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Research article

Generalization of some retarded integral inequalities with power in two independent variables and applications

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Abstract: In this work, some new kinds of nonlinear power integral inequalities in two-dimensional cases are introduced. With advanced analytical mathematical methods, many estimations are obtained in order to generalize to higher order, well known results in the literature for $p = 1$ and $p \in (0, 1]$. As an example, we apply these results to determine the boundedness of solutions of nonlinear bidimensional Volterra type integral equations for $p > 1$.

Keywords: boundedness; retarded integral inequality; power; Volterra integral equation
Mathematics Subject Classification: 26D10, 26D15, 45D05

1. Introduction

Applying integral inequalities to derive implicit bounds for unknown functions has become a crucial instrument in advancing the theory surrounding linear, nonlinear differential, and integral equations. These inequalities play an essential role in the analysis of the qualitative and quantitative properties of their solution (see [9, 10, 12]). Gronwall [11] was the pioneer in formalizing such an inequality with a single variable, which was later followed by the significant contributions of Bellman [3] and Bihari [4].

In recent years, there has been an increase in research focusing on integral inequalities that involve multiple independent variables, as seen in studies [2,5–7,17]. This rigorous approach helps to establish existence, uniqueness, and a stability analysis for solutions to numerous complex issues while also addressing practical challenges across diverse domains such as physics, engineering, and economics.

The following well-known fundamental Gronwall inequality [11] was used to estimate the solution of a linear differential equation:

$$u(x) \leq \int_{\alpha}^{x} (bu(s) + a)\, dx, \;\; x \in J, \tag{1.1}$$

where $J = [\alpha, \alpha + h]$.

The estimation of the unknown function u developed by Bellman [3] in 1943 for some constant $c \geq 0$ is given as follows:

$$u(x) \leq c + \int_{\alpha}^{x} f(s)ds, \ \ x \in J. \tag{1.2}$$

Furthermore, Pachpatte's generalization of the inequalities (1.1) and (1.2) found in [15] pertains to a single variable, as outlined below:

$$u(x) \leq c + \int_{0}^{x} [f(s)u(s) + h(s)]\, ds + \int_{0}^{x} f(s)\left(\int_{\alpha}^{s} g(\tau)u(\tau)d\tau\right) ds, \tag{1.3}$$

which occurred in the nonlinear context.

Still, the work of Abdeldaim and El-Deeb in [1] on an integral inequality (1.3) with a delay $\alpha(x)$ is outlined below:

$$u(x) \leq c + \int_{0}^{\alpha(x)} [f(s)u(s) + h(s)]\, ds + \int_{0}^{\alpha(x)} f(s)\left(\int_{\alpha}^{s} g(\tau)u(\tau)d\tau\right) ds, \tag{1.4}$$

where $\alpha \in C^1(R_+, R_+)$ is a nondecreasing function with $\alpha(x) \leq x$ and $\alpha(0) = 0$.

We additionally reviewed the paper of Li and Wang [13], where they introduced the power under the same conditions on $\alpha(x)$, as shown below:

$$u(x) \leq h(x) + \int_{0}^{\alpha(x)} f(s)\left[u^m(s) + \int_{\alpha}^{s} g(\tau)u^n(\tau)\, d\tau\right]^p ds, \tag{1.5}$$

where $m, n, p \in (0, 1]$.

Note that inequalities (1.3)–(1.5) have been proved in the cases $p = 1$ and $p \in (0, 1]$, respectively, though not $p > 1$? The aforementioned results are not covered, and it would also be interesting to generalize the inequalities considered in [8, 14, 16] to the more general nonlinearities. This document gives the sharp extensions to the nonlinear retarded integral inequalities with powers $p > 1$ (and $q > 1$), and furthermore in the bidimensional case. To keep our results in the context of the integral and differential equations, we state our results in the generalized types inequalities (1.6) and (1.7), as displayed below:

$$\begin{aligned} u(x, y) \ \ \leq \ \ & \eta(x, y) + \int_{0}^{\alpha(x)} \int_{0}^{\beta(y)} f(s, t)\Big(u^m(s, t) \\ & + \int_{0}^{s} \int_{0}^{t} g(r, l)u^n(r, l)\left[u(r, l) + \int_{0}^{r} \int_{0}^{l} h(\tau, \sigma)u^q(\tau, \sigma)d\sigma d\tau\right] dldr\Big)^p dtds, \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} u(x, y) \ \ \leq \ \ & \eta(x, y) + \int_{0}^{\alpha(x)} \int_{0}^{\beta(y)} f(s, t)\Bigg(u^m(s, t) + \int_{0}^{s} \int_{0}^{t} g(r, l)u^n(r, l) \\ & \left[u(r, l) + \int_{0}^{r} \int_{0}^{l} h(\tau, \sigma)u(\tau, \sigma)d\sigma d\tau\right]^q dldr\Bigg)^p dtds. \end{aligned} \tag{1.7}$$

To resolve the complex calculation methods developed in this work, we introduce and prove two fundamental lemmas, namely Lemmas 2.3 and 2.4, to accurately estimate the unknown function $u(x, y)$ in (1.6) and (1.7). As part of a direct application, an illustrative example is also provided to show the usefulness of our findings.

2. Main results

Throughout this paper, R denotes the set of real numbers, whereas $R_+ = [0, \infty)$ is the subset of R, and the derivative is presented through $(')$. Moreover, the sets of all continuous functions from R_+ into R_+ are denoted by $C(R_+, R_+)$.

Lemma 2.1. *[13] Let $a \geq 0$ and $m \geq n > 0$; then,*

$$a^{\frac{n}{m}} \leq \frac{n}{m}a + \frac{m-n}{m}.$$

Lemma 2.2. *[16] Assume that $u, v \geq 0$, and $p \geq 0$. Then, $(u+v)^p \leq k_p\,(u^p + v^p)$, where*

$$k_p = \begin{cases} 1, & 0 \leq p \leq 1, \\ 2^{p-1}, & p > 1. \end{cases}$$

Now, let us state and prove our first principal lemma, which will be used in Theorem 2.1.

