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Research article

Algebraic structure of some complex intuitionistic fuzzy subgroups and their homomorphism

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Abstract: The complex fuzzy environment is an innovative tool to deal with fuzzy situations in different mathematical problems. Aiming at the concept of complex intuitionistic fuzzy subgroups, this paper has introduced cut-subsets of complex intuitionistic fuzzy sets, and studied the relationship among the cut-subsets and complex intuitionistic fuzzy subgroups, complex intuitionistic fuzzy Abel subgroups, and complex intuitionistic fuzzy cyclic subgroups. Further, we gave the left and right cosets of complex intuitionistic fuzzy subgroups, defined complex intuitionistic fuzzy normal subgroups, and discussed some of their algebraic properties. Based on this thought, we proposed a new concept of $(\alpha_{1,2},\beta_{1,2})$-complex intuitionistic fuzzy subgroups, and proved that an $(\alpha_{1,2},\beta_{1,2})$-complex intuitionistic fuzzy subgroup is a general form of every complex intuitionistic fuzzy subgroup. At the same time, $(\alpha_{1,2},\beta_{1,2})$-complex intuitionistic fuzzy normal subgroups and their cosets were introduced. Finally, we established a general homomorphism of complex intuitionistic fuzzy subgroups, and studied the relationship between the image and pre-image of complex intuitionistic fuzzy subgroups and complex intuitionistic fuzzy normal subgroups, respectively, under group homomorphism.

Keywords: complex intuitionistic fuzzy set; complex intuitionistic fuzzy subgroups; complex intuitionistic fuzzy normal subgroups
Mathematics Subject Classifications: 03E72, 08A72, 26E50

1. Introduction

Group theory is the most important branch of mathematics, which has a wide range of applications in algebraic geometry, theoretical physics, and cryptography. Fuzzy set theory provides a mathematical way to grasp the ambiguity associated with brain processes such as human intelligence and thinking. This theory also guides us to better solve our daily life problems through proper decision-making procedures. Zadeh [1] proposed the concept of fuzzy sets in 1965. Rosenfeld [2] developed the connection between group theory and fuzzy sets and proposed the theory of fuzzy subgroups. In

1986, Atanassov [3] published his first article on intuitionistic fuzzy sets (IFSs). Biswas [4] introduced the algebraic structure of intuitionistic fuzzification and proposed the concept of intuitionistic fuzzy subgroups (IFSGs). Further, the development of IFSGs can be seen in [5–10]. In addition, along with the uncertainty and ambiguity of data in our daily lives, the process change periodicity of data also arises. Therefore, the current theory is not enough to take information into account, so there is information loss during processing. To solve this problem, Ramot et al. [11] proposed complex fuzzy sets that extend the range of membership functions from real numbers to complex numbers with a unit disk. Since the complex fuzzy sets only consider the membership degree of the data entity, and do not consider the non-membership degree of the data entity, the non-membership degree also occupies the same position in the decision-making process of the evaluation system. Alkouri and Salleh [12] extended the definition of complex non-membership functions, and studied their basic properties. Thus, the complex intuitionistic fuzzy set (CIFS) was introduced, and the complex fuzzy set and the complex intuitionistic fuzzy set were combined. The complex intuitionistic fuzzy set can effectively handle uncertainty and fuzziness, providing more comprehensive and accurate decision support, especially in disease diagnosis in medical decision-making. It expresses the doctor's assessment of symptoms through membership and non-membership degrees, thereby helping doctors more accurately determine whether a patient has a certain disease. Suppose we have a patient, and the doctor needs to determine whether the patient has heart disease based on three symptoms: whether they have chest pain, whether they have difficulty breathing, and whether the electrocardiogram is abnormal. The doctor assesses each symptom based on their experience and professional knowledge, and gives the membership and non-membership degree values, respectively. These values reflect the doctor's confidence in the correlation between each symptom and heart disease. Each symptom has a different importance in the diagnostic process, so weights need to be assigned. The doctor multiplies the membership and non-membership degree values of each symptom by the corresponding weight, and then adds the results to obtain the comprehensive membership and non-membership degrees. By comparing the two, it can be determined whether the patient has heart disease. The complex intuitionistic fuzzy set can more comprehensively express the doctor's assessment of symptoms, thereby improving the accuracy and reliability of the diagnosis. This method is particularly suitable for complex medical decision-making scenarios and can help doctors make more reasonable judgments in the face of uncertain information.

In 2016, Alhusbann and Salleh [13] introduced the concept of complex fuzzy groups. A year later, Alsarahead and Ahmed [14–16] derived different concepts of complex fuzzy groups, complex fuzzy subgroups, and complex fuzzy soft subgroups from the Rosenfeld and Liu methods [2,17]. In 2021, Alolaiyan et al. [18] introduced (α,β)-complex fuzzy sets ((α,β)-CFSs) to represent (α,β)-complex fuzzy normal subgroups ((α,β)-CFNSGs). On this basis, in 2023, Doaa Al-Sharoa [19] extended it to subgroups of complex intuitionistic fuzzy sets (CIFSs), proposed $(\alpha_{1,2},\beta_{1,2})$-complex intuitionistic fuzzy subgroups ($(\alpha_{1,2},\beta_{1,2})$-CIFSGs), and proved their algebraic structure. In addition, the Lagrange theorem of $(\alpha_{1,2},\beta_{1,2})$-CIFSGs was also deduced. At the same time, complex fuzzy subgroups were developed into complex intuitionistic fuzzy subgroups (CIFSGs). Therefore, in the field of complex numbers, Alhusban et al. [20,21] proposed the concepts of complex intuitionistic fuzzy groups (CIFGs) and complex intuitionistic fuzzy normal subgroups (CIFNSGs) in 2016 and 2017.

In decision theory, how to effectively represent and deal with uncertain information is a key problem. In recent years, with the increasing demand of complex system and multi-source information fusion, the traditional uncertainty processing methods have gradually shown their limitations. In order to describe and deal with uncertainty more precisely, researchers begin to explore new theoretical frameworks and technical means. Among them, the theory of complex evidence, as a new method of uncertainty treatment, has gradually attracted attention. In 2020, Xiao [22] proposed a new theoretical framework for complex evidence, defining generalized belief functions and generalized likelihood functions that take into account not only the uncertainty of the evidence, but also the phase information of the evidence, thereby enabling a more comprehensive representation of uncertainty. In addition, in 2023, Xiao [23] proposed a new quantum X-entropy for measuring the uncertainty of generalized quantum mass functions. Quantum X-entropy takes into account not only the randomness of the quantum mass function, but also its structural properties, enabling a more comprehensive measure of uncertainty. In the same year, Xiao and Pedrycz [24] also designed a variety of multi-source quantum information fusion algorithms based on quantum mass function negation and applied them to pattern classification. These algorithms perform well in processing complex uncertain information and provide strong support for quantum decision-making. An important part of decision theory is multi-criteria decision-making.

The main problem of the multi-criteria decision-making method is how to evaluate alternatives effectively under multi-criteria and select the best alternative from the alternative set. In 2015, J. Rezaei [25] proposed a new method for solving multi-criteria decision problems, the best-worst method. In 2022, S.P. Wan et al. [26] extended the best-worst approach to intuitionistic fuzzy environments to deal with multi-criteria decision problems with intuitionistic fuzzy preference relationships. In 2024, Dong and Wan [27] extended this method to interval-intuitionistic fuzzy environments, proposing a new interval-intuitional-fuzzy best-worst method that can effectively deal with multi-criteria decision problems while maintaining consistency. In the same year, Wan et al. [28] proposed a new intuitionistic fuzzy best-worst method to deal with group decision-making problems with intuitionistic fuzzy preference relations. Lu et al. [29] further developed an interactive iterative group decision-making method specifically for decision problems based on interval intuitionistic fuzzy preference relations. Wan et al. [30] proposed a new decision framework combining the trapezoidal cloud model and MULTIMOORA method to solve the routing problem of container multimodal transport. Recently, some researchers have begun to pay attention to rank-based methods of group consensus reaching. For example, in 2025, Wan et al. [31] proposed a dual-strategy consensus-reaching process based on probabilistic linguistic information to solve multi-criteria group decision-making problems. These studies lay a foundation for the application of fuzzy sets and intuitionistic fuzzy sets in algebraic structures. However, with the increase of complex information, if practical applications need to deal with more complex uncertainty problems, especially in scenarios where membership, non-membership, and periodicity characteristics need to be considered at the same time, more powerful tools are needed to deal with such information. Inspired by the above, this paper further extends the theoretical framework of complex intuitionistic fuzzy sets in algebraic structures by studying the concept of cut sets of CIFSGs and giving a new definition of $(\alpha_{1,2}, \beta_{1,2})$-CIFSGs, so as to deal with complex information better. Therefore, the main contributions of this paper can be summarized as follows:

- Based on the existing theoretical knowledge of CIFSGs and CIFNSGs, the cut set of a CIFSG is defined, and the relationships between the cut set and complex intuitionistic fuzzy subgroup, complex intuitionistic fuzzy Abelian subgroup, and complex intuitionistic fuzzy cyclic subgroup are studied. At the same time, the CIFNSG is combined with the conjugate class of the group.

- A new concept of an $(\alpha_{1,2}, \beta_{1,2})$-CIFSG is presented, and the proposed theory is applied to medical decision-making.
- The relationships between the image and the inverse image of complex intuitionistic fuzzy subgroups and complex intuitionistic fuzzy normal subgroups are studied, respectively, under group homomorphism, and some examples are given to illustrate.

We organize this article as follows. In Section 2, we first introduce some basic concepts and some important properties required for this article. In Section 3, the concepts of CIFSGs and their cut-subsets are introduced, and the relationship between the cut-subsets and some CIFSGs are studied. In Section 4, based on the left and right cosets of CIFSGs, the definition of CIFNSGs is proposed, and some of their algebraic properties are discussed. On this basis, a new concept of $(\alpha_{1,2}, \beta_{1,2})$ -CIFSGs is proposed in Section 5, which shows that an $(\alpha_{1,2}, \beta_{1,2})$-CIFSG is a general form of every CIFSG. At the same time, $(\alpha_{1,2}, \beta_{1,2})$-complex intuitionistic fuzzy normal subgroups ($(\alpha_{1,2}, \beta_{1,2})$-CIFNSGs) and their cosets are introduced. Finally, in Section 6, we establish a general homomorphism about CIFSGs, and prove that the homomorphism image and the pre-image of CIFSGs are still CIFSGs, and the homomorphism image and the pre-image of CIFNSGs are still CIFNSGs.

2. Preliminaries

In this section, we recall the basic definitions and related notions used in the paper.

Definition 2.1. [32] *We say that a nonempty set G forms a group for an algebraic operation called multiplication, if*

(1) G is called for this multiplication;

(2) The associative law is true: $a(bc) = (ab)c$. It is true for any three elements a, b, c of G;

(3) There is at least one left identity element e in G that holds $ea = a$. It is true for any element a of G;

(4) For every element a of G, there is at least one left inverse a^{-1} in G that holds $a^{-1}a = e$.

Definition 2.2. [32] *It is necessary and sufficient for a nonempty subset H of the group G to be a subgroup of G:*

(1) $a, b \in H \Rightarrow ab \in H$.

(2) $a \in H \Rightarrow a^{-1} \in H$.

In other words, the following (3) is equivalent to (1) and (2).

(3) $a, b \in H \Rightarrow ab^{-1} \in H$.

Definition 2.3. [3] *An IFS A of the universe of discourse U is of the form*

$$A = \{\langle x, \psi_A(x), \vartheta_A(x)\rangle | x \in U\},$$

where $\psi_A(x)$ and $\vartheta_A(x)$ provide the membership function and non-membership function of A, respectively, and $0 \le \psi_A(x) + \vartheta_A(x) \le 1$, for all $x \in U$.

Definition 2.4. [4] *If an IFS A of the group G satisfies the following conditions, then A is called an IFSG of the group G.*

(1) $\psi_A(xy) \geq \min\{\psi_A(x), \psi_A(y)\}$;
(2) $\psi_A(x^{-1}) \geq \psi_A(x)$;
(3) $\vartheta_A(xy) \leq \max\{\vartheta_A(x), \vartheta_A(y)\}$;
(4) $\vartheta_A(x^{-1}) \leq \vartheta_A(x)$, for all $x, y \in G$.

