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Research article

Existence and stability analysis of solutions for periodic conformable differential systems with non-instantaneous impulses

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Abstract: This paper focuses on analyzing the existence and stability of solutions for periodic conformable systems with non-instantaneous impulses. First, we define the notion of the conformable Cauchy matrix to present solutions and demonstrate fundamental characteristics including periodicity and exponential estimation. Moreover, the effect of the non-instantaneous impulses on the exponential stability is comprehensively analyzed. Next, by applying the constant variation method, we can derive the solution for the linear nonhomogeneous system with non-instantaneous impulses. In addition, the existence of periodic solutions for the given linear nonhomogeneous system is investigated. Further, the conditions required to guarantee the existence and uniqueness of the periodic solutions for nonlinear systems are provided.

Keywords: conformable Cauchy matrix; conformable derivative; exponential stability; periodic solution
Mathematics Subject Classification: 34A37, 34C25

1. Introduction

In the natural sciences and human social activities, impulsive and periodic phenomena are both widespread and of profound significance. Impulse systems, a prevalent mathematical model, are used to depict sudden change phenomena in practical problems across various fields such as economics, population dynamics systems, physics, machinery, and biotechnology. Not only can they fully consider the impact of perturbation or disturbance on the system state, but they can also accurately reflect the characteristics of the system itself. With the deepening of research, researchers have conducted extensive studies on the impulsive effect, such as [1–9].

Periodic theory is an attractive subject in the qualitative theory of differential equations. Applications in fields such as physics, mathematical biology, control theory, and other technical sciences are of great significance. There are many papers concerning the periodic systems, and they have yielded a variety of valuable outcomes, such as [10, 11]. Further, in biomathematics, periodic phenomena and impulsive phenomena often occur simultaneously within one system. Hence, many scholars focus on the periodic differential systems with impulsive effects, such as [12–17]. As an extension and expansion of integer order calculus, a large number of scholars have conducted research on different types of derivatives and obtained many excellent results, such as [18, 19]. Subsequently, following the introduction of the concept of conformable derivatives [20], many results have been published about conformable derivatives, such as [21–32]. Previous research has focused on the existence, uniqueness, and stability of solutions to non-instantaneous impulsive differential equations, as well as the existence and asymptotic stability of periodic solutions. However, the relationship between the solutions and the periodic solutions of periodic conformable differential systems with non-instantaneous impulses has not been investigated. Such research can help us establish a more complete theoretical system and further deepen our understanding of related issues. In this paper, we mainly analyze the existence and stability of solutions for the systems given as below:

$$\begin{cases} \mathfrak{D}_{\kappa}^{\varsigma_l}\gamma(\iota) = \mathcal{B}\gamma(\iota),\ \iota \in (\varsigma_l, \iota_{l+1}],\ l \in \mathbb{N}_0 := \{0, 1, 2, \cdots\},\ 0 < \kappa < 1, \\ \gamma(\iota_l^+) = (\mathcal{I} + C_l)\gamma(\iota_l^-),\ l \in \mathbb{N} := \{1, 2, \cdots\}, \\ \gamma(\iota) = (\mathcal{I} + C_l)\gamma(\iota_l^-),\ \iota \in (\iota_l, \varsigma_l],\ l \in \mathbb{N}, \\ \gamma(\varsigma_l^+) = \gamma(\varsigma_l^-),\ l \in \mathbb{N}, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \mathfrak{D}_{\kappa}^{\varsigma_l}\gamma(\iota) = \mathcal{B}\gamma(\iota) + a(\iota),\ \iota \in (\varsigma_l, \iota_{l+1}],\ l \in \mathbb{N}_0,\ 0 < \kappa < 1, \\ \gamma(\iota_l^+) = (\mathcal{I} + C_l)\gamma(\iota_l^-) + b_l,\ l \in \mathbb{N}, \\ \gamma(\iota) = (\mathcal{I} + C_l)\gamma(\iota_l^-) + b_l,\ \iota \in (\iota_l, \varsigma_l],\ l \in \mathbb{N}, \\ \gamma(\varsigma_l^+) = \gamma(\varsigma_l^-),\ l \in \mathbb{N}, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \mathfrak{D}_{\kappa}^{\varsigma_l}\gamma(\iota) = \mathcal{B}\gamma(\iota) + \mathcal{A}(\iota, \gamma(\iota)),\ \iota \in (\varsigma_l, \iota_{l+1}],\ l \in \mathbb{N}_0,\ 0 < \kappa < 1, \\ \gamma(\iota_l^+) = (\mathcal{I} + C_l)\gamma(\iota_l^-) + b_l,\ l \in \mathbb{N}, \\ \gamma(\iota) = (\mathcal{I} + C_l)\gamma(\iota_l^-) + b_l,\ \iota \in (\iota_l, \varsigma_l],\ l \in \mathbb{N}, \\ \gamma(\varsigma_l^+) = \gamma(\varsigma_l^-),\ l \in \mathbb{N}, \end{cases} \tag{1.3}$$

in which $\mathcal{B} \in \mathbb{R}^{n\times n}$ and $C_l \in \mathbb{R}^{n\times n}$, $a : \bigcup_{l=0}^{\infty}(\varsigma_l, \iota_{l+1}] \to \mathbb{R}^n$ and $\mathcal{A} : \bigcup_{l=0}^{\infty}(\varsigma_l, \iota_{l+1}] \times \mathbb{R}^n \to \mathbb{R}^n$. $\{\iota_l\}_{l\in\mathbb{N}_0}$ and $\{\varsigma_l\}_{l\in\mathbb{N}_0}$ satisfy $Q = \iota_0 = \varsigma_0 < \iota_1 \le \varsigma_1 < \cdots < \iota_l \le \varsigma_l < \iota_{l+1}$ and $\mathcal{B}(\mathcal{I} + C_l) = (\mathcal{I} + C_l)\mathcal{B}$ for $l \in \mathbb{N}$. What's more, $\mathcal{I}$ denotes the unit matrix.

We propose the following assumptions:

(G_1) There are $q \in \mathbb{N}$ and $\delta \in \mathbb{R}_+$ satisfy $\iota_{l+q} = \iota_l + \delta,\ l \in \mathbb{N}$ and $\varsigma_{l+q} = \varsigma_l + \delta,\ l \in \mathbb{N}_0$.

(G_2) For each $l \in \mathbb{N}$, $C_{l+q} = C_l$ and $b_{l+q} = b_l$.

Among the 6 sections of this paper, section 2 introduces the conformable Cauchy matrix $\Xi(\cdot, \cdot)$ and discusses its properties. Section 3 discusses the results concerning the periodicity and stability of the solution for (1.1). Section 4 gives the expression of the solution and sufficient conditions for the existence of a periodic solution for (1.2). Section 5 studies the periodic solution of (1.3). Last, examples are given to verify our results.

