Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(2): 4153–4172.
DOI:10.3934/math.2025193
Received: 22 November 2024
Revised: 12 February 2025
Accepted: 17 February 2025
Published: 27 February 2025

Research article

On pairs of equations with unequal powers of primes and powers of 2

Li Zhu*

School of Statistics and Mathematics, Shanghai Lixin University of Accounting and Finance,
No. 995, Shangchuan Road, Pudong New Area, Shanghai 201209, China

* **Correspondence:** Email: shulunzl@163.com.

Abstract: It is proved that every pair of sufficiently large even integers can be represented in the form of a pair of equations, each containing two squares of primes, two cubes of primes, two biquadrates of primes, and 30 powers of 2. Moreover, we also proved that every sufficiently large even integer can be expressed as the sum of two squares of primes, two cubes of primes, two biquadrates of primes, and 14 powers of 2. These two theorems constitute improvements upon the previous results.

Keywords: Waring–Goldbach problem; Hardy–Littlewood method; powers of 2; additive number theory; Linnik problem
Mathematics Subject Classification: 11P05, 11P55

1. Introduction

The Goldbach conjecture is one of the most famous problems, and numerous variations have been derived from it. In the 1950s, Linnik [9, 10] showed that every sufficiently large even integer can be represented as a sum of two primes and K powers of 2, where K is an absolute constant. In 1975, Gallagher [2] established an asymptotic formula for the number of such representations. In 1998, the explicit value of K was first obtained by Liu, Liu, and Wang [11]. They showed that $K = 54000$ is acceptable. Afterwards, many mathematicians improved the value of K (see [4, 7, 8, 12, 15, 16, 19]). The best result so far is due to Pintz and Ruzsa [16], who proved that $K = 8$ is acceptable.

In 2017, motivated by the works of Linnik [9, 10], Liu [13] studied a Goldbach–Linnik problem with unequal powers of primes. To be specific, he considered the problem on the representation of the large even integer N in the form

$$N = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4 + 2^{v_1} + \cdots + 2^{v_k}, \tag{1.1}$$

where p_i are prime numbers and v_j are positive integers. He proved that (1.1) is solvable for $k = 41$. Subsequently, the acceptable value of k was successively refined by Lü [14], Zhao [23], and Zhang [20].

Very recently, based on the work of [21,22], Zhu [24] further improved the result to $k = 17$.

On the other hand, in 2023, Huang [5] studied Eq (1.1) in an extended way. He attempted to simultaneously represent pairs of positive even integers N_1 and N_2 with $N_2 < N_1 \ll N_2$, in the form

$$\begin{cases} N_1 = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4 + 2^{v_1} + \cdots + 2^{v_k} \\ N_2 = p_7^2 + p_8^2 + p_9^3 + p_{10}^3 + p_{11}^4 + p_{12}^4 + 2^{v_1} + \cdots + 2^{v_k} \end{cases}. \tag{1.2}$$

In [5], he proved that the simultaneous equations (1.2) are solvable for $k = 105$. In 2024, Han, Liu, and Yue [3] improved the value of k to 36.

In this paper, we shall continue to improve the results of [3] and [24] and establish the following sharper results:

Theorem 1. *For $k = 30$, the simultaneous equations (1.2) are solvable for every sufficiently large positive even integers N_1 and N_2 satisfying $N_2 < N_1 \ll N_2$.*

Theorem 2. *For $k = 14$, the Eq (1.1) is solvable for every sufficiently large positive even integer N.*

2. Notation and outline of the method

In this paper, we assume that N_1 and N_2 are sufficiently large even integers satisfying $N_2 < N_1 \ll N_2$. We fix a positive constant η satisfying $\eta \leq 10^{-100}$. Let ε be an arbitrarily small positive number, and the value of ε may change from line to line. The letter p, with or without a subscript, is reserved for a prime number. As usual, we use $e(\alpha)$ to denote $e^{2\pi i\alpha}$, and $\varphi(n)$ stands for the Euler function. Moreover, we write

$$P_{i,j}^{+} = \left(\left(\frac{1}{6} + \eta\right)N_j\right)^{\frac{1}{i}},\ P_{i,j}^{-} = \left(\left(\frac{1}{6} - \eta\right)N_j\right)^{\frac{1}{i}},\ L = \frac{\log(N_1/\log N_1)}{\log 2},$$
$$S_{i,j}(\alpha) = \sum_{P_{i,j}^{-} \leq p \leq P_{i,j}^{+}} e(p^i\alpha)\log p,\ \ H(\alpha) = \sum_{1 \leq v \leq L} e(2^v\alpha).$$

In order to apply the circle method, we set

$$P_j = N_j^{\frac{3}{20}-2\varepsilon} \ \text{ and } \ Q_j = N_j^{\frac{17}{20}+\varepsilon}. \tag{2.1}$$

Then we can define

$$\mathfrak{M} = \mathfrak{M}_1 \times \mathfrak{M}_2 = \left\{(\alpha_1, \alpha_2) : \alpha_1 \in \mathfrak{M}_1, \alpha_2 \in \mathfrak{M}_2\right\},$$
$$\mathfrak{m} = \left[\frac{1}{Q_1}, 1 + \frac{1}{Q_1}\right] \times \left[\frac{1}{Q_2}, 1 + \frac{1}{Q_2}\right] \setminus \mathfrak{M}, \quad \mathfrak{m}_j = \left[\frac{1}{Q_j}, 1 + \frac{1}{Q_j}\right] \setminus \mathfrak{M}_j, \tag{2.2}$$

where

$$\mathfrak{M}_j = \bigcup_{q \leq P_j} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}_j(q,a),\ \mathfrak{M}_j(q,a) = \left(\frac{a}{q} - \frac{1}{qQ_j}, \frac{a}{q} + \frac{1}{qQ_j}\right]. \tag{2.3}$$

Now let

$$\mathcal{R}(k, N_1, N_2) = \sum (\log p_1)(\log p_2)\ldots(\log p_{12})$$

be the weighted number of solutions of (1.2) in $(p_1, \ldots, p_{12}, v_1, \ldots, v_k)$ with

$$\begin{aligned} &P^-_{2,1} \le p_1, p_2 \le P^+_{2,1}, P^-_{3,1} \le p_3, p_4 \le P^+_{3,1}, P^-_{4,1} \le p_5, p_6 \le P^+_{4,1}, \\ &P^-_{2,2} \le p_7, p_8 \le P^+_{2,2}, P^-_{3,2} \le p_9, p_{10} \le P^+_{3,2}, P^-_{4,2} \le p_{11}, p_{12} \le P^+_{4,2}, \\ &1 \le v_1, \ldots, v_k \le L. \end{aligned}$$

Then by orthogonality, we have

$$\begin{aligned} &\mathcal{R}(k, N_1, N_2) \\ = &\int_0^1 \int_0^1 \prod_{1\le j\le 2} \Big(S^2_{2,j}(\alpha_j) S^2_{3,j}(\alpha_j) S^2_{4,j}(\alpha_j) e(-\alpha_j N_j)\Big) H^k(\alpha_1 + \alpha_2) d\alpha_1 d\alpha_2 \\ = &\left(\iint\limits_{\mathfrak{M}} + \iint\limits_{\mathfrak{m}}\right) \prod_{1\le j\le 2} \Big(S^2_{2,j}(\alpha_j) S^2_{3,j}(\alpha_j) S^2_{4,j}(\alpha_j) e(-\alpha_j N_j)\Big) H^k(\alpha_1 + \alpha_2) d\alpha_1 d\alpha_2 \\ := &\mathcal{R}_1(k, N_1, N_2) + \mathcal{R}_2(k, N_1, N_2). \end{aligned} \tag{2.4}$$

For $k \ge 30$, we shall prove

$$\mathcal{R}_1(k, N_1, N_2) \ge 9.946 N_1^{\frac{7}{6}} N_2^{\frac{7}{6}} I^2 L^k, \ \ |\mathcal{R}_2(k, N_1, N_2)| \le 8.6372 N_1^{\frac{7}{6}} N_2^{\frac{7}{6}} I^2 L^k, \tag{2.5}$$

where I is the positive constant defined as (3.4).

3. The lower bound for $\mathcal{R}_1(k, N_1, N_2)$

We first state some auxiliary results. Let

$$\begin{aligned} &C_k(q, a) = \sum_{\substack{m=1 \\ (m,q)=1}}^{q} e\left(\frac{am^k}{q}\right), \ B(n, q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C_2^2(q, a) C_3^2(q, a) C_4^2(q, a) e\Big(-\frac{an}{q}\Big), \\ &A(n, q) = \frac{1}{\varphi^6(q)} B(n, q), \quad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n, q), \\ &g_k(\lambda) = \int_{\frac{1}{6}-\eta}^{\frac{1}{6}+\eta} e(t\lambda) t^{\frac{1}{k}-1} dt, \ \ \mathfrak{J}(n) = \int_{-\infty}^{\infty} g_2(\lambda)^2 g_3(\lambda)^2 g_4(\lambda)^2 e(-n\lambda) d\lambda. \end{aligned}$$

Lemma 3.1. *Let $\mathfrak{M}_j$ be defined as (2.3). Then we have*

$$\int_{\mathfrak{M}_j} S^2_{2,j}(\alpha) S^2_{3,j}(\alpha) S^2_{4,j}(\alpha) e(-n\alpha) d\alpha = \frac{\mathfrak{S}(n) \mathfrak{J}(\frac{n}{N_j}) N_j^{\frac{7}{6}}}{2^2 \cdot 3^2 \cdot 4^2} + O\Big(N_j^{\frac{7}{6}} L^{-1}\Big),$$

where $\mathfrak{S}(n) \gg 1$ for $n \equiv 0 \pmod 2$.