Lemma 2.3. *Let $p, q > 1$ be given constants, and consider the functions $u, \eta, f, g, h \in C\,(R_+, R_+)$, and $\alpha, \beta \in C^1\,(R_+, R_+)$. Assume that $\alpha(x)$ and $\beta(y)$ are nondecreasing functions such that $\alpha(x) \leq x$ and $\beta(y) \leq y$ for all $x, y \in R_+$. If u satisfies*

$$u(x,y) \leq \eta(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t)\Big[g(s,t)u^q(s,t) + h(s,t)\Big]^p dtds, \tag{2.1}$$

and

$$k^{1-pq}(x,y) + (1-pq)\,2^{pq-1} \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t)g^p(s,t)dtds > 0,$$

then

$$u(x,y) \leq \eta(x,y) + \left[k^{1-pq}(x,y) + (1-pq)\,2^{pq-1} \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t)g^p(s,t)dtds\right]^{\frac{1}{1-pq}}, \tag{2.2}$$

where

$$k(x,y) = 2^{p-1} \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t)\Big[2^{q-1}g(s,t)a^q(s,t) + h(s,t)\Big]^p dtds. \tag{2.3}$$

Proof. Consider the following function:

$$z(x,y) = \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t)\Big[g(s,t)u^q(s,t) + h(s,t)\Big]^p dtds,$$

with $z(x,0) = z(0,y) = 0$. Then, $z(x,y)$ is a nondecreasing function, and we have

$$u(x,y) \leq \eta(x,y) + z(x,y). \tag{2.4}$$

Using (2.4), and applying Lemma 2.2, we obtain the following:

$$
\begin{aligned}
z(x,y) &\le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[g(s,t)\Big(\eta(s,t)+z(s,t)\Big)^q + h(s,t)\Big]^p dtds \\
&\le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[g(s,t)2^{q-1}\Big(\eta^q(s,t)+z^q(s,t)\Big) + h(s,t)\Big]^p dtds \\
&\le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[2^{q-1}g(s,t)z^q(s,t) + 2^{q-1}g(s,t)\eta^q(s,t) + h(s,t)\Big]^p dtds;
\end{aligned}
$$

then,

$$
\begin{aligned}
z(x,y) &\le 2^{p-1}\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\left(2^{p(q-1)}g^p(s,t)z^{pq}(s,t) + \Big[2^{q-1}g(s,t)\eta^q(s,t)+h(s,t)\Big]^p\right)dtds \\
&\le k(x,y) + 2^{pq-1}\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)g^p(s,t)z^{pq}(s,t)dtds,
\end{aligned}
$$

given that $k(x,y)$ is a nondecreasing function. Then, for (X,Y) fixed, we obtain the following:

$$
z(x,y) \le k(X,Y) + 2^{pq-1}\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)g^p(s,t)z^{pq}(s,t)dtds, \tag{2.5}
$$

for $x \in [0,X]$, $y \in [0,Y]$. Define

$$
v(x,y) = k(X,Y) + 2^{pq-1}\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)g^p(s,t)z^{pq}(s,t)dtds, \tag{2.6}
$$

with $v(0,y) = v(x,0) = k(X,Y)$, and $v(x,y)$ is a nondecreasing function; then,

$$
z(x,y) \le v(x,y). \tag{2.7}
$$

Differentiating Eq (2.6) with respect to x, and using Eq (2.7), we obtain the following:

$$
\begin{aligned}
\frac{\partial}{\partial x}v(x,y) &\le 2^{pq-1}\alpha'(x)\int_0^{\beta(y)} f(\alpha(x),t)g^p(\alpha(x),t)v^{pq}(\alpha(x),t)dt \\
&\le 2^{pq-1}\alpha'(x)v^{pq}(\alpha(x),\beta(y))\int_0^{\beta(y)} f(\alpha(x),t)g^p(\alpha(x),t)dt \\
&\le 2^{pq-1}\alpha'(x)v^{pq}(x,y)\int_0^{\beta(y)} f(\alpha(x),t)g^p(\alpha(x),t)dt.
\end{aligned}
$$

Therefore,

$$
\frac{\frac{\partial}{\partial x}v(x,y)}{v^{pq}(x,y)} \le 2^{pq-1}\alpha'(x)\int_0^{\beta(y)} f(\alpha(x),t)g^p(\alpha(x),t)dt. \tag{2.8}
$$

From (2.8), we have the following:

$$
v^{-pq}(x,y)\frac{\partial}{\partial x}v(x,y) = -\frac{1}{pq-1}\left(\frac{\partial}{\partial x}v^{-(pq-1)}(x,y)\right) \le 2^{pq-1}\alpha'(x)\int_0^{\beta(y)} f(\alpha(x),t)g^p(\alpha(x),t)dt.
$$

By integrating both sides of the final inequality with respect to x, we obtain

$$v^{1-pq}(x,y) \geq k^{1-pq}(X,Y) + (1-pq)\,2^{pq-1}\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)g^p(s,t)dtds,$$

or

$$v(x,y) \leq \left[\frac{1}{k^{1-pq}(X,Y) + (1-pq)\,2^{pq-1}\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)g^p(s,t)dtds}\right]^{\frac{1}{pq-1}}.$$

Since X, Y are arbitrary, therefore,

$$v(x,y) \leq \left[k^{1-pq}(x,y) + (1-pq)\,2^{pq-1}\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)g^p(s,t)dtds\right]^{\frac{1}{1-pq}}. \tag{2.9}$$

From (2.4), (2.7), and (2.9), we obtain the desired result (2.2). □

Theorem 2.1. *Assume that m, n, p, and q are nonnegative constants satisfying m, n, p, and $q > 1$ with $m < n + q$, and let $u, \eta, f, g, h \in C(R_+, R_+)$, and α, β as in Lemma 2.3, and*

$$\begin{aligned} u(x,y) \;\leq\; & \eta(x,y) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Bigg(u^m(s,t) + \int_0^s\int_0^t g(r,l)u^n(r,l) \\ & \left[u(r,l) + \int_0^r\int_0^l h(\tau,\sigma)u(\tau,\sigma)d\sigma d\tau\right]^q dldr\Bigg)^p dtds. \end{aligned} \tag{2.10}$$

Then, for

$$l^{1-p(n+q)}(x,y) + \Big(1 - p(n+q)\Big)2^{p(n+q)-1}\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)R^p(s,t)dtds > 0,$$

we obtain

$$\begin{aligned} u(x,y) \;\leq\; & \eta(x,y) + \Big[l^{1-p(n+q)}(x,y) \\ & + \Big(1 - p(n+q)\Big)2^{p(n+q)-1}\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)R^p(s,t)dtds\Big]^{\frac{1}{1-p(n+q)}}, \end{aligned} \tag{2.11}$$

where

$$l(x,y) = 2^{p-1}\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)N^p(s,t)dtds, \tag{2.12}$$

$$\begin{aligned} N(s,t) \;=\; & M(s,t) + 2^{m-1}\frac{n+q-m}{n+q} \\ & + 2^{n+q-2}\frac{n}{n+q}\int_0^s\int_0^t g(r,l)\eta^n(r,l)\left(1 + \int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)^q dldr \\ & + 2^{n+q-2}\frac{q}{n+q}\int_0^s\int_0^t g(r,l)\left(\eta(r,l) + \int_0^r\int_0^l h(\tau,\sigma)\eta(\tau,\sigma)d\sigma d\tau\right)^q dldr, \end{aligned} \tag{2.13}$$

