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Research article

Persistence and stability in an SVIR epidemic model with relapse on timescales

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Abstract: In this paper, an Susceptible-Vaccinated-Infected-Recovered (SVIR) epidemic model incorporating relapse dynamics on a timescale was studied. Using the dynamic inequalities: $\mathsf{S}(\mathtt{r}) \leq \alpha^{\mathsf{U}}/(\alpha^{\mathsf{L}}+\gamma^{\mathsf{L}})+\epsilon$, $\mathsf{V}(\mathtt{r}) \leq \gamma^{\mathsf{U}}\ell_{11}/(\alpha^{\mathsf{L}}+\delta_1^{\mathsf{L}})+\epsilon$, $\mathsf{I}(\mathtt{r}) \leq \alpha^{\mathsf{U}}/\alpha^{\mathsf{L}}+\epsilon$, $\mathsf{R}(\mathtt{r}) \leq (\delta_1^{\mathsf{U}}\ell_{12}+\delta^{\mathsf{U}}\ell_{13})/(\alpha^{\mathsf{L}}+\mathsf{d}^{\mathsf{L}})+\epsilon$, $\mathsf{S}(\mathtt{r}) \geq \alpha^{\mathsf{L}}/(\alpha^{\mathsf{U}}+\beta^{\mathsf{U}}\ell_1+\gamma^{\mathsf{U}})+\epsilon$, $\mathsf{V}(\mathtt{r}) \geq \gamma^{\mathsf{L}}\ell_0/(\alpha^{\mathsf{U}}+\beta_1^{\mathsf{U}}\ell_1+\delta_1^{\mathsf{U}})+\epsilon$, $\mathsf{I}(\mathtt{r}) \geq \mathsf{d}^{\mathsf{L}}\ell_{03}/(\delta^{\mathsf{U}}+\alpha^{\mathsf{U}})+\epsilon$, $\mathsf{R}(\mathtt{r}) \geq \delta_1^{\mathsf{L}}\ell_{02}/(\alpha^{\mathsf{U}}+\mathsf{d}^{\mathsf{U}})+\epsilon$, and constructing an appropriate Lyapunov functional, sufficient conditions were determined to guarantee the permanence of the system. Additionally, the existence, uniqueness, and uniform asymptotic stability of globally attractive, almost periodic positive solutions were derived. Furthermore, an in-depth analysis highlighted the significance of relapse dynamics. Numerical simulations were included to validate the system's permanence, demonstrating that the disease persists under certain conditions. These simulations revealed that vaccination and relapse dynamics played a crucial role in controlling the epidemic. Specifically, as long as the infected population remained smaller than the susceptible population, the infection was controlled, keeping both the infected and recovered populations low. Their oscillatory behavior suggested that periodic vaccinations may be key to stabilizing disease dynamics. This study underscored the applicability of the proposed model in providing a robust theoretical foundation for understanding and managing the spread of infectious diseases.

Keywords: SVIR model; epidemic model; timescale; almost periodic positive solution
Mathematics Subject Classification: 34A12, 92D30, 34N05

1. Introduction

Diseases caused by pathogens are triggered by harmful microbes, including bacteria, viruses, fungi, and parasitic organisms, and can spread through direct or indirect contact between humans and animals. Despite tremendous advances in medical knowledge, these diseases continue to be a major cause of death globally and provide significant obstacles to economies and public health. In order to study the spread of these diseases and direct the development of control measures, mathematical frameworks have become extremely effective tools. Numerous studies have provided insightful information on epidemic modeling, including the work of Alsakaji et al. [1], who analyzed the stochastic dynamics of COVID-19 in the UAE using an extended SEIR epidemic model incorporating vaccination, time delays, and random noise. Their study demonstrated that stochastic perturbations and time delays significantly impact disease persistence and extinction. Chang et al. [2] investigated cross-diffusion-induced patterns in an SIR epidemic model on complex networks, highlighting how susceptible individuals avoid infected ones and providing insights into pattern formations and Turing instability in epidemiological networks. Li and Zhang [3] explored the dynamic behaviors of a modified SIR model with nonlinear incidence and recovery rates, revealing the existence of backward bifurcation and showing that the basic reproduction number is not always a threshold parameter, emphasizing the impact of hospital bed availability on disease control.

In recent years, various types of equations—differential, difference, and dynamic equations on measure chains—have emerged as essential tools for modeling diverse processes. Among these, dynamic equations on timescales, introduced by Stefan Hilger in his 1988 doctoral dissertation [4], stand out for their versatility and unifying nature. This theory bridges the gap between continuous and discrete systems, extending classical results while providing a framework for hybrid models.

Timescales, defined as closed, nonempty subsets of real numbers, generalize time into a single domain where continuous and discrete phenomena coexist. For instance, $\mathbb{T} = \mathbb{R}$ represents continuous time, $\mathbb{T} = \mathbb{Z}$ corresponds to discrete time, and hybrid combinations like $\mathbb{T} = \mathbb{R} \cup \mathbb{Z}$ model systems with both characteristics. This flexibility is invaluable for real-world applications where systems exhibit hybrid temporal behaviors, such as in population dynamics, control systems, and financial modeling. Central to this framework are the forward and backward jump operators, σ and ρ, which identify the smallest and largest points in the timescale greater or smaller than a given point, respectively. These operators are mathematically defined as

$$\sigma(t) = \inf\{\xi \in \mathbb{T} : \xi > t\}, \quad \rho(t) = \sup\{\xi \in \mathbb{T} : \xi < t\}.$$

The graininess function, $\mu(t) = \sigma(t) - t$, quantifies the spacing between consecutive points in the timescale, enabling the characterization of its discrete or continuous nature.

For example, points in a timescale can be categorized based on their neighborhood. A point $t \in \mathbb{T}$ is termed right-scattered if $\sigma(t) > t$, while it is left-scattered if $\rho(t) < t$. Conversely, a point is right-dense if $\sigma(t) = t$ and left-dense if $\rho(t) = t$. Points that are scattered on both sides are called isolated, while those dense on both sides are termed dense. Such classifications are instrumental in analyzing the behavior of dynamic systems on timescales.

In addition, the timescale's structure can be refined through operations like removing maximal or minimal elements. For instance, the non-maximal subset, $\mathbb{T}^\kappa$, is derived as follows:

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Similarly, repeated derivations such as $\mathbb{T}^{\kappa^2}$ and $\mathbb{T}^{\kappa^n}$ provide a hierarchical structure for analyzing timescales.

The graininess and backward graininess functions play a vital role in this theory. While the graininess function $\mu(t)$ measures the forward spacing, the backward graininess function, defined as $\nu(t) = t - \rho(t)$, captures the backward spacing, offering a comprehensive understanding of the timescale's geometry.

Furthermore, intervals within a timescale, such as $[c, d]_{\mathbb{T}} = \{t \in \mathbb{T} : c \le t \le d\}$, allow for precise definitions of domains over which dynamic equations are studied. This flexibility facilitates the modeling of hybrid systems, where both continuous and discrete behaviors coexist seamlessly.

We also define the concept of delta differentiation. A function u is said to be delta differentiable at $t \in \mathbb{T}^\kappa$ if there exists a number, denoted by $u^\Delta(t)$, such that for any $\varepsilon > 0$, there exists a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) satisfying

$$\left|(u(\sigma(t)) - u(s)) - u^\Delta(t)(\sigma(t) - s)\right| \le \varepsilon|\sigma(t) - s|$$

for all $s \in U$. The number $u^\Delta(t)$ is called the delta derivative of u at t. Moreover, if $u^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$, then u is delta differentiable on $\mathbb{T}^\kappa$, and the function $u^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is referred to as the (delta) derivative of u on $\mathbb{T}^\kappa$.

This unifying concept of differentiation extends the classical derivative to timescales, enabling the study of systems that evolve continuously, discretely, or in hybrid forms. For instance, in population dynamics, the growth of insect populations serves as a prime example. During favorable seasons, insects grow and reproduce continuously, while in colder months, populations decline, leaving dormant eggs that hatch in distinct, nonoverlapping generations. Classical differential or difference equations cannot fully capture such phenomena. However, timescale calculus, with its delta derivative, accommodates both continuous growth phases and discrete generational transitions seamlessly, offering more accurate and meaningful models.

Since Hilger's pioneering work, researchers have made notable contributions to this field [5–9]. Bohner and Peterson have advanced its theoretical foundations, while others have applied it to areas like control theory, signal processing, and quantum mechanics. These advancements underscore the broad applicability and potential of timescale calculus in addressing problems across scientific disciplines.

Jie et al. [10] investigated the qualitative dynamics of an Susceptible-Infected-Susceptible (SIS) model in 2010 that was represented by the equations:

$$\begin{aligned} \mathsf{S}^\Delta(t) &= \mathsf{I}(t)\Big[-\frac{\mathsf{a}}{\mathsf{N}}\mathsf{S}(t) + \mathsf{b}\Big], \quad \mathsf{S}(t) \ge 0, \\ \mathsf{I}^\Delta(t) &= \mathsf{I}(t)\Big[\frac{\mathsf{a}}{\mathsf{N}}\mathsf{S}(t) - \mathsf{b}\Big], \quad \mathsf{I}(t) \ge 0. \end{aligned}$$

In a different study later that year [11], they showed that the dynamic behavior of the system might change dramatically, going from relatively straightforward with steady-state solutions in continuous time to more intricate and chaotic behavior in discrete-time situations.

In 2016, Bohner and Streipert [12] analyzed an SIS model represented by the following system of equations:

$$\begin{aligned} \mathsf{S}^{\Delta}(t) &= \mathsf{I}(t)\left[-\mathsf{a}\mathsf{S}(\sigma(t)) + \mathsf{b}\right], \quad \mathsf{S}(t) > 0, \\ \mathsf{I}^{\Delta}(t) &= \mathsf{I}(t)\left[\mathsf{a}\mathsf{S}(\sigma(t)) - \mathsf{b}\right], \quad \mathsf{I}(t) \geq 0, \end{aligned}$$

where $\mathsf{a} > 0$ and $\mathsf{b} > 0$. They focused on investigating the stability properties of the model's steady states. Subsequently, Bohner, Streipert, and Torres studied the stability analysis of a time-dependent SIR system in [13], characterized by the following equations:

$$\begin{aligned} \mathsf{S}^{\Delta}(t) &= -\frac{\mathsf{a}(t)\mathsf{S}(t)\mathsf{I}(\sigma(t))}{\mathsf{S}(t) + \mathsf{I}(t)}, \\ \mathsf{I}^{\Delta}(t) &= \frac{\mathsf{a}(t)\mathsf{S}(t)\mathsf{I}(\sigma(t))}{\mathsf{S}(t) + \mathsf{I}(t)} - \mathsf{b}(t)\mathsf{I}(\sigma(t)), \\ \mathsf{R}^{\Delta}(t) &= \mathsf{b}(t)\mathsf{R}(\sigma(t)), \quad \mathsf{S}(t), \mathsf{I}(t) > 0. \end{aligned}$$

In practical scenarios, the almost periodic changes in surroundings significantly influence various living and ecological processes, often being more prevalent and encompassing than strictly periodic fluctuations. The notion of almost periodic timescales was introduced by Li and Wang [14]. Further studies on this concept can be found in [15, 16].

The stability analysis for a discrete SIS system, represented by the following system, was examined by Bohner and Streipert [17] in a recent study:

$$\begin{aligned} \Delta\mathsf{S}(t) &= -a(t)\mathsf{S}(t+1)\mathsf{I}(t) + \mathsf{b}(t)\mathsf{I}(t), \\ \Delta\mathsf{I}(t) &= a(t)\mathsf{S}(t+1)\mathsf{I}(t) - \mathsf{b}(t)\mathsf{I}(t). \end{aligned}$$

2. Model framework and basic concepts

Within this part, the mathematical framework that captures the dynamics of a population is presented, considering the interactions between susceptible, vaccinated, infected, and recovered individuals. The model accounts for various processes such as recruitment, disease transmission, recovery, relapse, and immunity. Building upon the continuous model explored in earlier studies like [18], we extend it to a timescale framework. This extension combines both continuous and discrete time dynamics, offering a more flexible approach to understanding the population's behavior over time.

The population dynamics are studied within the framework of the timescale domain $\mathbb{T}$, where $\mathtt{r} \in \mathbb{T}$ denotes a variable representing time. The population is categorized into four groups: Susceptible individuals $\mathsf{S}(\mathtt{r})$, vaccinated individuals $\mathsf{V}(\mathtt{r})$, infected individuals $\mathsf{I}(\mathtt{r})$, and recovered individuals $\mathsf{R}(\mathtt{r})$. The dynamics governing the transitions between these groups are depicted in Figure 1.

Taking into account the previous discussion, the following postulates are made regarding the group and the progression of the disease:

(1) The disease is transmitted through both direct contact between susceptible and infected individuals, as well as through vaccinated individuals, where the effectiveness of vaccination is modulated by $\beta_1(\mathtt{r})$.
(2) The recruitment rate $\alpha(\mathtt{r})$ and recovery rate $\gamma(\mathtt{r})$ are assumed to be periodic and can change over time.

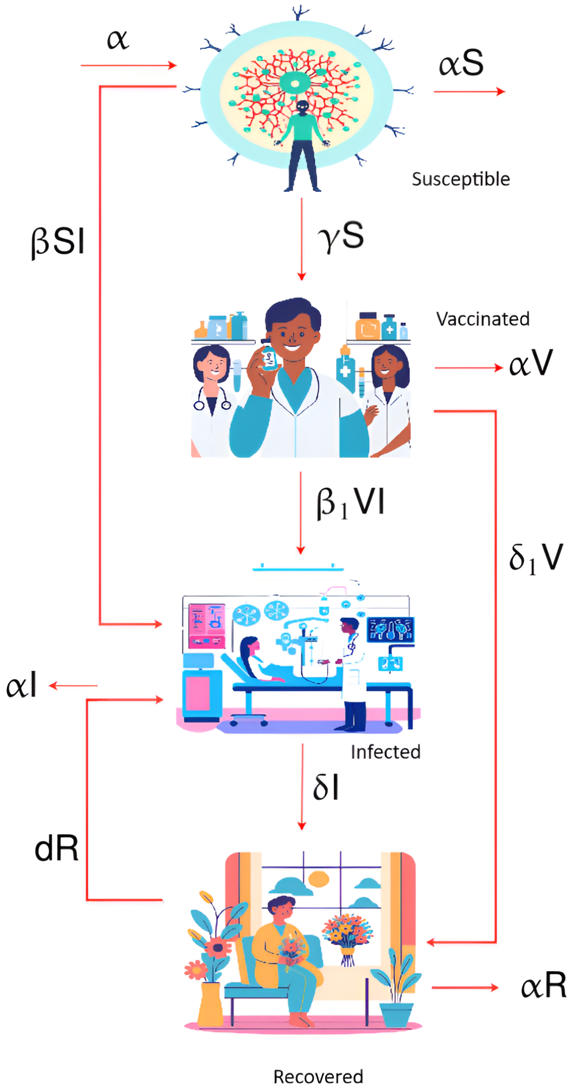

Figure 1. The transmission diagram.

(3) The rate of disease transmission $\beta(\mathtt{r})$ is assumed to depend on both the susceptible and infected populations, while $\beta_1(\mathtt{r})$ represents the transmission rate between vaccinated individuals and infected individuals.

(4) The recovery and immunity development processes are governed by $\delta(\mathtt{r})$ and $\delta_1(\mathtt{r})$, which represent the rates of recovery for infected individuals and the rate at which vaccinated individuals gain immunity, respectively.

(5) The relapse rate $\mathsf{d}(\mathtt{r})$ is incorporated to account for the potential recurrence of the disease in recovered individuals.