2. Preliminaries

Set $\mathbb{Q} = [Q, +\infty)$ and $PC(\mathbb{Q}, \mathbb{R}^n) := \{\gamma : \mathbb{Q} \to \mathbb{R}^n : \gamma \in C((\iota_l, \iota_{l+1}], \mathbb{R}^n),\ l \in \mathbb{N}_0$, there exists $\gamma(\iota_l^-)$ and $\gamma(\iota_l^+),\ l \in \mathbb{N}$ with $\gamma(\iota_l^-) = \gamma(\iota_l)\}$, where $C((\iota_l, \iota_{l+1}], \mathbb{R}^n)$ denotes the space of all continuous functions from $(\iota_l, \iota_{l+1}]$ into $\mathbb{R}^n$ with $\|\gamma\| = \sup_{\iota \in \mathbb{Q}} \|\gamma(\iota)\|$. One denotes $a = (a_1, \cdots, a_n)^\top \in \mathbb{R}^n$ with $\|a\| = \max_{1 \le i \le n} |a_i|$ and $\alpha \in \mathbb{R}^{n \times n}$ with $\|\alpha\| = \max_{1 \le i \le n} \sum_{j=1}^{n} |\alpha_{ij}|$.

To start, one presents the relevant concepts.

Definition 2.1. *(see [20, Definition 2.1]) The conformable derivative of a function $\gamma : \mathbb{Q} \to \mathbb{R}$ is*

$$
\begin{aligned}
\mathfrak{D}_\kappa^Q \gamma(\iota) &= \lim_{\varepsilon \to 0} \frac{\gamma(\iota + \varepsilon(\iota - Q)^{1-\kappa}) - \gamma(\iota)}{\varepsilon},\ \iota \ge Q,\ 0 < \kappa < 1, \\
\mathfrak{D}_\kappa^Q \gamma(Q) &= \lim_{\iota \to Q^+} \mathfrak{D}_\kappa^Q \gamma(\iota).
\end{aligned}
$$

Remark 2.2. *If $\gamma \in C^1(\mathbb{Q}, \mathbb{R})$, then $\mathfrak{D}_\kappa^Q \gamma(\iota) = (\iota - Q)^{1-\kappa} \gamma'(\iota)$.*

Definition 2.3. *(see [20, Notation]) The conformable integral of a function $\gamma : \mathbb{Q} \to \mathbb{R}$ is*

$$
\mathfrak{J}_\kappa^Q \gamma(\iota) = \int_Q^\iota \gamma(\varsigma) d_\kappa(\varsigma, Q) = \int_Q^\iota (\varsigma - Q)^{\kappa-1} \gamma(\varsigma) d\varsigma,\ \iota \ge Q,\ 0 < \kappa < 1,
$$

if $Q = 0$, then we write $d_\kappa(\varsigma, Q)$ as $d_\kappa(\varsigma)$.

In order to derive the solution for (1.1), we present this definition.

Definition 2.4. *Denote the conformable Cauchy matrix $\Xi(\cdot, \cdot)$ as*

$$
\Xi(\iota, \varsigma) = \prod_{l=\phi(Q,\varsigma)+1}^{\phi(Q,\iota)} (\mathcal{I} + C_l) e^{\mathcal{B}\left[\left(\frac{(\iota - \varsigma_{\phi(Q,\iota)})^\kappa}{\kappa}\right)_+ - \left(\frac{(\varsigma - \varsigma_{\phi(Q,\varsigma)})^\kappa}{\kappa}\right)_+ + \sum_{l=\phi(Q,\varsigma)}^{\phi(Q,\iota)-1} \frac{(\iota_{l+1} - \varsigma_l)^\kappa}{\kappa}\right]}, \tag{2.1}
$$

where $\phi(Q, \iota)$ denotes the number of ι_l existing in (Q, ι) and $z_+ := \max\{0, z\}$ for $z \in \mathbb{R}$. If $\phi(Q, \iota) = \phi(Q, \varsigma)$, then $\prod_{l=\phi(Q,\varsigma)+1}^{\phi(Q,\iota)} (\mathcal{I} + C_l) = \mathcal{I}$, $\sum_{l=\phi(Q,\varsigma)}^{\phi(Q,\iota)-1} = 0$.

Theorem 2.5. *The solution $\gamma(\iota, \varsigma; \gamma_\varsigma) \in PC(\mathbb{Q}, \mathbb{R}^n)$ of (1.1) with $\gamma(\varsigma) = \gamma_\varsigma$ is*

$$
\gamma(\iota, \varsigma; \gamma_\varsigma) = \Xi(\iota, \varsigma) \gamma_\varsigma,\ \iota \ge \varsigma \ge Q.
$$

Particularly,

$$
\begin{aligned}
\gamma(\iota; \gamma_Q) &:= \gamma(\iota, Q; \gamma_Q) \\
&= \Xi(\iota, Q) \gamma_Q \\
&= \prod_{l=1}^{\phi(Q,\iota)} (\mathcal{I} + C_l) e^{\mathcal{B}\left[\left(\frac{(\iota - \varsigma_{\phi(Q,\iota)})^\kappa}{\kappa}\right)_+ + \sum_{l=0}^{\phi(Q,\iota)-1} \frac{(\iota_{l+1} - \varsigma_l)^\kappa}{\kappa}\right]} \gamma_Q.
\end{aligned}
$$

Proof. There are many conditions to consider.

Condition 1: $\phi(Q,\iota)=\phi(Q,\varsigma)$.

(i) Set any $\iota,\varsigma\in(\varsigma_l,\iota_{l+1}]$, for $\iota\in[\varsigma_0,\iota_1]$, by Remark 2.2, we obtain

$$\gamma(\iota)=e^{\mathcal{B}\frac{(\iota-\varsigma_0)^\kappa}{\kappa}}\gamma_Q,$$

and when $\iota\in(\iota_1,\varsigma_1]$,

$$\begin{aligned}\gamma(\iota)&=(\mathcal{I}+C_1)\gamma(\iota_1^-)\\&=(\mathcal{I}+C_1)e^{\mathcal{B}\frac{(\iota_1-\varsigma_0)^\kappa}{\kappa}}\gamma_Q.\end{aligned}$$

For $\iota\in(\varsigma_1,\iota_2]$,

$$\begin{aligned}\gamma(\iota)&=e^{\mathcal{B}\frac{(\iota-\varsigma_1)^\kappa}{\kappa}}\gamma(\varsigma_1)\\&=e^{\mathcal{B}\frac{(\iota-\varsigma_1)^\kappa}{\kappa}}(\mathcal{I}+C_1)e^{\mathcal{B}\frac{(\iota_1-\varsigma_0)^\kappa}{\kappa}}\gamma_Q\\&=(\mathcal{I}+C_1)e^{\mathcal{B}\left[\frac{(\iota-\varsigma_1)^\kappa}{\kappa}+\frac{(\iota_1-\varsigma_0)^\kappa}{\kappa}\right]}\gamma_Q,\end{aligned}$$

and

$$\gamma(\varsigma)=(\mathcal{I}+C_1)e^{\mathcal{B}\left[\frac{(\varsigma-\varsigma_1)^\kappa}{\kappa}+\frac{(\iota_1-\varsigma_0)^\kappa}{\kappa}\right]}\gamma_Q.$$

So

$$\gamma(\iota)=e^{\mathcal{B}\left[\frac{(\iota-\varsigma_1)^\kappa}{\kappa}-\frac{(\varsigma-\varsigma_1)^\kappa}{\kappa}\right]}\gamma(\varsigma).$$

Then for a positive integer l, we suppose that the following equalities hold.