$$
\begin{aligned}
M(s,t) &= 2^{m-1}\eta^m(s,t) + 2^{q+n-2}\int_0^s\int_0^t g(r,l)\eta^n(r,l) \\
&\quad \left(\eta(r,l) + \int_0^r\int_0^l h(\tau,\sigma)\eta(\tau,\sigma)d\sigma d\tau\right)^q dldr, \qquad (2.14)
\end{aligned}
$$

and

$$
\begin{aligned}
R(s,t) &= 2^{m-1}\frac{m}{n+q} + 2^{n+q-2}\frac{q}{n+q}\int_0^s\int_0^t g(r,l)\eta^n(r,l)\left(1 + \int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)^q dldr \\
&\quad + 2^{n+q-2}\frac{n}{n+q}\int_0^s\int_0^t g(r,l)\left(\eta(r,l) + \int_0^r\int_0^l h(\tau,\sigma)\eta(\tau,\sigma)d\sigma d\tau\right)^q dldr \\
&\quad + 2^{q+n-2}\int_0^s\int_0^t g(r,l)\left(1 + \int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)^q dldr. \qquad (2.15)
\end{aligned}
$$

Proof. Let

$$
\begin{aligned}
z(x,y) &= \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\left(u^m(s,t) + \int_0^s\int_0^t g(r,l)u^n(r,l)\right. \\
&\quad \left.\times\left[u(r,l) + \int_0^r\int_0^l h(\tau,\sigma)u(\tau,\sigma)d\sigma d\tau\right]^q dldr\right)^p dtds,
\end{aligned}
$$

with $z(x,0) = z(0,y) = 0$. Then, $z(x,y)$ is a nonnegative and nondecreasing function, and

$$
u(x,y) \le \eta(x,y) + z(x,y). \qquad (2.16)
$$

Using Lemma 2.2, and from (2.16), we obtain the following:

$$
\begin{aligned}
z(x,y) &\le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[2^{m-1}\big(\eta^m(s,t) + z^m(s,t)\big) + 2^{n-1}\int_0^s\int_0^t g(r,l)\big(\eta^n(r,l) + z^n(r,l)\big) \\
&\quad \times\left[\eta(r,l) + z(r,l) + \int_0^r\int_0^l h(\tau,\sigma)\big(\eta(\tau,\sigma) + z(\tau,\sigma)\big)d\sigma d\tau\right]^q dldr\Big]^p dtds \\
&\le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[2^{m-1}\big(\eta^m(s,t) + z^m(s,t)\big) + 2^{n-1}\int_0^s\int_0^t g(r,l) \\
&\quad \big(\eta^n(r,l) + z^n(r,l)\big)2^{q-1}\left[\left(\eta(r,l) + \int_0^r\int_0^l h(\tau,\sigma)\eta(\tau,\sigma)d\sigma d\tau\right)^q\right. \\
&\quad \left.+\left(z(r,l) + \int_0^r\int_0^l h(\tau,\sigma)z(\tau,\sigma)d\sigma d\tau\right)^q\right]dldr\Big]^p dtds \\
&\le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[M(s,t) + 2^{m-1}z^m(s,t) \\
&\quad + 2^{q+n-2}z^q(s,t)\int_0^s\int_0^t g(r,l)\eta^n(r,l)\left(1 + \int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)^q dldr
\end{aligned}
$$

$$+2^{q+n-2}z^{n}(s,t)\int_0^s\int_0^t g(r,l)\left(\eta(r,l)+\int_0^r\int_0^l h(\tau,\sigma)\eta(\tau,\sigma)d\sigma d\tau\right)^q dldr$$
$$+2^{q+n-2}z^{n+q}(s,t)\int_0^s\int_0^t g(r,l)\left(1+\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)^q dldr\Bigg]^p dtds, \tag{2.17}$$

where $M(s,t)$ is defined by (2.14).

Let $z^{n+q}(x,y)=v(x,y)$; therefore, (2.17) can be reformulated as follows:

$$\begin{aligned}
v^{\frac{1}{n+q}}(x,y) \leq & \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[M(s,t)+2^{m-1}v^{\frac{m}{n+q}}(s,t)\\
&+2^{q+n-2}v^{\frac{q}{n+q}}(s,t)\int_0^s\int_0^t g(r,l)\eta^n(r,l)\left(1+\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)^q dldr\\
&+2^{q+n-2}v^{\frac{n}{n+q}}(s,t)\int_0^s\int_0^t g(r,l)\left(\eta(r,l)+\int_0^r\int_0^l h(\tau,\sigma)\eta(\tau,\sigma)d\sigma d\tau\right)^q dldr\\
&+2^{q+n-2}v(s,t)\int_0^s\int_0^t g(r,l)\left(1+\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)^q dldr\Big]^p dtds.
\end{aligned}$$

Using Lemma 2.1, we have the following:

$$\begin{aligned}
v^{\frac{1}{n+q}}(x,y) \leq & \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Bigg[M(s,t)+2^{m-1}\left(\frac{m}{n+q}v(s,t)+\frac{n+q-m}{n+q}\right)\\
&+2^{n+q-2}\left(\frac{q}{n+q}v(s,t)+\frac{n}{n+q}\right)\int_0^s\int_0^t g(r,l)\eta^n(r,l)\left(1+\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)^q dldr\\
&+2^{n+q-2}\left(\frac{n}{n+q}v(s,t)+\frac{q}{n+q}\right)\int_0^s\int_0^t g(r,l)\left(\eta(r,l)+\int_0^r\int_0^l h(\tau,\sigma)\eta(\tau,\sigma)d\sigma d\tau\right)^q dldr\\
&+2^{q+n-2}v(s,t)\int_0^s\int_0^t g(r,l)\left(1+\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)^q dldr\Bigg]^p dtds\\
\leq & \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Bigg[M(s,t)+2^{m-1}\frac{m}{n+q}v(s,t)+2^{m-1}\frac{n+q-m}{n+q}\\
&+2^{n+q-2}\frac{q}{n+q}v(s,t)\int_0^s\int_0^t g(r,l)\eta^n(r,l)\left(1+\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)^q dldr\\
&+2^{n+q-2}\frac{n}{n+q}\int_0^s\int_0^t g(r,l)\eta^n(r,l)\left(1+\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)^q dldr\\
&+2^{n+q-2}\frac{n}{n+q}v(s,t)\int_0^s\int_0^t g(r,l)\left(\eta(r,l)+\int_0^r\int_0^l h(\tau,\sigma)\eta(\tau,\sigma)d\sigma d\tau\right)^q dldr\\
&+2^{n+q-2}\frac{q}{n+q}\int_0^s\int_0^t g(r,l)\left(\eta(r,l)+\int_0^r\int_0^l h(\tau,\sigma)\eta(\tau,\sigma)d\sigma d\tau\right)^q dldr\\
&+2^{q+n-2}v(s,t)\int_0^s\int_0^t g(r,l)\left(1+\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)^q dldr\Bigg]^p dtds\\
\leq & \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[v(s,t)R(s,t)+N(s,t)\Big]^p dtds,
\end{aligned}$$

where N and R are defined by (2.13) and (2.15) respectively. Then,

$$z(x,y) \leq \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[R(s,t)z^{n+q}(s,t)+N(s,t)\Big]^p dtds. \tag{2.18}$$