The transition dynamics between these compartments are governed by the following system of dynamic equations on timescales:

$$\begin{cases} S^{\Delta}(r) = \alpha(r) - \beta(r)S(r)I(r) - (\alpha(r) + \gamma(r))S(r), \\ V^{\Delta}(r) = \gamma(r)S(r) - \beta_1(r)V(r)I(r) - (\alpha(r) + \delta_1(r))V(r), \\ I^{\Delta}(r) = \beta(r)S(r)I(r) + \beta_1(r)V(r)I(r) - \delta(r)I(r) - \alpha(r)I(r) + d(r)R(r), \\ R^{\Delta}(r) = \delta_1(r)V(r) + \delta(r)I(r) - \alpha(r)R(r) - d(r)R(r), \end{cases} \tag{2.1}$$

for $r \in \mathbb{T}$, with initial conditions is that all variables $S(0)$, $V(0)$, $I(0)$, and $R(0)$ are positive at $r = 0$. Also, $\alpha(r), \beta(r), \beta_1(r), \gamma(r), \delta(r)$, and $\delta_1(r)$ are positive almost periodic functions for $r \in \mathbb{T}$.

2.1. Motivation of relapse dynamics on timescales in the SVIR model

The inclusion of relapse dynamics in the SVIR model on timescales is motivated by both epidemiological realities and mathematical generalizations that aim to enhance the model's flexibility and applicability. Below is the rationale for our consideration:

2.1.1. Epidemiological motivation

Relapse dynamics are critical in capturing the behavior of certain infectious diseases where individuals, after recovery, may lose immunity and become susceptible or reinfected. Examples include diseases like tuberculosis, malaria, and certain viral infections. The relapse process is influenced by factors such as incomplete immunity, genetic variability of pathogens, or re-exposure. Modeling this phenomenon allows for a realistic representation of the disease dynamics, particularly in settings where relapses significantly contribute to the overall infection prevalence.

2.1.2. Timescale framework

Extending the model to timescales combines continuous and discrete dynamics, enabling the study of disease progression in environments where interactions occur at irregular intervals. For instance:

- Continuous time: Used to model real-time disease spread in densely populated areas.
- Discrete time: Captures periodic interventions like vaccination campaigns, health awareness programs, or seasonal variations in disease dynamics.

By incorporating timescales, the model can simulate hybrid scenarios, such as populations with continuous interactions punctuated by discrete interventions or fluctuations in environmental conditions that influence relapse rates.

2.1.3. Mathematical formulation of relapse dynamics

In the proposed system, the term $d(r)R(r)$ accounts for the transition of individuals from the recovered compartment back to the infected compartment due to relapse. The function $d(r)$ represents the relapse rate, which is considered to be an almost periodic function on the timescale. This assumption allows the relapse rate to vary systematically over time, reflecting periodic factors such as seasonal changes, environmental influences, or treatment effectiveness.

2.1.4. Significance in stability and permanence analysis

The inclusion of relapse dynamics adds complexity to the model but is essential for accurately capturing the long-term behavior of the epidemic. Specifically:

- Stability: The model investigates conditions under which the population compartments (susceptible, vaccinated, infected, and recovered) achieve a stable distribution. Relapse dynamics influence this stability by affecting the balance between infection and recovery rates.
- Permanence: The persistence of the disease in the population is strongly tied to relapse rates. By analyzing $\mathtt{d}(\mathtt{r})$, we provide insights into thresholds that determine whether the disease dies out or remains endemic.

2.1.5. Impact on control strategies

The relapse dynamics highlight the necessity of considering recovered individuals in vaccination and intervention strategies. For example:

- If relapse rates are high, boosting immunity through periodic vaccination or post-recovery treatment becomes critical.
- Relapse dynamics underscore the importance of reducing $\mathtt{d}(\mathtt{r})$ through medical advancements or behavioral changes.

The inclusion of relapse dynamics in the SVIR model provides a robust framework for studying the long-term behavior of epidemics and formulating effective control strategies. This addition underscores the importance of accounting for relapse in understanding disease dynamics on timescales.

The ecological meaning of the almost periodic functions is provided in Figure 2.

Ecological Interpretations of Functions

Function	Ecological Meaning
$\alpha(r)$	Recruitment and natural mortality rates.
$\gamma(r)$	Vaccination rate of the population.
$\beta(r)$	Transmission rate between susceptible and infected.
$\beta_1(r)$	Transmission rate between vaccinated and infected.
$\delta(r)$	Rate at which infected individuals recover.
$\delta_1(r)$	Rate at which vaccinated individuals develop immunity.
$d(r)$	Relapse rate of recovered individuals.

Figure 2. Ecological interpretations of key functions in the model, describing various rates and interactions within the population dynamics.

Remark 1. To explore the existence of almost periodic solutions for the SVIR model, which incorporates relapsed cases and periodic incidence rates alongside a relapsed treatment function, it is essential to examine both dynamic equations on continuous domains and their discrete analogs. By bridging these approaches, the study can be extended to a unified framework on timescales. However, this requires significant advancements in timescale theory, as analyzing (2.1) within this broader context is quite demanding.

Let $C = C([-\mathbb{k}, 0]_{\mathbb{T}}, \mathbb{R}^4)$ denote the Banach space of continuous functions defined on the timescale interval $[-\mathbb{k}, 0]_{\mathbb{T}}$ with values in $\mathbb{R}^4$. Each function $f \in C$ satisfies:

$$f : [-\mathbb{k}, 0]_{\mathbb{T}} \to \mathbb{R}^4,$$

where $[-\mathbb{k}, 0]_{\mathbb{T}}$ is a closed and bounded subset of the timescale $\mathbb{T}$, and $\mathbb{R}^4$ represents the four-dimensional Euclidean space.

The norm on C is defined as:

$$\|f\|_C = \sup_{t\in[-\mathbb{k},0]_{\mathbb{T}}} \|f(t)\|_{\mathbb{R}^4},$$

where $\|f(t)\|_{\mathbb{R}^4}$ denotes the Euclidean norm of $f(t)$ in $\mathbb{R}^4$.

Equipped with this norm, C is a Banach space, as it is complete with respect to the norm $\|\cdot\|_C$.

Assume the initial conditions for (2.1) are given as follows:

$$\mathsf{S}(i) = \psi_1(i), \quad \mathsf{V}(i) = \psi_2(i), \quad \mathsf{I}(i) = \psi_3(i), \quad \mathsf{R}(i) = \psi_4(i), \quad i \in [-\mathbb{k}, 0]_{\mathbb{T}},$$

where the functions satisfy $\psi_{\mathtt{j}}(i) \geq 0$ for $i \in [-\mathbb{k}, 0]_{\mathbb{T}}$, and $\psi_{\mathtt{j}}(i)$ is positive at $i = 0$ for each index $\mathtt{j}$=1,2,3,4. The tuple $(\psi_1, \psi_2, \psi_3, \psi_4)$ belongs to the space C.

To describe bounds for a function $\mathrm{h}(\mathtt{r})$ defined on $\mathbb{T}$, we use the following notations:

$$\mathrm{h}^{\mathsf{L}} = \min\{\mathrm{h}(\mathtt{r}) : \mathtt{r} \in \mathbb{T}\}, \quad \mathrm{h}^{\mathsf{U}} = \max\{\mathrm{h}(\mathtt{r}) : \mathtt{r} \in \mathbb{T}\}.$$

Throughout the paper, we take the following to be true:

- Let the bounded and almost periodic functions $\alpha, \beta, \beta_1, \gamma, \delta, \delta_1$, and $\mathsf{d} : \mathbb{T} \to [0, \infty]$ be defined, which satisfy $0 < \mathsf{x}^{\mathsf{L}} \leq \mathsf{x}(\mathtt{r}) \leq \mathsf{x}^{\mathsf{U}}$ for $\mathsf{x} \in \{\alpha, \beta, \beta_1, \gamma, \delta, \delta_1, \mathsf{d}\}$.

Next, we present a few important definitions and results that will be instrumental in the subsequent discussions.

3. Basics of timescales

This section introduces the foundational concepts of timescale calculus, adapted from the primer in [6]. In many dynamic systems, processes evolve continuously over time but are also subject to discrete changes. Traditional models, such as ordinary differential equations (ODEs), excel at describing continuous-time dynamics, while discrete processes are used to capture step-wise changes. However, real-world systems often exhibit a blend of these behaviors, necessitating a unified approach to effectively model and analyze them.

The theory of timescales provides such a unification by generalizing the concept of time to encompass both continuous and discrete domains. This approach not only bridges the gap between these two paradigms but also opens new avenues for studying systems with hybrid dynamics, enabling more robust and flexible modeling frameworks.

Definition 3.1. A function $\mathtt{p} : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)\mathtt{p}(t) \neq 0$ for all $t \in \mathbb{T}_\kappa$. The set of all regressive and rd-continuous functions $\mathtt{p} : \mathbb{T} \to \mathbb{R}$ is denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}).$$

We further define the subset

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{\mathtt{p} \in \mathcal{R} : 1 + \mu(t)\mathtt{p}(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

Definition 3.2. For $p \in \mathcal{R}$, the exponential function $e_p(t, t_0)$ on the timescale $\mathbb{T}$ is defined as

$$e_p(t, t_0) = \exp\left(\int_{t_0}^{t} \xi_{\mu(s)}(p(s))\, \Delta s\right),$$

with

$$\xi_h(z) = \begin{cases} \dfrac{Log(1 + hz)}{h} & \text{if } h > 0, \\ z & \text{if } h = 0, \end{cases}$$

where *Log* is the principal logarithm function.

Definition 3.3. For functions $\alpha, \beta \in \mathcal{R}(\mathbb{T}, \mathbb{R})$, unconventional operations are defined as follows:

$$\alpha \oplus \beta := \alpha + \beta + \mu\alpha\beta, \quad \ominus\alpha := \frac{-\alpha}{1 + \mu\alpha}, \quad \alpha \ominus \beta := \alpha \oplus (\ominus\beta).$$

Lemma 3.4. *Assume that* $\mathtt{q} : \mathbb{T} \to \mathbb{R}$ *is a regressive function, then*

- $e_0(\alpha, \beta) \equiv 1$ *for all* $\alpha, \beta \in \mathbb{T}$.
- $e_{\mathtt{q}}(\alpha, \beta) = e_{\ominus\mathtt{q}}(\beta, \alpha)$.
- $e_{\mathtt{q}}(\alpha, \beta) = \frac{1}{e_{\mathtt{q}}(\beta, \alpha)}$.
- $e_{\mathtt{q}}(\alpha, \gamma)e_{\mathtt{q}}(\gamma, \beta) = e_{\mathtt{q}}(\alpha, \beta)$.

Lemma 3.5. *[19] Assume that* $\alpha > 0$, $b > 0$, *and* $-\alpha \in \mathcal{R}^+$. *If*

$$y^\Delta(\mathtt{p}) \geq (\leq) b - \alpha y^\sigma(\mathtt{p}), \quad y(t) > 0, \quad \mathtt{p} \in [\mathtt{p}_0, \infty)_\mathbb{T},$$

then

$$y(\mathtt{p}) \geq (\leq) \frac{b}{\alpha}\left(1 + \frac{\alpha y(t_0)}{b} - 1\right) e_{-\alpha}(\mathtt{p}, \mathtt{p}_0), \quad \mathtt{p} \in [\mathtt{p}_0, \infty)_\mathbb{T}.$$

Comparable results for ODEs are presented in [20, 21].

Definition 3.6. A timescale $\mathbb{T}$ is called an almost periodic timescale if

$$\prod = \{\zeta \in \mathbb{R} : \mathtt{r} + \zeta \in \mathbb{T}, \text{ for all } \mathtt{r} \in \mathbb{T}\},$$

such that $\prod \neq \{0\}$. The set $\prod$ is referred to as the *translation set* of $\mathbb{T}$.

Characteristics of almost periodic timescales:

(1) Translation Invariance: A timescale $\mathbb{T}$ is almost periodic if it remains invariant under translations by elements of $\prod$. That is, shifting all points of $\mathbb{T}$ by $\zeta \in \prod$ does not remove any points from $\mathbb{T}$.
(2) Non-triviality of $\prod$: The condition $\prod \neq \{0\}$ ensures that the timescale exhibits a nondegenerate structure with translational symmetry, beyond a simple or purely discrete configuration.
(3) Connection to Periodicity: An almost periodic timescale generalizes the idea of periodicity. For example: If $\prod = \{nT : n \in \mathbb{Z}\}$ for some fixed $T > 0$, then $\mathbb{T}$ is strictly periodic with period T. For an almost periodic timescale, $\prod$ may include irregular spacings, accommodating a more generalized repetitive structure.
(4) Relation to Infimum and Supremum: For an almost periodic timescale $\mathbb{T}$, the bounds satisfy

$$\inf \mathbb{T} = -\infty \quad \text{and} \quad \sup \mathbb{T} = \infty,$$

reflecting the unbounded nature of such timescales.

Almost periodic timescales are particularly useful in the study of dynamic equations where solutions or behaviors exhibit quasi-periodicity or generalized repetitive structures, such as in control systems, biological models, and signal processing.

Example 3.7. Consider the timescale $\mathbb{T}$ defined as

$$\mathbb{T} = \bigcup_{n \in \mathbb{Z}} (n(p+q), n(p+q)+q),$$

where $p \neq -q$. Then, the forward jump operator $\rho(t)$ and the graininess function $\nu(t)$ are expressed as:

$$\rho(t) = \begin{cases} t & \text{if } t \in \bigcup_{n=0}^{\infty} (n(p+q), n(p+q)+q), \\ t+p & \text{if } t \in \bigcup_{n=0}^{\infty} \{n(p+q)+q\}, \end{cases}$$

$$\nu(t) = \begin{cases} 0 & \text{if } t \in \bigcup_{n=0}^{\infty} (n(p+q), n(p+q)+q), \\ p & \text{if } t \in \bigcup_{n=0}^{\infty} \{n(p+q)+q\}. \end{cases}$$

Here, $p + q \in \Omega \setminus \{0\}$, which confirms that $\mathbb{T}$ is an almost periodic timescale. If $q = 0$ and $p = 1$, then S corresponds to $\mathbb{Z}$. On the other hand, if $q = 1$ and $p = 0$, then $\mathbb{T}$ represents $\mathbb{R}$.

Definition 3.8. Let $\mathbb{T}$ be an almost periodic timescale. A function $x \in C(\mathbb{T}, \mathbb{R}^n)$ is called an almost periodic function if the ϵ-translation set of x, that is,

$$E\{\epsilon, x\} = \{\tau \in \prod : |x(\mathtt{r} + \zeta) - x(\mathtt{r})| < \epsilon \text{ for all } \mathtt{r} \in \mathbb{T}\},$$

is a relatively dense set in $\mathbb{T}$ for all $\epsilon > 0$, and there exists a constant $l(\epsilon) > 0$ such that each interval of length $l(\epsilon)$ contains a $\zeta \in E\{\epsilon, x\}$ for which $|x(\mathtt{r} + \zeta) - x(\mathtt{r})| < \epsilon$ for all $\mathtt{r} \in \mathbb{T}$. The value τ is known as the ϵ-translation number of x, and $l(\epsilon)$ is called the inclusion length of $E\{\epsilon, x\}$.