Proof. Let

$$J_j(n) = \sum_{\substack{m_1+m_2+\cdots+m_6=n \\ (\frac{1}{6}-\eta)N_j \le m_i \le (\frac{1}{6}+\eta)N_j \ (1\le i\le 6)}} (m_1 m_2)^{-\frac{1}{2}} (m_3 m_4)^{-\frac{2}{3}} (m_5 m_6)^{-\frac{3}{4}}.$$

It follows from [13, Lemma 2.1] that

$$\int_{\mathfrak{M}_j} S_{2,j}(\alpha)^2 S_{3,j}(\alpha)^2 S_{4,j}(\alpha)^2 e(-n\alpha)d\alpha = \frac{\mathfrak{S}(n)J_j(n)}{2^2\cdot 3^2\cdot 4^2} + O\Big(N_j^{\frac{7}{6}}L^{-1}\Big),$$

and $\mathfrak{S}(n) \gg 1$ for $n \equiv 0 \pmod 2$. Therefore, it suffices to show that

$$J_j(n) = \mathfrak{J}(\frac{n}{N_j})N_j^{\frac{7}{6}} + O\Big(N_j^{\frac{7}{6}}L^{-1}\Big). \tag{3.1}$$

Write

$$u_{k,j}(\lambda) = \int_{((\frac{1}{6}-\eta)N_j)^{\frac{1}{k}}}^{((\frac{1}{6}+\eta)N_j)^{\frac{1}{k}}} e(t^k\lambda)dt, \quad v_{k,j}(\lambda) = \frac{1}{k}\sum_{(\frac{1}{6}-\eta)N_j\le m\le(\frac{1}{6}+\eta)N_j} m^{\frac{1}{k}-1}e(m\lambda).$$

Then we can deduce from the orthogonality that

$$\begin{aligned}
\frac{J_j(n)}{2^2\cdot 3^2\cdot 4^2} &= \int_{-\frac{1}{2}}^{\frac{1}{2}} v_{2,j}(\lambda)^2 v_{3,j}(\lambda)^2 v_{4,j}(\lambda)^2 e(-n\lambda)d\lambda \\
&= \int_{|\lambda|\le\frac{\log N_j}{N_j}} v_{2,j}(\lambda)^2 v_{3,j}(\lambda)^2 v_{4,j}(\lambda)^2 e(-n\lambda)d\lambda + O\left(\int_{\frac{\log N_j}{N_j}}^{\frac{1}{2}} \frac{1}{\lambda^6 N_j^{\frac{23}{6}}}d\lambda\right) \\
&= \int_{|\lambda|\le\frac{\log N_j}{N_j}} u_{2,j}(\lambda)^2 u_{3,j}(\lambda)^2 u_{4,j}(\lambda)^2 e(-n\lambda)d\lambda + O(N_j^{\frac{7}{6}}\log^{-5}N_j),
\end{aligned} \tag{3.2}$$

where the elementary estimates

$$v_{k,j}(\lambda) \ll |\lambda|^{-1}N_j^{\frac{1-k}{k}}, \quad u_{k,j}(\lambda) = \frac{1}{k}\int_{(\frac{1}{6}-\eta)N_j}^{(\frac{1}{6}+\eta)N_j} e(t\lambda)t^{\frac{1}{k}-1}dt = v_{k,j}(\lambda) + O(1)$$

are used. Since $u_{k,j}(\lambda) = \frac{1}{k}N_j^{\frac{1}{k}}\int_{\frac{1}{6}-\eta}^{\frac{1}{6}+\eta} e(tN_j\lambda)t^{\frac{1}{k}-1}dt = \frac{1}{k}N_j^{\frac{1}{k}}g_k(N_j\lambda)$. Then we have

$$\begin{aligned}
&\int_{|\lambda|\le\frac{\log N_j}{N_j}} u_{2,j}(\lambda)^2 u_{3,j}(\lambda)^2 u_{4,j}(\lambda)^2 e(-n\lambda)d\lambda \\
&= \frac{N_j^{\frac{13}{6}}}{2^2\cdot 3^2\cdot 4^2}\int_{|\lambda|\le\frac{\log N_j}{N_j}} g_2(N_j\lambda)^2 g_3(N_j\lambda)^2 g_4(N_j\lambda)^2 e(-n\lambda)d\lambda \\
&= \frac{N_j^{\frac{7}{6}}}{2^2\cdot 3^2\cdot 4^2}\int_{|\lambda|\le\log N_j} g_2(\lambda)^2 g_3(\lambda)^2 g_4(\lambda)^2 e\Big(-\frac{n}{N_j}\lambda\Big)d\lambda \\
&= \frac{N_j^{\frac{7}{6}}\mathfrak{J}(\frac{n}{N_j})}{2^2\cdot 3^2\cdot 4^2} + O\left(N_j^{\frac{7}{6}}\int_{\log N_j}^{\infty}\frac{1}{\lambda^6}d\lambda\right) = \frac{N_j^{\frac{7}{6}}\mathfrak{J}(\frac{n}{N_j})}{2^2\cdot 3^2\cdot 4^2} + O(N_j^{\frac{7}{6}}\log^{-5}N_j),
\end{aligned} \tag{3.3}$$

where the bound $g_k(\lambda) \ll |\lambda|^{-1}$ (see [17, Lemma 4.3]) is used. Now (3.1) follows from (3.2) and (3.3).

Lemma 3.2. *Let*

$$I = \int\cdots\int_{\substack{-\eta<u_i<\eta\ (1\le i\le 5)\\ -\eta<u_1+u_2+u_3+u_4+u_5<\eta}} 1 du_1\ldots du_5. \tag{3.4}$$

Suppose that $\frac{n}{N_j} = 1 + O(\frac{1}{\log N_j})$. *Then we have*

$$\text{(i) } I \ge \Big(\frac{\eta}{5}\Big)^5 > 0, \ \text{ (ii) } \Im\Big(\frac{n}{N_j}\Big) \ge \Big(\frac{1}{6}+2\eta\Big)^{-\frac{23}{6}} I.$$

Proof. For (i), it is easy to see that

$$I \ge \int\cdots\int_{\substack{\frac{-\eta}{10}<u_i<\frac{\eta}{10}\ (1\le i\le 5)\\ -\eta<u_1+u_2+u_3+u_4+u_5<\eta}} 1 du_1\ldots du_5 = \int\cdots\int_{\frac{-\eta}{10}<u_i<\frac{\eta}{10}\ (1\le i\le 5)} 1 du_1\ldots du_5 = \Big(\frac{\eta}{5}\Big)^5 > 0.$$

For (ii), write $t_6 = s - \sum_{i=1}^{5} t_i$, then we have

$$\begin{aligned}
\Im\Big(\frac{n}{N_j}\Big) &= \int_{-\infty}^{\infty} g_2^2(\lambda) g_3^2(\lambda) g_4^2(\lambda) e\Big(-\frac{n}{N_j}\lambda\Big) d\lambda \\
&= \int_{\frac{1}{6}-\eta}^{\frac{1}{6}+\eta}\cdots\int_{\frac{1}{6}-\eta}^{\frac{1}{6}+\eta} (t_1t_2)^{-\frac{1}{2}}(t_3t_4)^{-\frac{2}{3}}(t_5t_6)^{-\frac{3}{4}} dt_1\ldots dt_6 \int_{-\infty}^{\infty} e\Big(\lambda\Big(\sum_{i=1}^{6} t_i - \frac{n}{N_j}\Big)\Big) d\lambda \\
&= \int_{1-6\eta}^{1+6\eta} \phi(s) ds \int_{-\infty}^{\infty} e\Big(\lambda\Big(s-\frac{n}{N_j}\Big)\Big) d\lambda,
\end{aligned}$$

where

$$\phi(s) = \int\cdots\int_{\substack{\frac{1}{6}-\eta<t_i<\frac{1}{6}+\eta\ (1\le i\le 5)\\ \frac{1}{6}-\eta<s-t_1-t_2-t_3-t_4-t_5<\frac{1}{6}+\eta}} (t_1t_2)^{-\frac{1}{2}}(t_3t_4)^{-\frac{2}{3}} t_5^{-\frac{3}{4}}\Big(s-\sum_{i=1}^{5} t_i\Big)^{-\frac{3}{4}} dt_1\ldots dt_5.$$

Note that $\int_{-R}^{R} e(\lambda u) d\lambda = \frac{\sin 2\pi R u}{\pi u}$. Thus

$$\Im\Big(\frac{n}{N_j}\Big) = \lim_{R\to\infty}\int_{1-6\eta}^{1+6\eta} \phi(s) \frac{\sin 2\pi R(s-\frac{n}{N_j})}{\pi(s-\frac{n}{N_j})} ds. \tag{3.5}$$

Since $\phi(s)$ is bounded and $1-6\eta < \frac{n}{N_j} = 1 + O(\frac{1}{\log N_j}) < 1+6\eta$. Hence, we can deduce from the Fourier's integral theorem (see [1, P.22]) that

$$\lim_{R\to\infty}\int_{1-6\eta}^{1+6\eta} \phi(s) \frac{\sin 2\pi R(s-\frac{n}{N_j})}{\pi(s-\frac{n}{N_j})} ds = \phi\Big(\frac{n}{N_j}\Big). \tag{3.6}$$