Now, applying Lemma 2.3 with $\eta(x,y)=0$, and $n+q$ instead of q to (2.18), we obtain the following:

$$z(x,y) \leq \left[l^{1-p(n+q)}(x,y)+\Big(1-p(n+q)\Big)2^{p(n+q)-1}\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)R^p(s,t)dtds\right]^{\frac{1}{1-p(n+q)}}. \tag{2.19}$$

From (2.16) and (2.19), we obtain the upper bound of $u(x,y)$ given in (2.11). □

Remark 2.1. *It is essential to note that we obtain the same form as (2.9) in Theorem 2.1 of [16] for $h=0$ in (2.10); however, in our context, it is given in the general nonlinear bidimensional case. Our results may also generalize those identified in [1] and [13].*

The next fundamental lemma which we shall use in the next theorem is as follows.

Lemma 2.4. *Suppose that $p,r>1$ are constants and $u,\eta,f,g,h,e\in C(R_+,R_+)$, and α,β are as in Lemma 2.3, and*

$$u(x,y)\leq \eta(x,y)+\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[g(s,t)u^2(s,t)+h(s,t)u(s,t)+e(s,t)\Big]^p dtds\right)^r. \tag{2.20}$$

Then, for

$$D^{1-2rp}(x,y)+(1-2rp)\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)B(s,t)dtds>0,$$

we have

$$u(x,y)\leq \eta(x,y)+\left(D^{1-2rp}(x,y)+(1-2rp)\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)B(s,t)dtds\right)^{\frac{r}{1-2rp}}, \tag{2.21}$$

where

$$D(x,y)=\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)C(s,t)dtds, \tag{2.22}$$

$$C(x,y)=2^{p-1}A^p(x,y)+2^{2p-3}h^p(x,y), \tag{2.23}$$

$$A(x,y)=2g(x,y)\eta^2(x,y)+h(x,y)\eta(x,y)+e(x,y), \tag{2.24}$$

and

$$B(x,y)=2^{3p-2}g^p(x,y)+2^{2p-3}h^p(x,y). \tag{2.25}$$

Proof. Take in (2.20) as follows:

$$z(x,y)=\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[g(s,t)u^2(s,t)+h(s,t)u(s,t)+e(s,t)\Big]^p dtds,$$

with $z(x,0)=z(0,y)=0$; then, $z(x,y)$ is a nondecreasing function, and

$$u(x,y)\leq \eta(x,y)+z^r(x,y). \tag{2.26}$$

From Lemma 2.2 and (2.26), we have the following:

$$\begin{aligned} z(x,y) &\le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[2g(s,t)\left[\eta^2(s,t)+z^{2r}(s,t)\right]+h(s,t)\left[\eta(s,t)+z^r(s,t)\right]+e(s,t)\Big]^p dtds \\ &\le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[A(s,t)+2g(s,t)z^{2r}(s,t)+h(s,t)z^r(s,t)\Big]^p dtds \\ &\le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)2^{p-1}\Big[A^p(s,t)+\Big[2g(s,t)z^{2r}(s,t)+h(s,t)z^r(s,t)\Big]^p\Big]dtds \\ &\le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)2^{p-1}\Big[A^p(s,t)+2^{p-1}\Big[2^p g^p(s,t)z^{2rp}(s,t)+h^p(s,t)z^{rp}(s,t)\Big]\Big]dtds \\ &\le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)2^{p-1}\Big[A^p(s,t)+2^{2p-1}g^p(s,t)z^{2rp}(s,t)+2^{p-1}h^p(s,t)z^{rp}(s,t)\Big]dtds, \end{aligned} \tag{2.27}$$

where $A(x,y)$ is defined by (2.24). Take in (2.28) as follows:

$$z^{2rp}(x,y) = v(x,y). \tag{2.28}$$

We obtain the following:

$$v^{\frac{1}{2rp}}(x,y) \le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)2^{p-1}\Big[A^p(s,t)+2^{2p-1}g^p(s,t)v(s,t)+2^{p-1}h^p(s,t)v^{\frac{1}{2}}(s,t)\Big]dtds. \tag{2.29}$$

Applying Lemma 2.1 to (2.29), we obtain the following:

$$\begin{aligned} v^{\frac{1}{2rp}}(x,y) &\le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)2^{p-1}\Big[A^p(s,t)+2^{2p-1}g^p(s,t)v(s,t)+2^{p-1}h^p(s,t)\left(\frac{1}{2}v(s,t)+\frac{1}{2}\right)\Big]dtds \\ &\le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[B(s,t)v(s,t)+g(s,t)\Big]dtds \\ &\le D(x,y)+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)B(s,t)v(s,t)dtds, \end{aligned} \tag{2.30}$$

where $B(x,y)$, $C(x,y)$, and $D(x,y)$ are defined by (2.25), (2.23), and (2.22), respectively.