Definition 2.5. [12] *A CIFS A is defined by $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in U\}$, where the membership function $\mu_A(x) = \eta_A(x) \cdot e^{i\varphi_A(x)}$ is defined as $\mu_A(x) : U \to \{z|z \in C, |z \leq 1\}$, and the non-membership function $\nu_A(x) = r_A(x) \cdot e^{is_A(x)}$ is defined as $\nu_A(x) : U \to \{z|z \in C, |z \leq 1\}$, where $|\mu_A(x) + \nu_A(x)| \leq 1$ and $i = \sqrt{-1}$, each of $\eta_A(x)$, $r_A(x)$ belong to $[0, 1]$, such that $0 \leq \eta_A(x) + r_A(x) \leq 1$, and $\varphi_A(x)$ and $s_A(x)$ are real-valued, $0 \leq \varphi_A(x) + s_A(x) \leq 2\pi$.*

Definition 2.6. [33] *Suppose that A and B are two CIFSs on the domain U, where $\mu_A(x) = \eta_A(x) \cdot e^{i\varphi_A(x)}$, $\mu_B(x) = \eta_B(x) \cdot e^{i\varphi_B(x)}$, $\nu_A(x) = r_A(x) \cdot e^{is_A(x)}$, $\nu_B(x) = r_B(x) \cdot e^{is_B(x)}$ are their membership functions and non-membership functions, respectively. Then the complex intuitionistic fuzzy Cartesian product of A and B is defined as:*

$$A \times B = \{\langle x, \mu_{A\times B}(x), \nu_{A\times B}(x)\rangle | x \in U\},$$

where

$$\begin{aligned} \mu_{A\times B}(x) &= \eta_{A\times B}(x) \cdot e^{i\varphi_{A\times B}(x)} \\ &= \min\{\eta_A(x), \eta_B(x)\} \cdot e^{i\min\{\varphi_A(x), \varphi_B(x)\}}, \\ \nu_{A\times B}(x) &= r_{A\times B}(x) \cdot e^{is_{A\times B}(x)} \\ &= \max\{r_A(x), r_B(x)\} \cdot e^{i\max\{s_A(x), s_B(x)\}}. \end{aligned}$$

3. Complex intuitionistic fuzzy subgroups

In this section, we review the definition of CIFSGs and prove that the Cartesian product of two CIFSGs is still a CIFSG. At the same time, we describe the cut sets of CIFSGs, and discuss the relationship between the cut sets and CIFSGs, complex intuitionistic fuzzy Abel subgroups (CIFASGs), and complex intuitionistic fuzzy cyclic subgroups (CIFCSGs).

Definition 3.1. [21] *If a CIFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in U\}$ of the group G, it is expressed as: $\mu_A(x) = \eta_A(x) \cdot e^{i\varphi_A(x)}$, $\nu_A(x) = r_A(x) \cdot e^{is_A(x)}$. Then, if the following four conclusions are satisfied, A is called a CIFSG of the group G.*

(1) $\eta_A(xy) \cdot e^{i\varphi_A(xy)} \geq \min\{\eta_A(x), \eta_A(y)\} \cdot e^{i\min\{\varphi_A(x), \varphi_A(y)\}}$;
(2) $\eta_A(x^{-1}) \cdot e^{i\varphi_A(x^{-1})} \geq \eta_A(x) \cdot e^{i\varphi_A(x)}$;
(3) $r_A(xy) \cdot e^{is_A(xy)} \leq \max\{r_A(x), r_A(y)\} \cdot e^{i\max\{s_A(x), s_A(y)\}}$;
(4) $r_A(x^{-1}) \cdot e^{is_A(x^{-1})} \leq r_A(x) \cdot e^{is_A(x)}$, for all $x, y \in G$.

Remark 3.1. *If A is a CIFSG of the group G, for any $x, y \in G$, then,*

$$\eta_A(x^{-1}y) \cdot e^{i\varphi_A(x^{-1}y)} \geq \min\{\eta_A(x) \cdot e^{i\varphi_A(x)}, \eta_A(y) \cdot e^{i\varphi_A(y)}\},$$

$$r_A(x^{-1}y) \cdot e^{is_A(x^{-1}y)} \leq \max\{r_A(x) \cdot e^{is_A(x)}, r_A(y) \cdot e^{is_A(y)}\}.$$

Theorem 3.1. *The Cartesian product of two CIFSGs of the group G is still a CIFSG.*

Proof: Suppose A and B are two CIFSGs of the group G. Then,

$$\begin{aligned}
\mu_{A\times B}(xy) &= \eta_{A\times B}(xy)\cdot e^{i\varphi_{A\times B}(xy)} \\
&\geq \min\{\eta_{A\times B}(x),\eta_{A\times B}(y)\}\cdot e^{i\min\{\varphi_{A\times B}(x),\varphi_{A\times B}(y)\}} \\
&= \min\{\eta_{A\times B}(x)\cdot e^{i\varphi_{A\times B}(x)},\eta_{A\times B}(y)\cdot e^{i\varphi_{A\times B}(y)}\} \\
&= \min\{\mu_{A\times B}(x),\mu_{A\times B}(y)\}, \\
\mu_{A\times B}(x^{-1}) &= \eta_{A\times B}(x^{-1})\cdot e^{i\varphi_{A\times B}(x^{-1})} \\
&\geq \eta_{A\times B}(x)\cdot e^{i\varphi_{A\times B}(x)} \\
&= \mu_{A\times B}(x), \\
\nu_{A\times B}(xy) &= r_{A\times B}(xy)\cdot e^{is_{A\times B}(xy)} \\
&\leq \max\{r_{A\times B}(x),r_{A\times B}(y)\}\cdot e^{i\max\{s_{A\times B}(x),s_{A\times B}(y)\}} \\
&= \max\{r_{A\times B}(x)\cdot e^{is_{A\times B}(x)},r_{A\times B}(y)\cdot e^{is_{A\times B}(y)}\} \\
&= \max\{\nu_{A\times B}(x),\nu_{A\times B}(y)\}, \\
\nu_{A\times B}(x^{-1}) &= r_{A\times B}(x^{-1})\cdot e^{is_{A\times B}(x^{-1})} \\
&\leq r_{A\times B}(x)\cdot e^{is_{A\times B}(x)} \\
&= \nu_{A\times B}(x).
\end{aligned}$$

Definition 3.2. [33] *Let $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle|x\in U\}$ be a CIFS on the domain U. For all $\rho,\lambda\in[0,1]$ and $\theta,\delta\in[0,2\pi]$, the cut-subset of the CIFS A is defined as:*

$$A_{(\rho,\theta)}^{(\lambda,\delta)}=\{x\in U:\eta_A(x)\geq\rho,\varphi_A(x)\geq\theta,r_A(x)\leq\lambda,s_A(x)\leq\delta\}.$$

When the phase item $\varphi_A(x)=s_A(x)=0$, CIFSs degenerate to IFSs. For $\theta=\delta=0$, we obtain the cut-subset $A_\rho^\lambda=\{x\in U:\eta_A(x)\geq\rho,r_A(x)\leq\lambda\}$ and for $\rho=\lambda=0$, $A_\theta^\delta=\{x\in U:\varphi_A(x)\geq\theta,s_A(x)\leq\delta\}$.

Theorem 3.2. *A is a CIFSG of the group G if and only if $A_{(\rho,\theta)}^{(\lambda,\delta)}$ is a subgroup of the group G.*

Proof: Let A be a CIFSG of the group G. Then, $A_{(\rho,\theta)}^{(\lambda,\delta)}$ satisfies

$$\eta_A(x)\geq\rho,\varphi_A(x)\geq\theta,r_A(x)\leq\lambda,s_A(x)\leq\delta,$$

and, apparently, e is the identity of $A_{(\rho,\theta)}^{(\lambda,\delta)}$ and makes it non-empty. Let x,y be any two elements of $A_{(\rho,\theta)}^{(\lambda,\delta)}$. Then,

$$\eta_A(x)\geq\rho,\varphi_A(x)\geq\theta,r_A(x)\leq\lambda,s_A(x)\leq\delta,$$
$$\eta_A(y)\geq\rho,\varphi_A(y)\geq\theta,r_A(y)\leq\lambda,s_A(y)\leq\delta,$$

$$\begin{aligned}
\eta_A(xy)\cdot e^{i\varphi_A(xy)} &= \mu_A(xy) \\
&\geq \min\{\mu_A(x),\mu_A(y)\} \\
&= \min\{\eta_A(x)\cdot e^{i\varphi_A(x)},\eta_A(y)\cdot e^{i\varphi_A(y)}\} \\
&= \min\{\eta_A(x),\eta_A(y)\}\cdot e^{i\min\{\varphi_A(x),\varphi_A(y)\}},
\end{aligned}$$

and further

$$\eta_A(xy) \geq \min\{\eta_A(x), \eta_A(y)\} \geq \min\{\rho, \rho\} = \rho,$$
$$\varphi_A(xy) \geq \min\{\varphi_A(x), \varphi_A(y)\} \geq \min\{\theta, \theta\} = \theta.$$

Thus, $\eta_A(xy) \geq \rho$, $\varphi_A(xy) \geq \theta$.

$$\begin{aligned} r_A(xy) \cdot e^{is_A(xy)} &= \nu_A(xy) \\ &\leq \max\{\nu_A(x), \nu_A(y)\} \\ &= \max\{r_A(x) \cdot e^{is_A(x)}, r_A(y) \cdot e^{is_A(y)}\} \\ &= \max\{r_A(x), r_A(y)\} \cdot e^{i \max\{s_A(x), s_A(y)\}}, \end{aligned}$$

and further

$$r_A(xy) \leq \max\{r_A(x), r_A(y)\} \leq \max\{\lambda, \lambda\} = \lambda,$$
$$s_A(xy) \leq \max\{s_A(x), s_A(y)\} \leq \max\{\delta, \delta\} = \delta.$$

Thus, $r_A(xy) \leq \lambda$, $s_A(xy) \leq \delta$, and we prove $xy \in A_{(\rho,\theta)}^{(\lambda,\delta)}$. Here we prove $x^{-1} \in A_{(\rho,\theta)}^{(\lambda,\delta)}$ in the same way:

$$\begin{aligned} \eta_A(x^{-1}) \cdot e^{i\varphi_A(x^{-1})} &= \mu_A(x^{-1}) \\ &\geq \mu_A(x) \\ &= \eta_A(x) \cdot e^{i\varphi_A(x)}, \end{aligned}$$

this is to say, $\eta_A(x^{-1}) \geq \eta_A(x) \geq \rho$, $\varphi_A(x^{-1}) \geq \varphi_A(x) \geq \theta$. For the same reason,

$$\begin{aligned} r_A(x^{-1}) \cdot s_A(x^{-1}) &= \nu_A(x^{-1}) \\ &\leq \nu_A(x) \\ &= r_A(x) \cdot e^{is_A(x)}, \end{aligned}$$

and thus, $r_A(x^{-1}) \leq r_A(x) \leq \lambda$, $s_A(x^{-1}) \leq s_A(x) \leq \delta$. To sum up, $\eta_A(x^{-1}) \geq \rho$, $\varphi_A(x^{-1}) \geq \theta$, $r_A(x^{-1}) \leq \lambda$, $s_A(x^{-1}) \leq \delta$, and therefore, $x^{-1} \in A_{(\rho,\theta)}^{(\lambda,\delta)}$. Hence, $A_{(\rho,\theta)}^{(\lambda,\delta)}$ is a subgroup of the group G. The proof is complete.

Example 3.1. *Let group $G = \{e, a, b, c\}$, where e is the identity element, and A is a CIFSG of G, $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in G\}$, where $\mu_A(x) = \eta_A(x) \cdot e^{i\varphi_A(x)}$, $\nu_A(x) = r_A(x) \cdot e^{is_A(x)}$. For any elements of group G, the membership and non-membership are defined as follows:*

$$\mu_A(e) = 0.8 \cdot e^{i0.4\pi}, \mu_A(a) = 0.6 \cdot e^{i0.3\pi}, \mu_A(b) = 0.4 \cdot e^{i0.3\pi}, \mu_A(c) = 0.2 \cdot e^{i0.4\pi},$$
$$\nu_A(e) = 0.8 \cdot e^{i0.1\pi}, \nu_A(a) = 0.2 \cdot e^{i0.1\pi}, \nu_A(b) = 0.3 \cdot e^{i0.2\pi}, \nu_A(c) = 0.4 \cdot e^{i0.3\pi}.$$

For any $\rho, \eta \in [0, 1], \theta, \delta \in [0, 2\pi]$, the cut set of A is $A_{(\rho,\theta)}^{(\lambda,\delta)}$, where

$$A_{(\rho,\theta)}^{(\lambda,\delta)} = \{x \in G : \eta_A(x) \geq \rho, \varphi_A(x) \geq \theta, r_A(x) \leq \lambda, s_A(x) \leq \delta\}.$$

Let $\rho = 0.5$, $\theta = 0.3\pi$, $\lambda = 0.4$, $\delta = 0.2\pi$, and then the cut set of A is $A_{(0.5,0.3\pi)}^{(0.4,0.2\pi)}$,

$$A_{(0.5,0.3\pi)}^{(0.4,0.2\pi)} = \{x \in G : \eta_A(x) \geq 0.5, \varphi_A(x) \geq 0.3\pi, r_A(x) \leq 0.4, s_A(x) \leq 0.2\pi\}.$$

We can see that the two elements e and a of the group G satisfy the cut set definition, i.e., $A_{(0.5,0.3\pi)}^{(0.4,0.2\pi)} = \{e, a\}$, and by Definition 2.2, $A_{(0.5,0.3\pi)}^{(0.4,0.2\pi)}$ is a subgroup of the group G.