For $\phi(\frac{n}{N_j})$, by the trivial estimate, we can obtain

$$\begin{aligned} \phi\Big(\frac{n}{N_j}\Big) &= \int\cdots\int_{\substack{\frac{1}{6}-\eta<t_i<\frac{1}{6}+\eta\ (1\le i\le 5)\\ \frac{1}{6}-\eta<\frac{n}{N_j}-t_1-t_2-t_3-t_4-t_5<\frac{1}{6}+\eta}} (t_1t_2)^{-\frac{1}{2}}(t_3t_4)^{-\frac{2}{3}}t_5^{-\frac{3}{4}}\Big(\frac{n}{N_j}-\sum_{i=1}^{5}t_i\Big)^{-\frac{3}{4}}dt_1\ldots dt_5 \\ &\geq \Big(\frac{1}{6}+\eta\Big)^{-\frac{23}{6}} \int\cdots\int_{\substack{\frac{1}{6}-\eta<t_i<\frac{1}{6}+\eta\ (1\le i\le 5)\\ \frac{1}{6}-\eta<\frac{n}{N_j}-t_1-t_2-t_3-t_4-t_5<\frac{1}{6}+\eta}} 1dt_1\ldots dt_5. \end{aligned} \tag{3.7}$$

Let $t_i = u_i + \frac{1}{6}$, then we have

$$\begin{aligned} \int\cdots\int_{\substack{\frac{1}{6}-\eta<t_i<\frac{1}{6}+\eta\ (1\le i\le 5)\\ \frac{1}{6}-\eta<\frac{n}{N_j}-t_1-t_2-t_3-t_4-t_5<\frac{1}{6}+\eta}} 1dt_1\ldots dt_5 &= \int\cdots\int_{\substack{-\eta<u_i<\eta\ (1\le i\le 5)\\ \frac{1}{6}-\eta<\frac{n}{N_j}-\frac{5}{6}-u_1-u_2-u_3-u_4-u_5<\frac{1}{6}+\eta}} 1du_1\ldots du_5 \\ &= \int\cdots\int_{\substack{-\eta<u_i<\eta\ (1\le i\le 5)\\ 1-\frac{n}{N_j}-\eta<u_1+u_2+u_3+u_4+u_5<1-\frac{n}{N_j}+\eta}} 1du_1\ldots du_5 \\ &= I+R, \end{aligned} \tag{3.8}$$

where I is defined as (3.4) and

$$R = \int\cdots\int_{\substack{-\eta<u_i<\eta\ (1\le i\le 5)\\ \eta<u_1+u_2+u_3+u_4+u_5<1-\frac{n}{N_j}+\eta}} 1du_1\ldots du_5 + \int\cdots\int_{\substack{-\eta<u_i<\eta\ (1\le i\le 5)\\ 1-\frac{n}{N_j}-\eta<u_1+u_2+u_3+u_4+u_5<-\eta}} 1du_1\ldots du_5.$$

From the condition $1-\frac{n}{N_j} = O(\frac{1}{\log N_j})$, we can obtain

$$R \ll \int_{-\eta}^{\eta}du_1\int_{-\eta}^{\eta}du_2\int_{-\eta}^{\eta}du_3\int_{-\eta}^{\eta}\Big(1-\frac{n}{N_j}\Big)du_4 \ll \frac{1}{\log N_j}. \tag{3.9}$$

Now by combining (3.5)–(3.9), we obtain

$$\mathfrak{I}\Big(\frac{n}{N_j}\Big) \geq \Big(\frac{1}{6}+\eta\Big)^{-\frac{23}{6}}\Big(I+O\Big(\frac{1}{\log N_j}\Big)\Big) \geq \Big(\frac{1}{6}+2\eta\Big)^{-\frac{23}{6}}I.$$

Lemma 3.3. *Suppose that* $(a,p)=1$. *Then we have*

$$\text{(i)}\,|C_j(p,a)| \le (j-1)p^{\frac{1}{2}}+1,\ \ \text{(ii)}\,C_3(p,a)=-1,\ \text{ if } p\equiv 2 \pmod 3.$$

Proof. It follows easily from [18, Lemma 4.3].

Lemma 3.4. *We have*

$$(1+A(n,17))\prod_{p\ge 23}(1+A(n,p)) \geq 0.9792.$$

Proof. For $p = 17$ or $23 \le p \le 199$, we can directly calculate $\min\limits_{1\le n\le p}(1+A(n,p))$ by computer and obtain that

$$1 + A(n,17) \ge 0.9994659,\ \ 1 + A(n,23) \ge 0.9999786, \dots,\ 1 + A(n,199) \ge 0.9999972.$$

Thus

$$(1 + A(n,17)) \prod_{23\le p\le 199} (1 + A(n,p)) \ge 0.994943. \tag{3.10}$$

For $199 < p \le 10^5$, if $p \equiv 2 \pmod 3$ and $(a,p) = 1$, then we can deduce from Lemma 3.3 (i) and (ii) that

$$1 + A(n,p) \ge 1 - \frac{\sum\limits_{a=1}^{p-1} |C_2^2(p,a)C_4^2(p,a)|}{(p-1)^6} \ge 1 - \frac{(\sqrt{p}+1)^2(3\sqrt{p}+1)^2}{(p-1)^5}. \tag{3.11}$$

If $p \equiv 1 \pmod 3$, then by Lemma 3.3 (i), we have

$$1 + A(n,p) \ge 1 - \frac{(\sqrt{p}+1)^2(2\sqrt{p}+1)^2(3\sqrt{p}+1)^2}{(p-1)^5}. \tag{3.12}$$

Combining (3.11) and (3.12), we can deduce from numerical calculation that

$$\begin{aligned} & \prod_{199<p\le 10^5} (1 + A(n,p)) \\ \ge\ & \prod_{\substack{199<p\le 10^5 \\ p\equiv 1 \pmod 3}} \left(1 - \frac{(\sqrt{p}+1)^2(2\sqrt{p}+1)^2(3\sqrt{p}+1)^2}{(p-1)^5}\right) \\ & \times \prod_{\substack{199<p\le 10^5 \\ p\equiv 2 \pmod 3}} \left(1 - \frac{(\sqrt{p}+1)^2(3\sqrt{p}+1)^2}{(p-1)^5}\right) \\ \ge\ & 0.98425 \times 0.999989 \ge 0.984239. \end{aligned} \tag{3.13}$$

For $p > 10^5$, it follows from [13, Section 3, P.443] that

$$\prod_{p>10^5} (1 + A(n,p)) \ \ge\ \prod_{p>10^5} \left(1 - \frac{1}{(p-1)^2}\right)^{37} \ge 0.99994. \tag{3.14}$$

Now we can conclude from (3.10) and (3.13)–(3.14) that

$$(1 + A(n,17)) \prod_{p\ge 23} (1 + A(n,p)) \ge 0.994943 \times 0.984239 \times 0.99994 \ge 0.9792.$$

Lemma 3.5. *Let*

$$\Xi(N_1,N_2,k) = \{(n_1,n_2) : \{ \begin{matrix} n_1 = N_1 - 2^{v_1} - 2^{v_2} \cdots - 2^{v_k} \\ n_2 = N_2 - 2^{v_1} - 2^{v_2} \cdots - 2^{v_k} \end{matrix} , 1 \le v_1, \dots, v_k \le L\},$$

$$\Xi(N_1, k) = \{n : n = N_1 - 2^{v_1} - 2^{v_2} \cdots - 2^{v_k}, 1 \le v_1, \ldots, v_k \le L\}.$$

Then for $N_1 \equiv N_2 \equiv 0 \pmod 2$*, we have*

$$\text{(i)} \quad \sum_{\substack{(n_1,n_2)\in\Xi(N_1,N_2,k)\\ n_1\equiv n_2\equiv 0 \pmod 2}} \mathfrak{S}(n_1)\mathfrak{S}(n_2) \ge 3.57L^k, \quad \text{if } k \ge 30,$$

$$\text{(ii)} \quad \sum_{\substack{n\in\Xi(N_1,k)\\ n\equiv 0 \pmod 2}} \mathfrak{S}(n) \ge 1.91267L^k, \quad \text{if } k \ge 14.$$

Proof. According to [13, (3.3)], we have

$$\mathfrak{S}(n) = \prod_{p\ge 2}(1 + A(n, p)). \tag{3.15}$$

Set $C = 0.9792$ and $\mathcal{P} = \{3, 5, 7, 11, 13, 19\}$. Then by applying Lemma 3.4, we obtain

$$\mathfrak{S}(n) \ge C(1 + A(n, 2)) \prod_{p\in\mathcal{P}}(1 + A(n, p)) = 2C \prod_{p\in\mathcal{P}}(1 + A(n, p)), \tag{3.16}$$

where the obvious fact $1+A(n, 2) = 2$ for $n \equiv 0 \pmod 2$ is used. Write $q = \prod_{p\in\mathcal{P}} p = 285285$, $t = N_2-N_1$. Thus

$$\begin{aligned}
&\sum_{\substack{(n_1,n_2)\in\Xi(N_1,N_2,k)\\ n_1\equiv n_2\equiv 0 \pmod 2}} \mathfrak{S}(n_1)\mathfrak{S}(n_2)\\
\ge\ & 4C^2 \sum_{\substack{(n_1,n_2)\in\Xi(N_1,N_2,k)\\ n_1\equiv n_2\equiv 0 \pmod 2}} \prod_{p\in\mathcal{P}}(1 + A(n_1, p))(1 + A(n_2, p))\\
=\ & 4C^2 \sum_{1\le j\le q} \sum_{\substack{(n_1,n_2)\in\Xi(N_1,N_2,k)\\ n_1\equiv n_2\equiv 0 \pmod 2\\ n_1\equiv j \pmod q\\ n_2\equiv t+j \pmod q}} \prod_{p\in\mathcal{P}}(1 + A(n_1, p))(1 + A(n_2, p))\\
=\ & 4C^2 \sum_{1\le j\le q} \prod_{p\in\mathcal{P}}(1 + A(j, p))(1 + A(t + j, p)) \sum_{\substack{(n_1,n_2)\in\Xi(N_1,N_2,k)\\ n_1\equiv n_2\equiv 0 \pmod 2\\ n_1\equiv j \pmod q\\ n_2\equiv t+j \pmod q}} 1.
\end{aligned} \tag{3.17}$$

