$-1 \leq \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^{+}_{e11},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^{+}_{enm}} 1-\prod_{\gimel=1}^{m}(\prod_{\beth=1}^{n}(1-\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\gimel)}})^{w_\beth})^{\psi_\gimel} \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^{-}_{e11},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^{-}_{enm}} -\prod_{\gimel=1}^{m}(\prod_{\beth=1}^{n}(|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\gimel)}}|)^{w_\beth})^{\psi_\gimel} \tag{11.1}$$

$$= \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^{+}_{e11},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^{+}_{enm}} 1-\prod_{\gimel=1}^{m}(\prod_{\beth=1}^{n}(1-\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\gimel)}})^{w_\beth})^{\psi_\gimel} \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^{-}_{e11},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^{-}_{enm}} -\prod_{\gimel=1}^{m}(\prod_{\beth=1}^{n}(|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\gimel)}}|)^{w_\beth})^{\psi_\gimel} \leq 1. \tag{11.2}$$

So, once more, a BHFSNs is the aggregate value that BHFSWA produced. □

Appendix B.

Proof. As $\tilde{h}_{e_{\beth\gimel}} = \tilde{h}_{e} = (\tilde{h}^{+}_{e_{\beth\gimel}}, \tilde{h}^{-}_{e_{\beth\gimel}})\ \forall\ \beth, \gimel$ so that

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^{+}_{e11},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^{+}_{enm}} 1-\prod_{\gimel=1}^{m}(\prod_{\beth=1}^{n}(1-\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\gimel)}})^{w_\beth})^{\psi_\gimel}, \\ \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^{-}_{e11},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^{-}_{enm}} -(\prod_{\gimel=1}^{m}(\prod_{\beth=1}^{n}(|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\gimel)}}|)^{w_\beth})^{\psi_\gimel}) \end{array} \right\}.$$

$=(\bigcup_{\tilde{\mu}^{+}\in\tilde{h}^{+}_{e}}\{(1-(1-\tilde{\mu}^{+}_{\tilde{h}_{e}})\}, \bigcup_{\tilde{\mu}^{-}\in\tilde{h}^{-}_{e}}\{-|\tilde{\mu}^{-}_{\tilde{h}_{e}}|\}) = \tilde{h}_{e} = (\tilde{h}^{+}_{e_{\beth\gimel}}, \tilde{h}^{-}_{e_{\beth\gimel}})$, which completes the prove. □

Appendix C.

Proof. By Definition 2.17, we have

$\lambda\tilde{h}_{e(\beth\gimel)} = (\bigcup_{\mu_{\tilde{\beth}\gimel}^{+}\in h^{+}_{e(\beth\gimel)}}\{(1-(1-\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\gimel)}})^{\lambda}\}, \bigcup_{\mu_{\tilde{\beth}\gimel}^{-}\in h^{-}_{e(\beth\gimel)}}\{-|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\gimel)}}|^{\lambda}\})$, thus,

$BHFSWA(\lambda\tilde{h}_{e_{11}}, \lambda\tilde{h}_{e_{12}}, \ldots, \lambda\tilde{h}_{e_{nm}})=$

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^{+}_{e11},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^{+}_{enm}} 1-\prod_{\gimel=1}^{m}(\prod_{\beth=1}^{n}(1-\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\gimel)}})^{\lambda w_\beth})^{\psi_\gimel}, \\ \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^{-}_{e11},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^{-}_{enm}} -(\prod_{\gimel=1}^{m}(\prod_{\beth=1}^{n}(|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\gimel)}}|)^{\lambda w_\beth})^{\psi_\gimel}) \end{array} \right\},$$

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^{+}_{e11},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^{+}_{enm}} 1-(\prod_{\gimel=1}^{m}(\prod_{\beth=1}^{n}(1-\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\gimel)}})^{w_\beth})^{\psi_\gimel})^{\lambda}, \\ \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^{-}_{e11},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^{-}_{enm}} -(\prod_{\gimel=1}^{m}(\prod_{\beth=1}^{n}(|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\gimel)}}|)^{w_\beth})^{\psi_\gimel})^{\lambda} \end{array} \right\},$$

$$=\lambda\{BHFSWA(\tilde{h}_{e_{11}}, \tilde{h}_{e_{12}}, \ldots, \tilde{h}_{e_{nm}})\},$$

which completes the proof. □

Appendix D.