Lemma 3.1. *The converse of Theorem* 3.2 *is also true, i.e., if* $A^{(\lambda,\delta)}_{(\rho,\theta)}$ *is a subgroup of the group G, then A is a CIFSG of the group G.*

Theorem 3.3. *Let A be a CIFSG of the group G. Then the cut-subsets* A^{λ}_{ρ} *and* A^{δ}_{θ} *are two subgroups of the group G, for all* $\rho, \lambda \in [0,1]$ *and* $\theta, \delta \in [0, 2\pi]$, *where e is an identity element of G.*

Proof: Note that A^{λ}_{ρ} is nonempty, as $e \in A^{\lambda}_{\rho}$. Let x, y be two elements of the cut-subset A^{λ}_{ρ}. Then,

$$\eta_A(x) \geq \rho, r_A(x) \leq \lambda, \eta_A(y) \geq \rho, r_A(y) \leq \lambda.$$

Since the equations $\mu_A(xy) = \eta_A(xy) \cdot e^{i\varphi_A(xy)}$ and $\nu_A(xy) = r_A(xy) \cdot e^{is_A(xy)}$ hold, we have

$$\begin{aligned}\eta_A(xy) &\geq \min\{\eta_A(x), \eta_A(y)\}\\ &\geq \min\{\rho, \rho\}\\ &= \rho,\\ r_A(xy) &\leq \max\{r_A(x), r_A(y)\}\\ &\leq \max\{\lambda, \lambda\}\\ &= \lambda.\end{aligned}$$

Therefore, $xy \in A^{\lambda}_{\rho}$. Next, $\eta_A(x^{-1}) \geq \eta_A(x) \geq \rho$, $r_A(x^{-1}) \leq r_A(x) \leq \lambda$. Thus, A^{λ}_{ρ} is a subgroup of G. Similarly, A^{δ}_{θ} is a subgroup of G. That proves the theorem.

Theorem 3.4. *Let A and B be two CIFSGs of two groups* G_1 *and* G_2, *respectively, for all* $\rho, \lambda \in [0,1]$ *and* $\theta, \delta \in [0, 2\pi]$, *and then,*

$$(A \times B)^{(\lambda,\delta)}_{(\rho,\theta)} = A^{(\lambda,\delta)}_{(\rho,\theta)} \times B^{(\lambda,\delta)}_{(\rho,\theta)}.$$

Proof: Let $(x, y) \in (A \times B)^{(\lambda,\delta)}_{(\rho,\theta)}$. Then, we just need to prove $(x, y) \in A^{(\lambda,\delta)}_{(\rho,\theta)} \times B^{(\lambda,\delta)}_{(\rho,\theta)}$. If $(x, y) \in (A \times B)^{(\lambda,\delta)}_{(\rho,\theta)}$, then

$$\eta_{A\times B}(x, y) \geq \rho, \varphi_{A\times B}(x, y) \geq \theta, r_{A\times B}(x, y) \leq \lambda, s_{A\times B}(x, y) \leq \delta.$$

Thus,

$$\begin{gathered}\min\{\eta_A(x), \eta_B(y)\} \geq \rho, \min\{\varphi_A(x), \varphi_B(y)\} \geq \theta,\\ \max\{r_A(x), r_B(y)\} \leq \lambda, \max\{s_A(x), s_B(y)\} \leq \delta,\\ \eta_A(x) \geq \rho, \eta_B(y) \geq \rho, \varphi_A(x) \geq \theta, \varphi_B(y) \geq \theta,\\ r_A(x) \leq \lambda, r_B(y) \leq \lambda, s_A(x) \leq \delta, s_B(y) \leq \delta.\end{gathered}$$

If we reverse the order of the above formulas, we can get

$$\begin{gathered}\eta_A(x) \geq \rho, \varphi_A(x) \geq \theta, r_A(x) \leq \lambda, s_A(x) \leq \delta,\\ x \in A^{(\lambda,\delta)}_{(\rho,\theta)},\\ \eta_B(y) \geq \rho, \varphi_B(y) \geq \theta, r_B(y) \leq \lambda, s_B(y) \leq \delta,\\ y \in B^{(\lambda,\delta)}_{(\rho,\theta)}.\end{gathered}$$

Then, we naturally get the following formula:

$$(x, y) \in A^{(\lambda,\delta)}_{(\rho,\theta)} \times B^{(\lambda,\delta)}_{(\rho,\theta)}.$$

Therefore, the conclusion is valid, that is

$$(A \times B)^{(\lambda,\delta)}_{(\rho,\theta)} = A^{(\lambda,\delta)}_{(\rho,\theta)} \times B^{(\lambda,\delta)}_{(\rho,\theta)}.$$

Example 3.2. *Let $G_1 = \{e_1, a_1, b_1\}, G_2 = \{e_2, a_2, b_2\}$, where e_1 and e_2 are the identity elements of G_1 and G_2, respectively, and A and B are two CIFSGs of G_1 and G_2, respectively, where the membership and non-membership are defined as follows:*

$$\mu_A(e_1) = 0.8 \cdot e^{i0.4\pi}, \mu_A(a_1) = 0.6 \cdot e^{i0.4\pi}, \mu_A(b_1) = 0.4 \cdot e^{i0.3\pi},$$

$$\nu_A(e_1) = 0.1 \cdot e^{i0.1\pi}, \nu_A(a_1) = 0.2 \cdot e^{i0.1\pi}, \nu_A(b_1) = 0.3 \cdot e^{i0.2\pi},$$

$$\mu_B(e_2) = 0.7 \cdot e^{i0.3\pi}, \mu_B(a_2) = 0.5 \cdot e^{i0.3\pi}, \mu_B(b_2) = 0.3 \cdot e^{i0.4\pi},$$

$$\nu_B(e_2) = 0.2 \cdot e^{i0.1\pi}, \nu_B(a_2) = 0.3 \cdot e^{i0.2\pi}, \nu_B(b_2) = 0.4 \cdot e^{i0.3\pi},$$

for any $\rho, \lambda \in [0, 1], \theta, \delta \in [0, 2\pi]$. By Definition 3.2, we have that the cut set of A is $A^{(\lambda,\delta)}_{(\rho,\theta)}$ and the cut set of B is $B^{(\lambda,\delta)}_{(\rho,\theta)}$. Let $\rho = 0.5$, $\theta = 0.3\pi$, $\lambda = 0.4$, $\delta = 0.2\pi$, and then,

$$A^{(0.4,0.2\pi)}_{(0.5,0.3\pi)} = \{e_1, a_1\}, B^{(0.4,0.2\pi)}_{(0.5,0.3\pi)} = \{e_2, a_2\}.$$

By Definition 2.6, naturally, we get

$$(A \times B)^{(0.4,0.2\pi)}_{(0.5,0.3\pi)} = \{(e_1, e_2), (e_1, a_2), (a_1, e_2), (a_1, a_2)\},$$

and then, by the properties of Cartesian products, we can get

$$A^{(0.4,0.2\pi)}_{(0.5,0.3\pi)} \times B^{(0.4,0.2\pi)}_{(0.5,0.3\pi)} = \{e_1, a_1\} \times \{e_2, a_2\} = \{(e_1, e_2), (e_1, a_2), (a_1, e_2), (a_1, a_2)\},$$

and therefore,

$$A^{(0.4,0.2\pi)}_{(0.5,0.3\pi)} \times B^{(0.4,0.2\pi)}_{(0.5,0.3\pi)} = (A \times B)^{(0.4,0.2\pi)}_{(0.5,0.3\pi)}.$$

Definition 3.3. *Let A be a CIFSG of the group G. Then, A is called a CIFASG of G if $A^{(\lambda,\delta)}_{(\rho,\theta)}$ is an Abel subgroup of G, for all $\rho, \lambda \in [0, 1]$ and $\theta, \delta \in [0, 2\pi]$.*

Remark 3.2. *Every subgroup of an Abel group is also an Abel group.*

Theorem 3.5. *If G is an Abel group, then every CIFSG of the group G is a CIFASG of the group G.*

Proof: Given that G is an Abel group, then $xy = yx$ holds for all $x, y \in G$. Since A is a CIFSG of the group G, by Theorem 3.2, we obtain that $A^{(\lambda,\delta)}_{(\rho,\theta)}$ is a subgroup of G. In the view of Remark 3.2, we know that $A^{(\lambda,\delta)}_{(\rho,\theta)}$ is an Abel subgroup of G. By Definition 3.3, we conclude that A is a CIFASG of the group G.

Remark 3.3. *The converse of Theorem* 3.5 *is not necessarily true.*

Theorem 3.6. *Let A and B be two CIFSGs of two groups G_1 and G_2, respectively. Then, $A \times B$ is a CIFASG of the group $G_1 \times G_2$ if and only if both A and B are two CIFASGs of two groups G_1 and G_2.*

Proof: Suppose that A and B are two CIFSGs of two groups G_1 and G_2, respectively. Then, $A^{(\lambda,\delta)}_{(\rho,\theta)}$ and $B^{(\lambda,\delta)}_{(\rho,\theta)}$ are two Abel subgroups of G_1 and G_2, respectively, for all $\rho, \theta \in [0, 1]$ and $\lambda, \delta \in [0, 2\pi]$, and $A^{(\lambda,\delta)}_{(\rho,\theta)} \times B^{(\lambda,\delta)}_{(\rho,\theta)}$ is an Abel subgroup of $G_1 \times G_2$. In view of Theorem 3.4, we have

$$(A \times B)^{(\lambda,\delta)}_{(\rho,\theta)} = A^{(\lambda,\delta)}_{(\rho,\theta)} \times B^{(\lambda,\delta)}_{(\rho,\theta)}.$$

Therefore, $(A \times B)_{(\rho,\theta)}^{(\lambda,\delta)}$ is an Abel subgroup of $G_1 \times G_2$, for all $\rho, \theta \in [0, 1]$ and $\lambda, \delta \in [0, 2\pi]$. $A \times B$ is a CIFASG of $G_1 \times G_2$. Conversely, let $A \times B$ be an Abel subgroup of $G_1 \times G_2$. Then, $(A \times B)_{(\rho,\theta)}^{(\lambda,\delta)}$ is an Abel subgroup of $G_1 \times G_2$. This implies that $A_{(\rho,\theta)}^{(\lambda,\delta)} \times B_{(\rho,\theta)}^{(\lambda,\delta)}$ is an Abel subgroup of $G_1 \times G_2$, and $A_{(\rho,\theta)}^{(\lambda,\delta)}$ and $B_{(\rho,\theta)}^{(\lambda,\delta)}$ are two Abel subgroups of $G_1 \times G_2$, respectively. Thus, A and B are two CIFSGs of two groups G_1 and G_2, respectively. Hence, the proof is complete.

Definition 3.4. *Let A be a CIFSG of the group G. Then A is called a CIFCSG of the group G, if* $A_{(\rho,\theta)}^{(\lambda,\delta)}$ *is a cyclic group, for all* $\rho, \theta \in [0, 1]$ *and* $\lambda, \delta \in [0, 2\pi]$.

Remark 3.4. *Every subgroup of a cyclic group is a cyclic group.*

Theorem 3.7. *If G is a cyclic group, then every CIFSG of G is a CIFCSG of G.*

Proof: Given that G is a cyclic group. Let A be a CIFSG of the group G. By Theorem 3.2, we have that $A_{(\rho,\theta)}^{(\lambda,\delta)}$ is a subgroup of G. In view of Remark 3.4, we know that $A_{(\rho,\theta)}^{(\lambda,\delta)}$ is a cyclic subgroup of G. By Definition 3.4, we conclude that A is a CIFCSG of the group G. The converse of the above stated result is not necessarily true.

4. Complex intuitionistic fuzzy normal subgroups

In this section, we define the left and right cosets of CIFSGs, and then describe the representation of CIFNSGs. At the same time, some basic characteristics of CIFNSGs are discussed.