Let S denote the innermost sum in (3.17). Since $N_1 \equiv N_2 \equiv 0 \pmod 2$, we have

$$S = \sum_{\substack{1\le v_1,\ldots,v_k\le L\\ N_1-2^{v_1}-\cdots-2^{v_k}\equiv j \pmod q\\ N_2-2^{v_1}-\cdots-2^{v_k}\equiv N_2-N_1+j \pmod q}} 1 = \sum_{\substack{1\le v_1,\ldots,v_k\le L\\ 2^{v_1}+\cdots+2^{v_k}\equiv N_1-j \pmod q}} 1.$$

Let $\rho(q)$ denote the smallest positive integer ρ such that $2^\rho \equiv 1 \pmod q$. Thus

$$S = \left(\frac{L}{\rho(q)} + O(1)\right)^k \sum_{\substack{1\le v_1,\ldots,v_k\le\rho(q)\\ 2^{v_1}+\cdots+2^{v_k}\equiv N_1-j \pmod q}} 1$$

$$= \left(\frac{L}{\rho(q)} + O(1)\right)^k \frac{1}{q} \sum_{r=1}^{q} e\left(\frac{r(j-N_1)}{q}\right)\left(\sum_{1\le v\le \rho(q)} e\left(\frac{r2^v}{q}\right)\right)^k. \tag{3.18}$$

When $q = 285285$ and $k \ge 30$, with the help of a computer, we can verify that $\rho(q) = 180$ and

$$\begin{aligned} & \sum_{r=1}^{q-1}\left(\frac{1}{\rho(q)}\left|\sum_{1\le v\le\rho(q)} e\left(\frac{r2^v}{q}\right)\right|\right)^k \\ \le & \sum_{r=1}^{285284}\left(\frac{1}{180}\left|\sum_{1\le v\le 180} e\left(\frac{r2^v}{285285}\right)\right|\right)^{30} \le 2.37\times 10^{-6}. \end{aligned} \tag{3.19}$$

Therefore, from (3.18) and (3.19), we have

$$\begin{aligned} S &\ge \left(\frac{L}{\rho(q)} + O(1)\right)^k \frac{1}{q}\left(\rho^k(q) - \sum_{r=1}^{q-1}\left|\sum_{1\le v\le\rho(q)} e\left(\frac{r2^v}{q}\right)\right|^k\right) \\ &\ge \frac{L^k}{q}\left(1 - \sum_{r=1}^{q-1}\left(\frac{1}{\rho(q)}\left|\sum_{1\le v\le\rho(q)} e\left(\frac{r2^v}{q}\right)\right|\right)^k\right) + O(L^{k-1}) \\ &\ge \frac{L^k}{q}\left(1 - 2.37\times 10^{-6}\right) + O(L^{k-1}) \ge 0.999997\frac{L^k}{q}. \end{aligned} \tag{3.20}$$

Inserting (3.20) into (3.17), we obtain

$$\begin{aligned} & \sum_{\substack{(n_1,n_2)\in\Xi(N_1,N_2,k) \\ n_1\equiv n_2\equiv 0 \pmod 2}} \mathfrak{S}(n_1)\mathfrak{S}(n_2) \\ \ge\ & 4C^2\times 0.999997\frac{L^k}{q}\sum_{1\le j\le q}\prod_{p\in\mathcal{P}}(1+A(j,p))(1+A(t+j,p)). \end{aligned} \tag{3.21}$$

Note that $q = \prod_{p\in\mathcal{P}} p$, we can deduce from the Chinese remainder theorem that

$$\begin{aligned} & \sum_{1\le j\le q}\prod_{p\in\mathcal{P}}(1+A(j,p))(1+A(j+t,p)) \\ =\ & \sum_{1\le j_1\le 3}\sum_{1\le j_2\le 5}\sum_{1\le j_3\le 7}\sum_{1\le j_4\le 11}\sum_{1\le j_5\le 13}\sum_{1\le j_6\le 19}(1+A(j_1,3))(1+A(j_1+t,3)) \\ & \times(1+A(j_2,5))(1+A(j_2+t,5))\ldots(1+A(j_6,19))(1+A(j_6+t,19)) \\ =\ & \prod_{p\in\mathcal{P}}\left(\sum_{1\le j\le p}(1+A(j,p))(1+A(j+t,p))\right) \\ \ge\ & \prod_{p\in\mathcal{P}}\min_{1\le t\le p}\left(\sum_{1\le j\le p}(1+A(j,p))(1+A(j+t,p))\right). \end{aligned} \tag{3.22}$$

By the numerical calculation, we can obtain

$$\prod_{p\in\mathcal{P}}\min_{1\le t\le p}\left(\sum_{1\le j\le p}(1+A(j,p))(1+A(j+t,p))\right) \ge 265611.695. \tag{3.23}$$

Thus, we can conclude from (3.21)–(3.23) that

$$\sum_{\substack{(n_1,n_2)\in\Xi(N_1,N_2,k)\\ n_1\equiv n_2\equiv 0 \pmod 2}} \mathfrak{S}(n_1)\mathfrak{S}(n_2) \geq 4C^2 \times 0.999997 \times 265611.695\frac{L^k}{q} \geq 3.57L^k. \tag{3.24}$$

The proof of (ii) is similar. From (3.16)–(3.17), we have

$$\begin{aligned}\sum_{\substack{n\in\Xi(N_1,k)\\ n\equiv 0 \pmod 2}} \mathfrak{S}(n) &\geq 2C \sum_{\substack{n\in\Xi(N_1,k)\\ n\equiv 0 \pmod 2}} \prod_{p\in\mathcal{P}}(1+A(n,p))\\ &= 2C\sum_{1\leq j\leq q}\prod_{p\in\mathcal{P}}(1+A(j,p)) \sum_{\substack{n\in\Xi(N_1,k)\\ n\equiv 0 \pmod 2\\ n\equiv j \pmod q}} 1. \end{aligned}\tag{3.25}$$

The innermost sum can be estimated using the same method as in (3.18)–(3.20), except by replacing $k \geq 30$ with $k \geq 14$. By numerical calculation, we have

$$\begin{aligned}\sum_{\substack{n\in\Xi(N_1,k)\\ n\equiv 0 \pmod 2\\ n\equiv j \pmod q}} 1 &\geq \frac{L^k}{q}\left(1-\sum_{r=1}^{285284}\left(\frac{1}{180}\left|\sum_{1\leq v\leq 180} e\Big(\frac{r2^v}{285285}\Big)\right|\right)^{14}\right)+O(L^{k-1})\\ &\geq \frac{L^k}{q}(1-0.023347)+O(L^{k-1}) \geq 0.976652\frac{L^k}{q}. \end{aligned}\tag{3.26}$$

Similar to (3.22), we have

$$\sum_{1\leq j\leq q}\prod_{p\in\mathcal{P}}(1+A(j,p)) = \prod_{p\in\mathcal{P}}\left(\sum_{1\leq j\leq p}(1+A(j,p))\right). \tag{3.27}$$

Moreover, by applying the bound $\sum_{1\leq j\leq p} e(-\frac{aj}{p}) = 0$ for $a \neq 0 \pmod p$, we can obtain

$$\begin{aligned}\sum_{1\leq j\leq p}(1+A(j,p)) &= p + \frac{\sum_{1\leq a\leq p-1} C_2^2(p,a)C_3^2(p,a)C_4^2(p,a)\sum_{1\leq j\leq p} e\left(-\frac{aj}{p}\right)}{(p-1)^6}\\ &= p. \end{aligned}\tag{3.28}$$

Now by combining (3.25)–(3.28), we can derive that

$$\begin{aligned}\sum_{\substack{n\in\Xi(N_1,k)\\ n\equiv 0 \pmod 2}} \mathfrak{S}(n) &\geq 2C\times 0.976652\frac{L^k}{q}\prod_{p\in\mathcal{P}}\left(\sum_{1\leq j\leq p}(1+A(j,p))\right)\\ &\geq 1.91267\frac{L^k}{q}\prod_{p\in\mathcal{P}} p = 1.91267L^k. \end{aligned}$$

Proposition 3.1. *Suppose that* $k \geq 30$ *and* $N_1 \equiv N_2 \equiv 0 \pmod 2$. *Then we have*

$$\mathcal{R}_1(k, N_1, N_2) \geq 9.946I^2 N_1^{\frac{7}{6}} N_2^{\frac{7}{6}} L^k. \tag{3.29}$$