When $\iota\in(\varsigma_l,\iota_{l+1}]$,

$$\begin{aligned}\gamma(\iota)&=e^{\mathcal{B}\frac{(\iota-\varsigma_l)^\kappa}{\kappa}}\gamma(\varsigma_l)\\&=(\mathcal{I}+C_l)(\mathcal{I}+C_{l-1})\cdots(\mathcal{I}+C_1)e^{\mathcal{B}\left[\frac{(\iota-\varsigma_l)^\kappa}{\kappa}+\frac{(\iota_l-\varsigma_{l-1})^\kappa}{\kappa}+\cdots+\frac{(\iota_1-\varsigma_0)^\kappa}{\kappa}\right]}\gamma_Q,\end{aligned}$$

and when $\iota\in(\iota_{l+1},\varsigma_{l+1}]$,

$$\begin{aligned}\gamma(\iota)&=(\mathcal{I}+C_{l+1})\gamma(\iota_{l+1}^-)\\&=(\mathcal{I}+C_{l+1})e^{\mathcal{B}\frac{(\iota_{l+1}-\varsigma_l)^\kappa}{\kappa}}\gamma(\varsigma_l)\\&=(\mathcal{I}+C_{l+1})(\mathcal{I}+C_l)\cdots(\mathcal{I}+C_1)e^{\mathcal{B}\left[\frac{(\iota_{l+1}-\varsigma_l)^\kappa}{\kappa}+\frac{(\iota_l-\varsigma_{l-1})^\kappa}{\kappa}+\cdots+\frac{(\iota_1-\varsigma_0)^\kappa}{\kappa}\right]}\gamma_Q.\end{aligned}$$

Thus, for $\iota\in(\varsigma_{l+1},\iota_{l+2}]$,

$$\begin{aligned}\gamma(\iota)&=e^{\mathcal{B}\frac{(\iota-\varsigma_{l+1})^\kappa}{\kappa}}\gamma(\varsigma_{l+1})\\&=(\mathcal{I}+C_{l+1})(\mathcal{I}+C_l)\cdots(\mathcal{I}+C_1)e^{\mathcal{B}\left[\frac{(\iota-\varsigma_{l+1})^\kappa}{\kappa}+\frac{(\iota_{l+1}-\varsigma_l)^\kappa}{\kappa}+\cdots+\frac{(\iota_1-\varsigma_0)^\kappa}{\kappa}\right]}\gamma_Q,\end{aligned}$$

and

$$\gamma(\varsigma)=(\mathcal{I}+C_{l+1})(\mathcal{I}+C_l)\cdots(\mathcal{I}+C_1)e^{\mathcal{B}\left[\frac{(\varsigma-\varsigma_{l+1})^\kappa}{\kappa}+\frac{(\iota_{l+1}-\varsigma_l)^\kappa}{\kappa}+\cdots+\frac{(\iota_1-\varsigma_0)^\kappa}{\kappa}\right]}\gamma_Q.$$

In conclusion,

$$\gamma(\iota) = e^{\mathcal{B}\left[\frac{(\iota-\varsigma_l)^\kappa}{\kappa} - \frac{(\varsigma-\varsigma_l)^\kappa}{\kappa}\right]}\gamma(\varsigma).$$

(ii) Set any $\iota, \varsigma \in (\iota_l, \varsigma_l]$, by

$$\gamma(\iota) = (\mathcal{I} + C_l)\gamma(\iota_l^-),$$

we know $\gamma(\iota) = \gamma(\varsigma)$.

(iii) Set any $\varsigma \in (\iota_{\phi(Q,\varsigma)}, \varsigma_{\phi(Q,\varsigma)}]$ and any $\iota \in (\varsigma_{\phi(Q,\iota)}, \iota_{\phi(Q,\iota)+1}]$,

$$\gamma(\iota) = e^{\mathcal{B}\frac{(\iota-\varsigma_{\phi(Q,\iota)})^\kappa}{\kappa}}\gamma(\varsigma_{\phi(Q,\iota)}),$$

then

$$\gamma(\iota) = e^{\mathcal{B}\frac{(\iota-\varsigma_{\phi(Q,\iota)})^\kappa}{\kappa}}\gamma(\varsigma).$$

Condition 2: $\phi(Q, \iota) = \phi(Q, \varsigma) + 1$.

(i) Set any $\varsigma \in (\varsigma_{\phi(Q,\varsigma)}, \iota_{\phi(Q,\varsigma)+1}]$ and any $\iota \in (\iota_{\phi(Q,\iota)}, \varsigma_{\phi(Q,\iota)}]$,

$$\begin{aligned}\gamma(\iota) &= (\mathcal{I} + C_{\phi(Q,\iota)})\gamma(\iota_{\phi(Q,\iota)}^-)\\ &= (\mathcal{I} + C_{\phi(Q,\iota)})e^{\mathcal{B}\left[\frac{(\iota_{\phi(Q,\iota)}-\varsigma_{\phi(Q,\varsigma)})^\kappa}{\kappa} - \frac{(\varsigma-\varsigma_{\phi(Q,\varsigma)})^\kappa}{\kappa}\right]}\gamma(\varsigma).\end{aligned}$$

(ii) Set any $\varsigma \in (\varsigma_{\phi(Q,\varsigma)}, \iota_{\phi(Q,\varsigma)+1}]$ and any $\iota \in (\varsigma_{\phi(Q,\iota)}, \iota_{\phi(Q,\iota)+1}]$,

$$\begin{aligned}\gamma(\iota) &= e^{\mathcal{B}\frac{(\iota-\varsigma_{\phi(Q,\iota)})^\kappa}{\kappa}}\gamma(\varsigma_{\phi(Q,\iota)})\\ &= e^{\mathcal{B}\frac{(\iota-\varsigma_{\phi(Q,\iota)})^\kappa}{\kappa}}(\mathcal{I} + C_{\phi(Q,\iota)})\gamma(\iota_{\phi(Q,\iota)}^-)\\ &= (\mathcal{I} + C_{\phi(Q,\iota)})e^{\mathcal{B}\left[\frac{(\iota-\varsigma_{\phi(Q,\iota)})^\kappa}{\kappa} - \frac{(\varsigma-\varsigma_{\phi(Q,\varsigma)})^\kappa}{\kappa} + \frac{(\iota_{\phi(Q,\iota)}-\varsigma_{\phi(Q,\varsigma)})^\kappa}{\kappa}\right]}\gamma(\varsigma).\end{aligned}$$