Definition 4.1. [19] *Let A be a CIFSG of the group G. Then the CIFS* $gA(x)$ *of G is called a complex intuitionistic fuzzy left coset of G, where* $gA(x)$ *is represented by A and g, for all* $x, g \in G$*, and we have*

$$gA(x) = \{\langle x, \eta_{gA}(x) \cdot e^{i\varphi_{gA}(x)}, r_{gA}(x) \cdot e^{is_{gA}(x)}\rangle | x \in G\},$$

$$\eta_{gA}(x) \cdot e^{i\varphi_{gA}(x)} = \eta_A(g^{-1}x) \cdot e^{i\varphi_A(g^{-1}x)},$$

$$r_{gA}(x) \cdot e^{is_{gA}(x)} = r_A(g^{-1}x) \cdot e^{is_A(g^{-1}x)}.$$

Similarly, for all $x, g \in G$, we can define a complex intuitionistic fuzzy right coset and it is described as:

$$Ag(x) = \{\langle x, \eta_{Ag}(x) \cdot e^{i\varphi_{Ag}(x)}, r_{Ag}(x) \cdot e^{is_{Ag}(x)}\rangle | x \in G\},$$

$$\eta_{Ag}(x) \cdot e^{i\varphi_{Ag}(x)} = \eta_A(xg^{-1}) \cdot e^{i\varphi_A(xg^{-1})},$$

$$r_{Ag}(x) \cdot e^{is_{Ag}(x)} = r_A(xg^{-1}) \cdot e^{is_A(xg^{-1})}.$$

Definition 4.2. [32] *Let G be a group. Then a subgroup A of the group G is a normal group if and only if* $xax^{-1} \in A$*, for all* $x \in G$ *and* $a \in A$.

Definition 4.3. [19] *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in U\}$ *be a CIFSG of the group G. Then A is called a CIFNSG if and only if* $\mu_A(xy) = \mu_A(yx)$ *and* $\nu_A(xy) = \nu_A(yx)$*. Or equivalently, A is called a CIFNSG of the group G if and only if* $\mu_A(x^{-1}yx) = \mu_A(y)$ *and* $\nu_A(x^{-1}yx) = \nu_A(y)$*, for all* $x, y \in G$.

Theorem 4.1. *The intersection of two CIFNSGs of the group G is still a CIFNSG of the group G.*

Proof: Let A, B be two CIFNSGs of the group G. Then, for any $x, y \in G$, we have

$$\mu_A(xy) = \mu_A(yx), \nu_A(xy) = \nu_A(yx),$$

$$\mu_B(xy) = \mu_B(yx), \nu_B(xy) = \nu_B(yx).$$

To prove that $A \cap B$ is a CIFNSG of the group G, we just have to prove $\mu_{A\cap B}(xy) = \mu_{A\cap B}(yx)$, $\nu_{A\cap B}(xy) = \nu_{A\cap B}(yx)$. Naturally, we can get

$$\begin{aligned}\mu_{A\cap B}(xy) &= \eta_{A\cap B}(xy) \cdot e^{i\varphi_{A\cap B}(xy)} \\ &\geq \min\{\eta_{A\cap B}(x), \eta_{A\cap B}(y)\} \cdot e^{i\min\{\varphi_{A\cap B}(x), \varphi_{A\cap B}(y)\}} \\ &= \min\{\eta_{A\cap B}(y), \eta_{A\cap B}(x)\} \cdot e^{i\min\{\varphi_{A\cap B}(y), \varphi_{A\cap B}(x)\}} \\ &= \eta_{A\cap B}(yx) \cdot e^{i\varphi_{A\cap B}(yx)} \\ &= \mu_{A\cap B}(yx).\end{aligned}$$

Therefore, we can get $\mu_{A\cap B}(xy) \geq \mu_{A\cap B}(yx)$. Similarly, $\mu_{A\cap B}(yx) \geq \mu_{A\cap B}(xy)$, and thus, $\mu_{A\cap B}(xy) = \mu_{A\cap B}(yx)$. By the same token, we have $\nu_{A\cap B}(xy) = \nu_{A\cap B}(yx)$. Thus, $A \cap B$ is a CIFNSG of the group G.

Example 4.1. *Let $G = \{e, a, b, c\}$, where e is the identity element, and A and B are two CIFNSGs of group G. For each element of the group G,*

$$\mu_A(e) = 0.8 \cdot e^{i0.1\pi}, \nu_A(e) = 0.1 \cdot e^{i0.1\pi}, \mu_B(e) = 0.7 \cdot e^{i0.2\pi}, \nu_B(e) = 0.2 \cdot e^{i0.1\pi},$$

$$\mu_A(a) = 0.6 \cdot e^{i0.2\pi}, \nu_A(a) = 0.2 \cdot e^{i0.1\pi}, \mu_B(a) = 0.5 \cdot e^{i0.3\pi}, \nu_B(a) = 0.3 \cdot e^{i0.2\pi},$$

$$\mu_A(b) = 0.4 \cdot e^{i0.3\pi}, \nu_A(b) = 0.3 \cdot e^{i0.2\pi}, \mu_B(b) = 0.3 \cdot e^{i0.4\pi}, \nu_B(b) = 0.4 \cdot e^{i0.3\pi},$$

$$\mu_A(c) = 0.2 \cdot e^{i0.4\pi}, \nu_A(c) = 0.4 \cdot e^{i0.3\pi}, \mu_B(c) = 0.1 \cdot e^{i0.5\pi}, \nu_B(c) = 0.5 \cdot e^{i0.4\pi},$$

$$\mu_{A\cap B}(e) = \min\{0.8 \cdot e^{i0.1\pi}, 0.7 \cdot e^{i0.2\pi}\} = 0.7 \cdot e^{i0.2\pi},$$

$$\nu_{A\cap B}(e) = \max\{0.1 \cdot e^{i0.1\pi}, 0.2 \cdot e^{i0.1\pi}\} = 0.2 \cdot e^{i0.1\pi},$$

$$\mu_{A\cap B}(a) = \min\{0.6 \cdot e^{i0.2\pi}, 0.5 \cdot e^{i0.3\pi}\} = 0.5 \cdot e^{i0.2\pi},$$

$$\nu_{A\cap B}(a) = \max\{0.2 \cdot e^{i0.1\pi}, 0.3 \cdot e^{i0.3\pi}\} = 0.3 \cdot e^{i0.3\pi},$$

$$\mu_{A\cap B}(b) = \min\{0.4 \cdot e^{i0.3\pi}, 0.3 \cdot e^{i0.4\pi}\} = 0.3 \cdot e^{i0.3\pi},$$

$$\nu_{A\cap B}(b) = \max\{0.3 \cdot e^{i0.2\pi}, 0.4 \cdot e^{i0.3\pi}\} = 0.4 \cdot e^{i0.3\pi},$$

$$\mu_{A\cap B}(c) = \min\{0.2 \cdot e^{i0.4\pi}, 0.1 \cdot e^{i0.5\pi}\} = 0.1 \cdot e^{i0.4\pi},$$

$$\nu_{A\cap B}(c) = \max\{0.4 \cdot e^{i0.3\pi}, 0.5 \cdot e^{i0.4\pi}\} = 0.5 \cdot e^{i0.4\pi}.$$

On account of A and B both being CIFNSGs, for any $x, y \in G$, they satisfy

$$\mu_A(xy) = \mu_A(yx), \nu_A(xy) = \nu_A(yx), \mu_B(xy) = \mu_B(yx), \nu_B(xy) = \nu_B(yx).$$

Through the above calculation, it is easy to know that the membership degree and non-membership degree of $A \cap B$ also satisfy the above conditions, so the intersection is still a CIFNSG.

Theorem 4.2. *Let A be a CIFSG of the group G. Then, A is a CIFNSG if and only if A is constant in the conjugate class of the group G.*

Proof: Suppose that A is a CIFNSG. Then for a conjugate x of the group G, for all $x, y \in G$, we get

$$\begin{aligned} \eta_A(y^{-1}xy) \cdot e^{i\varphi_A(y^{-1}xy)} &= \eta_A(xyy^{-1}) \cdot e^{i\varphi_A(xyy^{-1})} \\ &= \eta_A(x) \cdot e^{i\varphi_A(x)}, \\ r_A(y^{-1}xy) \cdot e^{is_A(y^{-1}xy)} &= r_A(xyy^{-1}) \cdot e^{is_A(xyy^{-1})} \\ &= r_A(x) \cdot e^{is_A(x)}. \end{aligned}$$

Conversely, suppose that A is constant in all conjugate classes of the group G. For all $x, y \in G$, we have

$$\begin{aligned} \eta_A(xy) \cdot e^{i\varphi_A(xy)} &= \eta_A(xyxx^{-1}) \cdot e^{i\varphi_A(xyxx^{-1})} \\ &= \eta_A(x(yx)x^{-1}) \cdot e^{i\varphi_A(x(yx)x^{-1})} \\ &= \eta_A((x^{-1})^{-1}(yx)x^{-1}) \cdot e^{i\varphi_A((x^{-1})^{-1}(yx)x^{-1})} \\ &= \eta_A(yx) \cdot e^{i\varphi_A(yx)}, \\ r_A(xy) \cdot e^{is_A(xy)} &= r_A(xyxx^{-1}) \cdot e^{is_A(xyxx^{-1})} \\ &= r_A(x(yx)x^{-1}) \cdot e^{is_A(x(yx)x^{-1})} \\ &= r_A((x^{-1})^{-1}(yx)x^{-1}) \cdot e^{is_A((x^{-1})^{-1}(yx)x^{-1})} \\ &= r_A(yx) \cdot e^{is_A(yx)}. \end{aligned}$$

Theorem 4.3. *If A is a CIFSG of the group G, for all $x, y \in G$, then,*

(1) $\eta_A(y) \cdot e^{i\varphi_A(y)} \leq \eta_A(e) \cdot e^{i\varphi_A(e)}$;
(2) $\eta_A(x^{-1}y) \cdot e^{i\varphi_A(x^{-1}y)} = \eta_A(e) \cdot e^{i\varphi_A(e)}$;
(3) $r_A(y) \cdot e^{is_A(y)} \geq r_A(e) \cdot e^{is_A(e)}$;
(4) $r_A(x^{-1}y) \cdot e^{is_A(x^{-1}y)} = r_A(e) \cdot e^{is_A(e)}$;

which implies that $\eta_A(x) \cdot e^{i\varphi_A(x)} = \eta_A(y) \cdot e^{i\varphi_A(y)}$ and $r_A(x) \cdot e^{is_A(x)} = r_A(y) \cdot e^{is_A(y)}$.

Theorem 4.4. *Let A be a complex intuitionistic fuzzy subgroupoid of a finite group G. Then, A is a CIFSG of the finite group G.*

Proof: Let G be a finite group. For any $x \in G$, we have $x^n = e$, where x has finite order, and e is the natural element of the group G. Then, we have $x^{-1} = x^{n-1}$.

$$\begin{aligned} \eta_A(x^{-1}) \cdot e^{i\varphi_A(x^{-1})} &= \eta_A(x^{n-1}) \cdot e^{i\varphi_A(x^{n-1})} \\ &= \eta_A(x^{n-2} \cdot x) \cdot e^{i\varphi_A(x^{n-2} \cdot x)} \\ &\geq \eta_A(x) \cdot e^{i\varphi_A(x)}, \\ r_A(x^{-1}) \cdot e^{is_A(x^{-1})} &= r_A(x^{n-1}) \cdot e^{is_A(x^{n-1})} \\ &= r_A(x^{n-2} \cdot x) \cdot e^{is_A(x^{n-2} \cdot x)} \\ &\leq r_A(x) \cdot e^{is_A(x)}, \end{aligned}$$

and, naturally, we get that the theorem is true.