Since $D(x,y)$ is a nondecreasing function, then for (X,Y) fixed and $0 \le x \le X$, $0 \le y \le Y$, we have the following:

$$v^{\frac{1}{2rp}}(x,y) \le D(X,Y)+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)B(s,t)v(s,t)dtds. \tag{2.31}$$

Take in (2.31) as follows:

$$j(x,y) = D(X,Y)+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)B(s,t)v(s,t)dtds, \tag{2.32}$$

with $j(x,0) = j(0,y) = D(X,Y)$. Then, $j(x,y)$ is a nondecreasing function, and

$$v^{\frac{1}{2rp}}(x,y) \le j(x,y). \tag{2.33}$$

Differentiating (2.32) with respect to x, and using (2.33), we obtain the following:

$$\begin{aligned}\frac{\partial}{\partial x}j(x,y) &= \alpha'(x)\int_0^{\beta(y)} f(\alpha(x),t)B(\alpha(x),t)v(\alpha(x),t)dt\\ &\leq \alpha'(x)v(\alpha(x),\beta(y))\int_0^{\beta(y)} f(\alpha(x),t)B(\alpha(x),t)dt\\ &\leq \alpha'(x)j^{2rp}(\alpha(x),\beta(y))\int_0^{\beta(y)} f(\alpha(x),t)B(\alpha(x),t)dt.\end{aligned}$$

The function $j(x,y)$ is positive and nondecreasing, and $p,r>1$ are constants; then, $j^{2rp}(x,y)$ is a positive and nondecreasing function. Moreover, we have $\alpha(x)\leq x,\beta(y)\leq y$; then, $j^{2rp}(\alpha(x),\beta(y))\leq j^{2rp}(x,y)$, so the last inequality can be rephrased as follows:

$$\frac{\partial}{\partial x}j(x,y) \leq \alpha'(x)j^{2rp}(x,y)\int_0^{\beta(y)} f(\alpha(x),t)B(\alpha(x),t)dt.$$

Therefore,

$$\frac{\frac{\partial}{\partial x}j(x,y)}{j^{2rp}(x,y)} \leq \alpha'(x)\int_0^{\beta(y)} f(\alpha(x),t)B(\alpha(x),t)dt.$$

Since

$$\frac{\frac{\partial}{\partial x}j(x,y)}{j^{2rp}(x,y)} = \frac{1}{1-2rp}\frac{\partial}{\partial x}j^{(1-2rp)}(x,y),$$

then,

$$\frac{\partial}{\partial x}j^{1-2rp}(x,y) \geq (1-2rp)\,\alpha'(x)\int_0^{\beta(y)} f(\alpha(x),t)B(\alpha(x),t)dt.$$

Integrating both sides of the last inequality with respect to s from 0 to x, we obtain the following:

$$j(x,y)\leq\left(D^{1-2rp}(X,Y)+(1-2rp)\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)B(s,t)dtds\right)^{\frac{1}{1-2rp}},$$

where

$$D^{1-2rp}(X,Y)+(1-2rp)\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)B(s,t)dtds>0.$$

Since (X,Y) is arbitrary, then,

$$j(x,y)\leq\left(D^{1-2rp}(x,y)+(1-2rp)\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)B(s,t)dtds\right)^{\frac{1}{1-2rp}}. \tag{2.34}$$

From (2.33), (2.34), (2.26), and (2.28), we obtain the desired result (2.21). □

Theorem 2.2. *Assume that m,n,p,q are as in Theorem 2.1, $u,\eta,f,g,h\in C(R_+,R_+)$, α,β are as in Lemma 2.3, and*

$$u(x,y) \leq \eta(x,y)+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\bigg(u^m(s,t)+\int_0^s\int_0^t g(r,l)u^n(r,l)$$

$$\left[u(r,l)+\int_0^r\int_0^l h(\tau,\sigma)u^q(\tau,\sigma)d\sigma d\tau\right]dldr\Bigg)^p dtds. \tag{2.35}$$

Then, for

$$F^{1-2(n+q)p}(x,y)+(1-2(n+q)p)\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)G(s,t)dtds>0,$$

we obtain

$$u(x,y)\le \eta(x,y)+\left(F^{1-2(n+q)p}(x,y)+(1-2(n+q)p)\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)G(s,t)dtds\right)^{\frac{1}{1-2(n+q)p}}, \tag{2.36}$$

where

$$F(x,y)=\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)H(s,t)dtds, \tag{2.37}$$

$$H(x,y)=2^{2p-3}R^p(x,y)+2^{p-1}Q^p(x,y), \tag{2.38}$$

$$G(x,y)=2^{3p-2}N^p(x,y)+2^{2p-3}R^p(x,y), \tag{2.39}$$

$$\begin{aligned} Q(s,t) &= M(s,t)+2^{m-1}\frac{n+q-m}{n+q}+2^{n-1}\frac{n+q-1}{n+q}\int_0^s\int_0^t g(r,l)\eta^n(r,l)dldr \\ &+2^{n+q-2}\frac{n}{n+q}\int_0^s\int_0^t g(r,l)\eta^n(r,l)\left(\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)dldr \\ &+2^{n-1}\frac{q}{n+q}\int_0^s\int_0^t g(r,l)\left(\eta(r,l)+2^{q-1}\int_0^r\int_0^l h(\tau,\sigma)\eta^q(\tau,\sigma)d\sigma d\tau\right)dldr \\ &+2^{n-1}\frac{q\,(n+q-1)}{(n+q)^2}\int_0^s\int_0^t g(r,l)dldr, \end{aligned} \tag{2.40}$$

$$\begin{aligned} M(s,t) &= 2^{m-1}\eta^m(s,t)+2^{n-1}\int_0^s\int_0^t g(r,l)\eta^n(r,l) \\ &\quad\times\left(\eta(r,l)+2^{q-1}\int_0^r\int_0^l h(\tau,\sigma)\eta^q(\tau,\sigma)d\sigma d\tau\right)dldr, \end{aligned} \tag{2.41}$$

$$\begin{aligned} R(s,t) &= 2^{m-1}\frac{m}{n+q}+2^{n-1}\frac{1}{n+q}\int_0^s\int_0^t g(r,l)\eta^n(r,l)dldr \\ &+2^{n+q-2}\frac{q}{n+q}\int_0^s\int_0^t g(r,l)\eta^n(r,l)\left(\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)dldr \\ &+2^{n-1}\frac{n}{n+q}\int_0^s\int_0^t g(r,l)\left(\eta(r,l)+2^{q-1}\int_0^r\int_0^l h(\tau,\sigma)\eta^q(\tau,\sigma)d\sigma d\tau\right)dldr \\ &+2^{n-1}\left(\frac{q+n\,(n+q-1)}{(n+q)^2}\right)\int_0^s\int_0^t g(r,l)dldr \\ &+2^{n+q-2}\int_0^s\int_0^t g(r,l)\left(\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)dldr, \end{aligned} \tag{2.42}$$

and

$$N(s,t)=\frac{n2^{n-1}}{(n+q)^2}\int_0^s\int_0^t g(r,l)dldr. \tag{2.43}$$

Proof. Let in (2.35) as follows:

$$\begin{aligned} z(x,y) &= \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\bigg(u^m(s,t)+\int_0^s\int_0^t g(r,l)u^n(r,l)\\ &\quad \left[u(r,l)+\int_0^r\int_0^l h(\tau,\sigma)u^q(\tau,\sigma)d\sigma d\tau\right]dldr\bigg)^p dtds, \end{aligned} \tag{2.44}$$