(iii) Set any $\varsigma \in (\iota_{\phi(Q,\varsigma)}, \varsigma_{\phi(Q,\varsigma)}]$ and any $\iota \in (\iota_{\phi(Q,\iota)}, \varsigma_{\phi(Q,\iota)}]$,

$$\begin{aligned}\gamma(\iota) &= (\mathcal{I} + C_{\phi(Q,\iota)})\gamma(\iota_{\phi(Q,\iota)}^-)\\ &= (\mathcal{I} + C_{\phi(Q,\iota)})e^{\mathcal{B}\frac{(\iota_{\phi(Q,\iota)}-\varsigma_{\phi(Q,\varsigma)})^\kappa}{\kappa}}\gamma(\varsigma_{\phi(Q,\varsigma)})\\ &= (\mathcal{I} + C_{\phi(Q,\iota)})e^{\mathcal{B}\frac{(\iota_{\phi(Q,\iota)}-\varsigma_{\phi(Q,\varsigma)})^\kappa}{\kappa}}\gamma(\varsigma).\end{aligned}$$

(iv) Set any $\varsigma \in (\iota_{\phi(Q,\varsigma)}, \varsigma_{\phi(Q,\varsigma)}]$ and any $\iota \in (\varsigma_{\phi(Q,\iota)}, \iota_{\phi(Q,\iota)+1}]$,

$$\begin{aligned}\gamma(\iota) &= e^{\mathcal{B}\frac{(\iota-\varsigma_{\phi(Q,\iota)})^\kappa}{\kappa}}\gamma(\varsigma_{\phi(Q,\iota)})\\ &= e^{\mathcal{B}\frac{(\iota-\varsigma_{\phi(Q,\iota)})^\kappa}{\kappa}}(\mathcal{I} + C_{\phi(Q,\iota)})\gamma(\iota_{\phi(Q,\iota)}^-)\\ &= e^{\mathcal{B}\frac{(\iota-\varsigma_{\phi(Q,\iota)})^\kappa}{\kappa}}(\mathcal{I} + C_{\phi(Q,\iota)})e^{\mathcal{B}\frac{(\iota_{\phi(Q,\iota)}-\varsigma_{\phi(Q,\varsigma)})^\kappa}{\kappa}}\gamma(\varsigma_{\phi(Q,\varsigma)})\\ &= e^{\mathcal{B}\frac{(\iota-\varsigma_{\phi(Q,\iota)})^\kappa}{\kappa}}(\mathcal{I} + C_{\phi(Q,\iota)})e^{\mathcal{B}\frac{(\iota_{\phi(Q,\iota)}-\varsigma_{\phi(Q,\varsigma)})^\kappa}{\kappa}}\gamma(\varsigma)\end{aligned}$$

$$= \quad (\mathcal{I} + C_{\phi(Q,\iota)})e^{\mathcal{B}\left[\frac{(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa}+\frac{(\iota_{\phi(Q,\iota)}-\varsigma_{\phi(Q,\varsigma)})^{\kappa}}{\kappa}\right]}\gamma(\varsigma).$$

Condition 3: general $\phi(Q,\iota)$ and $\phi(Q,\varsigma)$.
(i) Set any $\varsigma \in (\varsigma_{\phi(Q,\varsigma)}, \iota_{\phi(Q,\varsigma)+1}]$ and any $\iota \in (\iota_{\phi(Q,\iota)}, \varsigma_{\phi(Q,\iota)}]$,

$$\gamma(\iota) = (\mathcal{I} + C_{\phi(Q,\iota)})\cdots(\mathcal{I} + C_{\phi(Q,\varsigma)+1}) \times e^{\mathcal{B}\left[\frac{(\iota_{\phi(Q,\iota)}-\varsigma_{\phi(Q,\iota)-1})^{\kappa}}{\kappa}+\cdots+\frac{(\iota_{\phi(Q,\varsigma)+1}-\varsigma_{\phi(Q,\varsigma)})^{\kappa}}{\kappa}-\frac{(\varsigma-\varsigma_{\phi(Q,\varsigma)})^{\kappa}}{\kappa}\right]}\gamma(\varsigma).$$

(ii) Set any $\varsigma \in (\varsigma_{\phi(Q,\varsigma)}, \iota_{\phi(Q,\varsigma)+1}]$ and any $\iota \in (\varsigma_{\phi(Q,\iota)}, \iota_{\phi(Q,\iota)+1}]$,

$$\gamma(\iota) = (\mathcal{I} + C_{\phi(Q,\iota)})\cdots(\mathcal{I} + C_{\phi(Q,\varsigma)+1}) \times e^{\mathcal{B}\left[\frac{(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa}+\frac{(\iota_{\phi(Q,\iota)}-\varsigma_{\phi(Q,\iota)-1})^{\kappa}}{\kappa}+\cdots+\frac{(\iota_{\phi(Q,\varsigma)+1}-\varsigma_{\phi(Q,\varsigma)})^{\kappa}}{\kappa}-\frac{(\varsigma-\varsigma_{\phi(Q,\varsigma)})^{\kappa}}{\kappa}\right]}\gamma(\varsigma).$$

(iii) Set any $\varsigma \in (\iota_{\phi(Q,\varsigma)}, \varsigma_{\phi(Q,\varsigma)}]$ and any $\iota \in (\iota_{\phi(Q,\iota)}, \varsigma_{\phi(Q,\iota)}]$,

$$\gamma(\iota) = (\mathcal{I} + C_{\phi(Q,\iota)})\cdots(\mathcal{I} + C_{\phi(Q,\varsigma)+1}) \times e^{\mathcal{B}\left[\frac{(\iota_{\phi(Q,\iota)}-\varsigma_{\phi(Q,\iota)-1})^{\kappa}}{\kappa}+\cdots+\frac{(\iota_{\phi(Q,\varsigma)+1}-\varsigma_{\phi(Q,\varsigma)})^{\kappa}}{\kappa}\right]}\gamma(\varsigma).$$

(iv) Set any $\varsigma \in (\iota_{\phi(Q,\varsigma)}, \varsigma_{\phi(Q,\varsigma)}]$ and any $\iota \in (\varsigma_{\phi(Q,\iota)}, \iota_{\phi(Q,\iota)+1}]$,

$$\gamma(\iota) = (\mathcal{I} + C_{\phi(Q,\iota)})\cdots(\mathcal{I} + C_{\phi(Q,\varsigma)+1}) \times e^{\mathcal{B}\left[\frac{(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa}+\frac{(\iota_{\phi(Q,\iota)}-\varsigma_{\phi(Q,\iota)-1})^{\kappa}}{\kappa}+\cdots+\frac{(\iota_{\phi(Q,\varsigma)+1}-\varsigma_{\phi(Q,\varsigma)})^{\kappa}}{\kappa}\right]}\gamma(\varsigma).$$