Theorem 4.5. *If A is a CIFSG of the group G, for all $x \in G$, we have*

$$\eta_A(x) \cdot e^{i\varphi_A(x)} = \eta_A(e) \cdot e^{i\varphi_A(e)}, r_A(x) \cdot e^{is_A(x)} = r_A(e) \cdot e^{is_A(e)}.$$

Then, for all $y \in G$, the following formulas hold:

$$\eta_A(xy) \cdot e^{i\varphi_A(xy)} = \eta_A(y) \cdot e^{i\varphi_A(y)},$$

$$r_A(xy) \cdot e^{is_A(xy)} = r_A(y) \cdot e^{is_A(y)}.$$

Proof: Given that $\eta_A(x) \cdot e^{i\varphi_A(x)} = \eta_A(e) \cdot e^{i\varphi_A(e)}$, then by Theorem 4.3, for all $y \in G$, we have

$$\eta_A(y) \cdot e^{i\varphi_A(y)} \leq \eta_A(x) \cdot e^{i\varphi_A(x)},$$

$$\eta_A(xy) \cdot e^{i\varphi_A(xy)} \geq \min\{\eta_A(x) \cdot e^{i\varphi_A(x)}, \eta_A(y) \cdot e^{i\varphi_A(y)}\},$$

$$\eta_A(xy) \cdot e^{i\varphi_A(xy)} \geq \eta_A(y) \cdot e^{i\varphi_A(y)}. \tag{4.1}$$

Now, assume

$$\begin{aligned}\eta_A(y) \cdot e^{i\varphi_A(y)} &= \eta_A(x^{-1}xy) \cdot e^{i\varphi_A(x^{-1}xy)} \\ &\geq \min\{\eta_A(x^{-1}) \cdot e^{i\varphi_A(x^{-1})}, \eta_A(xy) \cdot e^{i\varphi_A(xy)}\} \\ &\geq \min\{\eta_A(x) \cdot e^{i\varphi_A(x)}, \eta_A(xy) \cdot e^{i\varphi_A(xy)}\}.\end{aligned}$$

Again, from Theorem 4.3, we have

$$\min\{\eta_A(x) \cdot e^{i\varphi_A(x)}, \eta_A(xy) \cdot e^{i\varphi_A(xy)}\} = \eta_A(xy) \cdot e^{i\varphi_A(xy)},$$

and therefore, we obtain

$$\eta_A(y) \cdot e^{i\varphi_A(y)} \geq \eta_A(xy) \cdot e^{i\varphi_A(xy)}. \tag{4.2}$$

From the Eqs (4.1) and (4.2), we have

$$\eta_A(xy) \cdot e^{i\varphi_A(xy)} = \eta_A(y) \cdot e^{i\varphi_A(y)}.$$

Assume that $r_A(x) \cdot e^{is_A(x)} = r_A(e) \cdot e^{is_A(e)}$, and then from Theorem 4.3, for all $y \in G$, we have

$$r_A(x) \cdot e^{is_A(x)} \leq r_A(y) \cdot e^{is_A(y)},$$

$$r_A(xy) \cdot e^{is_A(xy)} \leq \max\{r_A(x) \cdot e^{is_A(x)}, r_A(y) \cdot e^{is_A(y)}\},$$

$$r_A(xy) \cdot e^{is_A(xy)} \leq r_A(y) \cdot e^{is_A(y)}. \tag{4.3}$$

Now, assume

$$\begin{aligned}r_A(y) \cdot e^{is_A(y)} &= r_A(x^{-1}xy) \cdot e^{is_A(x^{-1}xy)} \\ &\leq \max\{r_A(x^{-1}) \cdot e^{is_A(x^{-1})}, r_A(xy) \cdot e^{is_A(xy)}\} \\ &\leq \max\{r_A(x) \cdot e^{is_A(x)}, r_A(xy) \cdot e^{is_A(xy)}\}.\end{aligned}$$

Again, from Theorem 4.3, we have

$$\max\{r_A(x)\cdot e^{is_A(x)}, r_A(xy)\cdot e^{is_A(xy)}\} = r_A(xy)\cdot e^{is_A(xy)},$$

and therefore, we obtain:

$$r_A(y)\cdot e^{is_A(y)} \le r_A(xy)\cdot e^{is_A(xy)}. \tag{4.4}$$

From the Eqs (4.3) and (4.4), we have

$$r_A(xy)\cdot e^{is_A(xy)} = r_A(y)\cdot e^{is_A(y)}.$$

5. $(\alpha_{1,2},\beta_{1,2})$-complex intuitionistic fuzzy subgroups

In this section, we propose a new concept of $(\alpha_{1,2},\beta_{1,2})$-CIFSs, and then characterize $(\alpha_{1,2},\beta_{1,2})$-CIFSGs and $(\alpha_{1,2},\beta_{1,2})$-CIFNSGs, give the left and right cosets of $(\alpha_{1,2},\beta_{1,2})$-CIFSGs, and prove that an $(\alpha_{1,2},\beta_{1,2})$-CIFSG is a general form of every CIFSG.

Definition 5.1. *Let $A = \{\langle x, \eta_A(x)\cdot e^{i\varphi_A(x)}, r_A(x)\cdot e^{is_A(x)}\rangle | x \in G\}$ be a CIFS of the group G, for any $\alpha_1,\alpha_2 \in [0,1]$ and $\beta_1,\beta_2 \in [0,2\pi]$. Then, the CIFS $A_{(\alpha_{1,2},\beta_{1,2})}$ is an $(\alpha_{1,2},\beta_{1,2})$-CIFS of the group G in regards to the CIFS A and it is defined as:*

$$\eta_{A_{\alpha_1}}(x)\cdot e^{i\varphi_{A_{\beta_1}}}(x) = \eta_A(x)\cdot e^{i\varphi_A(x)} \bullet \alpha_1\cdot e^{i\beta_1} = \max\{0, \eta_A(x)+\alpha_1-1\}\cdot e^{i\max\{0,\varphi_A(x)+\beta_1-2\pi\}},$$

$$r_{A_{\alpha_2}}(x)\cdot e^{is_{A_{\beta_2}}}(x) = r_A(x)\cdot e^{is_A(x)} \bullet \alpha_2\cdot e^{i\beta_2} = \min\{1, r_A(x)+\alpha_2\}\cdot e^{i\min\{2\pi, s_A(x)+\beta_2\}}.$$

Definition 5.2. [19] *Let $A_{(\alpha_{1,2},\beta_{1,2})}$ and $B_{(\alpha_{1,2},\beta_{1,2})}$ be two $(\alpha_{1,2},\beta_{1,2})$-CIFSs of the group G, and then*

(1) An $(\alpha_{1,2},\beta_{1,2})$-CIFS $A_{(\alpha_{1,2},\beta_{1,2})}$ is a homogeneous $(\alpha_{1,2},\beta_{1,2})$-CIFS, if for all $x,y \in G$, we have $\eta_{A_{\alpha_1}}(x) \le \eta_{A_{\alpha_1}}(y)$, $r_{A_{\alpha_2}}(x) \le r_{A_{\alpha_2}}(y)$ if and only if $\varphi_{A_{\beta_1}}(x) \le \varphi_{A_{\beta_1}}(y)$, $s_{A_{\beta_2}}(x) \le s_{A_{\beta_2}}(y)$.

(2) An $(\alpha_{1,2},\beta_{1,2})$-CIFS $A_{(\alpha_{1,2},\beta_{1,2})}$ is a homogeneous $(\alpha_{1,2},\beta_{1,2})$-CIFS with $B_{(\alpha_{1,2},\beta_{1,2})}$, if for all $x,y \in G$, we have $\eta_{A_{\alpha_1}}(x) \le \eta_{B_{\alpha_1}}(y)$, $r_{A_{\alpha_2}}(x) \le r_{B_{\alpha_2}}(y)$ if and only if $\varphi_{A_{\beta_1}}(x) \le \varphi_{B_{\beta_1}}(y)$, $s_{A_{\beta_2}}(x) \le s_{B_{\beta_2}}(y)$.

Remark 5.1. *All $(\alpha_{1,2},\beta_{1,2})$-CIFSs used in this paper are homogeneous.*

Remark 5.2. *If $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 2\pi$ in the above definition, $(\alpha_{1,2},\beta_{1,2})$-CIFSs degenerate classical CIFSs.*

Lemma 5.1. *Let $A_{(\alpha_{1,2},\beta_{1,2})}$ and $B_{(\alpha_{1,2},\beta_{1,2})}$ be two $(\alpha_{1,2},\beta_{1,2})$-CIFSs of the group G. Then,*

$$(A\cap B)_{(\alpha_{1,2},\beta_{1,2})} = A_{(\alpha_{1,2},\beta_{1,2})} \cap B_{(\alpha_{1,2},\beta_{1,2})}.$$

Example 5.1. *Consider a group of fourth-order $Z_4 = \{0,1,2,3\}$. Let $\alpha_1 = 0.7$, $\alpha_2 = 0.2$, $\beta_1 = 0.8\pi$, and $\beta_2 = 0.5\pi$, and a CIFS $A = \{\langle 0, 0.8\cdot e^{i1.4\pi}, 0.1\cdot e^{i0.3\pi}\rangle, \langle 1, 0.6\cdot e^{i1.3\pi}, 0.7\cdot e^{i0.5\pi}\rangle, \langle 2, 0.4\cdot e^{i0.4\pi}, 0.7\cdot e^{i0.3\pi}\rangle, \langle 3, 0.7\cdot e^{i1.6\pi}, 0.1\cdot e^{i0.5\pi}\rangle\}$ of the group Z_4. Then, the set $A_{(\alpha_{1,2},\beta_{1,2})}$ is called an $(\alpha_{1,2},\beta_{1,2})$-CIFS and it is defined as:*

$$A_{(\alpha_{1,2},\beta_{1,2})} = \{\langle 0, 0.5\cdot e^{i0.2\pi}, 0.3\cdot e^{i0.8\pi}\rangle, \langle 1, 0.3\cdot e^{i0.1\pi}, 0.9\cdot e^{i\pi}\rangle, \langle 2, 0.1\cdot e^{i0\pi}, 0.9\cdot e^{i0.8\pi}\rangle, \langle 3, 0.4\cdot e^{i0.4\pi}, 0.3\cdot e^{i\pi}\rangle\}.$$

Definition 5.3. [19] *Let $A_{(\alpha_{1,2},\beta_{1,2})}$ be an $(\alpha_{1,2},\beta_{1,2})$-CIFS of the group G. For all $\alpha_1, \alpha_2 \in [0,1]$ and $\beta_1, \beta_2 \in [0, 2\pi]$, then, $A_{(\alpha_{1,2},\beta_{1,2})}$ is called an $(\alpha_{1,2},\beta_{1,2})$-CIFSG of the group G if the following axioms hold:*

(1) $\eta_{A_{\alpha_1}}(xy) \cdot e^{i\varphi_{A_{\beta_1}}(xy)} \geq \min\{\eta_{A_{\alpha_1}}(x) \cdot e^{i\varphi_{A_{\beta_1}}(x)}, \eta_{A_{\alpha_1}}(y) \cdot e^{i\varphi_{A_{\beta_1}}(y)}\}$,

(2) $\eta_{A_{\alpha_1}}(x^{-1}) \cdot e^{i\varphi_{A_{\beta_1}}(x^{-1})} \geq \eta_{A_{\alpha_1}}(x) \cdot e^{i\varphi_{A_{\beta_1}}(x)}$,

(3) $r_{A_{\alpha_2}}(xy) \cdot e^{is_{A_{\beta_2}}(xy)} \leq \max\{r_{A_{\alpha_2}}(x) \cdot e^{is_{A_{\beta_2}}(x)}, r_{A_{\alpha_2}}(y) \cdot e^{is_{A_{\beta_2}}(y)}\}$,

(4) $r_{A_{\alpha_2}}(x^{-1}) \cdot e^{is_{A_{\beta_2}}(x^{-1})} \leq r_{A_{\alpha_2}}(x) \cdot e^{is_{A_{\beta_2}}(x)}$.

The following (5) and (6) are equivalent to (1), (2), (3), and (4),

(5) $\eta_{A_{\alpha_1}}(x^{-1}y) \cdot e^{i\varphi_{A_{\beta_1}}(x^{-1}y)} \geq \min\{\eta_{A_{\alpha_1}}(x) \cdot e^{i\varphi_{A_{\beta_1}}(x)}, \eta_{A_{\alpha_1}}(y) \cdot e^{i\varphi_{A_{\beta_1}}(y)}\}$,

(6) $r_{A_{\alpha_2}}(x^{-1}y) \cdot e^{is_{A_{\beta_2}}(x^{-1}y)} \leq \max\{r_{A_{\alpha_2}}(x) \cdot e^{is_{A_{\beta_2}}(x)}, r_{A_{\alpha_2}}(y) \cdot e^{is_{A_{\beta_2}}(y)}\}$.