Proof. Since $\tilde{h}_{e_{\beth\shortparallel}} = (\tilde{h}^{+}_{e_{\beth\shortparallel}}, \tilde{h}^{-}_{e_{\beth\shortparallel}})$, is a BHFSN, $\min_{\shortparallel}\min_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\} \leq \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}} \leq \max_{\shortparallel}\max_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\}$, which implies that, $1 - \max_{\shortparallel}\max_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\} \leq 1 - \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}} \leq 1 - \min_{\shortparallel}\min_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\}$

$$\Leftrightarrow (1 - \max_{\shortparallel}\max_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\})^{w_\beth} \leq (1 - \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}})^{w_\beth} \leq (1 - \min_{\shortparallel}\min_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\})^{w_\beth}$$

$$\Leftrightarrow (1 - \max_{\shortparallel}\max_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\})^{\sum_{\beth=1}^{n} w_\beth} \leq \prod_{\beth=1}^{n} (1 - \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}})^{w_\beth} \leq (1 - \min_{\shortparallel}\min_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\})^{\sum_{\beth=1}^{n} w_\beth}$$

$$\Leftrightarrow (1 - \max_{\shortparallel}\max_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\}) \leq \prod_{\beth=1}^{n} (1 - \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}})^{w_\beth} \leq (1 - \min_{\shortparallel}\min_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\})$$

$$\Leftrightarrow (1 - \max_{\shortparallel}\max_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\})^{\sum_{\shortparallel=1}^{m} \psi_\beth} \leq \prod_{\shortparallel=1}^{m} (\prod_{\beth=1}^{n} (1 - \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}})^{w_\beth})^{\psi_\shortparallel} \leq (1 - \min_{\shortparallel}\min_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\})^{\sum_{\shortparallel=1}^{m} \psi_\beth}$$

$$\Leftrightarrow (1 - \max_{\shortparallel}\max_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\}) \leq \prod_{\shortparallel=1}^{m} (\prod_{\beth=1}^{n} (1 - \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}})^{w_\beth})^{\psi_\shortparallel} \leq (1 - \min_{\shortparallel}\min_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\})^{\sum_{\beth=1}^{n} w_\beth}.$$

Hence,

$$\min_{\shortparallel}\min_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\} \leq 1 - \prod_{\shortparallel=1}^{m} (\prod_{\beth=1}^{n} (1 - \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}})^{w_\beth})^{\psi_\shortparallel} \leq \max_{\shortparallel}\max_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\}. \tag{11.3}$$

Similarly, $-\min_{\shortparallel}\min_{\beth}\{|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|\} \leq -|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}| \leq -\max_{\shortparallel}\max_{\beth}\{|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|\}$ .This means,

$$(- \min_{\shortparallel}\min_{\beth}\{|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|\})^{\sum_{\beth=1}^{n} w_\beth} \leq - \prod_{\beth=1}^{n} (|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|)^{w_\beth} \leq (- \max_{\shortparallel}\max_{\beth}\{|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|\})^{\sum_{\beth=1}^{n} w_\beth}$$

$$\Leftrightarrow (- \min_{\shortparallel}\min_{\beth}\{|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|\}) \leq - \prod_{\beth=1}^{n} (|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|)^{w_\beth} \leq (- \max_{\shortparallel}\max_{\beth}\{|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}\}|)$$

$$\Leftrightarrow (- \min_{\shortparallel}\min_{\beth}\{|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|\})^{\psi_\shortparallel} \leq (- \prod_{\beth=1}^{n} (|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|)^{w_\beth})^{\psi_\shortparallel} \leq (- \max_{\shortparallel}\max_{\beth}\{|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|\})^{\psi_\shortparallel}$$

$$\Leftrightarrow (- \min_{\shortparallel}\min_{\beth}\{|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|\})^{\sum_{\shortparallel=1}^{m} \psi_\shortparallel} \leq - \prod_{\shortparallel=1}^{m} (\prod_{\beth=1}^{n} (|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|)^{w_\beth})^{\psi_\shortparallel} \leq (- \max_{\shortparallel}\max_{\beth}\{|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|\})^{\sum_{\shortparallel=1}^{m} \psi_\shortparallel}.$$

Therefore,

$$\Leftrightarrow (- \min_{\shortparallel}\min_{\beth}\{|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|\}) \leq - \prod_{\shortparallel=1}^{m} (\prod_{\beth=1}^{n} (|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|)^{w_\beth})^{\psi_\shortparallel} \leq (- \max_{\shortparallel}\max_{\beth}\{|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|\}). \tag{11.4}$$

Let $\alpha = BHFSWA(\tilde{h}_{e_{11}}, \tilde{h}_{e_{12}}, ..., \tilde{h}_{e_{nm}}) = (\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)\alpha}}, \tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)\alpha}})$, then from Eqs 16 and 17, $\min_{\shortparallel}\min_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\} \leq \tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}} \leq \max_{\shortparallel}\max_{\beth}\{\tilde{\mu}^{+}_{\tilde{h}_{e(\beth\shortparallel)}}\}$ and $-\min_{\shortparallel}\min_{\beth}\{|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|\} \leq -|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}| \leq -\max_{\shortparallel}\max_{\beth}\{|\tilde{\mu}^{-}_{\tilde{h}_{e(\beth\shortparallel)}}|\}$. Hence, by the definition of 2.9, we get,

$$\tilde{h}^{-}_{e_{\beth\shortparallel}} \leq BHFSWA(\tilde{h}_{e_{11}}, \tilde{h}_{e_{12}}, ..., \tilde{h}_{e_{nm}}) \leq \tilde{h}^{+}_{e_{\beth\shortparallel}}. \tag{11.5}$$

Hence, the property holds. □

Appendix E.