Proof. Note that $H^k(\alpha_1+\alpha_2)e(-N_1\alpha_1-\alpha_2N_2)=\sum\limits_{\substack{(n_1,n_2)\in\Xi(N_1,N_2,k)\\ n_1\equiv n_2\equiv 0 \pmod 2}} e(-n_1\alpha_1)e(-n_2\alpha_2)$, then by Lemma 3.1, Lemma 3.2 (ii) and Lemma 3.5 (i), we have

$$\begin{aligned}
&\mathcal{R}_1(k,N_1,N_2)\\
=\ &\iint\limits_{\mathfrak{M}}\prod_{1\le j\le 2} S_{2,j}^2(\alpha_j)S_{3,j}^2(\alpha_j)S_{4,j}^2(\alpha_j)H^k(\alpha_1+\alpha_2)e(-\alpha_1N_1-\alpha_2N_2)d\alpha_1d\alpha_2\\
=\ &\sum_{\substack{(n_1,n_2)\in\Xi(N_1,N_2,k)\\ n_1\equiv n_2\equiv 0 \pmod 2}}\int_{\mathfrak{M}_1} S_{2,1}^2(\alpha_1)S_{3,1}^2(\alpha_1)S_{4,1}^2(\alpha_1)e(-n_1\alpha_1)d\alpha_1\\
&\times\int_{\mathfrak{M}_2} S_{2,2}^2(\alpha_2)S_{3,2}^2(\alpha_2)S_{4,2}^2(\alpha_2)e(-n_2\alpha_2)d\alpha_2\\
=\ &\sum_{\substack{(n_1,n_2)\in\Xi(N_1,N_2,k)\\ n_1\equiv n_2\equiv 0 \pmod 2}}\left(\mathfrak{S}(n_1)\mathfrak{S}(n_2)\mathfrak{J}(\frac{n_1}{N_1})\mathfrak{J}(\frac{n_2}{N_2})\frac{N_1^{\frac{7}{6}}N_2^{\frac{7}{6}}}{2^4\cdot 3^4\cdot 4^4}+O(N_1^{\frac{7}{6}}N_2^{\frac{7}{6}}L^{-1})\right)\\
\ge\ &\frac{(\frac{1}{6}+2\eta)^{-\frac{23}{3}}I^2N_1^{\frac{7}{6}}N_2^{\frac{7}{6}}}{2^4\cdot 3^4\cdot 4^4}\sum_{\substack{(n_1,n_2)\in\Xi(N_1,N_2,k)\\ n_1\equiv n_2\equiv 0 \pmod 2}}\mathfrak{S}(n_1)\mathfrak{S}(n_2)+O(N_1^{\frac{7}{6}}N_2^{\frac{7}{6}}L^{k-1})\\
\ge\ &9.946I^2N_1^{\frac{7}{6}}N_2^{\frac{7}{6}}L^k,
\end{aligned}\tag{3.30}$$

where the trivial bound $\sum\limits_{\substack{(n_1,n_2)\in\Xi(N_1,N_2,k)\\ n_1\equiv n_2\equiv 0 \pmod 2}} 1\ll L^k$ is used.

4. The upper bound for $\mathcal{R}_2(k,N_1,N_2)$

In this section, we will give the upper bound for $\mathcal{R}_2(k,N_1,N_2)$. For this purpose, we need to introduce a further division of the minor arcs $\mathfrak{m}$. Let

$$\mathcal{E}(u)=\{(\alpha_1,\alpha_2)\in\mathfrak{m}:|H(\alpha_1+\alpha_2)|\ge uL\}.$$

Then we have

$$\begin{aligned}
&|\mathcal{R}_2(k,N_1,N_2)|\\
\le\ &\iint\limits_{\mathfrak{m}}\left|\prod_{1\le j\le 2} S_{2,j}^2(\alpha_j)S_{3,j}^2(\alpha_j)S_{4,j}^2(\alpha_j)H^k(\alpha_1+\alpha_2)\right|d\alpha_1d\alpha_2\\
=\ &\left(\iint\limits_{\mathfrak{m}\setminus\mathcal{E}(u)}+\iint\limits_{\mathfrak{m}\bigcap\mathcal{E}(u)}\right)\left|\prod_{1\le j\le 2} S_{2,j}^2(\alpha_j)S_{3,j}^2(\alpha_j)S_{4,j}^2(\alpha_j)H^k(\alpha_1+\alpha_2)\right|d\alpha_1d\alpha_2\\
:=\ &\mathcal{R}_3(k,N_1,N_2,u)+\mathcal{R}_4(k,N_1,N_2,u).
\end{aligned}\tag{4.1}$$

In order to estimate $\mathcal{R}_3(k,N_1,N_2,u)$, we define

$$\mathfrak{M}_j^*(q,a)=\left(\frac{a}{q}-\frac{\log N_j}{N_j},\frac{a}{q}+\frac{\log N_j}{N_j}\right],\ \ \mathfrak{M}_j^*=\bigcup_{q\le\log N_j}\bigcup_{\substack{a=1\\(a,q)=1}}^{q}\mathfrak{M}_j^*(q,a),$$

$$\mathfrak{m}_j^* = \left[\frac{1}{Q_j}, 1 + \frac{1}{Q_j}\right] \backslash \mathfrak{M}_j^*, \quad \mathfrak{J}^* = \int_{-\infty}^{\infty} |g_2(\lambda)^2 g_3(\lambda)^2 g_4(\lambda)^2| d\lambda,$$

$$A^*(q) = \frac{1}{\varphi^6(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |C_2^2(q,a) C_3^2(q,a) C_4^2(q,a)|, \quad \mathfrak{S}^* = \sum_{q=1}^{\infty} A^*(q).$$

Lemma 4.1. *We have*

$$\mathfrak{S}^* \leq 3.394.$$

Proof. See [22, Lemma 3.1].

Lemma 4.2. *We have*

$$\int_{\mathfrak{M}_j^*} |S_{2,j}(\alpha)^2 S_{3,j}(\alpha)^2 S_{4,j}(\alpha)^2| d\alpha = \frac{\mathfrak{S}^* \mathfrak{J}^* N_j^{\frac{7}{6}}}{2^2 \cdot 3^2 \cdot 4^2} + O(N_j^{\frac{7}{6}} L^{-1}).$$

Proof. It follows from the standard major arcs techniques in the Waring-Goldbach problem.

Lemma 4.3. *We have*

$$\int_0^1 |S_{2,j}(\alpha)^2 S_{3,j}(\alpha)^2 S_{4,j}(\alpha)^2| d\alpha \leq \frac{(8+2\eta)\mathfrak{S}^* \mathfrak{J}^* N_j^{\frac{7}{6}}}{2^2 \cdot 3^2 \cdot 4^2}.$$

Proof. Let

$$J_j^* = \sum_{\substack{m_1 - m_2 + m_3 - m_4 + m_5 - m_6 = 0 \\ (\frac{1}{6} - \eta) N_j \leq m_i \leq (\frac{1}{6} + \eta) N_j \ (1 \leq i \leq 6)}} (m_1 m_2)^{-\frac{1}{2}} (m_3 m_4)^{-\frac{2}{3}} (m_5 m_6)^{-\frac{3}{4}}.$$

Then from [22, Proof of Lemma 4.1, P.417] with $k = 4$, we have

$$\int_0^1 |S_{2,j}(\alpha)^2 S_{3,j}(\alpha)^2 S_{4,j}(\alpha)^2| d\alpha \leq \frac{(8+\eta)\mathfrak{S}^* J_j^*}{2^2 \cdot 3^2 \cdot 4^2}. \tag{4.2}$$

For J_j^*, it follows from the same argument leading to (3.2) and (3.3) that

$$\begin{aligned} \frac{J_j^*}{2^2 \cdot 3^2 \cdot 4^2} &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| v_{2,j}(\lambda)^2 v_{3,j}(\lambda)^2 v_{4,j}(\lambda)^2 \right| d\lambda \\ &= \int_{|\lambda| \leq \frac{\log N_j}{N_j}} \left| u_{2,j}(\lambda)^2 u_{3,j}(\lambda)^2 u_{4,j}(\lambda)^2 \right| d\lambda + O\left(\frac{N_j^{\frac{7}{6}}}{\log^5 N_j} \right) \\ &= \frac{\mathfrak{J}^* N_j^{\frac{7}{6}}}{2^2 \cdot 3^2 \cdot 4^2} + O\left(\frac{N_j^{\frac{7}{6}}}{\log^5 N_j} \right). \end{aligned} \tag{4.3}$$

Now by substituting (4.3) into (4.2), we obtain

$$\int_0^1 |S_{2,j}(\alpha)^2 S_{3,j}(\alpha)^2 S_{4,j}(\alpha)^2| d\alpha \leq \frac{(8+\eta)\mathfrak{S}^* \mathfrak{J}^* N_j^{\frac{7}{6}}}{2^2 \cdot 3^2 \cdot 4^2} + O\left(\frac{N_j^{\frac{7}{6}}}{\log^5 N_j} \right) \leq \frac{(8+2\eta)\mathfrak{S}^* \mathfrak{J}^* N_j^{\frac{7}{6}}}{2^2 \cdot 3^2 \cdot 4^2}.$$

Lemma 4.4. *We have*

$$\int_{\mathfrak{m}_j^*} |S_{2,j}(\alpha)^2 S_{3,j}(\alpha)^2 S_{4,j}(\alpha)^2| d\alpha \leq \frac{(7+3\eta)\mathfrak{S}^*\mathfrak{J}^* N_j^{\frac{7}{6}}}{2^2\cdot 3^2\cdot 4^2}.$$

Proof. By Lemmas 4.2 and 4.3, we have

$$\begin{aligned}
&\int_{\mathfrak{m}_j^*} |S_{2,j}(\alpha)^2 S_{3,j}(\alpha)^2 S_{4,j}(\alpha)^2| d\alpha \\
= &\int_0^1 |S_{2,j}(\alpha)^2 S_{3,j}(\alpha)^2 S_{4,j}(\alpha)^2| d\alpha - \int_{\mathfrak{M}_j^*} |S_{2,j}(\alpha)^2 S_{3,j}(\alpha)^2 S_{4,j}(\alpha)^2| d\alpha \\
\leq &\frac{(8+2\eta)-1}{2^2\cdot 3^2\cdot 4^2}\mathfrak{S}^*\mathfrak{J}^* N_j^{\frac{7}{6}} + O(N_j^{\frac{7}{6}}L^{-1}) \leq \frac{(7+3\eta)\mathfrak{S}^*\mathfrak{J}^* N_j^{\frac{7}{6}}}{2^2\cdot 3^2\cdot 4^2}.
\end{aligned}$$

Lemma 4.5. *Let I be defined as (3.4). Then we have*

$$\mathfrak{J}^* \leq \left(\frac{1}{6}-\eta\right)^{-\frac{23}{6}} I. \tag{4.4}$$