To sum up,

$$\gamma(\iota) \quad = \quad \prod_{l=\phi(Q,\varsigma)+1}^{\phi(Q,\iota)} (\mathcal{I} + C_l)e^{\mathcal{B}\left[\left(\frac{(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa}\right)_+ - \left(\frac{(\varsigma-\varsigma_{\phi(Q,\varsigma)})^{\kappa}}{\kappa}\right)_+ + \sum_{l=\phi(Q,\varsigma)}^{\phi(Q,\iota)-1}\frac{(\iota_{l+1}-\varsigma_l)^{\kappa}}{\kappa}\right]}\gamma(\varsigma). \tag{2.2}$$

Let

$$\Xi(\iota,\varsigma) = \prod_{l=\phi(Q,\varsigma)+1}^{\phi(Q,\iota)} (\mathcal{I} + C_l)e^{\mathcal{B}\left[\left(\frac{(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa}\right)_+ - \left(\frac{(\varsigma-\varsigma_{\phi(Q,\varsigma)})^{\kappa}}{\kappa}\right)_+ + \sum_{l=\phi(Q,\varsigma)}^{\phi(Q,\iota)-1}\frac{(\iota_{l+1}-\varsigma_l)^{\kappa}}{\kappa}\right]},$$

then (2.2) can be written as

$$\gamma(\iota,\varsigma;\gamma_\varsigma) = \Xi(\iota,\varsigma)\gamma_\varsigma.$$

Particularly, when $\varsigma = Q$,

$$\gamma(\iota,Q;\gamma_Q) = \Xi(\iota,Q)\gamma_Q,\ \iota \geq Q,$$

and

$$\Xi(\iota,Q) = \prod_{l=1}^{\phi(Q,\iota)} (\mathcal{I} + C_l)e^{\mathcal{B}\left[\left(\frac{(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa}\right)_+ + \sum_{l=0}^{\phi(Q,\iota)-1}\frac{(\iota_{l+1}-\varsigma_l)^{\kappa}}{\kappa}\right]}.$$

□

Following this, we introduce the definitions and lemma that are used in this work.

Definition 2.6. *If* $\gamma(\iota) = \gamma(\iota+\delta),\ \iota \geq Q$, *then* $\gamma(\cdot)$ *is* δ*-periodic.*

Definition 2.7. *If there are constants $L \geq 1$ and $v > 0$ satisfying*

$$\|\Xi(\iota, \varsigma)\| \leq Le^{-v(\iota-\varsigma)},\ Q \leq \varsigma \leq \iota,$$

then system (1.1) is exponentially stable.

Lemma 2.8. *(see [33]) Let $\mathcal{B} \in \mathbb{R}^{n\times n}$ and $\varphi(\mathcal{B}) = \max\{\Re\ell | \ell \in \sigma(\mathcal{B})\}$. Then for any $\theta > 0$, there is $L_\theta \geq 1$ such that*

$$\|e^{\mathcal{B}\iota}\| \leq L_\theta e^{(\varphi(\mathcal{B})+\theta)\iota},$$

for any $\iota \geq 0$. Here $\sigma(\mathcal{B})$ is the spectrum of $\mathcal{B}$.

Lemma 2.9. *(Gronwall inequality, see [34]) Set $x(\cdot)$, $f(\cdot)$ as the nonnegative continuous function on $[t_0, \infty)$. If*

$$x(t) \leqslant x_0 + \int_{t_0}^{t} f(s)x(s)ds,\ x_0 \geq 0,\ t \geqslant t_0,$$

then

$$x(t) \leqslant x_0 e^{\int_{t_0}^{t} f(s)ds},\ t \geqslant t_0.$$

Next, we present the properties of $\Xi(\iota, \varsigma)$.

For the following results, we assume

$$\varrho = \sup_{l\in\mathbb{N}} \|\mathcal{I} + C_l\| < \infty,\ \lambda_1 = \inf_{l\in\mathbb{N}_0} \frac{(\iota_{l+1} - \varsigma_l)^\kappa}{\kappa} < \infty,\ \lambda_2 = \sup_{l\in\mathbb{N}} \frac{(\iota_{l+1} - \varsigma_l)^\kappa}{\kappa} < \infty,$$

and let

$$\lambda = \begin{cases} \lambda_1,\ \varphi(\mathcal{B}) + \theta < 0, \\ \lambda_2,\ \varphi(\mathcal{B}) + \theta \geq 0. \end{cases}$$

Theorem 2.10. *When $Q \leq \varsigma \leq \iota$, there are*

$$\|\Xi(\iota, \varsigma)\| \leq L_\theta e^{(\varphi(\mathcal{B})+\theta)\lambda} e^{\phi(\varsigma,\iota)(\ln\varrho + (\varphi(\mathcal{B})+\theta)\lambda)},$$

or

$$\|\Xi(\iota, \varsigma)\| \leq L_\theta e^{\phi(\varsigma,\iota)(\ln\varrho + (\varphi(\mathcal{B})+\theta)\lambda)}.$$

Proof. For any $\theta > 0$, with $\varsigma \in (\varsigma_{\phi(Q,\varsigma)},\ \iota_{\phi(Q,\varsigma)+1}]$, $\iota \in (\varsigma_{\phi(Q,\iota)}, \iota_{\phi(Q,\iota)+1}]$, by taking the norm of (2.1), we can get

$$\begin{aligned} \|\Xi(\iota, \varsigma)\| &= \|\prod_{l=\phi(Q,\varsigma)+1}^{\phi(Q,\iota)} (\mathcal{I} + C_l) e^{\mathcal{B}\left[\frac{(\iota-\varsigma_{\phi(Q,\iota)})^\kappa}{\kappa} - \frac{(\varsigma-\varsigma_{\phi(Q,\varsigma)})^\kappa}{\kappa} + \sum_{l=\phi(Q,\varsigma)}^{\phi(Q,\iota)-1} \frac{(\iota_{l+1}-\varsigma_l)^\kappa}{\kappa}\right]}\| \\ &\leq e^{\phi(\varsigma,\iota)\ln\varrho} L_\theta e^{(\varphi(\mathcal{B})+\theta)\left[\frac{(\iota-\varsigma_{\phi(Q,\iota)})^\kappa}{\kappa} + \phi(\varsigma,\iota)\lambda\right]} \leq L_\theta e^{(\varphi(\mathcal{B})+\theta)\lambda} e^{\phi(\varsigma,\iota)(\ln\varrho + (\varphi(\mathcal{B})+\theta)\lambda)}. \end{aligned}$$