Definition 3.9. Let D be an open set in $\mathbb{R}^n$ and let $\mathbb{T}$ be a positive almost periodic timescale. A function $f \in C(\mathbb{T} \times D, \mathbb{R}^n)$ is called an almost periodic function in $\mathtt{r} \in \mathbb{T}$ uniformly for $x \in D$ if the ϵ-translation set of f,

$$E\{\epsilon, f, S\} = \{\tau \in \prod : |f(\mathtt{r} + \zeta, x) - f(\mathtt{r}, x)| < \epsilon, \text{ for all } (\mathtt{r}, x) \in \mathbb{T} \times S\},$$

is a relatively dense set in $\mathbb{T}$ for all $\epsilon > 0$. Moreover, for each compact subset S of D, there exists a constant $l(\epsilon, S) > 0$ such that each interval of length $l(\epsilon, S)$ contains a $\zeta(\epsilon, S) \in E\{\epsilon, f, S\}$ for which

$$|f(\mathtt{r} + \zeta, x) - f(\mathtt{r}, x)| < \epsilon, \quad \text{for all } (\mathtt{r}, x) \in \mathbb{T} \times S.$$

Next, consider the following system:

$$z^{\Delta}(\mathtt{r}) = g(\mathtt{r}, z), \quad \mathtt{r} \in \mathbb{T}^+, \tag{3.1}$$

where $g : \mathbb{T}^+ \times \mathcal{S}_{\mathtt{M}} \to \mathbb{R}$, $\mathcal{S}_{\mathtt{M}} = \{z \in \mathbb{R}^n : \|z\| < \mathtt{M}\}$, $\|z\| = \sup_{\mathtt{r}\in\mathbb{T}} |z(\mathtt{r})|$, $\mathbb{T}^+$ is a nonempty closed subset of $\mathbb{R}^+ = [0, +\infty)$, and $\mathtt{M}$ is a positive real number. The function $g(\mathtt{r}, z)$ is assumed to be almost periodic in $\mathtt{r}$, uniformly for $z \in \mathcal{S}_{\mathtt{M}}$, and continuous in z. To solve the Eq (3.1), we consider the following system of equations as the product system:

$$z^{\Delta}(\mathtt{r}) = g(\mathtt{r}, z), \quad y^{\Delta}(\mathtt{r}) = g(\mathtt{r}, y),$$

The question of existence of a unique almost periodic solution $\phi(\mathtt{r}) \in \mathcal{S}$ of (3.1), which is uniformly asymptotically stable, is investigated [22]. For our model, we obtain the following result.

Lemma 3.10. *[22] Let $\mathcal{V}(\mathtt{r}, z, y)$ be a Lyapunov function defined on $\mathbb{T}^+ \times \mathcal{S}_{\mathtt{M}} \times \mathcal{S}_{\mathtt{M}}$, which satisfies the following conditions:*

(i) $a(\|z - y\|) \le \mathcal{V}(\mathtt{r}, z, y) \le b(\|z - y\|)$, *where* $a, b \in \mathcal{P}$, *and* $\mathcal{P} = \{g : \mathbb{R}^+ \to \mathbb{R}^+ \mid g$ *is continuous, increasing, and* $g(0) = 0\}$.

(ii) A constant $\mathcal{L} > 0$ *exists such that for all* z, z_1, y, y_1, *the inequality*

$$\left|\mathcal{V}(\mathtt{r}, z, y) - \mathcal{V}(\mathtt{r}, z_1, y_1)\right| \le \mathcal{L}(\|z - z_1\| + \|y - y_1\|),$$

holds true.

(iii) The function $\mathcal{V}$ *satisfies the following differential inequality:*

$$\mathcal{D}^+\mathcal{V}^{\Delta}(\mathtt{r}, z, y) \le -\lambda \mathcal{V}(\mathtt{r}, z, y),$$

where $\lambda > 0$ *is a constant and* $-\lambda \in \mathbb{R}^+$.

Furthermore, if the system (3.1) admits a solution $z(\mathtt{r}) \in \mathcal{S}$ *for* $\mathtt{r} \in \mathbb{T}^+$, *where* $\mathcal{S} \subset \mathcal{S}_{\mathtt{M}}$ *is compact, then the system has a unique almost periodic solution* $\mathtt{p}(\mathtt{r}) \in \mathcal{S}$. *This solution is uniformly asymptotically stable. Additionally, if* $g(\mathtt{r}, z)$ *is uniformly periodic in* $\mathtt{r}$ *for all* $z \in \mathcal{S}_{\mathtt{M}}$, *then the solution* $\mathtt{p}(\mathtt{r})$ *is periodic as well.*

Comparable results for ODEs are presented in [23, 24].

4. Permanence and uniform asymptotic stability

We begin by introducing the notion of permanence of solutions.

Definition 4.1. Let ℓ_0, ℓ_1 be positive real numbers such that

$$\ell_0 \le \liminf_{r\to\infty} S(r) \le \limsup_{r\to\infty} S(r) \le \ell_1,\ \ell_0 \le \liminf_{r\to\infty} V(r) \le \limsup_{r\to\infty} V(r) \le \ell_1,$$
$$\ell_0 \le \liminf_{r\to\infty} I(r) \le \limsup_{r\to\infty} I(r) \le \ell_1,\ \ell_0 \le \liminf_{r\to\infty} R(r) \le \limsup_{r\to\infty} R(r) \le \ell_1,$$

for any solution $(S(r), V(r), I(r), R(r))$ of system (2.1). Then, the system (2.1) is called permanent.

Lemma 4.2. *Assume that* $(S(r), V(r), I(r), R(r))$ *is a positive solution of system* (2.1). *If* $-\alpha^L$, $-(\alpha^L + \gamma^L)$, $(\alpha^L + \delta_1^L)$, $-(\alpha^L + d^L) \in \mathcal{R}^+$, *there are* $I_4 > 0$ *and* $\ell_1 > 0$ *for which*

$$S(r) \le \ell_1,\ V(r) \le \ell_1,\ I(r) \le \ell_1,\ R(r) \le \ell_1 \quad for \ \ r \in [I_4, \infty)_{\mathbb{T}}.$$

Proof. Suppose $(S(r), V(r), I(r), R(r))$ represents any positive solution of (2.1). Then, the first equation of system (2.1) implies that

$$\begin{aligned} S^\Delta(r) &= \alpha(r) - \beta(r)S(r)I(r) - (\alpha(r) + \gamma(r))S(r) \\ &\le \alpha(r) - (\alpha(r) + \gamma(r))S(r) \\ &\le \alpha^U - (\alpha^L + \gamma^L)S(r). \end{aligned}$$

Consequently, invoking Lemma 3.5, for any positive number ϵ as small as desired, one can identify a value $I_1 > 0$ satisfying:

$$S(r) \le \frac{\alpha^U}{\alpha^L + \gamma^L} + \epsilon = \ell_{11}, \quad r \in [I_1, \infty)_{\mathbb{T}}. \tag{4.1}$$

Proceeding further, based on the second equation of system (2.1) along with (4.1), the condition remains valid for $r \in [I_1, \infty)$:

$$\begin{aligned} V^\Delta(r) &= \gamma(r)S(r) - \beta_1(r)V(r)I(r) - (\alpha(r) + \delta_1(r))V(r) \\ &\le \gamma(r)S(r) - (\alpha(r) + \delta_1(r))V(r) \\ &\le \gamma^U \ell_{11} - (\alpha^L + \delta_1^L)V(r). \end{aligned}$$

According to Lemma 3.5, for any $\epsilon > 0$, no matter how small, one can find a $I_2 > I_1$ meeting the condition that

$$V(r) \le \frac{\gamma^U \ell_{11}}{\alpha^L + \delta_1^L} + \epsilon = \ell_{12}, \quad r \in [I_2, \infty)_{\mathbb{T}}. \tag{4.2}$$

Let $N(r) = S(r) + V(r) + I(r) + R(r)$. Then, first order forward Hilger derivative of $N(r)$ with respect to r is

$$N^\Delta(r) = S^\Delta(r) + V^\Delta(r) + I^\Delta(r) + R^\Delta(r).$$

Adding all the equations of (2.1), we get

$$S^\Delta(r) + V^\Delta(r) + I^\Delta(r) + R^\Delta(r) = \alpha(r) - \alpha(r)(S(r) + V(r) + I(r) + R(r)).$$

Therefore,

$$\begin{aligned}\mathsf{N}^{\Delta}(\mathtt{r}) &= \alpha(\mathtt{r}) - \alpha(\mathtt{r})\mathsf{N}(\mathtt{r}) \\ &\le \alpha^{\mathsf{U}} - \alpha^{\mathsf{L}}\mathsf{N}(\mathtt{r}).\end{aligned}$$

By Lemma 3.5, we get

$$\limsup_{\mathtt{r}\to\infty} \mathsf{N}(\mathtt{r}) \le \frac{\alpha^{\mathsf{U}}}{\alpha^{\mathsf{L}}} := \mathsf{K}_3.$$

It is evident that

$$\limsup_{\mathtt{r}\to\infty} \mathsf{I}(\mathtt{r}) \le \limsup_{\mathtt{r}\to\infty} \mathsf{N}(\mathtt{r}).$$

So, we get

$$\limsup_{\mathtt{r}\to\infty} \mathsf{I}(\mathtt{r}) \le \ell_{13}.$$

Thus, there exists a sufficiently small $\epsilon > 0$, and we can find a $I_3 > I_2$ satisfying

$$\mathsf{I}(\mathtt{r}) \le \ell_{13}, \quad \mathtt{r} \in [I_3, \infty)_{\mathbb{T}}. \tag{4.3}$$

Lastly, from the last equation of system (2.1), and using (4.1)–(4.3) for $\mathtt{r} \in [I_3, \infty)$,

$$\begin{aligned}\mathsf{R}^{\Delta}(\mathtt{r}) &= \delta_1(\mathtt{r})\mathsf{V}(\mathtt{r}) + \delta(\mathtt{r})\mathsf{I}(\mathtt{r}) - \alpha(\mathtt{r})\mathsf{R}(\mathtt{r}) - \mathsf{d}(\mathtt{r})\mathsf{R}(\mathtt{r}) \\ &\le \delta_1^{\mathsf{U}}\ell_{12} + \delta^{\mathsf{U}}\ell_{13} - (\alpha^{\mathsf{L}} + \mathsf{d}^{\mathsf{L}})\mathsf{R}(\mathtt{r}).\end{aligned}$$

According to Lemma 3.5, for any $\epsilon > 0$ that is sufficiently close to zero, a value I_4 can be found with $I_4 > I_3$, which satisfies:

$$\mathsf{R}(\mathtt{r}) \le \frac{\delta_1^{\mathsf{U}}\ell_{12} + \delta^{\mathsf{U}}\ell_{13}}{\alpha^{\mathsf{L}} + \mathsf{d}^{\mathsf{L}}} + \epsilon = K_4, \quad \mathtt{r} \in [I_4, \infty).$$

Let $\ell_1 > \max\{\ell_{11}, \ell_{12}, \ell_{13}, \ell_{14}\}$, then

$$\mathsf{S}(\mathtt{r}) \le \ell_1,\ \mathsf{V}(\mathtt{r}) \le \ell_1,\ \mathsf{I}(\mathtt{r}) \le \ell_1,\ \mathsf{R}(\mathtt{r}) \le \ell_1 \ \ for \ \ \mathtt{r} \in [I_4, \infty)_{\mathbb{T}}.$$

Therefore, the argument is concluded. □

Lemma 4.3. *Let* $(\mathsf{S}(\mathtt{r}), \mathsf{V}(\mathtt{r}), \mathsf{I}(\mathtt{r}), \mathsf{R}(\mathtt{r}))$ *be a positive solution to the system described in* (2.1)*. If the following conditions hold for the parameters:*

$$-(\alpha^{\mathsf{U}} + \mathsf{d}^{\mathsf{U}}),\ -(\delta^{\mathsf{U}} + \alpha^{\mathsf{U}}),\ -(\alpha^{\mathsf{U}} + \beta^{\mathsf{U}}\ell_1 + \gamma^{\mathsf{U}}),\ -(\alpha^{\mathsf{U}} + \beta_1^{\mathsf{U}}\ell_1 + \delta_1^{\mathsf{U}}) \in \mathcal{R}^+,$$

then there exist constants $I_8 > 0$ *and* $\ell_0 > 0$ *for which, for all* $\mathtt{r} \in [I_8, \infty)_{\mathbb{T}}$*, the following inequalities are satisfied:*

$$\mathsf{S}(\mathtt{r}) \ge \ell_0, \quad \mathsf{V}(\mathtt{r}) \ge \ell_0, \quad \mathsf{I}(\mathtt{r}) \ge \ell_0, \quad \mathsf{R}(\mathtt{r}) \ge \ell_0.$$

Proof. Consider the tuple $(\mathsf{S}(\mathtt{r}), \mathsf{V}(\mathtt{r}), \mathsf{I}(\mathtt{r}), \mathsf{R}(\mathtt{r}))$ as a positive solution to system (2.1). Referring to the initial equation in system (2.1) and leveraging the results established in Lemma 4.2, we deduce that for all $\mathtt{r} \in [I_4, \infty)$:

$$\begin{aligned}\mathsf{S}^{\Delta}(\mathtt{r}) &= \alpha(\mathtt{r}) - \beta(\mathtt{r})\mathsf{S}(\mathtt{r})\mathsf{I}(\mathtt{r}) - (\alpha(\mathtt{r}) + \gamma(\mathtt{r}))\mathsf{S}(\mathtt{r}) \\ &\ge \alpha^{\mathsf{L}} - \beta^{\mathsf{U}}\mathsf{S}(\mathtt{r})\ell_1 - (\alpha^{\mathsf{U}} + \gamma^{\mathsf{U}})\mathsf{S}(\mathtt{r}) \\ &= \alpha^{\mathsf{L}} - (\alpha^{\mathsf{U}} + \beta^{\mathsf{U}}\ell_1 + \gamma^{\mathsf{U}})\mathsf{S}(\mathtt{r}).\end{aligned}$$

By Lemma 3.5, for any $\epsilon > 0$ chosen as small as desired, one can find an $I_5 > 0$ satisfying

$$S(r) \geq \frac{\alpha^L}{\alpha^U + \beta^U \ell_1 + \gamma^U} + \epsilon = \ell_{01}, \quad r \in [I_5, \infty)_{\mathbb{T}}. \tag{4.4}$$

From (2.1), we find that

$$\begin{aligned} V^{\Delta}(r) &= \gamma(r)S(r) - \beta_1(r)V(r)I(r) - (\alpha(r) + \delta_1(r))V(r) \\ &\geq \gamma^L \ell_0 - \beta_1^U V(r)\ell_1 - (\alpha^U + \delta_1^U)V(r) \\ &= \gamma^L \ell_0 - (\alpha^U + \beta_1^U \ell_1 + \delta_1^U)V(r). \end{aligned}$$

By Lemma 3.5, for any $\epsilon > 0$ chosen as small as desired, one can find an $I_6 > I_5$ satisfying

$$V(r) \geq \frac{\gamma^L \ell_0}{\alpha^U + \beta_1^U \ell_1 + \delta_1^U} + \epsilon = \ell_{02}, \quad r \in [I_6, \infty)_{\mathbb{T}}. \tag{4.5}$$

From the final equation of system (2.1), in conjunction with (4.4) and (4.5), it follows that for $r \in [I_6, \infty)$, we obtain

$$\begin{aligned} R^{\Delta}(r) &= \delta_1(r)V(r) + \delta(r)I(r) - \alpha(r)R(r) - d(r)R(r) \\ &\geq \delta_1(r)V(r) - (\alpha(r) + d(r))R(r) \\ &\geq \delta_1^L \ell_{02} - (\alpha^U + d^U)R(r). \end{aligned} \tag{4.6}$$

From (4.6) and by Lemma 3.5, for any $\epsilon > 0$ chosen as small as desired, one can determine an $I_7 > I_6$ fulfilling the condition that $r \in [I_7, \infty)$:

$$R(r) \geq \frac{\delta_1^L \ell_{02}}{\alpha^U + d^U} + \epsilon = \ell_{03}.$$