Proof. The proof of Lemma 4.5 is similar to that of Lemma 3.2 (ii). Let $t_6 = t_1 - t_2 + t_3 - t_4 + t_5 - s$, then we can obtain

$$\mathfrak{J}^* = \int_{-\infty}^{\infty} |g_2^2(\lambda) g_3^2(\lambda) g_4^2(\lambda)| d\lambda = \int_{-6\eta}^{6\eta} \phi^*(s) ds \int_{-\infty}^{\infty} e(\lambda s) d\lambda, \tag{4.5}$$

where

$$\phi^*(s) = \underset{\substack{\frac{1}{6}-\eta<t_i<\frac{1}{6}+\eta\ (1\leq i\leq 5)\\ \frac{1}{6}-\eta<t_1-t_2+t_3-t_4+t_5-s<\frac{1}{6}+\eta}}{\int\cdots\int} (t_1t_2)^{-\frac{1}{2}}(t_3t_4)^{-\frac{2}{3}} t_5^{-\frac{3}{4}} (t_1-t_2+t_3-t_4+t_5-s)^{-\frac{3}{4}} dt_1\ldots dt_5.$$

Applying the Fourier integral theorem, we can obtain

$$\int_{-6\eta}^{6\eta} \phi^*(s) ds \int_{-\infty}^{\infty} e(\lambda s) d\lambda = \lim_{R\to\infty} \int_{-6\eta}^{6\eta} \phi^*(s) \frac{\sin 2\pi R s}{\pi s} ds = \phi^*(0). \tag{4.6}$$

Write $t_i = u_i + \frac{1}{6}$. Thus

$$\begin{aligned}
\phi^*(0) &= \underset{\substack{\frac{1}{6}-\eta<t_i<\frac{1}{6}+\eta\ (1\leq i\leq 5)\\ \frac{1}{6}-\eta<t_1-t_2+t_3-t_4+t_5<\frac{1}{6}+\eta}}{\int\cdots\int} (t_1t_2)^{-\frac{1}{2}}(t_3t_4)^{-\frac{2}{3}} t_5^{-\frac{3}{4}} (t_1-t_2+t_3-t_4+t_5)^{-\frac{3}{4}} dt_1\ldots dt_5 \\
&\leq \left(\frac{1}{6}-\eta\right)^{-\frac{23}{6}} \underset{\substack{-\eta<u_i<\eta\ (1\leq i\leq 5)\\ -\eta<u_1-u_2+u_3-u_4+u_5<\eta}}{\int\cdots\int} 1 du_1\ldots du_5.
\end{aligned} \tag{4.7}$$

Making the change of variables $u_2 = -s_2,\ \ u_4 = -s_4$, we find that

$$\underset{\substack{-\eta<u_i<\eta\ (1\le i\le 5)\\ -\eta<u_1-u_2+u_3-u_4+u_5<\eta}}{\int\cdots\int} 1du_1du_2\dots du_5 = \underset{\substack{-\eta<u_1,s_2,u_3,s_4,u_5<\eta\\ -\eta<u_1+s_2+u_3+s_4+u_5<\eta}}{\int\cdots\int} 1du_1ds_2\dots du_5 = I. \tag{4.8}$$

Now the desired result follows from (4.5)–(4.8).
Proposition 4.1. *We have*

$$\mathcal{R}_3(k, N_1, N_2, u) \le 2021.835u^k I^2 N_1^{\frac{7}{6}} N_2^{\frac{7}{6}} L^k.$$

Proof. Note that $|H(\alpha_1 + \alpha_2)| < uL$ for $(\alpha_1, \alpha_2) \in \mathfrak{m}\backslash\mathcal{E}(u)$. Then we have

$$\mathcal{R}_3(k, N_1, N_2, u) \le (uL)^k \iint\limits_{\mathfrak{m}\backslash\mathcal{E}(u)} \prod_{1\le j\le 2} \Big|S_{2,j}^2(\alpha_j)S_{3,j}^2(\alpha_j)S_{4,j}^2(\alpha_j)\Big| d\alpha_1 d\alpha_2. \tag{4.9}$$

It is easy to see that $\mathfrak{m}_1 \subseteq \mathfrak{m}_1^*, \mathfrak{m}_2 \bigcup(\mathfrak{M}_2\backslash\mathfrak{M}_2^*) \subseteq \mathfrak{m}_2^*$ and $\mathfrak{m}_2 \bigcap(\mathfrak{M}_2\backslash\mathfrak{M}_2^*) = \emptyset$. Hence

$$\begin{aligned}
\iint\limits_{\mathfrak{m}} 1d\alpha_1 d\alpha_2 &= \int\limits_{\mathfrak{M}_1} 1d\alpha_1 \int\limits_{\mathfrak{m}_2} 1d\alpha_2 + \int\limits_{\mathfrak{m}_1} 1d\alpha_1 \int\limits_{\mathfrak{m}_2} 1d\alpha_2 + \int\limits_{\mathfrak{m}_1} 1d\alpha_1 \int\limits_{\mathfrak{M}_2} 1d\alpha_2\\
&= \int_0^1 1d\alpha_1 \int\limits_{\mathfrak{m}_2} 1d\alpha_2 + \int\limits_{\mathfrak{m}_1} 1d\alpha_1 \int\limits_{\mathfrak{M}_2} 1d\alpha_2\\
&= \int_0^1 1d\alpha_1 \int\limits_{\mathfrak{m}_2} 1d\alpha_2 + \int\limits_{\mathfrak{m}_1} 1d\alpha_1 \int\limits_{\mathfrak{M}_2\backslash\mathfrak{M}_2^*} 1d\alpha_2 + \int\limits_{\mathfrak{m}_1} 1d\alpha_1 \int\limits_{\mathfrak{M}_2^*} 1d\alpha_2\\
&\le \int_0^1 1d\alpha_1 \int\limits_{\mathfrak{m}_2} 1d\alpha_2 + \int_0^1 1d\alpha_1 \int\limits_{\mathfrak{M}_2\backslash\mathfrak{M}_2^*} 1d\alpha_2 + \int\limits_{\mathfrak{m}_1^*} 1d\alpha_1 \int\limits_{\mathfrak{M}_2^*} 1d\alpha_2\\
&\le \int_0^1 1d\alpha_1 \int\limits_{\mathfrak{m}_2^*} 1d\alpha_2 + \int\limits_{\mathfrak{m}_1^*} 1d\alpha_1 \int\limits_{\mathfrak{M}_2^*} 1d\alpha_2.
\end{aligned} \tag{4.10}$$

From (4.9)–(4.10) and Lemmas 4.1–4.5, we can obtain

$$\begin{aligned}
&\mathcal{R}_3(k, N_1, N_2, u)\\
\le\ & (uL)^k \int_0^1 |S_{2,1}^2(\alpha_1)S_{3,1}^2(\alpha_1)S_{4,1}^2(\alpha_1)| d\alpha_1 \int\limits_{\mathfrak{m}_2^*} |S_{2,2}^2(\alpha_2)S_{3,2}^2(\alpha_2)S_{4,2}^2(\alpha_2)| d\alpha_2\\
& + (uL)^k \int\limits_{\mathfrak{m}_1^*} |S_{2,1}^2(\alpha_1)S_{3,1}^2(\alpha_1)S_{4,1}^2(\alpha_1)| d\alpha_1 \int\limits_{\mathfrak{M}_2^*} |S_{2,2}^2(\alpha_2)S_{3,2}^2(\alpha_2)S_{4,2}^2(\alpha_2)| d\alpha_2\\
\le\ & (uL)^k \Big(\frac{(8+2\eta)(7+3\eta)}{2^4\cdot 3^4\cdot 4^4}(\mathfrak{S}^*\mathfrak{J}^*)^2 N_1^{\frac{7}{6}} N_2^{\frac{7}{6}} + \frac{(7+3\eta)(1+\eta)}{2^4\cdot 3^4\cdot 4^4}(\mathfrak{S}^*\mathfrak{J}^*)^2 N_1^{\frac{7}{6}} N_2^{\frac{7}{6}}\Big)\\
\le\ & \Big(\frac{1}{6} - \eta\Big)^{-\frac{23}{3}} \frac{(63+50\eta)}{2^4\cdot 3^4\cdot 4^4} \times 3.394^2 u^k N_1^{\frac{7}{6}} N_2^{\frac{7}{6}} I^2 L^k
\end{aligned}$$

$$\leq \quad 2021.835u^kN_1^{\frac{7}{6}}N_2^{\frac{7}{6}}I^2L^k. \tag{4.11}$$

Lemma 4.6. *We have*

$$\text{(i)} \int_{\mathfrak{m}_j} |S_{2,j}(\alpha)^2S_{3,j}(\alpha)^3S_{4,j}(\alpha)^2|d\alpha \ll N_j^{\frac{35}{24}+\varepsilon}, \text{(ii)} \int_0^1 |S_{2,j}(\alpha)^2S_{4,j}(\alpha)^4|d\alpha \ll N_j^{1+\varepsilon},$$

$$\text{(iii)} \int_0^1 |S_{2,j}(\alpha)^2S_{3,j}(\alpha)^2S_{4,j}(\alpha)^2|d\alpha \ll N_j^{\frac{7}{6}+\varepsilon}.$$

Proof. See [24, Lemma 4.5 and Lemma 4.1].

Lemma 4.7. *Let*

$$\mathcal{E}^*(u) = \{\alpha \in (0,1] : |H(\alpha)| \geq uL\}.$$

Write meas$(\mathcal{E}^*(u))$ *for the measure of the set* $\mathcal{E}^*(u)$*. Then we have*

$$meas(\mathcal{E}^*(0.83372131685)) \leq N_1^{-\frac{2}{3}-10^{-20}}.$$

Proof. See [6, Lemma 5 and (3.10)].