With $\varsigma \in (\varsigma_{\phi(Q,\varsigma)}, \iota_{\phi(Q,\varsigma)+1}]$, $\iota \in (\iota_{\phi(Q,\iota)}, \varsigma_{\phi(Q,\iota)}]$, there is

$$\begin{aligned} \|\Xi(\iota,\varsigma)\| &= \|\prod_{l=\phi(Q,\varsigma)+1}^{\phi(Q,\iota)} (\mathcal{I}+C_l) e^{\mathcal{B}\left[-\frac{(\varsigma-\varsigma_{\phi(Q,\varsigma)})^\kappa}{\kappa}+\sum_{l=\phi(Q,\varsigma)}^{\phi(Q,\iota)-1}\frac{(\iota_{l+1}-\varsigma_l)^\kappa}{\kappa}\right]}\| \\ &\le e^{\phi(\varsigma,\iota)\ln\varrho} L_\theta e^{(\varphi(\mathcal{B})+\theta)\phi(\varsigma,\iota)\lambda} \\ &\le L_\theta e^{\phi(\varsigma,\iota)(\ln\varrho+(\varphi(\mathcal{B})+\theta)\lambda)}. \end{aligned}$$

□

Theorem 2.11. *Suppose that* (G_1) *and* (G_2) *hold, we have*

$$\Xi(\iota,\varsigma) = \Xi(\iota,\eta)\Xi(\eta,\varsigma),\ Q \le \varsigma < \eta < \iota.$$

Proof. By the form of (2.1), we have

$$\begin{aligned} \Xi(\iota,\eta)\Xi(\eta,\varsigma) &= e^{\mathcal{B}\left[\left(\frac{(\iota-\varsigma_{\phi(Q,\iota)})^\kappa}{\kappa}\right)_+-\left(\frac{(\eta-\varsigma_{\phi(Q,\eta)})^\kappa}{\kappa}\right)_++\sum_{l=\phi(Q,\eta)}^{\phi(Q,\iota)-1}\frac{(\iota_{l+1}-\varsigma_l)^\kappa}{\kappa}\right]} \prod_{l=\phi(Q,\eta)+1}^{\phi(Q,\iota)} (\mathcal{I}+C_l) \\ &\times \prod_{l=\phi(Q,\varsigma)+1}^{\phi(Q,\eta)} (\mathcal{I}+C_l) e^{\mathcal{B}\left[\left(\frac{(\eta-\varsigma_{\phi(Q,\eta)})^\kappa}{\kappa}\right)_+-\left(\frac{(\varsigma-\varsigma_{\phi(Q,\varsigma)})^\kappa}{\kappa}\right)_++\sum_{l=\phi(Q,\varsigma)}^{\phi(Q,\eta)-1}\frac{(\iota_{l+1}-\varsigma_l)^\kappa}{\kappa}\right]} \\ &= \Xi(\iota,\varsigma). \end{aligned}$$

□

Theorem 2.12. *If* (G_1) *and* (G_2) *hold, we can obtain*

$$\Xi(\iota+\delta,\varsigma+\delta) = \Xi(\iota,\varsigma),\ Q < \varsigma < \iota.$$

Proof. By using (2.1), there is

$$\begin{aligned} \Xi(\iota+\delta,\varsigma+\delta) &= \prod_{l=\phi(Q,\varsigma+\delta)+1}^{\phi(Q,\iota+\delta)} (\mathcal{I}+C_l) e^{\mathcal{B}\left[\left(\frac{(\iota+\delta-\varsigma_{\phi(Q,\iota+\delta)})^\kappa}{\kappa}\right)_+-\left(\frac{(\varsigma+\delta-\varsigma_{\phi(Q,\varsigma+\delta)})^\kappa}{\kappa}\right)_++\sum_{l=\phi(Q,\varsigma+\delta)}^{\phi(Q,\iota+\delta)-1}\frac{(\iota_{l+1}-\varsigma_l)^\kappa}{\kappa}\right]} \\ &= \prod_{l=\phi(Q,\varsigma)+q+1}^{\phi(Q,\iota)+q} (\mathcal{I}+C_l) e^{\mathcal{B}\left[\left(\frac{(\iota+\delta-(\varsigma_{\phi(Q,\iota)}+\delta))^\kappa}{\kappa}\right)_+-\left(\frac{(\varsigma+\delta-(\varsigma_{\phi(Q,\varsigma)}+\delta))^\kappa}{\kappa}\right)_++\sum_{l=\phi(Q,\varsigma)+q}^{\phi(Q,\iota)+q-1}\frac{(\iota_{l+1}-\varsigma_l)^\kappa}{\kappa}\right]} \\ &= \prod_{l=\phi(Q,\varsigma)+1}^{\phi(Q,\iota)} (\mathcal{I}+C_l) e^{\mathcal{B}\left[\left(\frac{(\iota-\varsigma_{\phi(Q,\iota)})^\kappa}{\kappa}\right)_+-\left(\frac{(\varsigma-\varsigma_{\phi(Q,\varsigma)})^\kappa}{\kappa}\right)_++\sum_{l=\phi(Q,\varsigma)}^{\phi(Q,\iota)-1}\frac{(\iota_{l+1}-\varsigma_l)^\kappa}{\kappa}\right]} \\ &= \Xi(\iota,\varsigma). \end{aligned}$$

□

Theorem 2.13. *Let* (G_1) *and* (G_2) *be satisfied, for* $N \in \mathbb{N}$, *then*

$$\Xi(\iota+N\delta,Q) = \Xi(\iota,Q)[\Xi(Q+\delta,Q)]^N.$$

Proof. Theorems 2.11 and 2.12 obtain

$$\begin{aligned}
\Xi(\iota + N\delta, Q) &= \prod_{l=1}^{\phi(Q,\iota)+Nq} (\mathcal{I} + C_l) e^{\mathcal{B}\left[\left(\frac{(\iota+N\delta-\varsigma_{\phi(Q,\iota+N\delta)})^{\kappa}}{\kappa}\right)_{+}+\sum_{l=0}^{\phi(Q,\iota)+Nq-1} \frac{(\iota_{l+1}-\varsigma_l)^{\kappa}}{\kappa}\right]} \\
&= \prod_{l=1}^{\phi(Q,\iota)+Nq} (\mathcal{I} + C_l) e^{\mathcal{B}\left[\left(\frac{(\iota+N\delta-(\varsigma_{\phi(Q,\iota)}+N\delta))^{\kappa}}{\kappa}\right)_{+}+\sum_{l=0}^{\phi(Q,\iota)+Nq-1} \frac{(\iota_{l+1}-\varsigma_l)^{\kappa}}{\kappa}\right]} \\
&= \prod_{l=Nq+1}^{\phi(Q,\iota)+Nq} (\mathcal{I} + C_l) e^{\mathcal{B}\left[\left(\frac{(\iota+N\delta-(\varsigma_{\phi(Q,\iota)}+N\delta))^{\kappa}}{\kappa}\right)_{+}+\sum_{l=Nq}^{\phi(Q,\iota)+Nq-1} \frac{(\iota_{l+1}-\varsigma_l)^{\kappa}}{\kappa}\right]} \times \prod_{l=1}^{Nq} (\mathcal{I} + C_l) e^{\mathcal{B}\sum_{l=0}^{Nq-1} \frac{(\iota_{l+1}-\varsigma_l)^{\kappa}}{\kappa}} \\
&= \prod_{l=1}^{\phi(Q,\iota)} (\mathcal{I} + C_l) e^{\mathcal{B}\left[\left(\frac{(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa}\right)_{+}+\sum_{l=0}^{\phi(Q,\iota)-1} \frac{(\iota_{l+1}-\varsigma_l)^{\kappa}}{\kappa}\right]} \times \left[\prod_{l=1}^{q} (\mathcal{I} + C_l) e^{\mathcal{B}\sum_{l=0}^{q-1} \frac{(\iota_{l+1}-\varsigma_l)^{\kappa}}{\kappa}}\right]^N \\
&= \Xi(\iota, Q)[\Xi(Q + \delta, Q)]^N.
\end{aligned}$$

□

3. Linear homogeneous problem

The focus of this section is on the linear homogeneous problem.