From the third equation of (2.1), we have for $r \in [I_7, \infty)$,

$$\begin{aligned} I^{\Delta}(r) &= \beta(r)S(r)I(r) + \beta_1(r)V(r)I(r) - \delta(r)I(r) - \alpha(r)I(r) + d(r)R(r) \\ &\geq d(r)R(r) - (\delta(r) + \alpha(r))I(r) \\ &\geq d^L \ell_{03} - (\delta^U + \alpha^U)I(r). \end{aligned}$$

By Lemma 3.5, for any $\epsilon > 0$ chosen as small as desired, one can determine an $I_8 > I_7$ fulfilling the condition that $r \in [I_8, \infty)$:

$$I(r) \geq \frac{d^L \ell_{03}}{\delta^U + \alpha^U} + \epsilon = \ell_{04}.$$

Let $0 < \ell_0 < \min\{\ell_{01}, \ell_{02}, \ell_{03}, \ell_{04}\}$, then

$$S(r) \geq \ell_0,\ V(r) \geq \ell_0,\ I(r) \geq \ell_0,\ R(r) \geq \ell_0$$

for $r \in [I_8, \infty)_{\mathbb{T}}$. □

Theorem 4.4. *Suppose the hypotheses of Lemmas 4.2 and 4.3 are satisfied. Under these assumptions, system* (2.1) *exhibits permanence.*

Proof. In combination with Lemmas 4.2 and 4.3, the intended conclusion can be derived.
Define

$$\mho = \Big\{(\mathsf{S}(\mathtt{r}), \mathsf{V}(\mathtt{r}), \mathsf{I}(\mathtt{r}), \mathsf{R}(\mathtt{r})) : (\mathsf{S}(\mathtt{r}), \mathsf{V}(\mathtt{r}), \mathsf{I}(\mathtt{r}), \mathsf{R}(\mathtt{r})) \text{ to be a solution of (2.1)},$$

$$0 < \mathsf{S}^- \leq \mathsf{S}(\mathtt{r}) \leq \mathsf{S}^+,\ 0 < \mathsf{V}^- \leq \mathsf{V}(\mathtt{r}) \leq \mathsf{V}^+,\ 0 < \mathsf{I}^- \leq \mathsf{I}(\mathtt{r}) \leq \mathsf{I}^+,\ 0 < \mathtt{r}^- \leq \mathsf{R}(\mathtt{r}) \leq \mathtt{r}^+ \Big\}.$$

It is evident that $\mho$ remains invariant under the dynamics of system (2.1). □

Lemma 4.5. *If the conditions of Lemmas 4.2 and 4.3 are satisfied, it follows that* $\mho$ *is nonempty.*

Proof. Since the conditions of Lemmas 4.2 and 4.3 are satisfied, it follows from Theorem 4.4 that $\mho$ is nonempty. □

Now, we establish sufficient conditions for the existence of a unique positive almost periodic solution to system (2.1) that is uniform asymptotically stable.
Consider the following assumption, which will hold in the next result:

($\mathscr{A}$) Let the bounded and almost periodic functions $\alpha, \beta, \beta_1, \gamma, \delta, \delta_1$, and $\mathsf{d} : \mathbb{T} \to [0, \infty]$ be defined, which satisfy $0 < \mathsf{x}^{\mathsf{L}} \leq \mathsf{x}(\mathtt{r}) \leq \mathsf{x}^{\mathsf{U}}$ for $\mathsf{x} \in \{\alpha, \beta, \beta_1, \gamma, \delta, \delta_1, \mathsf{d}\}$.

($\mathscr{B}$) There exist a positive constant $\lambda > 0$ and $-\lambda \in \mathcal{R}^+$, where $\lambda = \min\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and

$$\begin{aligned}
\lambda_1 &= \alpha^{\mathsf{L}} + \beta^{\mathsf{L}} + \gamma^{\mathsf{L}} - \gamma^{\mathsf{U}} - \beta^{\mathsf{U}}\mathsf{I}^+,\\
\lambda_2 &= \alpha^{\mathsf{L}} + \beta_1^{\mathsf{L}}\mathsf{I}^- + \delta_1^{\mathsf{L}} - \beta_1^{\mathsf{U}}\mathsf{I}^+ - \delta_1^{\mathsf{U}},\\
\lambda_3 &= \alpha^{\mathsf{L}} - \beta^{\mathsf{U}}\mathsf{S}^+ - \beta_1^{\mathsf{U}}\mathsf{V}^+ + \delta^{\mathsf{L}} + \beta^{\mathsf{L}}\mathsf{S}^- + \delta_1^{\mathsf{L}}\mathsf{V}^- - \delta^{\mathsf{U}},\\
\lambda_4 &= \alpha^{\mathsf{L}} + \mathsf{d}^{\mathsf{L}} - \mathsf{d}^{\mathsf{U}}.
\end{aligned}$$

Theorem 4.6. *Let* ($\mathscr{A}$) *and* ($\mathscr{B}$) *hold, then the dynamic system* (2.1) *admits a unique almost periodic solution* $(\mathsf{S}(\mathtt{r}), \mathsf{V}(\mathtt{r}), \mathsf{I}(\mathtt{r}), \mathsf{R}(\mathtt{r}))$ *contained in* $\mho$*, which is uniformly asymptotically stable.*

Proof. Based on Theorem 4.4 and Lemma 4.5, any solution of the system (2.1), denoted as $(\mathsf{S}(\mathtt{r}), \mathsf{V}(\mathtt{r}), \mathsf{I}(\mathtt{r}), \mathsf{R}(\mathtt{r}))$, satisfies the following bounds:

$$\begin{aligned}
0 < \mathsf{S}^- &\leq \mathsf{S}(\mathtt{r}) \leq \mathsf{S}^+,\\
0 < \mathsf{V}^- &\leq \mathsf{V}(\mathtt{r}) \leq \mathsf{V}^+,\\
0 < \mathsf{I}^- &\leq \mathsf{I}(\mathtt{r}) \leq \mathsf{I}^+,\\
0 < \mathtt{r}^- &\leq \mathsf{R}(\mathtt{r}) \leq \mathtt{r}^+.
\end{aligned}$$

Consequently, the absolute values of these components are constrained as follows:

$$|\mathsf{S}(\mathtt{r})| \leq \mathsf{J}_1, |\mathsf{V}(\mathtt{r})| \leq \mathsf{J}_2, |\mathsf{I}(\mathtt{r})| \leq \mathsf{J}_3, |\mathsf{R}(\mathtt{r})| \leq \mathsf{J}_4,$$

where the constants J_1, J_2, J_3, and J_4 are given by:

$$\begin{aligned}
\mathsf{J}_1 &= \max\{|\mathsf{S}^-|, |\mathsf{S}^+|\},\\
\mathsf{J}_2 &= \max\{|\mathsf{V}^-|, |\mathsf{V}^+|\},\\
\mathsf{J}_3 &= \max\{|\mathsf{I}^-|, |\mathsf{I}^+|\},\\
\mathsf{J}_4 &= \max\{|\mathtt{r}^-|, |\mathtt{r}^+|\}.
\end{aligned}$$

This ensures that the solutions of the system are bounded within well-defined intervals for each component.

$$\text{Denote } \|(\mathsf{S}(\mathrm{r}), \mathsf{V}(\mathrm{r}), \mathsf{I}(\mathrm{r}), \mathsf{R}(\mathrm{r}))\| = \sup_{\mathrm{r}\in\mathbb{T}^+} |\mathsf{S}(\mathrm{r})| + \sup_{\mathrm{r}\in\mathbb{T}^+} |\mathsf{V}(\mathrm{r})| + \sup_{\mathrm{r}\in\mathbb{T}^+} |\mathsf{I}(\mathrm{r})| + \sup_{\mathrm{r}\in\mathbb{T}^+} |\mathsf{R}(\mathrm{r})|.$$

Let any two positive solutions of (2.1) be $y = (\mathsf{S}(\mathrm{r}), \mathsf{V}(\mathrm{r}), \mathsf{I}(\mathrm{r}), \mathsf{R}(\mathrm{r}))$, $\widehat{y} = (\widehat{\mathsf{S}}(\mathrm{r}), \widehat{\mathsf{V}}(\mathrm{r}), \widehat{\mathsf{I}}(\mathrm{r}), \widehat{\mathsf{R}}(\mathrm{r}))$. Then

$$\|y\| \leq \mathsf{J}_1 + \mathsf{J}_2 + \mathsf{J}_3 + \mathsf{J}_4 \quad \text{and} \quad \|\widehat{y}\| \leq \mathsf{J}_1 + \mathsf{J}_2 + \mathsf{J}_3 + \mathsf{J}_4.$$

From the structure of the system (2.1), it follows that

$$\left.\begin{aligned}
\mathsf{S}^{\Delta}(\mathrm{r}) &= \alpha(\mathrm{r}) - \beta(\mathrm{r})\mathsf{S}(\mathrm{r})\mathsf{I}(\mathrm{r}) - (\alpha(\mathrm{r}) + \gamma(\mathrm{r}))\mathsf{S}(\mathrm{r}),\\
\mathsf{V}^{\Delta}(\mathrm{r}) &= \gamma(\mathrm{r})\mathsf{S}(\mathrm{r}) - \beta_1(\mathrm{r})\mathsf{V}(\mathrm{r})\mathsf{I}(\mathrm{r}) - (\alpha(\mathrm{r}) + \delta_1(\mathrm{r}))\mathsf{V}(\mathrm{r}),\\
\mathsf{I}^{\Delta}(\mathrm{r}) &= \beta(\mathrm{r})\mathsf{S}(\mathrm{r})\mathsf{I}(\mathrm{r}) + \beta_1(\mathrm{r})\mathsf{V}(\mathrm{r})\mathsf{I}(\mathrm{r}) - \delta(\mathrm{r})\mathsf{I}(\mathrm{r}) - \alpha(\mathrm{r})\mathsf{I}(\mathrm{r}) + \mathsf{d}(\mathrm{r})\mathsf{R}(\mathrm{r}),\\
\mathsf{R}^{\Delta}(\mathrm{r}) &= \delta_1(\mathrm{r})\mathsf{V}(\mathrm{r}) + \delta(\mathrm{r})\mathsf{I}(\mathrm{r}) - \alpha(\mathrm{r})\mathsf{R}(\mathrm{r}) - \mathsf{d}(\mathrm{r})\mathsf{R}(\mathrm{r}),\\
\widehat{\mathsf{S}}^{\Delta}(\mathrm{r}) &= \alpha(\mathrm{r}) - \beta(\mathrm{r})\widehat{\mathsf{S}}(\mathrm{r})\widehat{\mathsf{I}}(\mathrm{r}) - (\alpha(\mathrm{r}) + \gamma(\mathrm{r}))\widehat{\mathsf{S}}(\mathrm{r}),\\
\widehat{\mathsf{V}}^{\Delta}(\mathrm{r}) &= \gamma(\mathrm{r})\widehat{\mathsf{S}}(\mathrm{r}) - \beta_1(\mathrm{r})\widehat{\mathsf{V}}(\mathrm{r})\widehat{\mathsf{I}}(\mathrm{r}) - (\alpha(\mathrm{r}) + \delta_1(\mathrm{r}))\widehat{\mathsf{V}}(\mathrm{r}),\\
\widehat{\mathsf{I}}^{\Delta}(\mathrm{r}) &= \beta(\mathrm{r})\widehat{\mathsf{S}}(\mathrm{r})\widehat{\mathsf{I}}(\mathrm{r}) + \beta_1(\mathrm{r})\widehat{\mathsf{V}}(\mathrm{r})\widehat{\mathsf{I}}(\mathrm{r}) - \delta(\mathrm{r})\widehat{\mathsf{I}}(\mathrm{r}) - \alpha(\mathrm{r})\widehat{\mathsf{I}}(\mathrm{r}) + \mathsf{d}(\mathrm{r})\widehat{\mathsf{R}}(\mathrm{r}),\\
\widehat{\mathsf{R}}^{\Delta}(\mathrm{r}) &= \delta_1(\mathrm{r})\widehat{\mathsf{V}}(\mathrm{r}) + \delta(\mathrm{r})\widehat{\mathsf{I}}(\mathrm{r}) - \alpha(\mathrm{r})\widehat{\mathsf{R}}(\mathrm{r}) - \mathsf{d}(\mathrm{r})\widehat{\mathsf{R}}(\mathrm{r}).
\end{aligned}\right\} \tag{4.7}$$

Let the Lyapunov function $\mathcal{V}(\mathrm{r}, y, \widehat{y})$ be defined on $\mathbb{T}^+ \times \mho \times \mho$ as

$$\mathcal{V}(\mathrm{r}, y, \widehat{y}) = |\mathsf{S}(\mathrm{r}) - \widehat{\mathsf{S}}(\mathrm{r})| + |\mathsf{V}(\mathrm{r}) - \widehat{\mathsf{V}}(\mathrm{r})| + |\mathsf{I}(\mathrm{r}) - \widehat{\mathsf{I}}(\mathrm{r})| + |\mathsf{R}(\mathrm{r}) - \widehat{\mathsf{R}}(\mathrm{r})|.$$

Define the norm

$$\begin{aligned}
\|y(\mathrm{r}) - \widehat{y}(\mathrm{r})\| = &\sup_{\mathrm{r}\in\mathbb{T}^+} |\mathsf{S}(\mathrm{r}) - \widehat{\mathsf{S}}(\mathrm{r})| + \sup_{\mathrm{r}\in\mathbb{T}^+} |\mathsf{V}(\mathrm{r}) - \widehat{\mathsf{V}}(\mathrm{r})|\\
&+ \sup_{\mathrm{r}\in\mathbb{T}^+} |\mathsf{I}(\mathrm{r}) - \widehat{\mathsf{I}}(\mathrm{r})| + \sup_{\mathrm{r}\in\mathbb{T}^+} |\mathsf{R}(\mathrm{r}) - \widehat{\mathsf{R}}(\mathrm{r})|.
\end{aligned} \tag{4.8}$$

It can be readily observed that there are two constants $l > 0$, $m > 0$ for which

$$l\|y(\mathrm{r}) - \widehat{y}(\mathrm{r})\| \leq \mathcal{V}(\mathrm{r}, y, \widehat{y}) \leq m\|y(\mathrm{r}) - \widehat{y}(\mathrm{r})\|.$$