with $z(x,0)=z(0,y)=0$. Then, $z(x,y)$ is a nondecreasing function, and

$$u(x,y)\le \eta(x,y)+z(x,y). \tag{2.45}$$

Using (2.45) in (2.44), and applying Lemma 2.2, we obtain the following:

$$\begin{aligned} z(x,y) &\le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[2^{m-1}\Big(\eta^m(s,t)+z^m(s,t)\Big)\\ &\quad +2^{n-1}\int_0^s\int_0^t g(r,l)\Big(\eta^n(r,l)+z^n(r,l)\Big)\Big[\eta(r,l)+z(r,l)\\ &\quad +2^{q-1}\int_0^r\int_0^l h(\tau,\sigma)\Big(\eta^q(\tau,\sigma)+z^q(\tau,\sigma)\Big)d\sigma d\tau\Big]dldr\Big]^p dtds\\ &\le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[2^{m-1}\eta^m(s,t)+2^{m-1}z^m(s,t)\\ &\quad +2^{n-1}\int_0^s\int_0^t\Big(g(r,l)\eta^n(r,l)+g(r,l)z^n(r,l)\Big)\times\bigg[\bigg(\eta(r,l)+2^{q-1}\int_0^r\int_0^l h(\tau,\sigma)\eta^q(\tau,\sigma)d\sigma d\tau\bigg)\\ &\quad +\bigg(z(r,l)+2^{q-1}\int_0^r\int_0^l h(\tau,\sigma)z^q(\tau,\sigma)d\sigma d\tau\bigg)\bigg]dldr\Big]^p dtds\\ &\le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[M(s,t)+2^{m-1}z^m(s,t)+2^{n-1}\int_0^s\int_0^t g(r,l)\eta^n(r,l)z(r,l)dldr\\ &\quad +2^{n+q-2}\int_0^s\int_0^t g(r,l)\eta^n(r,l)\bigg(\int_0^r\int_0^l h(\tau,\sigma)z^q(\tau,\sigma)d\sigma d\tau\bigg)dldr\\ &\quad +2^{n-1}\int_0^s\int_0^t g(r,l)z^n(r,l)\bigg(\eta(r,l)+2^{q-1}\int_0^r\int_0^l h(\tau,\sigma)\eta^q(\tau,\sigma)d\sigma d\tau\bigg)dldr\\ &\quad +2^{n-1}\int_0^s\int_0^t g(r,l)z^{n+1}(r,l)dldr\\ &\quad +2^{n+q-2}\int_0^s\int_0^t g(r,l)z^n(r,l)\bigg(\int_0^r\int_0^l h(\tau,\sigma)z^q(\tau,\sigma)d\sigma d\tau\bigg)dldr\Big]^p dtds\\ &\le \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\bigg[M(s,t)+2^{m-1}z^m(s,t)+2^{n-1}z(s,t)\int_0^s\int_0^t g(r,l)\eta^n(r,l)dldr\\ &\quad +2^{n+q-2}z^q(s,t)\int_0^s\int_0^t g(r,l){a_0}^n(r,l)\bigg(\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\bigg)dldr \end{aligned}$$

$$+2^{n-1}z^{n}(s,t)\int_0^s\int_0^t g(r,l)\left(\eta(r,l)+2^{q-1}\int_0^r\int_0^l h(\tau,\sigma)\eta^{q}(\tau,\sigma)d\sigma d\tau\right)dldr$$
$$+2^{n-1}z^{n+1}(s,t)\int_0^s\int_0^t g(r,l)dldr$$
$$+2^{n+q-2}z^{n+q}(s,t)\int_0^s\int_0^t g(r,l)\left(\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)dldr\Bigg]^p dtds,$$

where $M(s,t)$ is defined by (2.41).

Let $z^{n+q}(x,y)=v(x,y)$ in the last inequality; then, we obtain the following:

$$\begin{aligned}
v^{\frac{1}{n+q}}(x,y) \leq\ & \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[M(s,t)+2^{m-1}v^{\frac{m}{n+q}}(s,t)\\
&+2^{n-1}v^{\frac{1}{n+q}}(s,t)\int_0^s\int_0^t g(r,l)\eta^{n}(r,l)dldr\\
&+2^{n+q-2}v^{\frac{q}{n+q}}(s,t)\int_0^s\int_0^t g(r,l)\eta^{n}(r,l)\left(\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)dldr\\
&+2^{n-1}v^{\frac{n}{n+q}}(s,t)\int_0^s\int_0^t g(r,l)\left(\eta(r,l)+2^{q-1}\int_0^r\int_0^l h(\tau,\sigma)\eta^{q}(\tau,\sigma)d\sigma d\tau\right)dldr\\
&+2^{n-1}v^{\frac{n+1}{n+q}}(s,t)\int_0^s\int_0^t g(r,l)dldr\\
&+2^{n+q-2}v(s,t)\int_0^s\int_0^t g(r,l)\left(\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)dldr\Bigg]^p dtds.
\end{aligned}$$

Using Lemma 2.1, we have the following:

$$\begin{aligned}
v^{\frac{1}{n+q}}(x,y) \leq\ & \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Bigg[M(s,t)+2^{m-1}\left(\frac{m}{n+q}v(s,t)+\frac{n+q-m}{n+q}\right)\\
&+2^{n-1}\left(\frac{1}{n+q}v(s,t)+\frac{n+q-1}{n+q}\right)\int_0^s\int_0^t g(r,l)\eta^{n}(r,l)dldr\\
&+2^{n+q-2}\left(\frac{q}{n+q}v(s,t)+\frac{n}{n+q}\right)\int_0^s\int_0^t g(r,l)\eta^{n}(r,l)\\
&\left(\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)dldr+2^{n-1}\left(\frac{n}{n+q}v(s,t)+\frac{q}{n+q}\right)\\
&\int_0^s\int_0^t g(r,l)\left(\eta(r,l)+2^{q-1}\int_0^r\int_0^l h(\tau,\sigma)\eta^{q}(\tau,\sigma)d\sigma d\tau\right)dldr\\
&+2^{n-1}v^{\frac{n}{n+q}}(s,t)v^{\frac{1}{n+q}}(s,t)\int_0^s\int_0^t g(r,l)dldr\\
&+2^{n+q-2}v(s,t)\int_0^s\int_0^t g(r,l)\left(\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)dldr\Bigg]^p dtds\\
\leq\ & \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Bigg[M(s,t)+2^{m-1}\frac{m}{n+q}v(s,t)+2^{m-1}\frac{n+q-m}{n+q}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2^{n-1}}{n+q}v(s,t)\int_0^s\int_0^t g(r,l)\eta^n(r,l)dldr\\
&+2^{n-1}\frac{n+q-1}{n+q}\int_0^s\int_0^t g(r,l)\eta^n(r,l)dldr\\
&+2^{n+q-2}\frac{q}{n+q}v(s,t)\int_0^s\int_0^t g(r,l)\eta^n(r,l)\left(\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)dldr\\
&+2^{n+q-2}\frac{n}{n+q}\int_0^s\int_0^t g(r,l)\eta^n(r,l)\left(\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)dldr\\
&+2^{n-1}\frac{n}{n+q}v(s,t)\int_0^s\int_0^t g(r,l)\left(\eta(r,l)+2^{q-1}\int_0^r\int_0^l h(\tau,\sigma)\eta^q(\tau,\sigma)d\sigma d\tau\right)dldr\\
&+2^{n-1}\frac{q}{n+q}\int_0^s\int_0^t g(r,l)\left(\eta(r,l)+2^{q-1}\int_0^r\int_0^l h(\tau,\sigma)\eta^q(\tau,\sigma)d\sigma d\tau\right)dldr\\
&+2^{n-1}\left(\frac{n}{n+q}v(s,t)+\frac{q}{n+q}\right)\left(\frac{1}{n+q}v(s,t)+\frac{n+q-1}{n+q}\right)\int_0^s\int_0^t g(r,l)dldr\\
&+2^{n+q-2}v(s,t)\int_0^s\int_0^t g(r,l)\left(\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)dldr\Bigg]^p dtds,\\
\leq\ &\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Bigg[M(s,t)+2^{m-1}\frac{m}{n+q}v(s,t)+2^{m-1}\frac{n+q-m}{n+q}\\
&+2^{n-1}\frac{1}{n+q}v(s,t)\int_0^s\int_0^t g(r,l)\eta^n(r,l)dldr\\
&+2^{n-1}\frac{n+q-1}{n+q}\int_0^s\int_0^t g(r,l)\eta^n(r,l)dldr\\
&+2^{n+q-2}\frac{q}{n+q}v(s,t)\int_0^s\int_0^t g(r,l)\eta^n(r,l)\left(\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)dldr\\
&+2^{n+q-2}\frac{n}{n+q}\int_0^s\int_0^t g(r,l)\eta^n(r,l)\left(\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)dldr\\
&+2^{n-1}\frac{n}{n+q}v(s,t)\int_0^s\int_0^t g(r,l)\left(\eta(r,l)+2^{q-1}\int_0^r\int_0^l h(\tau,\sigma)\eta^q(\tau,\sigma)d\sigma d\tau\right)dldr\\
&+2^{n-1}\frac{q}{n+q}\int_0^s\int_0^t g(r,l)\left(\eta(r,l)+2^{q-1}\int_0^r\int_0^l h(\tau,\sigma)\eta^q(\tau,\sigma)d\sigma d\tau\right)dldr\\
&+2^{n-1}\left(\frac{n}{(n+q)^2}v^2(s,t)+\left(\frac{q}{(n+q)^2}+\frac{n\,(n+q-1)}{(n+q)^2}\right)v(s,t)+\frac{q\,(n+q-1)}{(n+q)^2}\right)\\
&\int_0^s\int_0^t g(r,l)dldr+2^{n+q-2}v(s,t)\int_0^s\int_0^t g(r,l)\left(\int_0^r\int_0^l h(\tau,\sigma)d\sigma d\tau\right)dldr\Bigg]^p dtds.
\end{aligned}
$$

Then,

$$
v^{\frac{1}{n+q}}(x,y)\leq\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[Q(s,t)+R(s,t)v(s,t)+N(s,t)v^2(s,t)\Big]^p dtds,
$$

or

$$v(x,y) \leq \left[\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\Big[Q(s,t)+R(s,t)v(s,t)+N(s,t)v^2(s,t)\Big]^p dtds\right]^{n+q}, \tag{2.46}$$

where $N(s,t)$, $R(s,t)$, and $Q(s,t)$ are defined by (2.43), (2.42), and (2.40), respectively.

By applying Lemma 2.4 with $a(x,y) = 0$, $f(s,t)$ instead of $b(s,t)$, $N(s,t)$ instead of $c(s,t)$, $R(s,t)$ instead of $d(s,t)$, $Q(s,t)$ instead of $e(s,t)$, and $n+q > 1$ instead of $r > 1$ to (2.46), we obtain the following:

$$v(x,y) \leq \left(F^{1-2(n+q)p}(x,y) + (1-2(n+q)p)\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)G(s,t)dtds\right)^{\frac{n+q}{1-2(n+q)p}}. \tag{2.47}$$

Finally, using (2.45), (2.47), and the fact that $z^{n+q}(x,y) = v(x,y)$, we obtain (2.36) and the proof is complete. □

Remark 2.2. *Inequality (2.35) enhances the results found in [1, 13, 15, 16] by introducing a more general bidimensional nonlinear context.*

Remark 2.3. *Note that Theorems 2.1 and 2.2 generalize the results obtained in [1, 15], which were proven in the case $p = 1$. Additionally, they generalize the findings from [13, 14] in the case $p \in (0,1]$, and they provide a generalization of the findings from [8, 16] in the case $p > 1$.*

3. Application

In this section, we apply our results to study the boundedness of the solution of a retarded integral equation of a bidimensional Volterra type, which arises in various problems.

Example 3.1. We consider the following retarded integral equation of Volterra type:

$$\begin{aligned}\chi(x,y) &+ \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\chi^6(s,t)dtds + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\left(\int_0^s\int_0^t g(r,l)\chi^7(r,l)dldr\right.\\ &\left.+\int_0^s\int_0^t g(r,l)\chi^4(r,l)\left(\int_0^r\int_0^l h(\tau,\sigma)\chi(\tau,\sigma)d\sigma d\tau\right)^3 dldr\right)^2 dtds = c(x,y),\end{aligned} \tag{3.1}$$

where χ, f, g, h, and c be continuous functions on R, and produces the following result.