Proposition 4.2. *Let* $u = 0.83372131685$*. Then we have*

$$\mathcal{R}_4(k, N_1, N_2, u) \ll N_1^{\frac{7}{6}}N_2^{\frac{7}{6}}L^{k-1}.$$

Proof. For brevity, we write

$$F_2(\alpha_1) = \int_{\substack{\alpha_2\in\mathfrak{m}_2\\|H(\alpha_1+\alpha_2)|\geq uL}} \left|S_{2,2}^2(\alpha_2)S_{3,2}^2(\alpha_2)S_{4,2}^2(\alpha_2)\right|d\alpha_2,$$

$$F_1(\alpha_2) = \int_{\substack{\alpha_1\in\mathfrak{m}_1\\|H(\alpha_1+\alpha_2)|\geq uL}} \left|S_{2,1}^2(\alpha_1)S_{3,1}^2(\alpha_1)S_{4,1}^2(\alpha_1)\right|d\alpha_1.$$

From the definition of $\mathfrak{m}$ and $\mathcal{E}(u)$, we have

$$\begin{aligned}
&\mathcal{R}_4(k, N_1, N_2, u)\\
\ll\ & L^k\left(\iint\limits_{\substack{(\alpha_1,\alpha_2)\in[0,1]\times\mathfrak{m}_2\\|H(\alpha_1+\alpha_2)|\geq uL}} + \iint\limits_{\substack{(\alpha_1,\alpha_2)\in\mathfrak{m}_1\times[0,1]\\|H(\alpha_1+\alpha_2)|\geq uL}}\right)\left|\prod_{1\leq j\leq 2} S_{2,j}^2(\alpha_j)S_{3,j}^2(\alpha_j)S_{4,j}^2(\alpha_j)\right|d\alpha_1d\alpha_2\\
=\ & L^k\int_0^1\left|S_{2,1}^2(\alpha_1)S_{3,1}^2(\alpha_1)S_{4,1}^2(\alpha_1)F_2(\alpha_1)\right|d\alpha_1\\
&+L^k\int_0^1\left|S_{2,2}^2(\alpha_2)S_{3,2}^2(\alpha_2)S_{4,2}^2(\alpha_2)F_1(\alpha_2)\right|d\alpha_2,
\end{aligned} \tag{4.12}$$

where the trivial bound $H(\alpha_1+\alpha_2) \ll L$ is used. By applying Hölder's inequality, Hua's inequality and Lemma 4.6 (i), (ii), we can obtain

$$F_2(\alpha_1) \ll \left(\int_0^1 |S_{2,2}^2(\alpha_2)S_{4,2}^4(\alpha_2)|d\alpha_2\right)^{\frac{1}{6}}\left(\int_0^1 |S_{2,2}^4(\alpha_2)|d\alpha_2\right)^{\frac{1}{12}}$$

$$\times\left(\int_{\mathfrak{m}_2}|S_{2,2}^2(\alpha_2)S_{3,2}^3(\alpha_2)S_{4,2}^2(\alpha_2)|d\alpha_2\right)^{\frac{2}{3}}\left(\int_{\substack{\alpha_2\in[0,1]\\|H(\alpha_1+\alpha_2)|\geq uL}}1d\alpha_2\right)^{\frac{1}{12}}$$
$$\ll N_2^{\frac{11}{9}+\varepsilon}\left(\int_{\substack{\alpha_2\in[0,1]\\|H(\alpha_1+\alpha_2)|\geq uL}}1d\alpha_2\right)^{\frac{1}{12}}. \tag{4.13}$$

Thus

$$\int_0^1\left|S_{2,1}^2(\alpha_1)S_{3,1}^2(\alpha_1)S_{4,1}^2(\alpha_1)F_2(\alpha_1)\right|d\alpha_1$$
$$\ll N_2^{\frac{11}{9}+\varepsilon}\int_0^1\left|S_{2,1}^2(\alpha_1)S_{3,1}^2(\alpha_1)S_{4,1}^2(\alpha_1)\right|\left(\int_{\substack{\alpha_2\in[0,1]\\|H(\alpha_1+\alpha_2)|\geq uL}}1d\alpha_2\right)^{\frac{1}{12}}d\alpha_1$$
$$= N_2^{\frac{11}{9}+\varepsilon}\int_0^1\left|S_{2,1}^2(\alpha_1)S_{3,1}^2(\alpha_1)S_{4,1}^2(\alpha_1)\right|\left(\int_{\substack{\beta\in[\alpha_1,1+\alpha_1]\\|H(\beta)|\geq uL}}1d\beta\right)^{\frac{1}{12}}d\alpha_1, \tag{4.14}$$

where we used the integral transformation $\beta=\alpha_1+\alpha_2$. Note that $H(\beta)$ is of period one. Hence

$$\int_{\substack{\beta\in[\alpha_1,1+\alpha_1]\\|H(\beta)|\geq uL}}1d\beta=\int_{\substack{\beta\in[0,1]\\|H(\beta)|\geq uL}}1d\beta. \tag{4.15}$$

On substituting (4.15) into (4.14), we obtain

$$\int_0^1\left|S_{2,1}^2(\alpha_1)S_{3,1}^2(\alpha_1)S_{4,1}^2(\alpha_1)F_2(\alpha_1)\right|d\alpha_1$$
$$\ll N_2^{\frac{11}{9}+\varepsilon}\int_0^1\left|S_{2,1}^2(\alpha_1)S_{3,1}^2(\alpha_1)S_{4,1}^2(\alpha_1)\right|\left(\int_{\substack{\beta\in[0,1]\\|H(\beta)|\geq uL}}1d\beta\right)^{\frac{1}{12}}d\alpha_1$$
$$\ll N_2^{\frac{11}{9}+\varepsilon}N_1^{-\frac{1}{18}-10^{-22}}\int_0^1\left|S_{2,1}^2(\alpha_1)S_{3,1}^2(\alpha_1)S_{4,1}^2(\alpha_1)\right|d\alpha_1\ll N_2^{\frac{11}{9}+\varepsilon}N_1^{\frac{10}{9}-10^{-23}}, \tag{4.16}$$

where Lemma 4.7 and Lemma 4.6 (iii) are used. In a similar manner, we have

$$\int_0^1\left|S_{2,2}^2(\alpha_2)S_{3,2}^2(\alpha_2)S_{4,2}^2(\alpha_2)F_1(\alpha_2)\right|d\alpha_2$$
$$\ll N_1^{\frac{11}{9}+\varepsilon}\int_0^1\left|S_{2,2}^2(\alpha_2)S_{3,2}^2(\alpha_2)S_{4,2}^2(\alpha_2)\right|\left(\int_{\substack{\beta\in[0,1]\\|H(\beta)|\geq uL}}1d\beta\right)^{\frac{1}{12}}d\alpha_2$$
$$\ll N_1^{\frac{7}{6}-10^{-22}+\varepsilon}\int_0^1\left|S_{2,2}^2(\alpha_2)S_{3,2}^2(\alpha_2)S_{4,2}^2(\alpha_2)\right|d\alpha_2\ll N_1^{\frac{7}{6}-10^{-23}}N_2^{\frac{7}{6}+\varepsilon}. \tag{4.17}$$

Since $N_2<N_1\ll N_2$, then we can conclude from (4.12) and (4.16)–(4.17) that

$$\mathcal{R}_4(k,N_1,N_2,u)\ll L^kN_2^{\frac{11}{9}+\varepsilon}N_1^{\frac{10}{9}-10^{-23}}+L^kN_1^{\frac{7}{6}-10^{-23}}N_2^{\frac{7}{6}+\varepsilon}\ll N_1^{\frac{7}{6}}N_2^{\frac{7}{6}}L^{k-1}.$$

Now we turn to give the estimate for $|\mathcal{R}_2(k, N_1, N_2)|$. Suppose that $k \geq 30$. Then by combining (4.1) and Propositions 4.1–4.2 with $u = 0.83372131685$, we have

$$\begin{aligned} |\mathcal{R}_2(k, N_1, N_2)| &\leq \mathcal{R}_3(k, N_1, N_2, u) + \mathcal{R}_4(k, N_1, N_2, u) \\ &\leq 2021.835 \times 0.83372131685^{30} I^2 N_1^{\frac{7}{6}} N_2^{\frac{7}{6}} L^k + O\left(N_1^{\frac{7}{6}} N_2^{\frac{7}{6}} L^{k-1}\right) \\ &\leq 8.6372 I^2 N_1^{\frac{7}{6}} N_2^{\frac{7}{6}} L^k. \end{aligned} \tag{4.18}$$

5. Proof of Theorem 1

When $k \geq 30$, by combining Proposition 3.1, (2.4), and (4.18), we can obtain

$$\begin{aligned} \mathcal{R}(k, N_1, N_2) &\geq \mathcal{R}_1(k, N_1, N_2) - |\mathcal{R}_2(k, N_1, N_2)| \\ &\geq (9.946 - 8.6372) I^2 N_1^{\frac{7}{6}} N_2^{\frac{7}{6}} L^k \\ &> 1.308 I^2 N_1^{\frac{7}{6}} N_2^{\frac{7}{6}} L^k > 0, \end{aligned} \tag{5.1}$$

where Lemma 3.2 (i) is used in the last step. Now the proof of Theorem 1 is completed.