Consider $\mathfrak{B}, \ell \in C(\mathbb{R}^+, \mathbb{R}^+)$ to be defined as $\mathfrak{B}(z) = lz$ and $\ell(z) = mz$. With these definitions, condition (i) of Lemma 3.10 is fulfilled. Furthermore, we observe that

$$\begin{aligned}
&\Big|\mathcal{V}(\mathrm{r},y(\mathrm{r}),\widehat{y}(\mathrm{r})) - \mathcal{V}(\mathrm{r},y^\star(\mathrm{r}),\widehat{y}^\star(\mathrm{r}))\Big| \\
&\quad = \Big||\mathsf{S}(\mathrm{r}) - \widehat{\mathsf{S}}(\mathrm{r})| + |\mathsf{V}(\mathrm{r}) - \widehat{\mathsf{V}}(\mathrm{r})| + |\mathsf{I}(\mathrm{r}) - \widehat{\mathsf{I}}(\mathrm{r})| + |\mathsf{R}(\mathrm{r}) - \widehat{\mathsf{R}}(\mathrm{r})| \\
&\quad - |\mathsf{S}^\star(\mathrm{r}) - \widehat{\mathsf{S}}^\star(\mathrm{r})| - |\mathsf{V}^\star(\mathrm{r}) - \widehat{\mathsf{V}}^\star(\mathrm{r})| - |\mathsf{I}^\star(\mathrm{r}) - \widehat{\mathsf{I}}^\star(\mathrm{r})| - |\mathrm{r}^\star(\mathrm{r}) - \widehat{\mathsf{R}}^\star(\mathrm{r})|\Big| \\
&\quad \le \Big||\mathsf{S}(\mathrm{r}) - \widehat{\mathsf{S}}(\mathrm{r})| - |\mathsf{S}^\star(\mathrm{r}) - \widehat{\mathsf{S}}^\star(\mathrm{r})|\Big| + \Big||\mathsf{V}(\mathrm{r}) - \widehat{\mathsf{V}}(\mathrm{r})| - |\mathsf{V}^\star(\mathrm{r}) - \widehat{\mathsf{V}}^\star(\mathrm{r})|\Big| \\
&\quad + \Big||\mathsf{I}(\mathrm{r}) - \widehat{\mathsf{I}}(\mathrm{r})| - |\mathsf{I}^\star(\mathrm{r}) - \widehat{\mathsf{I}}^\star(\mathrm{r})|\Big| + \Big||\mathsf{R}(\mathrm{r}) - \widehat{\mathsf{R}}(\mathrm{r})| - |\mathrm{r}^\star(\mathrm{r}) - \widehat{\mathsf{R}}^\star(\mathrm{r})|\Big| \\
&\quad \le \Big|[\mathsf{S}(\mathrm{r}) - \widehat{\mathsf{S}}(\mathrm{r})] - [\mathsf{S}^\star(\mathrm{r}) - \widehat{\mathsf{S}}^\star(\mathrm{r})]\Big| + \Big|[\mathsf{V}(\mathrm{r}) - \widehat{\mathsf{V}}(\mathrm{r})] - [\mathsf{V}^\star(\mathrm{r}) - \widehat{\mathsf{V}}^\star(\mathrm{r})]\Big| \\
&\quad + \Big|[\mathsf{I}(\mathrm{r}) - \widehat{\mathsf{I}}(\mathrm{r})] - [\mathsf{I}^\star(\mathrm{r}) - \widehat{\mathsf{I}}^\star(\mathrm{r})]\Big| + \Big|[\mathsf{R}(\mathrm{r}) - \widehat{\mathsf{R}}(\mathrm{r})] - [\mathrm{r}^\star(\mathrm{r}) - \widehat{\mathsf{R}}^\star(\mathrm{r})]\Big| \\
&\quad \le \Big|[\mathsf{S}(\mathrm{r}) - \mathsf{S}^\star(\mathrm{r})] - [\widehat{\mathsf{S}}(\mathrm{r}) - \widehat{\mathsf{S}}^\star(\mathrm{r})]\Big| + \Big|[\mathsf{V}(\mathrm{r}) - \mathsf{V}^\star(\mathrm{r})] - [\widehat{\mathsf{V}}(\mathrm{r}) - \widehat{\mathsf{V}}^\star(\mathrm{r})]\Big| \\
&\quad + \Big|[\mathsf{I}(\mathrm{r}) - \mathsf{I}^\star(\mathrm{r})] - [\widehat{\mathsf{I}}(\mathrm{r}) - \widehat{\mathsf{I}}^\star(\mathrm{r})]\Big| + \Big|[\mathsf{R}(\mathrm{r}) - \mathrm{r}^\star(\mathrm{r})] - [\widehat{\mathsf{R}}(\mathrm{r}) - \widehat{\mathsf{R}}^\star(\mathrm{r})]\Big| \\
&\quad \le |\mathsf{S}(\mathrm{r}) - \mathsf{S}^\star(\mathrm{r})| + |\widehat{\mathsf{S}}(\mathrm{r}) - \widehat{\mathsf{S}}^\star(\mathrm{r})| + |\mathsf{V}(\mathrm{r}) - \mathsf{V}^\star(\mathrm{r})| + |\widehat{\mathsf{V}}(\mathrm{r}) - \widehat{\mathsf{V}}^\star(\mathrm{r})| \\
&\quad + |\mathsf{I}(\mathrm{r}) - \mathsf{I}^\star(\mathrm{r})| + |\widehat{\mathsf{I}}(\mathrm{r}) - \widehat{\mathsf{I}}^\star(\mathrm{r})| + |\mathsf{R}(\mathrm{r}) - \mathrm{r}^\star(\mathrm{r})| + |\widehat{\mathsf{R}}(\mathrm{r}) - \widehat{\mathsf{R}}^\star(\mathrm{r})| \\
&\quad \le |\mathsf{S}(\mathrm{r}) - \mathsf{S}^\star(\mathrm{r})| + |\mathsf{V}(\mathrm{r}) - \mathsf{V}^\star(\mathrm{r})| + |\mathsf{I}(\mathrm{r}) - \mathsf{I}^\star(\mathrm{r})| + |\mathsf{R}(\mathrm{r}) - \mathsf{R}^\star(\mathrm{r})| \\
&\quad + |\widehat{\mathsf{S}}(\mathrm{r}) - \widehat{\mathsf{S}}^\star(\mathrm{r})| + |\widehat{\mathsf{V}}(\mathrm{r}) - \widehat{\mathsf{V}}^\star(\mathrm{r})| + |\widehat{\mathsf{I}}(\mathrm{r}) - \widehat{\mathsf{I}}^\star(\mathrm{r})| + |\widehat{\mathsf{R}}(\mathrm{r}) - \widehat{\mathsf{R}}^\star(\mathrm{r})| \\
&\quad \le \Big[\sup_{\mathrm{r}\in\mathbb{T}^+} |\mathsf{S}(\mathrm{r}) - \mathsf{S}^\star(\mathrm{r})| + \sup_{\mathrm{r}\in\mathbb{T}^+} |\mathsf{V}(\mathrm{r}) - \mathsf{V}^\star(\mathrm{r})| + \sup_{\mathrm{r}\in\mathbb{T}^+} |\mathsf{I}(\mathrm{r}) - \mathsf{I}^\star(\mathrm{r})| + \sup_{\mathrm{r}\in\mathbb{T}^+} |\mathsf{R}(\mathrm{r}) - \mathsf{R}^\star(\mathrm{r})|\Big] \\
&\quad + \Big[\sup_{\mathrm{r}\in\mathbb{T}^+} |\widehat{\mathsf{S}}(\mathrm{r}) - \widehat{\mathsf{S}}^\star(\mathrm{r})| + \sup_{\mathrm{r}\in\mathbb{T}^+} |\widehat{\mathsf{V}}(\mathrm{r}) - \widehat{\mathsf{V}}^\star(\mathrm{r})| + \sup_{\mathrm{r}\in\mathbb{T}^+} |\widehat{\mathsf{I}}(\mathrm{r}) - \widehat{\mathsf{I}}^\star(\mathrm{r})| + \sup_{\mathrm{r}\in\mathbb{T}^+} |\widehat{\mathsf{R}}(\mathrm{r}) - \widehat{\mathsf{R}}^\star(\mathrm{r})|\Big].
\end{aligned}$$

From the definition of the Norm (4.8), we get

$$\Big|\mathcal{V}(\mathrm{r},y(\mathrm{r}),\widehat{y}(\mathrm{r})) - \mathcal{V}(\mathrm{r},y^\star(\mathrm{r}),\widehat{y}^\star(\mathrm{r}))\Big| \le \mathsf{L}[\|y - y^\star\| + \|\widehat{y} - \widehat{y}^\star\|],$$

where $\mathsf{L} = 1$, so condition (ii) of Lemma 3.10 is satisfied. Now consider a function $\mathcal{W}(\mathrm{r}) = \mathcal{W}_1(\mathrm{r}) + \mathcal{W}_2(\mathrm{r}) + \mathcal{W}_3(\mathrm{r}) + \mathcal{W}_4(\mathrm{r})$, where

$$\mathcal{W}_1(\mathrm{r}) = |\mathsf{S}(\mathrm{r}) - \widehat{\mathsf{S}}(\mathrm{r})|,\ \mathcal{W}_2(\mathrm{r}) = |\mathsf{V}(\mathrm{r}) - \widehat{\mathsf{V}}(\mathrm{r})|,$$

$$\mathcal{W}_3(\mathrm{r}) = |\mathsf{I}(\mathrm{r}) - \widehat{\mathsf{I}}(\mathrm{r})|,\ \mathcal{W}_4(\mathrm{r}) = |\mathsf{R}(\mathrm{r}) - \widehat{\mathsf{R}}(\mathrm{r})|.$$

For $\mathrm{r} \in \mathbb{T}^+$, calculating the Dini derivative and using [25, Lemma 4.2], it follows that $D^+\mathcal{W}_1(\mathrm{r})^\Delta$ of $W_1(\mathrm{r})$ along system (4.7),

$$
\begin{aligned}
D^{+}\mathcal{W}_1^{\Delta}(\mathrm{r}) &\leq sign(\mathsf{S}(\sigma(\mathrm{r})) - \widehat{\mathsf{S}}(\sigma(\mathrm{r})))[\mathsf{S}(\mathrm{r}) - \widehat{\mathsf{S}}(\mathrm{r})]^{\Delta} \\
&\leq sign(\mathsf{S}(\sigma(\mathrm{r})) - \widehat{\mathsf{S}}(\sigma(\mathrm{r})))\Big[(\alpha(\mathrm{r}) - \beta(\mathrm{r})\mathsf{S}(\mathrm{r})\mathsf{I}(\mathrm{r}) - (\alpha(\mathrm{r}) + \gamma(\mathrm{r}))\mathsf{S}(\mathrm{r})) \\
&\qquad - (\alpha(\mathrm{r}) - \beta(\mathrm{r})\widehat{\mathsf{S}}(\mathrm{r})\widehat{\mathsf{I}}(\mathrm{r}) - (\alpha(\mathrm{r}) + \gamma(\mathrm{r}))\widehat{\mathsf{S}}(\mathrm{r}))\Big] \\
&\leq sign(\mathsf{S}(\sigma(\mathrm{r})) - \widehat{\mathsf{S}}(\sigma(\mathrm{r})))\Big[\beta(\mathrm{r})\widehat{\mathsf{S}}(\mathrm{r})\widehat{\mathsf{I}}(\mathrm{r}) - \beta(\mathrm{r})\mathsf{S}(\mathrm{r})\mathsf{I}(\mathrm{r}) \\
&\qquad - (\alpha(\mathrm{r}) + \gamma(\mathrm{r}))(\mathsf{S}(\mathrm{r}) - \widehat{\mathsf{S}}(\mathrm{r}))\Big] \\
&\leq sign(\mathsf{S}(\sigma(\mathrm{r})) - \widehat{\mathsf{S}}(\sigma(\mathrm{r})))\Big[- \beta(\mathrm{r})(\mathsf{S}(\mathrm{r}) - \widehat{\mathsf{S}}(\mathrm{r}))\mathsf{I}(\mathrm{r}) - \beta(\mathrm{r})\widehat{\mathsf{S}}(\mathrm{r})(\mathsf{I}(\mathrm{r}) - \widehat{\mathsf{I}}(\mathrm{r})) \\
&\qquad - (\alpha(\mathrm{r}) + \gamma(\mathrm{r}))(\mathsf{S}(\mathrm{r}) - \widehat{\mathsf{S}}(\mathrm{r}))\Big] \\
&\leq sign(\mathsf{S}(\sigma(\mathrm{r})) - \widehat{\mathsf{S}}(\sigma(\mathrm{r})))\Big[- (\alpha(\mathrm{r}) + \beta(\mathrm{r}) + \gamma(\mathrm{r}))(\mathsf{S}(\mathrm{r}) - \widehat{\mathsf{S}}(\mathrm{r})) \\
&\qquad - \beta(\mathrm{r})\widehat{\mathsf{S}}(\mathrm{r})(\mathsf{I}(\mathrm{r}) - \widehat{\mathsf{I}}(\mathrm{r}))\Big] \\
&\leq -\Big[\alpha^{\mathsf{L}} + \beta^{\mathsf{L}} + \gamma^{\mathsf{L}}\Big]\Big|\mathsf{S}(\mathrm{r}) - \widehat{\mathsf{S}}(\mathrm{r})\Big| - \beta^{\mathsf{L}}\mathsf{S}^{-}\Big|\mathsf{I}(\mathrm{r}) - \widehat{\mathsf{I}}(\mathrm{r})\Big|.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
D^{+}\mathcal{W}_2^{\Delta}(\mathrm{r}) &\leq sign(\mathsf{V}(\sigma(\mathrm{r})) - \widehat{\mathsf{V}}(\sigma(\mathrm{r})))[\mathsf{V}(\mathrm{r}) - \widehat{\mathsf{V}}(\mathrm{r})]^{\Delta} \\
&\leq sign(\mathsf{V}(\sigma(\mathrm{r})) - \widehat{\mathsf{V}}(\sigma(\mathrm{r})))\Big[\gamma(\mathrm{r})\mathsf{S}(\mathrm{r}) - \beta_1(\mathrm{r})\mathsf{V}(\mathrm{r})\mathsf{I}(\mathrm{r}) - (\alpha(\mathrm{r}) + \delta_1(\mathrm{r}))\mathsf{V}(\mathrm{r}) \\
&\qquad - (\gamma(\mathrm{r})\widehat{\mathsf{S}}(\mathrm{r}) - \beta_1(\mathrm{r})\widehat{\mathsf{V}}(\mathrm{r})\widehat{\mathsf{I}}(\mathrm{r}) - (\alpha(\mathrm{r}) + \delta_1(\mathrm{r}))\widehat{\mathsf{V}}(\mathrm{r}))\Big] \\
&\leq sign(\mathsf{V}(\sigma(\mathrm{r})) - \widehat{\mathsf{V}}(\sigma(\mathrm{r})))\Big[\gamma(\mathrm{r})(\mathsf{S}(\mathrm{r}) - \widehat{\mathsf{S}}(\mathrm{r})) - \beta_1(\mathrm{r})\mathsf{V}(\mathrm{r})\mathsf{I}(\mathrm{r}) \\
&\qquad + \beta_1(\mathrm{r})\widehat{\mathsf{V}}(\mathrm{r})\widehat{\mathsf{I}}(\mathrm{r}) - (\alpha(\mathrm{r}) + \delta_1(\mathrm{r}))(\mathsf{V}(\mathrm{r}) - \widehat{\mathsf{V}}(\mathrm{r}))\Big] \\
&\leq sign(\mathsf{V}(\sigma(\mathrm{r})) - \widehat{\mathsf{V}}(\sigma(\mathrm{r})))\Big[\gamma(\mathrm{r})(\mathsf{S}(\mathrm{r}) - \widehat{\mathsf{S}}(\mathrm{r})) - \beta_1(\mathrm{r})\mathsf{I}(\mathrm{r})(\mathsf{V}(\mathrm{r}) - \widehat{\mathsf{V}}(\mathrm{r})) \\
&\qquad - \beta_1(\mathrm{r})\widehat{\mathsf{V}}(\mathrm{r})(\mathsf{I}(\mathrm{r}) - \widehat{\mathsf{I}}(\mathrm{r})) - (\alpha(\mathrm{r}) + \delta_1(\mathrm{r}))(\mathsf{V}(\mathrm{r}) - \widehat{\mathsf{V}}(\mathrm{r}))\Big] \\
&\leq \gamma^{\mathsf{U}}\Big|\mathsf{S}(\mathrm{r}) - \widehat{\mathsf{S}}(\mathrm{r})\Big| - (\alpha^{\mathsf{L}} + \beta_1^{\mathsf{L}}\mathsf{I}^{-} + \delta_1^{\mathsf{L}})\Big|\mathsf{V}(\mathrm{r}) - \widehat{\mathsf{V}}(\mathrm{r})\Big| - \delta_1^{\mathsf{L}}\mathsf{V}^{-}\Big|\mathsf{I}(\mathrm{r}) - \widehat{\mathsf{I}}(\mathrm{r})\Big|.
\end{aligned}
$$