Theorem 3.1. *Let (G_1) and (G_2) be satisfied; one of the results follows:*
(i) (1.1) has the unique trivial $\delta-$periodic solution iff $rank(\mathcal{I} - \Xi(Q + \delta, Q)) = n$.
(ii) (1.1) has at least one nontrivial $\delta-$periodic solution iff $rank(\mathcal{I} - \Xi(Q + \delta, Q)) < n$.

Proof. Sufficiency: (i) If $rank(\mathcal{I} - \Xi(Q + \delta, Q)) = n$, then $(\mathcal{I} - \Xi(Q + \delta, Q))\gamma = 0$ only has the zero solution, thus the solution of (1.1) are trivial.

(ii) If $rank(\mathcal{I} - \Xi(Q + \delta, Q)) < n$, then there exist nonzero solutions of $(\mathcal{I} - \Xi(Q + \delta, Q))\gamma = 0$ and (1.1) has nontrivial $\delta-$periodic solutions.

Next, we prove the necessity via the method of proof by contradiction:

(i) If $rank(\mathcal{I} - \Xi(Q+\delta, Q)) < n$, then (1.1) has nontrivial $\delta-$periodic solutions, which is contradictory to the given condition. Thus, $rank(\mathcal{I} - \Xi(Q + \delta, Q)) = n$.

(ii) If $rank(\mathcal{I} - \Xi(Q + \delta, Q)) = n$, then (1.1) has the unique trivial $\delta-$periodic solution, which contradicts the fact that (1.1) has at least one nontrivial $\delta-$periodic solution. Thus, $rank(\mathcal{I} - \Xi(Q + \delta, Q)) < n$. □

Before the discussion about the stability, we introduce this relationship.

Theorem 3.2. *Let (G_1) be satisfied; one has*

$$\lim_{\iota-\varsigma\to\infty} \frac{\phi(\varsigma, \iota)}{\iota - \varsigma} = \frac{q}{\delta}.$$

Proof. Using (G_1), with $\varsigma \in [\mathcal{M}\delta, (\mathcal{M} + 1)\delta]$ and $\iota \in [\mathcal{N}\delta, (\mathcal{N} + 1)\delta]$ where $\mathcal{M} \le \mathcal{N}$, there are

$$(\mathcal{N} - \mathcal{M} - 1)\delta \le \iota - \varsigma \le (\mathcal{N} + 1 - \mathcal{M})\delta,$$

and

$$(\mathcal{N}-\mathcal{M}-1)q \leq \phi(\varsigma,\iota) \leq (\mathcal{N}+1-\mathcal{M})q.$$

Hence,

$$\frac{(\mathcal{N}-\mathcal{M}-1)q}{(\mathcal{N}+1-\mathcal{M})\delta} \leq \frac{\phi(\varsigma,\iota)}{\iota-\varsigma} \leq \frac{(\mathcal{N}+1-\mathcal{M})q}{(\mathcal{N}-\mathcal{M}-1)\delta}.$$

It obviously holds that $\iota-\varsigma \to \infty$ iff $\mathcal{N}-\mathcal{M} \to \infty$. So,

$$\frac{q}{\delta} \leq \lim_{\iota-\varsigma\to\infty} \frac{\phi(\varsigma,\iota)}{\iota-\varsigma} \leq \frac{q}{\delta},$$

and

$$\lim_{\iota-\varsigma\to\infty} \frac{\phi(\varsigma,\iota)}{\iota-\varsigma} = \frac{q}{\delta}.$$

□

Next, we consider the stability of (1.1).

Theorem 3.3. *Suppose that* (G_1) *and* (G_2) *are satisfied. If* $\ln\varrho + (\varphi(\mathcal{B})+\theta)\lambda < 0$*, then system (1.1) is exponentially stable.*

Proof. Theorem 3.2 implies that for an arbitrarily small $\varepsilon > 0$, one has

$$\left|\frac{\phi(\varsigma,\iota)}{\iota-\varsigma} - \frac{q}{\delta}\right| < \varepsilon,$$

and

$$(\frac{q}{\delta}-\varepsilon)(\iota-\varsigma) \leq \phi(\varsigma,\iota) \leq (\frac{q}{\delta}+\varepsilon)(\iota-\varsigma).$$

For $\iota \in (\varsigma_{\phi(Q,\iota)}, \iota_{\phi(Q,\iota)+1}]$, with any $\varepsilon \in (0, \frac{q}{\delta})$, Theorem 2.10 implies

$$\begin{aligned} \|\Xi(\iota,Q)\| &\leq L_\theta e^{(\varphi(\mathcal{B})+\theta)\lambda} e^{\phi(Q,\iota)(\ln\varrho+(\varphi(\mathcal{B})+\theta)\lambda)} \\ &\leq L_\theta e^{(\varphi(\mathcal{B})+\theta)\lambda} e^{(\frac{q}{\delta}-\varepsilon)(\ln\varrho+(\varphi(\mathcal{B})+\theta)\lambda)(\iota-Q)} \\ &\leq L_\theta e^{|\varphi(\mathcal{B})+\theta|\lambda} e^{-(\varepsilon-\frac{q}{\delta})(\ln\varrho+(\varphi(\mathcal{B})+\theta)\lambda)(\iota-Q)}, \end{aligned}$$

in which $L = L_\theta e^{|\varphi(\mathcal{B})+\theta|\lambda} \geq 1$ and $v = (\varepsilon - \frac{q}{\delta})(\ln\varrho + (\varphi(\mathcal{B})+\theta)\lambda) > 0$.