6. Proof of Theorem 2

We sketch the proof of Theorem 2, since the idea of the proof is similar to that of Theorem 1. We only give the changes that are necessary for our Theorem 2. Let

$$\mathcal{R}(k, N_1) = \sum_{\substack{N_1 = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4 + 2^{\nu_1} + \cdots + 2^{\nu_k} \\ P_{2,1}^- \leq p_1, p_2 \leq P_{2,1}^+,\ P_{3,1}^- \leq p_3, p_4 \leq P_{3,1}^+, \\ P_{4,1}^- \leq p_5, p_6 \leq P_{4,1}^+,\ 1 \leq \nu_1, \ldots, \nu_k \leq L}} (\log p_1) \ldots (\log p_6).$$

Suppose that $k \geq 14$ and $u = 0.83372131685$. Then from the orthogonality, we have

$$\begin{aligned} &\mathcal{R}(k, N_1) \\ = &\left(\int_{\mathfrak{M}_1} + \int_{\mathfrak{m}_1 \setminus \mathcal{E}^*(u)} + \int_{\mathfrak{m}_1 \bigcap \mathcal{E}^*(u)} \right) S_{2,1}(\alpha)^2 S_{3,1}(\alpha)^2 S_{4,1}(\alpha)^2 H(\alpha)^k e(-N_1 \alpha) d\alpha \\ := &\ \mathcal{R}_1(k, N_1) + \mathcal{R}_2(k, N_1, u) + \mathcal{R}_3(k, N_1, u). \end{aligned} \tag{6.1}$$

Applying Lemmas 3.1–3.2 and Lemma 3.5 (ii), we have

$$\begin{aligned} \mathcal{R}_1(k, N_1) &= \sum_{\substack{n \in \Xi(N_1, k) \\ n \equiv 0 \pmod 2}} \left(\frac{\mathfrak{S}(n) \mathfrak{J}\left(\frac{n}{N_1}\right) N_1^{\frac{7}{6}}}{2^2 \cdot 3^2 \cdot 4^2} + O\left(N_1^{\frac{7}{6}} L^{-1}\right) \right) \\ &\geq \frac{(\frac{1}{6} + 2\eta)^{-\frac{23}{6}} I N_1^{\frac{7}{6}}}{2^2 \cdot 3^2 \cdot 4^2} \sum_{\substack{n \in \Xi(N_1, k) \\ n \equiv 0 \pmod 2}} \mathfrak{S}(n) + O\left(N_1^{\frac{7}{6}} L^{-1} \sum_{\substack{n \in \Xi(N_1, k) \\ n \equiv 0 \pmod 2}} 1 \right) \\ &\geq 3.192498 I N_1^{\frac{7}{6}} L^k + O\left(N_1^{\frac{7}{6}} L^{k-1}\right) \geq 3.19249 I N_1^{\frac{7}{6}} L^k. \end{aligned} \tag{6.2}$$

Since $\mathfrak{m}_1 \backslash \mathcal{E}^*(u) \subset \mathfrak{m}_1^*$, we can deduce from Lemma 4.1 and Lemmas 4.4–4.5 that

$$\begin{aligned}\mathcal{R}_2(k, N_1, u) &\leq (uL)^k \int_{\mathfrak{m}_1 \backslash \mathcal{E}^*(u)} |S_{2,1}(\alpha)^2 S_{3,1}(\alpha)^2 S_{4,1}(\alpha)^2| d\alpha \\ &\leq \frac{(7+3\eta)u^k L^k}{2^2 \cdot 3^2 \cdot 4^2} \mathfrak{S}^* \mathfrak{J}^* N_1^{\frac{7}{6}} \leq 39.6553 u^k I N_1^{\frac{7}{6}} L^k. \end{aligned} \tag{6.3}$$

Moreover, by Hölder's inequality, Hua's inequality, and Lemmas 4.6–4.7, we have

$$\begin{aligned}\mathcal{R}_3(k, N_1, u) &\ll L^k \left(\int_0^1 |S_{2,1}(\alpha)^2 S_{4,1}(\alpha)^4| d\alpha \right)^{\frac{1}{6}} \left(\int_0^1 |S_{2,1}^4(\alpha)| d\alpha \right)^{\frac{1}{12}} \\ &\quad \times \left(\int_{\mathfrak{m}_1} |S_{2,1}(\alpha)^2 S_{3,1}(\alpha)^3 S_{4,1}(\alpha)^2| d\alpha \right)^{\frac{2}{3}} \left(\int_{\mathcal{E}^*(0.83372131685)} 1 d\alpha \right)^{\frac{1}{12}} \\ &\ll N_1^{\frac{1}{6}+\frac{1}{12}+\frac{35}{36}-\frac{1}{18}-10^{-22}+\varepsilon} \ll N_1^{\frac{7}{6}-\varepsilon}. \end{aligned} \tag{6.4}$$

From (6.1)–(6.4), we can conclude that

$$\begin{aligned}\mathcal{R}(k, N_1) &\geq \mathcal{R}_1(k, N_1) - |\mathcal{R}_2(k, N_1, u)| - |\mathcal{R}_3(k, N_1, u)| \\ &\geq (3.19249 - 39.6553 \times 0.83372131685^{14}) I N_1^{\frac{7}{6}} L^k + O(N_1^{\frac{7}{6}-\varepsilon}) \\ &> 0.08 I N_1^{\frac{7}{6}} L^k. \end{aligned}$$

Now the proof of Theorem 2 is complete.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author thanks the referees for their time and comments. The author would like to express the most sincere gratitude to Professor Yingchun Cai for his valuable advice and constant encouragement.

Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

References

1. H. Davenport, *Analytic methods for Diophantine equations and Diophantine inequalities*, 2 Eds., Cambridge: Cambridge University Press, 2005. https://doi.org/10.1017/CBO9780511542893
2. P. X. Gallagher, Primes and powers of 2, *Invent. Math.*, **29** (1975), 125–142. https://doi.org/10.1007/BF01390190

3. X. Han, H. F. Liu, R. Y. Yue, Goldbach-Linnik type problems involving unequal powers of primes and powers of 2, *Bull. Belg. Math. Soc. Simon Stevin*, **31** (2024), 220–237. https://doi.org/10.36045/j.bbms.231113
4. D. R. Heath-Brown, J.-C. Puchta, Integers represented as a sum of primes and powers of two, *Asian J. Math.*, **6** (2002), 535–565. https://doi.org/10.4310/AJM.2002.v6.n3.a7
5. E. X. Huang, On pairs of equations with unequal powers of primes, *Czech. Math. J.*, **73** (2023), 1219–1228. https://doi.org/10.21136/CMJ.2023.0470-22
6. A. Languasco, A. Zaccagnini, On a Diophantine problem with two primes and s powers of two, *Acta Arith.*, **145** (2010), 193–208. https://doi.org/10.4064/aa145-2-7
7. H. Z. Li, The number of powers of 2 in a representation of large even integers by sums of such powers and of two primes, *Acta Arith.*, **92** (2000), 229–237. https://doi.org/10.4064/aa-92-3-229-237
8. H. Z. Li, The number of powers of 2 in a representation of large even integers by sums of such powers and of two primes II, *Acta Arith.*, **96** (2001), 369–379. https://doi.org/10.4064/aa96-4-7
9. Yu. V. Linnik, Prime numbers and powers of two, (Russian), *Trudy Matematicheskogo Instituta imeni VA Steklova* **38** (1951), 151–169.
10. Yu. V. Linnik, Addition of prime numbers with powers of one and the same number, (Russian), *Matematicheskii Sbornik*, **32** (1953), 3–60.
11. J. Y. Liu, M. C. Liu, T. Z. Wang, The number of powers of two in a representation of large even integers II, *Sci. China Ser. A-Math.*, **41** (1998), 1255–1271. https://doi.org/10.1007/BF02882266
12. Z. X. Liu, G. S. Lü, Density of two squares of primes and powers of 2, *Int. J. Number Theory*, **7** (2011), 1317–1329. https://doi.org/10.1142/S1793042111004605
13. Z. X. Liu, Goldbach-Linnik type problems with unequal powers of primes, *J. Number Theory*, **176** (2017), 439–448. https://doi.org/10.1016/j.jnt.2016.12.009
14. X. D. Lü, On unequal powers of primes and powers of 2, *Ramanujan J.*, **50** (2019), 111–121. https://doi.org/10.1007/s11139-018-0128-2
15. J. Pintz, I. Z. Ruzsa, On Linnik's approximation to Goldbach's problem. I, *Acta Arith.*, **109** (2003), 169–194. https://doi.org/10.4064/aa109-2-6
16. J. Pintz, I. Z. Ruzsa, On Linnik's approximation to Goldbach's problem. II., *Acta Math. Hungar.*, **161** (2020), 569–582. https://doi.org/10.1007/s10474-020-01077-8
17. E. C. Titchmarsh, D. R. Heath-Brown, *The theory of the Riemann zeta-function*, 2 Eds., Oxford: Oxford University Press, 1986.
18. R. C. Vaughan, *The Hardy–Littlewood method*, 2 Eds., Cambridge: Cambridge University Press, 1997. https://doi.org/10.1017/CBO9780511470929
19. T. Z. Wang, On Linnik's almost Goldbach theorem, *Sci. China Ser. A-Math.*, **42** (1999), 1155–1172. https://doi.org/10.1007/BF02875983
20. R. Zhang, Goldbach-Linnik type problems with mixed powers of primes, *Ramanujan J.*, **60** (2023), 1069–1080. https://doi.org/10.1007/s11139-022-00662-5

21. L. L. Zhao, On the Waring-Goldbach problem for fourth and sixth powers, *Proc. Lond. Math. Soc.*, **108** (2014), 1593–1622. https://doi.org/10.1112/plms/pdt072

22. L. L. Zhao, On unequal powers of primes and powers of 2, *Acta Math. Hungar.*, **146** (2015), 405–420. https://doi.org/10.1007/s10474-015-0535-4

23. X. D. Zhao, Goldbach-Linnik type problems on cubes of primes, *Ramanujan J.*, **57** (2022), 239–251. https://doi.org/10.1007/s11139-020-00303-9

24. L. Zhu, Goldbach-Linnik type problems with unequal powers of primes, *J. Korean Math. Soc.*, **59** (2022), 407–420. https://doi.org/10.4134/JKMS.j210371