Example 5.2. *Recalling that $A_{(\alpha_{1,2},\beta_{1,2})} = \{\langle 0, 0.5 \cdot e^{i0.2\pi}, 0.3 \cdot e^{i0.8\pi}\rangle, \langle 1, 0.3 \cdot e^{i0.1\pi}, 0.9 \cdot e^{i\pi}\rangle, \langle 2, 0.1 \cdot e^{i0\pi}, 0.9 \cdot e^{i0.8\pi}\rangle, \langle 3, 0.4 \cdot e^{i0.4\pi}, 0.3 \cdot e^{i\pi}\rangle\}$ from Example 5.1 satisfies all axioms of Definition 5.3 for all elements in the group Z_4. For example, let $x = 1$ and $y = 2$, so $xy = 3$, $x^{-1} = 3$ in Z_4. Then,*

$$\begin{aligned}
\eta_{A_{\alpha_1}}(xy) \cdot e^{i\varphi_{A_{\beta_1}}(xy)} &= \eta_{A_{0.7}}(3) \cdot e^{i\varphi_{A_{0.8\pi}}(3)} \\
&= 0.4 \cdot e^{i0.4\pi} \\
&\geq \min\{\eta_{A_{0.7}}(1) \cdot e^{i\varphi_{A_{0.8\pi}}(1)}, \eta_{A_{0.7}}(2) \cdot e^{i\varphi_{A_{0.8\pi}}(2)}\} \\
&= \min\{0.3 \cdot e^{i0.1\pi}, 0.1 \cdot e^{i0\pi}\} \\
&= 0.1 \cdot e^{i0\pi}, \\
\eta_{A_{\alpha_1}}(x^{-1}) \cdot e^{i\varphi_{A_{\beta_1}}(x^{-1})} &= \eta_{A_{0.7}}(3) \cdot e^{i\varphi_{A_{0.8\pi}}(3)} \\
&= 0.4 \cdot e^{i0.4\pi} \\
&\geq \eta_{A_{0.7}}(1) \cdot e^{i\varphi_{A_{0.8\pi}}(1)} \\
&= 0.3 \cdot e^{i0.1\pi}, \\
r_{A_{\alpha_2}}(xy) \cdot e^{is_{A_{\beta_2}}(xy)} &= r_{A_{0.2}}(3) \cdot e^{is_{A_{0.8\pi}}(3)} \\
&= 0.3 \cdot e^{i\pi} \\
&\leq \max\{r_{A_{0.2}}(1) \cdot e^{is_{A_{0.5\pi}}(1)}, r_{A_{0.2}}(2) \cdot e^{is_{A_{0.5\pi}}(2)}\} \\
&= \max\{0.9 \cdot e^{i\pi}, 0.9 \cdot e^{i0.8\pi}\} \\
&= 0.9 \cdot e^{i\pi}, \\
r_{A_{\alpha_2}}(x^{-1}) \cdot e^{is_{A_{\beta_2}}(x^{-1})} &= r_{A_{0.2}}(3) \cdot e^{is_{A_{0.5\pi}}(3)} \\
&= 0.3 \cdot e^{i\pi} \\
&\leq r_{A_{0.2}}(1) \cdot e^{is_{A_{0.5\pi}}(1)} \\
&= 0.9 \cdot e^{i\pi}.
\end{aligned}$$

Theorem 5.1. *Let G be a group. If A is a CIFSG of G, then $A_{(\alpha_{1,2},\beta_{1,2})}$ is an $(\alpha_{1,2},\beta_{1,2})$-CIFSG of the group G.*

Proof: Assume that A is a CIFSG of the group G. For all $x, y \in G$, we have

$$\eta_{A_{\alpha_1}}(xy) \cdot e^{i\varphi_{A_{\beta_1}}}(xy) = \eta_A(xy) \cdot e^{i\varphi_A(xy)} \bullet \alpha_1 \cdot e^{i\beta_1}$$

$$
\begin{aligned}
&\geq \min\{\eta_A(x)\cdot e^{i\varphi_A(x)}, \eta_A(y)\cdot e^{i\varphi_A(y)}\} \bullet \alpha_1 \cdot e^{i\beta_1}\\
&= \min\{\eta_A(x)\cdot e^{i\varphi_A(x)} \bullet \alpha_1 \cdot e^{i\beta_1}, \eta_A(y)\cdot e^{i\varphi_A(y)} \bullet \alpha_1 \cdot e^{i\beta_1}\}\\
&= \min\{\eta_{A_{\alpha_1}}(x)\cdot e^{i\varphi_{A_{\beta_1}}}(x), \eta_{A_{\alpha_1}}(y)\cdot e^{i\varphi_{A_{\beta_1}}}(y)\},\\
\eta_{A_{\alpha_1}}(x^{-1})\cdot e^{i\varphi_{A_{\beta_1}}}(x^{-1}) &= \eta_A(x^{-1})\cdot e^{i\varphi_A(x^{-1})} \bullet \alpha_1 \cdot e^{i\beta_1}\\
&\geq \eta_A(x)\cdot e^{i\varphi_A(x)} \bullet \alpha_1 \cdot e^{i\beta_1}\\
&= \eta_{A_{\alpha_1}}(x)\cdot e^{i\varphi_{A_{\beta_1}}}(x),\\
r_{A_{\alpha_2}}(xy)\cdot e^{is_{A_{\beta_2}}}(xy) &= r_A(xy)\cdot e^{is_A(xy)} \bullet \alpha_2 \cdot e^{i\beta_2}\\
&\leq \max\{r_A(x)\cdot e^{is_A(x)}, r_A(y)\cdot e^{is_A(y)}\} \bullet \alpha_2 \cdot e^{i\beta_2}\\
&= \max\{r_A(x)\cdot e^{is_A(x)} \bullet \alpha_2 \cdot e^{i\beta_2}, r_A(y)\cdot e^{is_A(y)} \bullet \alpha_2 \cdot e^{i\beta_2}\}\\
&= \max\{r_{A_{\alpha_2}}(x)\cdot e^{is_{A_{\beta_2}}}(x), r_{A_{\alpha_2}}(y)\cdot e^{is_{A_{\beta_2}}}(y)\},\\
r_{A_{\alpha_2}}(x^{-1})\cdot e^{is_{A_{\beta_2}}}(x^{-1}) &= r_A(x^{-1})\cdot e^{is_A(x^{-1})} \bullet \alpha_2 \cdot e^{i\beta_2}\\
&\leq r_A(x)\cdot e^{is_A(x)} \bullet \alpha_2 \cdot e^{i\beta_2}\\
&= r_{A_{\alpha_2}}(x)\cdot e^{is_{A_{\beta_2}}}(x).
\end{aligned}
$$

Example 5.3. *Let $G = Z_4 = \{0, 1, 2, 3\}$ represent the status of patients at different time points. Define an $(\alpha_{1,2}, \beta_{1,2})$-CIFSG $A_{(\alpha_{1,2},\beta_{1,2})}$, where the membership function and non-membership function are as follows:*

$$\eta_{A_{\alpha_1}}(x)\cdot e^{i\varphi_{A_{\beta_1}}}(x) = \eta_A(x)\cdot e^{i\varphi_A(x)} \bullet \alpha_1 \cdot e^{i\beta_1} = \max\{0, \eta_A(x)+\alpha_1-1\}\cdot e^{i\max\{0,\varphi_A(x)+\beta_1-2\pi\}},$$

$$r_{A_{\alpha_2}}(x)\cdot e^{is_{A_{\beta_2}}}(x) = r_A(x)\cdot e^{is_A(x)} \bullet \alpha_2 \cdot e^{i\beta_2} = \min\{1, r_A(x)+\alpha_2\}\cdot e^{i\min\{2\pi,s_A(x)+\beta_2\}}.$$

Let $\alpha_1 = 0.7, \alpha_2 = 0.2, \beta_1 = 0.8\pi, \beta_2 = 0.5\pi$. Define the degree of membership and non-membership of each element as follow:

$$
\begin{aligned}
A(0) &= \langle 0.8\cdot e^{i1.4\pi}, 0.1\cdot e^{i0.3\pi}\rangle,\\
A(1) &= \langle 0.5\cdot e^{i1.3\pi}, 0.5\cdot e^{i0.6\pi}\rangle,\\
A(2) &= \langle 0.4\cdot e^{i1.6\pi}, 0.7\cdot e^{i0.3\pi}\rangle,\\
A(3) &= \langle 0.7\cdot e^{i1.4\pi}, 0.1\cdot e^{i0.5\pi}\rangle,\\
A_{(0.7,0.2,0.4\pi,0.5\pi)}(0) &= \langle 0.5\cdot e^{i0.2\pi}, 0.3\cdot e^{i0.8\pi}\rangle,\\
A_{(0.7,0.2,0.4\pi,0.5\pi)}(1) &= \langle 0.2\cdot e^{i0.1\pi}, 0.7\cdot e^{i1.1\pi}\rangle,\\
A_{(0.7,0.2,0.4\pi,0.5\pi)}(2) &= \langle 0.1\cdot e^{i0.4\pi}, 0.9\cdot e^{i0.8\pi}\rangle,\\
A_{(0.7,0.2,0.4\pi,0.5\pi)}(3) &= \langle 0.4\cdot e^{i0.2\pi}, 0.3\cdot e^{i\pi}\rangle.
\end{aligned}
$$

By Theorem 5.1, we know that $A_{(0.7,0.2,0.4\pi,0.5\pi)}$ is a subgroup of G. Assuming that state "0" represents "health" and state "3" represents "severe fever", doctors can reduce the misdiagnosis rate by adjusting paraments α_1 and β_1. If α_1 is increased (e.g., 0.8), then it will only be determined as abnormal when the membership degree is higher. This is applicable to a conservative diagnosis strategy.

Definition 5.4. *Let $A_{(\alpha_{1,2},\beta_{1,2})}$ be an $(\alpha_{1,2},\beta_{1,2})$-CIFSG of the group G, where $\alpha_1,\alpha_2 \in [0,1]$ and $\beta_1,\beta_2 \in [0,2\pi]$. Then the $(\alpha_{1,2},\beta_{1,2})$-CIFS $gA_{(\alpha_{1,2},\beta_{1,2})}(x)$ of G is called an $(\alpha_{1,2},\beta_{1,2})$-complex intuitionistic fuzzy left coset of G, where $(\alpha_{1,2},\beta_{1,2})$-CIFS $gA_{(\alpha_{1,2},\beta_{1,2})}(x)$ is determined by $A_{(\alpha_{1,2},\beta_{1,2})}$ and g. Then for all $x, g \in G$, we have*

$$gA_{(\alpha_{1,2},\beta_{1,2})}(x) = \{\langle x, \eta_{gA_{\alpha_1}}(x)\cdot e^{i\varphi_{gA_{\beta_1}}(x)}, r_{gA_{\alpha_2}}(x)\cdot e^{is_{gA_{\beta_2}}(x)}\rangle | x \in G\},$$

where

$$\begin{aligned}
\eta_{gA_{\alpha_1}}(x)\cdot e^{i\varphi_{gA_{\beta_1}}(x)} &= \eta_{A_{\alpha_1}}(g^{-1}x)\cdot e^{i\varphi_{A_{\beta_1}}(g^{-1}x)} \\
&= \eta_A(g^{-1}x)\cdot e^{i\varphi_A(g^{-1}x)} \bullet \alpha_1 \cdot e^{i\beta_1} \\
&= \max\{0, \eta_A(g^{-1}x)+\alpha_1-1\}\cdot e^{i\max\{0,\varphi_A(g^{-1}x)+\beta_1-2\pi\}}, \\
r_{gA_{\alpha_2}}(x)\cdot e^{is_{gA_{\beta_2}}(x)} &= r_{A_{\alpha_2}}(g^{-1}x)\cdot e^{is_{A_{\beta_2}}(g^{-1}x)} \\
&= r_A(g^{-1}x)\cdot e^{is_A(g^{-1}x)} \bullet \alpha_2 \cdot e^{i\beta_2} \\
&= \min\{1, r_A(g^{-1}x)+\alpha_2\}\cdot e^{i\min\{2\pi, s_A(g^{-1}x)+\beta_2\}}.
\end{aligned}$$

Similarly, for all $x, g \in G$, we can define an $(\alpha_{1,2},\beta_{1,2})$-complex intuitionistic fuzzy right coset and it is described as:

$$\begin{aligned}
\eta_{A_{\alpha_1}g}(x)\cdot e^{i\varphi_{A_{\beta_1}g}(x)} &= \eta_{A_{\alpha_1}}(xg^{-1})\cdot e^{i\varphi_{A_{\beta_1}}(xg^{-1})} \\
&= \eta_A(xg^{-1})\cdot e^{i\varphi_A(xg^{-1})} \bullet \alpha_1 \cdot e^{i\beta_1} \\
&= \max\{0, \eta_A(xg^{-1})+\alpha_1-1\}\cdot e^{i\max\{0,\varphi_A(xg^{-1})+\beta_1-2\pi\}}, \\
r_{A_{\alpha_2}g}(x)\cdot e^{is_{A_{\beta_2}g}(x)} &= r_{A_{\alpha_2}}(xg^{-1})\cdot e^{is_{A_{\beta_2}}(xg^{-1})} \\
&= r_A(xg^{-1})\cdot e^{is_A(xg^{-1})} \bullet \alpha_2 \cdot e^{i\beta_2} \\
&= \min\{1, r_A(xg^{-1})+\alpha_2\}\cdot e^{i\min\{2\pi, s_A(xg^{-1})+\beta_2\}}.
\end{aligned}$$