Proof. As $\tilde{h}_{e_{\beth\shortparallel}} = \tilde{h}_e = (\tilde{h}^+_{e_{\beth\shortparallel}}, \tilde{h}^-_{e_{\beth\shortparallel}})\ \forall\ \beth, \shortparallel$, so that

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^+_{e_{11}},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^+_{e_{nm}}} \prod_{\shortparallel=1}^{m}(\prod_{\beth=1}^{n}(\tilde{\mu}^+_{\tilde{h}_{e(\beth\shortparallel)}})^{w_\beth})^{\psi_\shortparallel}, \\ \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^-_{e_{11}},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^-_{e_{nm}}} -1 + \prod_{\shortparallel=1}^{m}(\prod_{\beth=1}^{n}(1+\tilde{\mu}^-_{\tilde{h}_{e(\beth\shortparallel)}})^{w_\beth})^{\psi_\shortparallel} \end{array} \right\},$$

$$=(\bigcup_{\tilde{\mu}^+\in\tilde{h}^+_e}\{\tilde{\mu}^+_{\tilde{h}_e}\}, \bigcup_{\tilde{\mu}^-\in\tilde{h}^-_e}\{-1+(1+\tilde{\mu}^-_{\tilde{h}_e})\} = \tilde{h}_e = (\tilde{h}^+_{e_{\beth\shortparallel}}, \tilde{h}^-_{e_{\beth\shortparallel}}),$$

which completes the prove. □

Appendix F.

Proof. By Definition 2.17 we have $\tilde{h}^\lambda_{e(\beth\shortparallel)} = (\bigcup_{\mu_{\tilde{\beth}_\shortparallel}{}^+\in h^+_{\tilde{e}(\beth\shortparallel)}}\{(\tilde{\mu}^+_{\tilde{h}_{e(\beth\shortparallel)}})^\lambda\}, \bigcup_{\mu_{\tilde{\beth}_\shortparallel}{}^-\in h^-_{\tilde{e}(\beth\shortparallel)}}\{(-1+(1+\tilde{\mu}^+_{\tilde{h}_{e(\beth\shortparallel)}})^\lambda\})$ thus,

$BHFSWG(\tilde{h}^\lambda_{e_{11}}, \tilde{h}^\lambda_{e_{12}}, \ldots, \tilde{h}^\lambda_{e_{nm}})$=

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^+_{e_{11}},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^+_{e_{nm}}} \prod_{\shortparallel=1}^{m}(\prod_{\beth=1}^{n}(\tilde{\mu}^+_{\tilde{h}_{e(\beth\shortparallel)}})^{\lambda w_\beth})^{\psi_\shortparallel}, \\ \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^-_{e_{11}},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^-_{e_{nm}}} -1 + \prod_{\shortparallel=1}^{m}(\prod_{\beth=1}^{n}(1+\tilde{\mu}^-_{\tilde{h}_{e(\beth\shortparallel)}})^{\lambda w_\beth})^{\psi_\shortparallel} \end{array} \right\},$$

$$= \left\{ \begin{array}{c} \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^+_{e_{11}},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^+_{e_{nm}}} (\prod_{\shortparallel=1}^{m}(\prod_{\beth=1}^{n}(\tilde{\mu}^+_{\tilde{h}_{e(\beth\shortparallel)}})^{w_\beth})^{\psi_\shortparallel})^\lambda, \\ \bigcup_{\tilde{\mu}_{11}\in\tilde{h}^-_{e_{11}},\ldots,\tilde{\mu}_{nm}\in\tilde{h}^-_{e_{nm}}} -1 + (\prod_{\shortparallel=1}^{m}(\prod_{\beth=1}^{n}(1+\tilde{\mu}^-_{\tilde{h}_{e(\beth\shortparallel)}})^{w_\beth})^{\psi_\shortparallel})^\lambda \end{array} \right\},$$

$$=\{BHFSWG(\tilde{h}_{e_{11}}, \tilde{h}_{e_{12}}, \ldots, \tilde{h}_{e_{nm}})\}^\lambda,$$

which completes the prove. □

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