Theorem 3.1. *Let $\alpha(x)$ and $\beta(y)$ be continuous, differentiable, and nondecreasing functions on $[0,+\infty)$ with $\alpha(x) \leq x$, and $\beta(y) \leq y$. If $\chi(x,y)$ satisfies (3.1), and for*

$$l^{-13}(x,y) - 13\times 2^{12}\int_0^{\alpha(x)}\int_0^{\beta(y)} |f(s,t)|\,R^2(s,t)dtds > 0,$$

we have

$$|\chi(x,y)| \leq |c(x,y)| + \left[l^{-13}(x,y) - 13\times 2^{12}\int_0^{\alpha(x)}\int_0^{\beta(y)} |f(s,t)|\,R^2(s,t)dtds\right]^{-\frac{1}{13}}, \tag{3.2}$$

then,

$$l(x,y) = \int_0^{\alpha(x)} \int_0^{\beta(y)} |f(s,t)| N^2(s,t) dt ds, \tag{3.3}$$

$$\begin{aligned} N(s,t) &= M(s,t) + \frac{16}{7} + \frac{32}{7} \int_0^s \int_0^t |g(r,l)| \, c^4(r,l) \left(1 + \int_0^r \int_0^l |h(\tau,\sigma)| \, d\sigma d\tau\right)^3 dl dr \\ &+ \frac{24}{7} \int_0^s \int_0^t |g(r,l)| \left(|c(r,l)| + \int_0^r \int_0^l |h(\tau,\sigma)| \, |c(\tau,\sigma)| \, d\sigma d\tau\right)^3 dl dr, \end{aligned} \tag{3.4}$$

$$\begin{aligned} M(s,t) &= 4\left|c^3(s,t)\right| + 8 \int_0^s \int_0^t |g(r,l)| \, c^4(r,l) \\ &\times \left(|c(r,l)| + \int_0^r \int_0^l |h(\tau,\sigma)| \, |c(\tau,\sigma)| \, d\sigma d\tau\right)^3 dl dr, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} R(s,t) &= \frac{12}{7} + \frac{24}{7} \int_0^s \int_0^t |g(r,l)| \, c^4(r,l) \left(1 + \int_0^r \int_0^l |h(\tau,\sigma)| \, d\sigma d\tau\right)^3 dl dr \\ &+ \frac{32}{7} \int_0^s \int_0^t |g(r,l)| \left(|c(r,l)| + \int_0^r \int_0^l |h(\tau,\sigma)| \, |c(\tau,\sigma)| \, d\sigma d\tau\right)^3 dl dr \\ &+ 8 \int_0^s \int_0^t |g(r,l)| \left(1 + \int_0^r \int_0^l |h(\tau,\sigma)| \, d\sigma d\tau\right)^3 dl dr. \end{aligned} \tag{3.6}$$

Proof. By using Lemma 2.2 on the left hand side of (3.1), we obtain the following:

$$\begin{aligned} &\chi(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t) \left(\chi^6(s,t) + \left(\int_0^s \int_0^t g(r,l)\right.\right. \\ &\times \left.\left.\left(\chi^7(r,l) + \chi^4(r,l)\left(\left(\int_0^r \int_0^l h(\tau,\sigma)\chi(\tau,\sigma) d\sigma d\tau\right)^3\right) dl dr\right)^2\right) dt ds \\ =\ & \chi(x,y) + \frac{1}{2} \int_0^{\alpha(x)} \int_0^{\beta(y)} 2f(s,t) \left(\left(\chi^3(s,t)\right)^2 + \left(\int_0^s \int_0^t g(r,l)\right.\right. \\ &\times \left.\left.\left(\chi^7(r,l) + \chi^4(r,l)\left(\left(\int_0^r \int_0^l h(\tau,\sigma)\chi(\tau,\sigma) d\sigma d\tau\right)^3\right) dl dr\right)^2\right) dt ds \\ \geq\ & \chi(x,y) + \frac{1}{2} \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t) \left[\chi^3(s,t) + \int_0^s \int_0^t g(r,l)\right. \\ &\times \left.\left(\chi^7(r,l) + \chi^4(r,l)\left(\left(\int_0^r \int_0^l h(\tau,\sigma)\chi(\tau,\sigma) d\sigma d\tau\right)^3\right) dl dr\right]^2 dt ds \\ \geq\ & \chi(x,y) + \frac{1}{2} \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t) \left[\chi^3(s,t) + \frac{1}{4} \int_0^s \int_0^t g(r,l)\chi^4(r,l)\right. \end{aligned}$$

$$\times\left[2^2\left(\chi^3(r,l)+\left(\int_0^r\int_0^l h(\tau,\sigma)\chi(\tau,\sigma)d\sigma d\tau\right)^3\right)\right]dldr\Bigg]^2 dtds$$
$$\geq \chi(x,y)+\frac{1}{2}\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\left[\chi^3(s,t)+\int_0^s\int_0^t \frac{1}{4}g(r,l)\chi^4(r,l)\right.$$
$$\left.\times\left[\chi(r,l)+\int_0^r\int_0^l h(\tau,\sigma)\chi(\tau,\sigma)d\sigma d\tau\right]^3 dldr\right]^2 dtds.$$

From (3.1) and the last inequality, we obtain the following:

$$\chi(x,y) \leq c(x,y)-\frac{1}{2}\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\left[\chi^3(s,t)+\int_0^s\int_0^t \frac{1}{4}g(r,l)\chi^4(r,l)\right.$$
$$\left.\times\left[\chi(r,l)+\int_0^r\int_0^l h(\tau,\sigma)\chi(\tau,\sigma)d\sigma d\tau\right]^3 dldr\right]^2 dtds.$$

Therefore,

$$|\chi(x,y)| \leq |c(x,y)|+\int_0^{\alpha(x)}\int_0^{\beta(y)} \frac{1}{2}|f(s,t)|\Big[|\chi^3(s,t)|$$
$$+\int_0^s\int_0^t \frac{1}{4}|g(r,l)|\chi^4(r,l)\left[|\chi(r,l)|+\int_0^r\int_0^l |h(\tau,\sigma)|\,|\chi(\tau,\sigma)|\,d\sigma d\tau\right]^3 dldr\Bigg]^2 dtds.$$

Then, an application of Theorem 2.1 to the last inequality yields (3.2), and the proof is complete. □

4. Conclusions

Integral inequalities play a robust role in the development of mathematical sciences. Most integral inequalities are useful to study the qualitative properties of solutions to differential and integral equations. The Gronwall-Bellman inequality plays a considerable role in the study of qualitative properties of the solutions of certain differential equations. This inequality has attracted and continues to attract considerable attention in the literature. Recently, many authors have been interested in generalizing the Gronwall-Bellman inequality to other forms such as nonlinear integral inequalities with a delay, of the Volterra-Fredholm type, and nonlinear retarded integral inequalities with power. Following this trend and to develop the study of integral inequalities, we proved some new nonlinear integral inequalities with power in two variables, which generalized certain results given in [1,8,13–16] in a more general context. The obtained results can be employed to study the boundedness and uniqueness of solutions of some integral equation with power. As an application, an illustrative example was presented to study the boundedness of solution.

Author contributions

Mili Selma: Software, Writing–original draft, Writing–review and editing, Conceptualization, Methodology; Ammar Boudeliou: Supervision, Validation, Data curation, Software, Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there are no conflicts of interest.

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