Definition 5.5. [19] *Let $A_{(\alpha_{1,2},\beta_{1,2})}$ be an $(\alpha_{1,2},\beta_{1,2})$-CIFSG of the group G, where $\alpha_1,\alpha_2 \in [0,1]$ and $\beta_1,\beta_2 \in [0,2\pi]$. Then $A_{(\alpha_{1,2},\beta_{1,2})}$ is called an $(\alpha_{1,2},\beta_{1,2})$-CIFNSG if $A_{(\alpha_{1,2},\beta_{1,2})}(xy) = A_{(\alpha_{1,2},\beta_{1,2})}(yx)$. Equivalently, an $(\alpha_{1,2},\beta_{1,2})$-CIFSG $A_{(\alpha_{1,2},\beta_{1,2})}$ is called an $(\alpha_{1,2},\beta_{1,2})$-CIFNSG of the group G if $A_{(\alpha_{1,2},\beta_{1,2})}x(y) = xA_{(\alpha_{1,2},\beta_{1,2})}(y)$, for all $x, y \in G$.*

Remark 5.3. *Let $A_{(\alpha_{1,2},\beta_{1,2})}$ be an $(\alpha_{1,2},\beta_{1,2})$-CIFNSG of the group G. Then, $A_{(\alpha_{1,2},\beta_{1,2})}(y^{-1}xy) = A_{(\alpha_{1,2},\beta_{1,2})}(x)$, for all $x, y \in G$.*

Theorem 5.2. *If A is a CIFNSG of the group G, then $A_{(\alpha_{1,2},\beta_{1,2})}$ is an $(\alpha_{1,2},\beta_{1,2})$-CIFNSG of the group G.*

Proof: Suppose x and g are any two elements of the group G. Then, for the membership function, we have

$$\eta_A(g^{-1}x)\cdot e^{i\varphi_A(g^{-1}x)} = \eta_A(xg^{-1})\cdot e^{i\varphi_A(xg^{-1})}.$$

This implies that

$$\eta_A(g^{-1}x)\cdot e^{i\varphi_A(g^{-1}x)} \bullet \alpha_1 \cdot e^{i\beta_1} = \eta_A(xg^{-1})\cdot e^{i\varphi_A(xg^{-1})} \bullet \alpha_1 \cdot e^{i\beta_1},$$

which implies that

$$\eta_{gA_{\alpha_1}}(x) \cdot e^{i\varphi_{gA_{\beta_1}}(x)} = \eta_{A_{\alpha_1}g}(x) \cdot e^{i\varphi_{A_{\beta_1}g}(x)}.$$

Now, for the non-membership function, we have

$$r_A(g^{-1}x) \cdot e^{is_A(g^{-1}x)} = r_A(xg^{-1}) \cdot e^{is_A(xg^{-1})},$$

which implies that

$$r_A(g^{-1}x) \cdot e^{is_A(g^{-1}x)} \bullet \alpha_2 \cdot e^{i\beta_2} = r_A(xg^{-1}) \cdot e^{is_A(xg^{-1})} \bullet \alpha_2 \cdot e^{i\beta_2},$$

which implies that

$$r_{gA_{\alpha_2}}(x) \cdot e^{is_{gA_{\beta_2}}(x)} = r_{A_{\alpha_2}g}(x) \cdot e^{is_{A_{\beta_2}g}(x)},$$

which implies that

$$gA_{(\alpha_{1,2},\beta_{1,2})}(x) = A_{(\alpha_{1,2},\beta_{1,2})}g(x),$$

and therefore, $A_{(\alpha_{1,2},\beta_{1,2})}$ is an $(\alpha_{1,2},\beta_{1,2})$-CIFNSG of the group G.

6. Homomorphism of complex intuitionistic fuzzy subgroups

In this section, we establish a general homomorphism of CIFSGs, and study the relationship between the image and pre-image of CIFSGs and CIFNSGs under this group homomorphism, respectively.

Definition 6.1. [33] *Let $f : H \to G$ be a homomorphism from the group H to the group G. Let A and B be two CIFSGs of two groups H and G, respectively, for all $m \in H$, $n \in G$. The set $f(A)(n) = \{\langle n, f(\mu_A)(n), f(\nu_A)(n)\rangle\}$ is the image of A, where*

$$f(\mu_A)(n) = \begin{cases} \sup\{\mu_A(m), & \textit{if } f(m) = n, f^{-1}(n) \neq \emptyset\}, \\ 0, & \textit{otherwise}, \end{cases}$$

$$f(\nu_A)(n) = \begin{cases} \inf\{\nu_A(m), & \textit{if } f(m) = n, f^{-1}(n) \neq \emptyset\}, \\ 0, & \textit{otherwise}. \end{cases}$$

Then, the set $f^{-1}(B)(m) = \{\langle m, f^{-1}(\mu_B)(m), f^{-1}(\nu_B)(m)\rangle\}$ is called the pre-image of B, where for all $m \in H$, we have

$$f^{-1}(\mu_B)(m) = (\mu_B)(f(m)),$$

$$f^{-1}(\nu_B)(m) = (\nu_B)(f(m)).$$

Remark 6.1. *Let $f : H \to G$ be a homomorphism from the group H to the group G. Let A and B be two CIFSGs of two groups H and G, respectively. Then,*

(1) $f(\mu_A)(n) = f(\eta_A)(n) \cdot e^{if(\varphi_A)(n)}$,

(2) $f(\nu_A)(n) = f(r_A)(n) \cdot e^{if(s_A)(n)}$,

(3) $f^{-1}(\mu_B)(m) = f^{-1}(\eta_B)(m) \cdot e^{if^{-1}(\varphi_B)(m)}$,

(4) $f^{-1}(\nu_B)(m) = f^{-1}(r_B)(m) \cdot e^{if^{-1}(s_B)(m)}$.

Definition 6.2. [32] *Let G be a CIFSG, and then the group formed by a coset of a normal subgroup N of G is called a quotient group, which we denote by the symbol G/N.*

Remark 6.2. *Since the expoent of N is the number of cosets of N, we obviously have that the number of elements of the quotient group G/N is equal to the exponent of N.*

Theorem 6.1. *Let G be a CIFSG, and then a group G is homomorphic to each of its quotient groups G/N.*

Theorem 6.2. *Suppose that we have two CIFSGs G_1 and G_2, and that G_1 and G_2 are homomorphic. Then below this homomorphism surjective f:*

(1) The image $f(A)$ of a subgroup A of G_1 is a subgroup of G_2,

(2) The image $f(B)$ of a normal subgroup B of G_1 is a normal subgroup of G_2.

Proof: (1) For all $m, n \in G_2$, $f(A)(mn) = (\mu_{f(A)}(mn), \nu_{f(A)}(mn))$,

$$\begin{aligned}
\mu_{f(A)}(mn) &= \eta_{f(A)}(mn) \cdot e^{i\varphi_{f(A)}(mn)} \\
&\geq \min\{\eta_{f(A)}(m), \eta_{f(A)}(n)\} \cdot e^{i\min\{\varphi_{f(A)}(m), \varphi_{f(A)}(n)\}} \\
&= \min\{\eta_{f(A)}(m) \cdot e^{i\varphi_{f(A)}(m)}, \eta_{f(A)}(n) \cdot e^{i\varphi_{f(A)}(n)}\} \\
&= \min\{\mu_{f(A)}(m), \mu_{f(A)}(n)\}, \\
\nu_{f(A)}(mn) &= r_{f(A)}(mn) \cdot e^{is_{f(A)}(mn)} \\
&\leq \max\{r_{f(A)}(m), r_{f(A)}(n)\} \cdot e^{i\max\{s_{f(A)}(m), s_{f(A)}(n)\}} \\
&= \max\{r_{f(A)}(m) \cdot e^{is_{f(A)}(m)}, r_{f(A)}(n) \cdot e^{is_{f(A)}(n)}\} \\
&= \max\{\nu_{f(A)}(m), \nu_{f(A)}(n)\}.
\end{aligned}$$

Therefore, $mn \in f(A)$, $f(A)(m^{-1}) = (\mu_{f(A)}(m^{-1}), \nu_{f(A)}(m^{-1}))$,

$$\begin{aligned}
\mu_{f(A)}(m^{-1}) &= \eta_{f(A)}(m^{-1}) \cdot e^{i\varphi_{f(A)}(m^{-1})} \\
&\geq \eta_{f(A)}(m) \cdot e^{i\varphi_{f(A)}(m)} \\
&= \mu_{f(A)}(m), \\
\nu_{f(A)}(m^{-1}) &= r_{f(A)}(m^{-1}) \cdot e^{is_{f(A)}(m^{-1})} \\
&\leq r_{f(A)}(m) \cdot e^{is_{f(A)}(m)} \\
&= \nu_{f(A)}(m),
\end{aligned}$$

and thus, $m^{-1} \in f(A)$. Hence, $f(A)$ is a subgroup of G_2.

(2) It is shown that the image $f(B)$ of a normal subgroup B of G_1 is a normal subgroup of G_2, for all $m \in G_1$, $n \in G_2$, $f : G_1 \to G_2$, and we have

$$f(B)(n) = \{\langle n, f(\mu_B(n)), f(\nu_B(n))\rangle\},$$

$$f(\mu_B)(n) = \begin{cases} \sup\{\mu_B(m), & \text{if } f(m) = n, f^{-1}(n) \neq \emptyset\}, \\ 0, & \text{otherwise,} \end{cases}$$

$$f(\nu_B)(n) = \begin{cases} \inf\{\nu_B(m), & \text{if } f(m) = n, f^{-1}(n) \neq \emptyset\}, \\ 0, & \text{otherwise,} \end{cases}$$

for all $b \in f(B)$, $g \in G_2$.

To prove that $f(B)$ is a normal subgroup of G_2, it is only necessary to prove $gbg^{-1} \in f(B)$, that is to say

$$f(B)(gbg^{-1}) = \{\langle gbg^{-1}, f(\mu_B)(gbg^{-1}), f(\nu_B)(gbg^{-1})\rangle\},$$

$$f(\mu_B)(gbg^{-1}) = \begin{cases} \sup\{\mu_B(gbg^{-1}), & \text{if } f(gbg^{-1}) = n, f^{-1}(n) \neq \emptyset\}, \\ 0, & \text{otherwise}, \end{cases}$$

$$f(\nu_B)(gbg^{-1}) = \begin{cases} \inf\{\nu_B(gbg^{-1}), & \text{if } f(gbg^{-1}) = n, f^{-1}(n) \neq \emptyset\}, \\ 0, & \text{otherwise}. \end{cases}$$

Since the CIFSG B is a normal subgroup of the CIFSG G_1, then

$$\mu_B(gbg^{-1}) = \mu_B(b),$$

$$\nu_B(gbg^{-1}) = \nu_B(b),$$

$$f(\mu_B)(gbg^{-1}) = \begin{cases} \sup\{\mu_B(b), & \text{if } f(b) = n, f^{-1}(n) \neq \emptyset\}, \\ 0, & \text{otherwise}, \end{cases}$$

$$f(\nu_B)(gbg^{-1}) = \begin{cases} \inf\{\nu_B(b), & \text{if } f(b) = n, f^{-1}(n) \neq \emptyset\}, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, $f(\mu_B)(gbg^{-1}) = f(\mu_B)(b)$, $f(\nu_B)(gbg^{-1}) = f(\nu_B)(b)$. Naturally, $f(B)(gbg^{-1}) = f(B)(b)$, and thus, $f(B)$ is a normal subgroup of the CIFSG G_2. The conclusion is tenable.

Theorem 6.3. *Suppose that we have two CIFSGs G_1 and G_2, and that G_1 and G_2 are homomorphic. Then below this homomorphism surjective f:*

(1) The inverse image $f^{-1}(A)$ of a subgroup $f(A)$ of G_2 is a subgroup of G_1,

(2) The inverse image $f^{-1}(B)$ of a normal subgroup $f(B)$ of G_2 is a normal subgroup of G_1.