$$
\begin{aligned}
D^{+}\mathcal{W}_3^{\Delta}(r) &\le sign(\mathsf{I}(\sigma(r)) - \widehat{\mathsf{I}}(\sigma(r)))[\mathsf{I}(r) - \widehat{\mathsf{I}}(r)]^{\Delta} \\
&\le sign(\mathsf{I}(\sigma(r)) - \widehat{\mathsf{I}}(\sigma(r)))\Big[(\beta(r)\mathsf{S}(r)\mathsf{I}(r) + \beta_1(r)\mathsf{V}(r)\mathsf{I}(r) - \delta(r)\mathsf{I}(r) \\
&\qquad - \alpha(r)\mathsf{I}(r) + \mathsf{d}(r)\mathsf{R}(r)) - (\beta(r)\widehat{\mathsf{S}}(r)\widehat{\mathsf{I}}(r) + \beta_1(r)\widehat{\mathsf{V}}(r)\widehat{\mathsf{I}}(r) \\
&\qquad - \delta(r)\widehat{\mathsf{I}}(r) - \alpha(r)\widehat{\mathsf{I}}(r) + \mathsf{d}(r)\widehat{\mathsf{R}}(r))\Big] \\
&\le sign(\mathsf{V}(\sigma(r)) - \widehat{\mathsf{V}}(\sigma(r)))\Big[\beta(r)(\mathsf{S}(r)\mathsf{I}(r) - \widehat{\mathsf{S}}(r)\widehat{\mathsf{I}}(r)) \\
&\qquad + \beta_1(r)(\mathsf{V}(r)\mathsf{I}(r) - \widehat{\mathsf{V}}(r)\widehat{\mathsf{I}}(r)) \\
&\qquad - (\alpha(r) + \delta(r))(\mathsf{I}(r) - \widehat{\mathsf{I}}(r)) + \mathsf{d}(r)(\mathsf{R}(r) - \widehat{\mathsf{R}}(r))\Big] \\
&\le sign(\mathsf{V}(\sigma(r)) - \widehat{\mathsf{V}}(\sigma(r)))\Big[\beta(r)\mathsf{I}(r)(\mathsf{S}(r) - \widehat{\mathsf{S}}(r)) + \beta(r)\widehat{\mathsf{S}}(r)(\mathsf{I}(r) - \widehat{\mathsf{I}}(r)) \\
&\qquad + \beta_1(r)\mathsf{I}(r)(\mathsf{V}(r) - \widehat{\mathsf{V}}(r)) + \beta_1(r)\widehat{\mathsf{V}}(r)(\mathsf{I}(r) - \widehat{\mathsf{I}}(r)) \\
&\qquad - (\alpha(r) + \delta(r))(\mathsf{I}(r) - \widehat{\mathsf{I}}(r)) + \mathsf{d}(r)(\mathsf{R}(r) - \widehat{\mathsf{R}}(r))\Big] \\
&\le sign(\mathsf{V}(\sigma(r)) - \widehat{\mathsf{V}}(\sigma(r)))\Big[\beta(r)\mathsf{I}(r)(\mathsf{S}(r) - \widehat{\mathsf{S}}(r)) \\
&\qquad - (\alpha(r) - \beta(r)\widehat{\mathsf{S}}(r) - \beta_1(r)\widehat{\mathsf{V}}(r) \\
&\qquad + \delta(r))(\mathsf{I}(r) - \widehat{\mathsf{I}}(r)) + \beta_1(r)\mathsf{I}(r)(\mathsf{V}(r) - \widehat{\mathsf{V}}(r)) + \mathsf{d}(r)(\mathsf{R}(r) - \widehat{\mathsf{R}}(r))\Big] \\
&\le \beta^{\mathsf{U}}\mathsf{I}^{+}\big|\mathsf{S}(r) - \widehat{\mathsf{S}}(r)\big| + \beta_1^{\mathsf{U}}\mathsf{I}^{+}\big|\mathsf{V}(r) - \widehat{\mathsf{V}}(r)\big| - (\alpha^{\mathsf{L}} - \beta^{\mathsf{U}}\mathsf{S}^{+} - \beta_1^{\mathsf{U}}\mathsf{V}^{+} + \delta^{\mathsf{L}})\big|\mathsf{I}(r) - \widehat{\mathsf{I}}(r)\big| \\
&\qquad + \mathsf{d}^{\mathsf{U}}\big|\mathsf{R}(r) - \widehat{\mathsf{R}}(r)\big|,
\end{aligned}
$$

and

$$
\begin{aligned}
D^{+}\mathcal{W}_4^{\Delta}(r) &\le sign(R(\sigma(r)) - \hat{R}(\sigma(r)))[R(r) - \hat{R}(r)]^{\Delta} \\
&\le sign(R(\sigma(r)) - \hat{R}(\sigma(r)))\Big[(\delta_1(r)\mathsf{V}(r) + \delta(r)\mathsf{I}(r) - \alpha(r)\mathsf{R}(r) \\
&\qquad - \mathsf{d}(r)\mathsf{R}(r)) - (\delta_1(r)\widehat{\mathsf{V}}(r) + \delta(r)\widehat{\mathsf{I}}(r) - \alpha(r)\widehat{\mathsf{R}}(r) - \mathsf{d}(r)\widehat{\mathsf{R}}(r))\Big] \\
&\le sign(R(\sigma(r)) - \hat{R}(\sigma(r)))\Big[\delta_1(r)(\mathsf{V}(r)-)\widehat{\mathsf{V}}(r)) + \delta(r)(\mathsf{I}(r) - \widehat{\mathsf{I}}(r)) \\
&\qquad - (\alpha(r) + \mathsf{d}(r))(\mathsf{R}(r) - \widehat{\mathsf{R}}(r))\Big] \\
&\le \delta_1^{\mathsf{U}}\big|\mathsf{V}(r) - \widehat{\mathsf{V}}(r)\big| + \delta^{\mathsf{U}}\big|\mathsf{I}(r) - \widehat{\mathsf{I}}(r)\big| - (\alpha^{\mathsf{L}} + \mathsf{d}^{\mathsf{L}})\big|\mathsf{R}(r) - \widehat{\mathsf{R}}(r)\big|.
\end{aligned}
$$

Given that $\mathcal{V}(\mathrm{r}) \leq \mathcal{W}(\mathrm{r})$ for $\mathrm{r} \in \mathbb{T}^+$ and based on condition ($\mathscr{B}$), it can be inferred as a result

$$\begin{aligned}
D^+(\mathcal{V}(\mathrm{r}))^\Delta &\leq D^+(\mathcal{W}(\mathrm{r}))^\Delta = D^+(\mathcal{V}_1(\mathrm{r}) + \mathcal{V}_2(\mathrm{r}) + \mathcal{V}_3(\mathrm{r}) + \mathcal{V}_4(\mathrm{r}))^\Delta \\
&\leq -\left[\alpha^L + \beta^L + \gamma^L\right]\left|S(\mathrm{r}) - \widehat{S}(\mathrm{r})\right| - \beta^L S^-\left|I(\mathrm{r}) - \widehat{I}(\mathrm{r})\right| \\
&\quad + \gamma^U\left|S(\mathrm{r}) - \widehat{S}(\mathrm{r})\right| - (\alpha^L + \beta_1^L I^- + \delta_1^L)\left|V(\mathrm{r}) - \widehat{V}(\mathrm{r})\right| - \delta_1^L V^-\left|I(\mathrm{r}) - \widehat{I}(\mathrm{r})\right| \\
&\quad + \beta^U I^+\left|S(\mathrm{r}) - \widehat{S}(\mathrm{r})\right| + \beta_1^U I^+\left|V(\mathrm{r}) - \widehat{V}(\mathrm{r})\right| - (\alpha^L - \beta^U S^+ - \beta_1^U V^+ + \delta^L)\left|I(\mathrm{r}) - \widehat{I}(\mathrm{r})\right| \\
&\quad + d^U\left|R(\mathrm{r}) - \widehat{R}(\mathrm{r})\right| + \delta_1^U\left|V(\mathrm{r}) - \widehat{V}(\mathrm{r})\right| + \delta^U\left|I(\mathrm{r}) - \widehat{I}(\mathrm{r})\right| \\
&\quad - (\alpha^L + d^L)\left|R(\mathrm{r}) - \widehat{R}(\mathrm{r})\right| \\
&\leq -\left[\alpha^L + \beta^L + \gamma^L - \gamma^U - \beta^U I^+\right]\left|S(\mathrm{r}) - \widehat{S}(\mathrm{r})\right| \\
&\quad - \left[\alpha^L + \beta_1^L I^- + \delta_1^L - \beta_1^U I^+ - \delta_1^U\right]\left|V(\mathrm{r}) - \widehat{V}(\mathrm{r})\right| \\
&\quad - (\alpha^L - \beta^U S^+ - \beta_1^U V^+ + \delta^L + \beta^L S^- + \delta_1^L V^- - \delta^U)\left|I(\mathrm{r}) - \widehat{I}(\mathrm{r})\right| \\
&\quad - (\alpha^L + d^L - d^U)\left|R(\mathrm{r}) - \widehat{R}(\mathrm{r})\right| \\
&\leq -\lambda_1\left|S(\mathrm{r}) - \widehat{S}(\mathrm{r})\right| - \lambda_2\left|V(\mathrm{r}) - \widehat{V}(\mathrm{r})\right| - \lambda_3\left|I(\mathrm{r}) - \widehat{I}(\mathrm{r})\right| - \lambda_4\left|R(\mathrm{r}) - \widehat{R}(\mathrm{r})\right| \\
&\leq -\lambda\mathcal{V}(\mathrm{r}).
\end{aligned}$$

Condition ($\mathscr{B}$) ensures that Condition (*iii*) of Lemma 3.10 holds. Consequently, by Lemma 3.10, there exists a unique uniformly asymptotically stable almost periodic solution $(S(\mathrm{r}), V(\mathrm{r}), I(\mathrm{r}), R(\mathrm{r}))$ within $\mho$. Therefore, the argument is complete. □

5. Numerical simulations

In this section, numerical simulations are presented to illustrate the results obtained in the preceding sections on two different almost periodic timescales:
(1) Let $b_1, b_2 > 0$ and consider the timescale

$$\mathbb{P}_{b_1,b_2} = \bigcup_{j=0}^{\infty} [j(b_1 + b_2), j(b_1 + b_2) + b_1].$$

This timescale remains unchanged under translation by integer multiples of $b_1 + b_2$, meaning its translation set is $\prod = \{k(b_1 + b_2) \mid k \in \mathbb{Z}\}$. It includes the point 0 and is frequently employed in modeling the population behavior of species with defined lifespans, where the lifespan parameters are denoted by b_1 and b_2.
(2) Consider a timescale $\mathbb{T} = \{t_n : t_n = nT, n \in \mathbb{Z}\}$, where T is a constant. This timescale consists of discrete points that are spaced periodically by T. The translation set in this case will be $\prod = \{T\}$, as adding T to any point results in another valid point in the timescale.

Example 5.1. Consider the dynamic SVIR model (2.1) with relapse on timescale:

$$\begin{cases}
S^\Delta(\mathrm{r}) = \alpha(\mathrm{r}) - \beta(\mathrm{r})S(\mathrm{r})I(\mathrm{r}) - (\alpha(\mathrm{r}) + \gamma(\mathrm{r}))S(\mathrm{r}), \\
V^\Delta(\mathrm{r}) = \gamma(\mathrm{r})S(\mathrm{r}) - \beta_1(\mathrm{r})V(\mathrm{r})I(\mathrm{r}) - (\alpha(\mathrm{r}) + \delta_1(\mathrm{r}))V(\mathrm{r}), \\
I^\Delta(\mathrm{r}) = \beta(\mathrm{r})S(\mathrm{r})I(\mathrm{r}) + \beta_1(\mathrm{r})V(\mathrm{r})I(\mathrm{r}) - \delta(\mathrm{r})I(\mathrm{r}) - \alpha(\mathrm{r})I(\mathrm{r}) + d(\mathrm{r})R(\mathrm{r}), \\
R^\Delta(\mathrm{r}) = \delta_1(\mathrm{r})V(\mathrm{r}) + \delta(\mathrm{r})I(\mathrm{r}) - \alpha(\mathrm{r})R(\mathrm{r}) - d(\mathrm{r})R(\mathrm{r}),
\end{cases} \tag{5.1}$$

where $\alpha(r) = 10 + 0.1|\sin\sqrt{5}r|$, $\beta = 3 + |\cos\sqrt{2}r|$, $\beta_1(r) = 0.04 + |\sin(\pi r)|$, $\gamma(r) = 0.01 + |\sin(4\pi r)|$, $\delta(r) = 3\times10^{-3} + 2\times10^{-4}|\cos((\pi/3)r)|$, $\delta_1(r) = 0.02 + 2|\sin(5\pi r)|$, $d(r) = 0.01 + |\cos(7\pi r)|$.

We have adopted periodic functions similar to those used in [26], where the specific functional forms and properties were detailed.

By direct calculations, we obtain $\alpha^L = 10, \alpha^U = 10.1, \beta^L = 3, \beta^U = 4, \beta_1^L = 0.04, \beta_1^U = 1.04$, $\gamma^L = 0.01, \gamma^U = 1.01, \delta^L = 3\times10^{-3}, \delta^U = 3\times10^{-3}+2\times10^{-4}, \delta_1^L = 0.02, \delta_1^U = 2.02, d^L = 0.01, d^U = 1.01$. Thus, $S^+ \approx 1.009, V^+ \approx 0.1017, I^+ \approx 1.01, r^+ \approx 0.0208, S^- \approx 0.66, V^- \approx 0.0005, r^- \approx 9.02203149\times10^{-7}$, and $I^- \approx 8.92987518031\times10^{-10}$. Therefore, by Theorem 4.4, (5.1) is permanent.

Using the above values, we find $\lambda_1 = 7.96, \lambda_2 = 6.9496, \lambda_3 = 7.8383, \lambda_4 = 9$. So, $\lambda = \min\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\} = \lambda_2 > 0$. By Theorem 4.6, (5.1) has a unique almost periodic solution $(S(r), V(r), I(r), R(r)) \in \mho$ and is uniformly asymptotically stable. From Figure 3, it is evident that system (5.1) admits a positive almost periodic solution, represented by $(S^\star(r), V^\star(r), I^\star(r), r^\star(r))$. Furthermore, Figure 4 illustrates that any positive solution $(S(r), V(r), I(r), R(r))$ converges to this almost periodic solution $(S^\star(r), V^\star(r), I^\star(r), r^\star(r))$.

Additionally, Figure 4 demonstrate that varying initial conditions lead the disease dynamics to approach different almost periodic solutions. These results are analyzed on the timescale $\mathbb{P}_{b_1,b_2}$ with parameters $b_1 = 0$, $b_2 = 2$, and intervals $j = 10$. This suggests that, in addition to implementing appropriate control strategies, modifying initial conditions could influence the progression of the disease.