Proof: (1) For all $x, y \in G_1$, $f^{-1}(A)(xy) = \{\langle xy, \mu_{f^{-1}(A)}(xy), \nu_{f^{-1}(A)}(xy)\rangle\}$,

$$\begin{aligned}
\mu_{f^{-1}(A)}(xy) &= \eta_{f^{-1}(A)}(xy) \cdot e^{i\varphi_{f^{-1}(A)}(xy)} \\
&\geq \min\{\eta_{f^{-1}(A)}(x), \eta_{f^{-1}(A)}(y)\} \cdot e^{i\min\{\varphi_{f^{-1}(A)}(x), \varphi_{f^{-1}(A)}(y)\}} \\
&= \min\{\eta_{f^{-1}(A)}(x) \cdot e^{i\varphi_{f^{-1}(A)}(x)}, \eta_{f^{-1}(A)}(y) \cdot e^{i\varphi_{f^{-1}(A)}(y)}\} \\
&= \min\{\mu_{f^{-1}}(x), \mu_{f^{-1}}(y)\}, \\
\nu_{f^{-1}(A)}(xy) &= r_{f^{-1}(A)}(xy) \cdot e^{is_{f^{-1}(A)}(xy)} \\
&\leq \max\{r_{f^{-1}(A)}(x), r_{f^{-1}(A)}(y)\} \cdot e^{i\max\{s_{f^{-1}(A)}(x), s_{f^{-1}(A)}(y)\}} \\
&= \max\{r_{f^{-1}(A)}(x) \cdot e^{is_{f^{-1}(A)}(x)}, r_{f^{-1}(A)}(y) \cdot e^{is_{f^{-1}(A)}(y)}\} \\
&= \max\{\nu_{f^{-1}}(x), \nu_{f^{-1}}(y)\}.
\end{aligned}$$

Therefore, $xy \in f^{-1}(A)$, $f^{-1}(A)(x^{-1}) = \{\langle x^{-1}, \mu_{f^{-1}(A)}(x^{-1}), \nu_{f^{-1}(A)}(x^{-1})\rangle\}$,

$$\mu_{f^{-1}(A)}(x^{-1}) = \eta_{f^{-1}(A)}(x^{-1}) \cdot e^{i\varphi_{f^{-1}(A)}(x^{-1})}$$

$$\begin{aligned}
&\geq \eta_{f^{-1}(A)}(x) \cdot e^{i\varphi_{f^{-1}(A)}(x)} \\
&= \mu_{f^{-1}(A)}(x), \\
\nu_{f^{-1}(A)}(x^{-1}) &= r_{f^{-1}(A)}(x^{-1}) \cdot e^{is_{f^{-1}(A)}(x^{-1})} \\
&\leq r_{f^{-1}(A)}(x) \cdot e^{is_{f^{-1}(A)}(x)} \\
&= \nu_{f^{-1}(A)}(x),
\end{aligned}$$

and thus, $x^{-1} \in f^{-1}(A)$. Hence, $f^{-1}(A)$ is a subgroup of G_1.

(2) It is shown that the inverse image $f^{-1}(B)$ of a normal subgroup $f(B)$ of G_2 is a normal subgroup of G_1, for all $x \in G_1$, $y \in G_2$, $a \in f^{-1}(B)$, $g \in G_1$, $f : G_1 \to G_2$, and we have

$$f^{-1}(B)(y) = \{\langle y, f^{-1}(\mu_B(y)), f^{-1}(\nu_B(y))\rangle\},$$

$$f^{-1}(\mu_B)(y) = \begin{cases} \sup\{\mu_{f(B)}(x), & \text{if } f^{-1}(y) = x, f(x) \neq \emptyset\}, \\ 0, & \text{otherwise,} \end{cases}$$

$$f^{-1}(\nu_B)(y) = \begin{cases} \inf\{\nu_{f(B)}(x), & \text{if } f^{-1}(y) = x, f(x) \neq \emptyset\}, \\ 0, & \text{otherwise.} \end{cases}$$

To prove that $f^{-1}(B)$ is a normal subgroup of G_1, it is only necessary to prove $gag^{-1} \in f^{-1}(B)$, that is to say

$$f^{-1}(B)(gag^{-1}) = \{\langle gag^{-1}, f^{-1}(\mu_B)(gag^{-1}), f^{-1}(\nu_B)(gag^{-1})\rangle\},$$

$$f^{-1}(\mu_B)(gag^{-1}) = \begin{cases} \sup\{\mu_{f(B)}(gag^{-1}), & \text{if } f^{-1}(gag^{-1}) = x, f(x) \neq \emptyset\}, \\ 0, & \text{otherwise,} \end{cases}$$

$$f^{-1}(\nu_B)(gag^{-1}) = \begin{cases} \inf\{\nu_{f(B)}(gag^{-1}), & \text{if } f^{-1}(gag^{-1}) = x, f(x) \neq \emptyset\}, \\ 0, & \text{otherwise.} \end{cases}$$

Since the CIFSG $f(B)$ is a normal subgroup of the CIFSG G_2, then

$$\mu_{f(B)}(gag^{-1}) = \mu_{f(B)}(a),$$

$$\nu_{f(B)}(gag^{-1}) = \nu_{f(B)}(a),$$

$$f^{-1}(\mu_B)(gag^{-1}) = \begin{cases} \sup\{\mu_{f(B)}(a), & \text{if } f^{-1}(a) = x, f(x) \neq \emptyset\}, \\ 0, & \text{otherwise,} \end{cases}$$

$$f^{-1}(\nu_B)(gag^{-1}) = \begin{cases} \inf\{\nu_{f(B)}(a), & \text{if } f^{-1}(a) = x, f(x) \neq \emptyset\}, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, $f^{-1}(\mu_B)(gag^{-1}) = f^{-1}(\mu_B)(a)$, $f^{-1}(\nu_B)(gag^{-1}) = f^{-1}(\nu_B)(a)$, and thus, $f^{-1}(B)(gag^{-1}) = f^{-1}(B)(a)$. Therefore, $f^{-1}(B)$ is a normal subgroup of the CIFSG G_1. The conclusion is tenable.

Example 6.1. *Suppose we have two groups $G_1 = \{e_1, a_1, b_1, c_1\}$, $G_2 = \{e_2, a_2, b_2\}$, where e_1 and e_2 are the identity elements of G_1 and G_2, respectively, and we define a homomorphism $f : G_1 \to G_2$ as follows:*

$$f(e_1) = e_2, f(a_1) = a_2, f(b_1) = b_2, f(c_1) = e_2.$$

Let A be a subgroup of group G_1, $A = \{\langle m, \mu_A(m), \nu_A(m)\rangle \mid m \in G_1\}$, that satisfies:

$$\mu_A(e_1) = 0.8 \cdot e^{i0.1\pi}, \mu_A(a_1) = 0.6 \cdot e^{i0.2\pi}, \mu_A(b_1) = 0.4 \cdot e^{i0.3\pi}, \mu_A(c_1) = 0.2 \cdot e^{i0.4\pi},$$

$$\nu_A(e_1) = 0.2 \cdot e^{i0.1\pi}, \nu_A(a_1) = 0.2 \cdot e^{i0.1\pi}, \nu_A(b_1) = 0.3 \cdot e^{i0.2\pi}, \nu_A(c_1) = 0.4 \cdot e^{i0.3\pi}.$$

From Definition 6.1, we have that the image of A under f is $f(A)$,

$$f(A)(n) = \{n, f(\mu_A)(n), f(\nu_A)(n) \mid n \in G_2\},$$

and for the elements of G_2, we have

$$f(\mu_A)(e_2) = \sup\{\mu_A(e_1), \mu_A(c_1)\} = \sup\{0.8 \cdot e^{i0.1\pi}, 0.2 \cdot e^{i0.4\pi}\} = 0.8 \cdot e^{i0.1\pi},$$
$$f(\mu_A)(a_2) = \mu_A(a_1) = 0.6 \cdot e^{i0.2\pi},$$
$$f(\mu_A)(b_2) = \mu_A(b_1) = 0.4 \cdot e^{i0.3\pi},$$
$$f(\nu_A)(e_2) = \inf\{\nu_A(e_1), \nu_A(c_1)\} = \inf\{0.1 \cdot e^{i0.1\pi}, 0.4 \cdot e^{i0.3\pi}\} = 0.1 \cdot e^{i0.1\pi},$$
$$f(\nu_A)(a_2) = \nu_A(a_1) = 0.2 \cdot e^{i0.1\pi},$$
$$f(\nu_A)(b_2) = \nu_A(b_1) = 0.3 \cdot e^{i0.2\pi}.$$

Similarly, if we have a CIFSG B of G_2, $B = \{\langle n, \mu_B(n), \nu_B(n)\rangle \mid n \in G_2\}$, then it satisfies:

$$\mu_B(e_2) = 0.7 \cdot e^{i0.2\pi}, \mu_B(a_2) = 0.5 \cdot e^{i0.3\pi}, \mu_B(b_2) = 0.4 \cdot e^{i0.4\pi},$$

$$\nu_B(e_2) = 0.2 \cdot e^{i0.1\pi}, \nu_B(a_2) = 0.3 \cdot e^{i0.2\pi}, \nu_B(b_2) = 0.4 \cdot e^{i0.3\pi}.$$

Then the inverse image of B under f is $f^{-1}(B)$,

$$f^{-1}(B)(m) = \{\langle m, f^{-1}(\mu_B)(m), f^{-1}(\nu_B)(m)\rangle \mid m \in G_1\}.$$

For the elements of G_1, we have

$$f^{-1}(\mu_B)(e_1) = (\mu_B)(f(e_1)) = \mu_B(e_2) = 0.7 \cdot e^{i0.2\pi},$$
$$f^{-1}(\mu_B)(a_1) = (\mu_B)(f(a_1)) = \mu_B(a_2) = 0.5 \cdot e^{i0.3\pi},$$
$$f^{-1}(\mu_B)(b_1) = (\mu_B)(f(b_1)) = \mu_B(b_2) = 0.4 \cdot e^{i0.4\pi},$$
$$f^{-1}(\mu_B)(c_1) = (\mu_B)(f(c_1)) = \mu_B(c_2) = 0.7 \cdot e^{i0.2\pi},$$
$$f^{-1}(\nu_B)(e_1) = (\nu_B)(f(e_1)) = \nu_B(e_2) = 0.2 \cdot e^{i0.1\pi},$$
$$f^{-1}(\nu_B)(a_1) = (\nu_B)(f(a_1)) = \nu_B(a_2) = 0.3 \cdot e^{i0.2\pi},$$
$$f^{-1}(\nu_B)(b_1) = (\nu_B)(f(b_1)) = \nu_B(b_2) = 0.4 \cdot e^{i0.3\pi},$$
$$f^{-1}(\nu_B)(c_1) = (\nu_B)(f(c_1)) = \nu_B(e_2) = 0.2 \cdot e^{i0.1\pi}.$$

For arbitrary $x, y \in G_2$, $f(A)(xy) = (\ \mu_{f(A)}(xy), \nu_{f(A)}(xy)\ )$,
and it is easy to prove by Theorem 6.6,

$$\mu_{f(A)}(xy) \geq \min\{\mu_{f(A)}(x), \mu_{f(A)}(y)\}, \nu_{f(A)}(xy) \leq \max\{\nu_{f(A)}(x), \nu_{f(A)}(y)\},$$

$$\mu_{f(A)}(x^{-1}) \geq \mu_{f(A)}(x), \nu_{f(A)}(x^{-1}) \leq \mu_{f(A)}(x).$$

Then the image of $f(A)$ of subgroup A of G_1 is a subgroup of G_2, and the inverse image of subgroup B of G_2 is also a subgroup of G_1.

7. Conclusions and remarks

In this paper, first of all, we introduce cut-subsets of CIFSGs, and study the relationships between the cut-subsets, CIFSGs, CIFASGs, and CIFCSGs. Second, the concept of cosets of CIFSGs is given. On this basis, the CIFNSGs are described and some algebraic properties of the subgroups are discussed. Again, the concept of $(\alpha_{1,2}, \beta_{1,2})$-CIFSGs is a valuable extension of classical CIFSs. This paper presents a new concept of $(\alpha_{1,2}, \beta_{1,2})$-CIFSGs and proves that an $(\alpha_{1,2}, \beta_{1,2})$-CIFSG is a general form of every CIFSG. Finally, we establish a general homomorphism of CIFSGs, and study the relationships between the image and the pre-image of CIFSGs and CIFNSGs, respectively, under this group homomorphism. In addition, with the help of this newly defined $(\alpha_{1,2}, \beta_{1,2})$-CIFSG, we will continue to discuss the algebraic structure of some CIFSGs related to it. The main limitation of this study is that the new concept of $(\alpha_{1,2}, \beta_{1,2})$-CIFSGs presented in this paper is relatively complex, involving intuitionistic fuzzy sets over complex domains, and complex membership and non-membership functions. This complexity can make it difficult to understand and manipulate in real-world applications, especially in scenarios that require quick decisions or work with large amounts of data. Therefore, this paper mainly supports its views through mathematical proof and theoretical derivation, but lacks experimental verification or practical testing of the theory. Future research can be further extended the theoretical framework of $(\alpha_{1,2}, \beta_{1,2})$-CIFSGs, for example by introducing more parameters or considering other types of uncertainty. We can further explore the application of $(\alpha_{1,2}, \beta_{1,2})$-CIFSGs in specific fields, through the combination with other fields, and can develop more powerful tools and methods to solve more complex practical problems, such as medical diagnosis, financial risk assessment, intelligent manufacturing, etc. Through concrete case analysis, the application effects and applications of the theory in practical problems can be demonstrated. The ultimate goal will be to enhance the utility and applicability of the tool in real-world environments.

Author contributions

Zhuonan Wu: Conceptualization, methodology, writing original draft, editing; Zengtai Gong: Conceptualization, formal analysis, investigation, methodology, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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