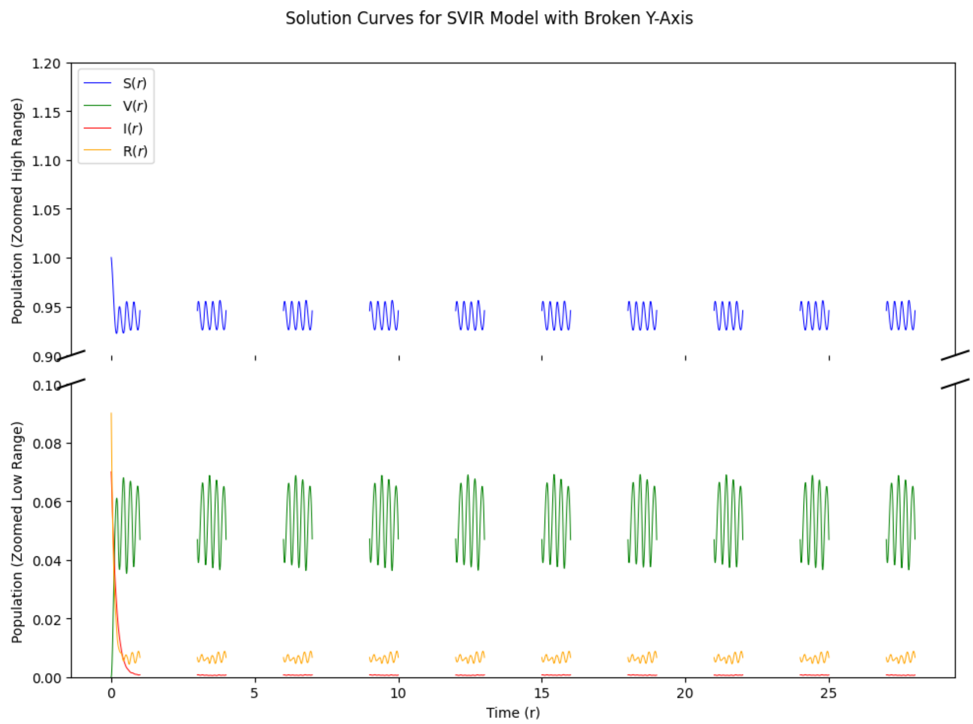

Figure 3. The plot highlights the relative stability of the compartments and the dominant role of the susceptible group with initial conditions $V(0) = 0$, $I(0) = 0.0009$, $R(0) = 0$.

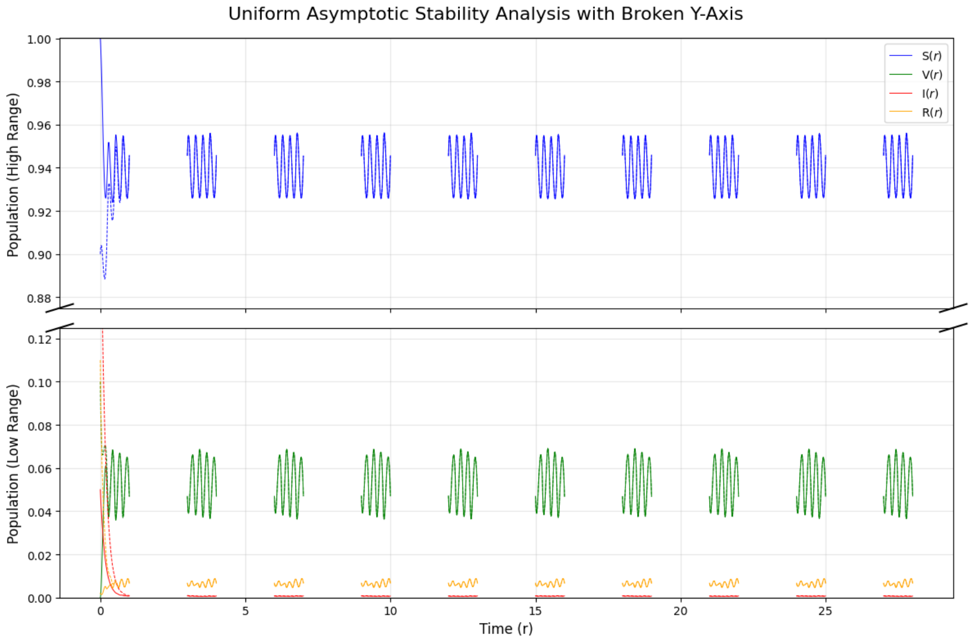

Figure 4. Uniform asymptotic stability of the SVIR model (5.1). The time series of $(\mathsf{S}(\mathtt{r}), \mathsf{V}(\mathtt{r}), \mathsf{I}(\mathtt{r}), \mathsf{R}(\mathtt{r}))$ and $(\mathsf{S}^{\star}(\mathtt{r}), \mathsf{V}^{\star}(\mathtt{r}), \mathsf{I}^{\star}(\mathtt{r}), \mathsf{R}^{\star}(\mathtt{r}))$ with initial values $\mathsf{S}(0) = 1, \mathsf{V}(0) = 0, \mathsf{I}(0) = 0.05, \mathsf{R}(0) = 0.002$, and $\mathsf{S}^{\star}(0) = 0.9, \mathsf{V}^{\star}(0) = 0.1, \mathsf{I}^{\star}(0) = 0.2, \mathtt{r}^{\star}(0) = 0.11$.

Observations

Susceptible Population $\mathsf{S}(\mathtt{r})$: The blue curve in the Figure 5 shows a periodic oscillations in the susceptible population. In the zoomed high range (top panel), the amplitude oscillates slightly below 1.0 but remains above 0.9, indicating most of the population remains susceptible. The oscillations suggest periodic variations in susceptibility, possibly due to external factors like seasonal changes or intervention measures.

Vaccinated Population $\mathsf{V}(\mathtt{r})$: The green curve in the Figure 6 represents the vaccinated population. In the zoomed low range (bottom panel), the amplitude is much lower compared to $\mathsf{S}(\mathtt{r})$, oscillating between 0.02 and 0.07. This indicates a smaller proportion of the population is vaccinated, with periodic variations implying vaccination campaigns or waning immunity.

Infectious Population $\mathsf{I}(\mathtt{r})$: The red curve in the Figure 7 representing the infectious population, is near zero in the bottom panel. Minimal infections suggest successful control measures, like vaccination or immunity.

Recovered Population $\mathsf{R}(\mathtt{r})$: The purple curve for recovered individuals also shows a low amplitude in the bottom panel in the Figure 8. Recovery oscillates minimally, indicating either a low infection rate or strong immunity within the vaccinated population.

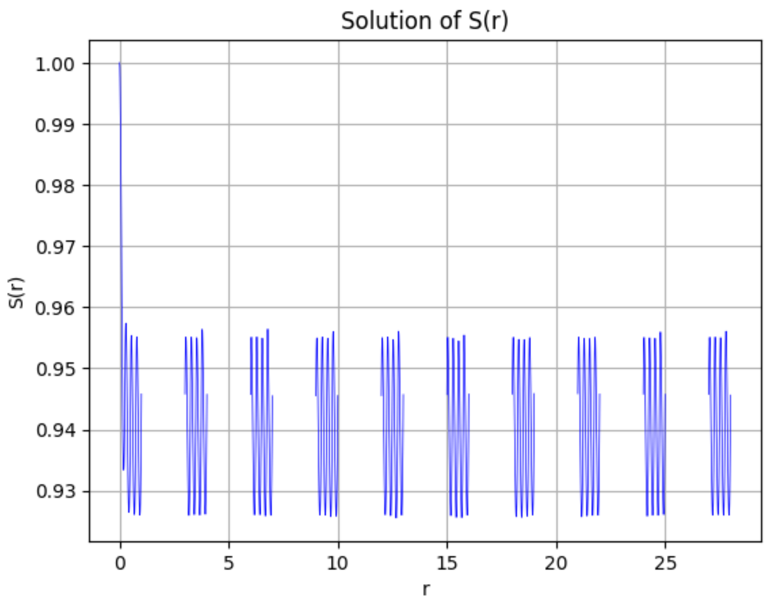

Figure 5. The plot illustrates the dynamics of the susceptible population over time, showing oscillations converging to an almost periodic positive solution with the initial condition S(0) = 1.

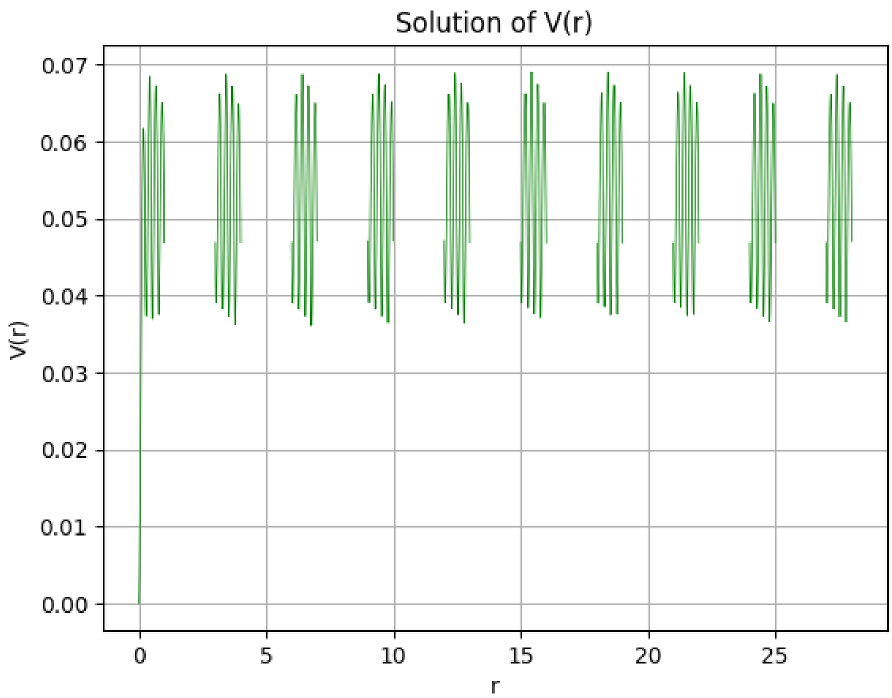

Figure 6. The plot depicts the growth and stabilization of the vaccinated population over time, converging to an almost periodic positive solution with the initial condition V(0) = 0.

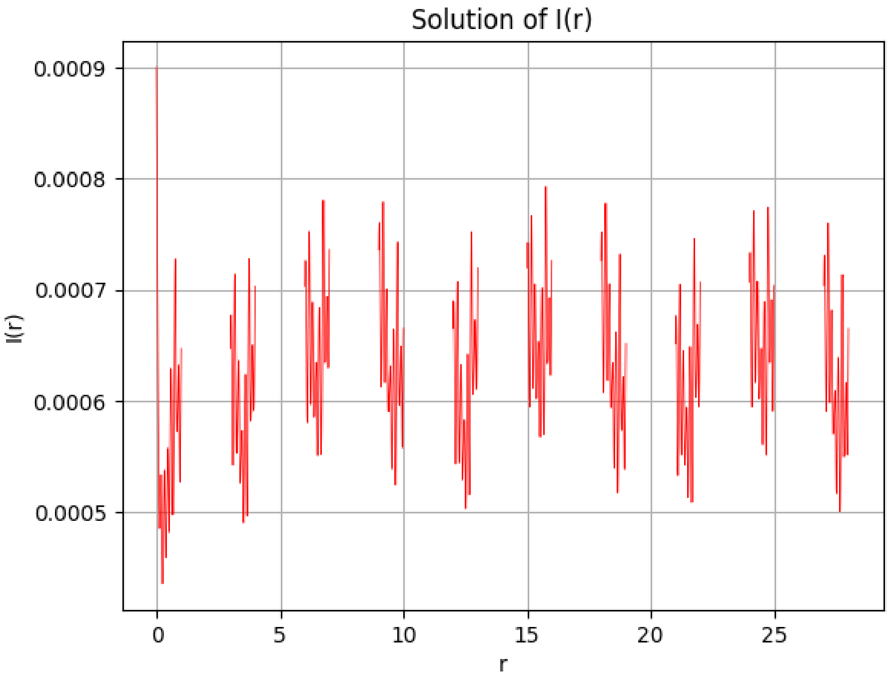

Figure 7. The plot shows the transient behavior of the infected population, which decreases and stabilizes into an almost periodic positive solution with the initial condition $\mathsf{I}(0) = 0.0009$.

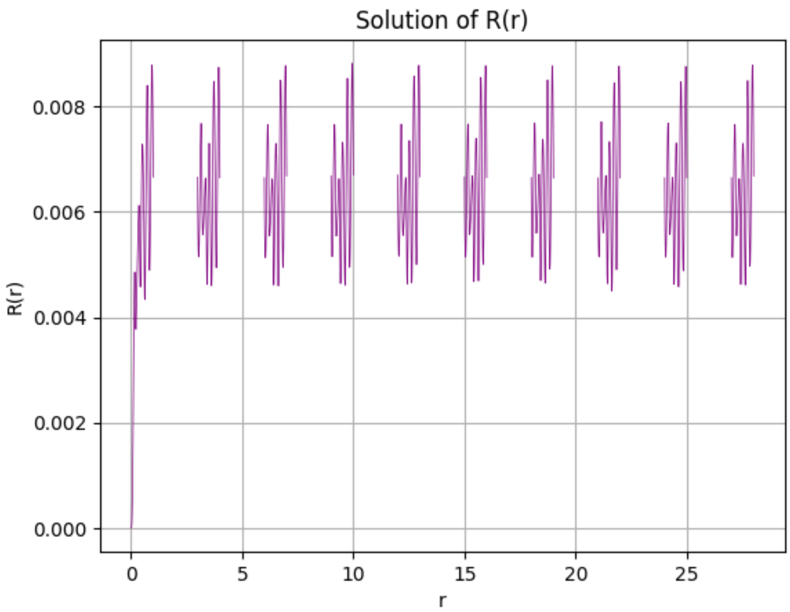

Figure 8. The plot highlights the evolution of the recovered population over time, stabilizing into an almost periodic positive solution with the initial condition $\mathsf{R}(0) = 0$.

Broken Y-axis plot

- The combined plot with a broken y-axis distinctly separates the behavior of different compartments.
- It highlights the dominant role of the susceptible group and emphasizes the relative stability of other compartments in maintaining almost periodic behavior.

Model implications

- Permanence: The bounded and oscillatory nature of all compartments confirms the permanence of the epidemic system under the given conditions.
- Vaccination Efficiency: The periodicity in $\mathsf{V}(\mathtt{r})$ suggests a consistent vaccination strategy significantly impacts the control of the infection.
- Stability: The uniform oscillatory nature and lack of divergence in the population dynamics suggest uniform asymptotic stability. This stability is likely influenced by the chosen parameter values, including transmission rates, recovery rates, and vaccination rates. The specific parameters used ensure that the system remains within a bounded region of the phase space, thereby maintaining equilibrium-like behavior.
- Epidemic Control: Lower values in $\mathsf{I}(\mathtt{r})$ and $\mathsf{R}(\mathtt{r})$, coupled with the interaction between $\mathsf{S}(\mathtt{r})$ and $\mathsf{V}(\mathtt{r})$, highlight the importance of vaccination and relapse reduction in controlling disease spread.
- Parameter Sensitivity: The observed stability and permanence are potentially sensitive to variations in model parameters. Future work should explore parameter ranges and their effects on system behavior to confirm the robustness of these results.

Uniform asymptotic stability of almost periodic solutions:
The plot demonstrates that all compartments: $\mathsf{S}(\mathtt{r}), \mathsf{V}(\mathtt{r}), \mathsf{I}(\mathtt{r})$, and $\mathsf{R}(\mathtt{r})$ stabilize into periodic or almost periodic oscillations after transient fluctuations. This behavior validates the uniform asymptotic stability of almost periodic solutions, as each compartment consistently returns to a regular oscillatory pattern despite initial deviations.

In previous analyses, the system was studied on the timescale $\mathbb{P}_{\mathsf{b}_1,\mathsf{b}_2}$, where the dynamic behavior of the epidemic compartments exhibited bounded oscillations, indicating permanence, stability, and the impact of periodic vaccination strategies. To further analyze the epidemic dynamics under discrete periodic time steps, we now examine the system on the discrete timescale $\mathbb{T} = \{t_n : t_n = nT, n \in \mathbb{Z}\}$, where each time point is spaced by a fixed interval $T = 1.5$.

This formulation allows us to study how the epidemic evolves when disease transmission, vaccination, and recovery occur at discrete intervals rather than continuously, reflecting real-world scenarios such as periodic health interventions or time-structured contact patterns. Unlike previous results on $\mathbb{P}_{\mathsf{b}_1,\mathsf{b}_2}$, where the system was analyzed over a generalized timescale, the discrete structure here ensures that all solution trajectories align exactly with the timescale points.

Susceptible Population: The plot of $\mathsf{S}(t_n)$ in the Figure 9 exhibits a damped initial peak followed by oscillatory behavior with small fluctuations around a nearly steady state. The pattern suggests quasi-periodic variations, indicating a tendency toward almost periodicity.

Vaccination Efficiency: From the Figure 10, the periodic fluctuations in $\mathsf{V}(t_n)$ suggest that consistent vaccination efforts significantly impact infection control. The interaction between $\mathsf{S}(t_n)$ and $\mathsf{V}(t_n)$ highlights the role of vaccination in shaping long-term epidemic trends on this timescale.

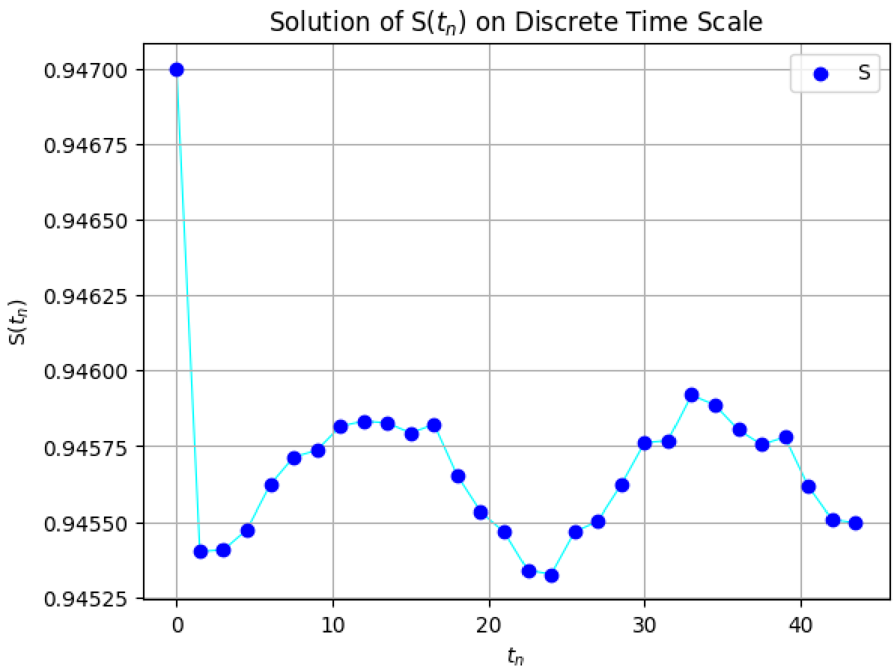

Figure 9. The plot illustrates the dynamics of the susceptible population over time, showing oscillations converging to an almost periodic positive solution with the initial condition S(0) = 0.947.

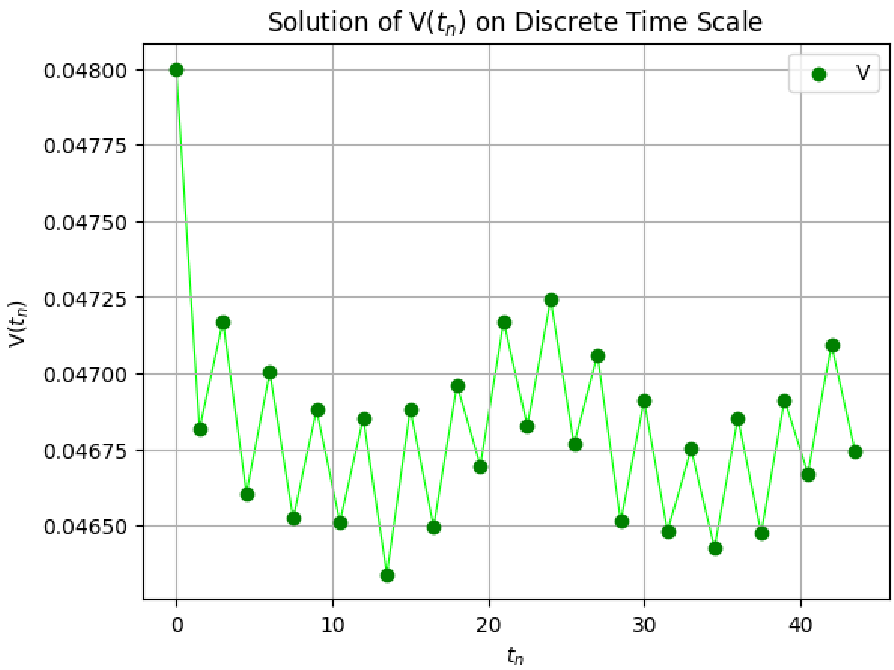

Figure 10. The periodic nature of V(t_n) with the initial condition V(0) = 0.048 suggests a stable vaccination impact, with fluctuations influenced by both vaccination rates and disease prevalence.

Infected Population: The plot of $I(t_n)$ in the Figure 11 shows an initial sharp decline followed by persistent oscillations with small amplitude variations. This pattern suggests a dynamic equilibrium with quasi-periodic fluctuations, indicating an almost periodic tendency.

Recovered Population: The plot of $R(t_n)$ in the Figure 12 initially increases and then exhibits sustained oscillations with alternating peaks and troughs. This repeating pattern suggests a long-term dynamic balance with almost periodic variations.

Stability: From the Figure 13, the uniform oscillatory nature of the solutions and the absence of divergence confirm uniform asymptotic stability. The bounded trajectories indicate that the system remains within a constrained region of phase space, preventing uncontrolled outbreaks or disease extinction under the given parameter choices.

Permanence: The bounded and oscillatory nature of all compartments confirms the persistence of the epidemic system over the discrete timescale. The system does not collapse to zero or diverge, indicating that the disease remains present in a dynamic but controlled manner.

Epidemic Control: The interaction between $S(t_n)$, $V(t_n)$, and the lower oscillatory values of $I(t_n)$ and $R(t_n)$ emphasize the importance of vaccination and relapse reduction in disease control. The discrete nature of the model demonstrates how periodic interventions influence infection levels at each step.

Parameter Sensitivity: The observed stability and permanence suggest that the system's behavior is sensitive to changes in model parameters (e.g., transmission rates, vaccination rates, and recovery rates). Future research should analyze the effects of varying these parameters to assess the robustness of these findings under different epidemic conditions.

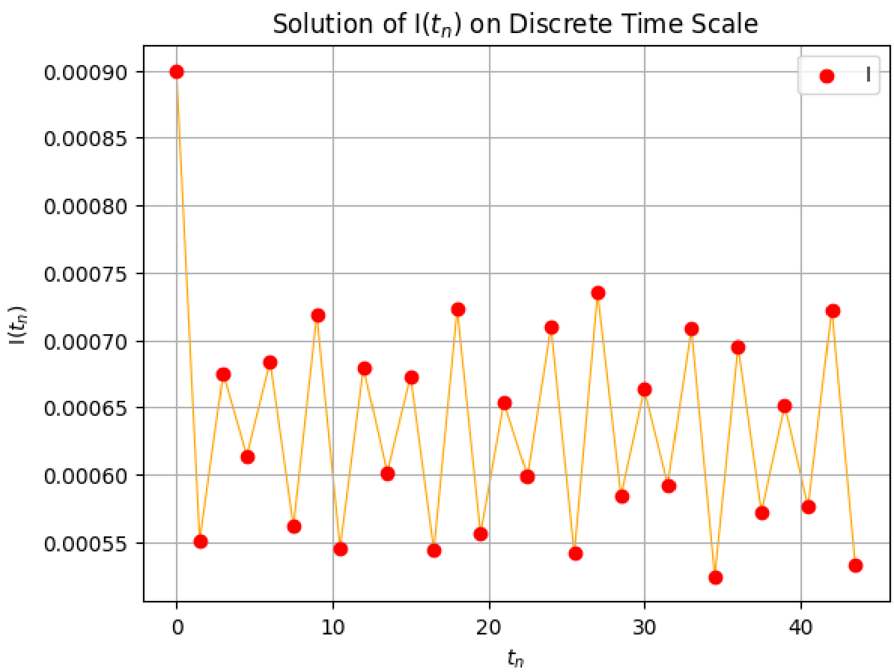

Figure 11. The oscillatory but non-divergent behavior of $I(t_n)$ with the initial condition $I(0) = 0.0009$ confirms disease persistence and stability, influenced by vaccination and relapse mechanisms. The periodic pattern suggests structured outbreaks that recur at fixed intervals.

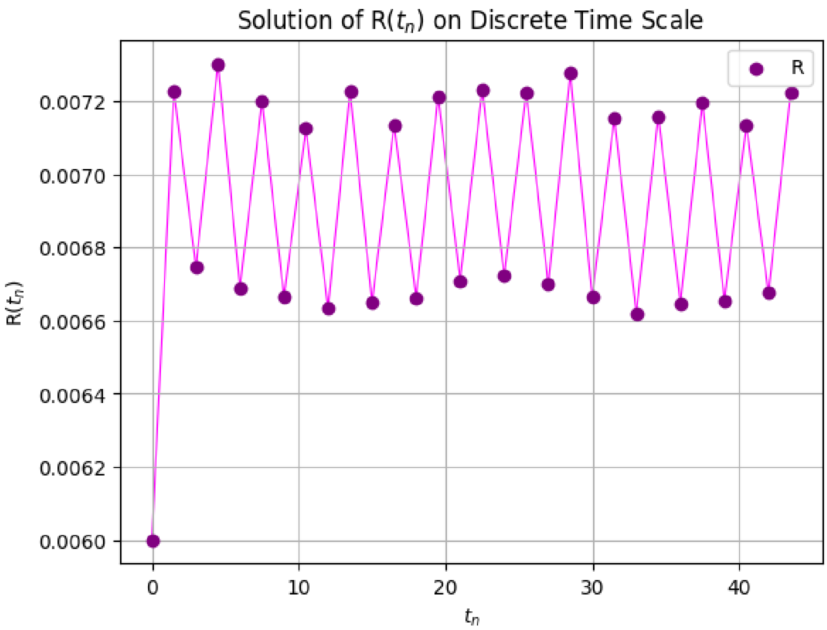

Figure 12. The bounded oscillations indicate that recovery and reinfection mechanisms interact cyclically, reinforcing the system's long-term stability under the given parameters.

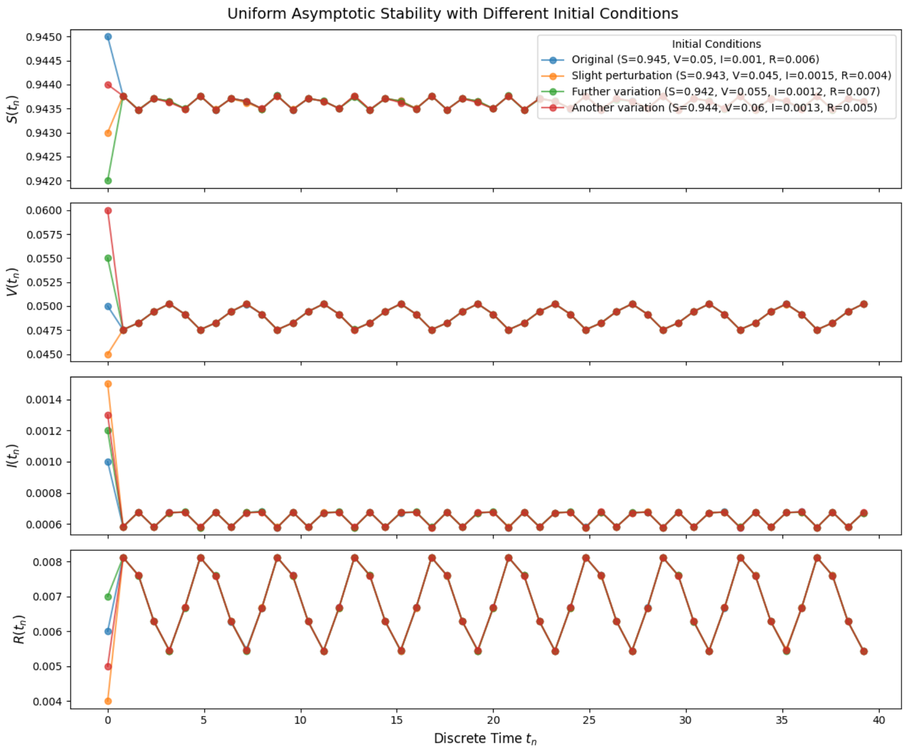

Figure 13. Plot showing the uniform asymptotic stability of the system with different initial conditions for the susceptible (S), vaccinated (V), infected (I), and recovered (R) populations. The results demonstrate the system's tendency to return to equilibrium over time, with slight variations in initial conditions.

6. Conclusions

Throughout this study, we developed and analyzed a timescale dynamic SVIR epidemic model incorporating relapse dynamics. By investigating the interplay between vaccination, infection, recovery, and relapse, we provided a rigorous theoretical framework for understanding the system's behavior over time. The model's mathematical analysis demonstrated the existence of almost periodic solutions and established their uniform asymptotic stability, highlighting the system's ability to maintain predictable, long-term dynamics under certain conditions.

Our numerical simulations confirmed the theoretical results, illustrating the system's permanence and revealing the critical roles played by vaccination and relapse dynamics in epidemic control. Notably, we observed that as long as the infected population remains smaller than the susceptible population, the infection can be effectively controlled, keeping both the infected and recovered populations at relatively low levels. These findings emphasize the importance of periodic vaccinations, which contribute to stabilizing disease dynamics through their influence on oscillatory behaviors.

Figures demonstrated the almost periodic solutions for each compartment, with periodic oscillations suggesting the stabilization of the system over time. The oscillatory behavior reflects the dynamic nature of disease spread and control. A combined plot of all four compartments emphasized the dominant role of the susceptible population, with other compartments exhibiting relative stability. Furthermore, the convergence of the system toward the almost periodic solution confirmed the uniform asymptotic stability of the model, ensuring that the system's behavior stabilizes regardless of initial conditions.

While this study develops substantial insight into the dynamics of the SVIR model with relapse, several avenues for further research remain open:

1. Extension to Multi-Strain Models: One of the future works might consider the extension of the present model by considering multi-strains of disease with their respective relapse and vaccination parameters. This will help in modeling more complicated disease systems, for instance, those involving antigenic variation or multiple types of vaccines.

2. Incorporation of Stochastic Elements: The model presented here assumes deterministic dynamics, while real-world systems are often governed by uncertainty and randomness. Incorporating stochastic elements into the model could provide a more realistic representation of disease transmission, especially in small or heterogeneous populations.

3. Impact of External Interventions: Further extensions could be made for external interventions such as quarantine measures, travel restrictions, or behavioral changes—say, keeping a distance from each other. The study of stability of such systems and the efficacy of diseases in view of these interventions would be an interesting area of future research. Spatial heterogeneity might give more insights, taking into consideration the spatial dynamics where individuals are distributed across different regions with different infection rates and vaccination strategies. In fact, this might be really useful for modeling pandemics across countries or even continents.

This finally brings us to the end of our insight into the study of the SVIR model with relapse, which provides further understanding in epidemic dynamics, especially on vaccination and relapse. Periodic control measures are of immense importance, and it gives an example for future disease management strategies. Future work will expand these results in more complex models and with real-world applications.

Author contributions

Sabbavarapu Nageswara Rao: Conceptualization, validation, formal analysis, investigation, resources, writing-review and editing, supervision, funding acquisition; Mahammad Khuddush: Conceptualization, methodology, software, formal analysis, investigation, data curation, writing-original draft preparation, visualization, supervision; Ahmed H. Msmali: Conceptualization, methodology, validation, data curation, writing-review and editing, supervision; Ali H. Hakami: Conceptualization, validation, writing-review and editing, supervision, project administration. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

It is declared that authors has no competing interests.

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