For $\iota \in (\iota_{\phi(Q,\iota)}, \varsigma_{\phi(Q,\iota)}]$, one has

$$\begin{aligned} \|\Xi(\iota,Q)\| &\leq L_\theta e^{\phi(Q,\iota)(\ln\varrho+(\varphi(\mathcal{B})+\theta)\lambda)} \\ &\leq L_\theta e^{(\frac{q}{\delta}-\varepsilon)(\ln\varrho+(\varphi(\mathcal{B})+\theta)\lambda)(\iota-Q)} \\ &\leq L_\theta e^{-(\varepsilon-\frac{q}{\delta})(\ln\varrho+(\varphi(\mathcal{B})+\theta)\lambda)(\iota-Q)}, \end{aligned}$$

in which $L = L_\theta$ and $v = (\varepsilon - \frac{q}{\delta})(\ln\varrho + (\varphi(\mathcal{B})+\theta)\lambda) > 0$.

Hence, (1.1) is exponentially stable.

□

Theorem 3.4. *Suppose that* (G_1) *and* (G_2) *are satisfied. If there is a* $\zeta > 0$ *such that*

$$\varphi(\mathcal{B}) + \theta + \frac{1}{\overline{\lambda}} \ln \varrho \leq -\zeta < 0, \tag{3.1}$$

where

$$\overline{\lambda} = \begin{cases} \lambda_1, \ \zeta + \varphi(\mathcal{B}) + \theta < 0, \\ \lambda_2, \ \zeta + \varphi(\mathcal{B}) + \theta \geq 0, \end{cases}$$

then system (1.1) is exponentially stable.

Proof. It is clear that

$$\Big(\frac{(\iota - \varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa}\Big)_{+} + \sum_{l=0}^{\phi(Q,\iota)-1} \frac{(\iota_{l+1} - \varsigma_l)^{\kappa}}{\kappa} \geq (\phi(Q,\iota) - 1)\lambda_1, \tag{3.2}$$

$$\Big(\frac{(\iota - \varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa}\Big)_{+} + \sum_{l=0}^{\phi(Q,\iota)-1} \frac{(\iota_{l+1} - \varsigma_l)^{\kappa}}{\kappa} \leq (\phi(Q,\iota) + 1)\lambda_2. \tag{3.3}$$

Combining (3.2) with (3.3), we obtain

$$(\phi(Q,\iota) - 1)\lambda_1 \leq \Big(\frac{(\iota - \varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa}\Big)_{+} + \sum_{l=0}^{\phi(Q,\iota)-1} \frac{(\iota_{l+1} - \varsigma_l)^{\kappa}}{\kappa} \leq (\phi(Q,\iota) + 1)\lambda_2,$$

and

$$\begin{aligned} & \frac{1}{\lambda_2}\Big[\Big(\frac{(\iota - \varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa}\Big)_{+} + \sum_{l=0}^{\phi(Q,\iota)-1} \frac{(\iota_{l+1} - \varsigma_l)^{\kappa}}{\kappa}\Big] - 1 \leq \phi(Q,\iota) \\ \leq \ & \frac{1}{\lambda_1}\Big[\Big(\frac{(\iota - \varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa}\Big)_{+} + \sum_{l=0}^{\phi(Q,\iota)-1} \frac{(\iota_{l+1} - \varsigma_l)^{\kappa}}{\kappa}\Big] + 1. \end{aligned} \tag{3.4}$$

So

$$\begin{aligned} & -\overline{\lambda}(\zeta + \varphi(\mathcal{B}) + \theta)\phi(Q,\iota) \\ \leq & \begin{cases} -\lambda_1(\zeta + \varphi(\mathcal{B}) + \theta)[\frac{1}{\lambda_1}((\frac{(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa})_{+} + \sum_{l=0}^{\phi(Q,\iota)-1} \frac{(\iota_{l+1}-\varsigma_l)^{\kappa}}{\kappa}) + 1] \\ = -(\zeta + \varphi(\mathcal{B}) + \theta)[(\frac{(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa})_{+} + \sum_{l=0}^{\phi(Q,\iota)-1} \frac{(\iota_{l+1}-\varsigma_l)^{\kappa}}{\kappa}] - \lambda_1(\zeta + \varphi(\mathcal{B}) + \theta), \ \zeta + \varphi(\mathcal{B}) + \theta < 0, \\ -\lambda_2(\zeta + \varphi(\mathcal{B}) + \theta)[\frac{1}{\lambda_2}((\frac{(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa})_{+} + \sum_{l=0}^{\phi(Q,\iota)-1} \frac{(\iota_{l+1}-\varsigma_l)^{\kappa}}{\kappa}) - 1] \\ = -(\zeta + \varphi(\mathcal{B}) + \theta)[(\frac{(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa})_{+} + \sum_{l=0}^{\phi(Q,\iota)-1} \frac{(\iota_{l+1}-\varsigma_l)^{\kappa}}{\kappa}] + \lambda_2(\zeta + \varphi(\mathcal{B}) + \theta), \ \zeta + \varphi(\mathcal{B}) + \theta \geq 0, \end{cases} \\ = & -(\zeta + \varphi(\mathcal{B}) + \theta)\Big[\Big(\frac{(\iota - \varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa}\Big)_{+} + \sum_{l=0}^{\phi(Q,\iota)-1} \frac{(\iota_{l+1} - \varsigma_l)^{\kappa}}{\kappa}\Big] + \overline{\lambda}|\zeta + \varphi(\mathcal{B}) + \theta|. \end{aligned} \tag{3.5}$$

Equation (3.1) implies

$$-\overline{\lambda}(\zeta + \varphi(\mathcal{B}) + \theta)\phi(Q,\iota) \geq \phi(Q,\iota) \ln \varrho \geq \sum_{l=1}^{\phi(Q,\iota)} \ln \|\mathcal{I} + C_l\|. \tag{3.6}$$

With (3.4)–(3.6), we obtain

$$\begin{aligned}
&(\varphi(\mathcal{B})+\theta)\Big[\Big(\frac{(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa}\Big)_{+}+\sum_{l=0}^{\phi(Q,\iota)-1}\frac{(\iota_{l+1}-\varsigma_{l})^{\kappa}}{\kappa}\Big]+\sum_{l=1}^{\phi(Q,\iota)}\ln\|\mathcal{I}+C_{l}\| \\
\leq\;& -\zeta\Big[\Big(\frac{(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa}\Big)_{+}+\sum_{l=0}^{\phi(Q,\iota)-1}\frac{(\iota_{l+1}-\varsigma_{l})^{\kappa}}{\kappa}\Big]+\overline{\lambda}|\zeta+\varphi(\mathcal{B})+\theta| \\
\leq\;& -\zeta\lambda_1\phi(Q,\iota)+\overline{\lambda}|\zeta+\varphi(\mathcal{B})+\theta|.
\end{aligned}$$

Finally, we know

$$\begin{aligned}
\|\Xi(\iota,Q)\| &\leq L_{\theta}e^{\sum_{l=1}^{\phi(Q,\iota)}\ln\|\mathcal{I}+C_{l}\|+(\varphi(\mathcal{B})+\theta)\big[\big(\frac{(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa}\big)_{+}+\sum_{l=0}^{\phi(Q,\iota)-1}\frac{(\iota_{l+1}-\varsigma_{l})^{\kappa}}{\kappa}\big]} \\
&\leq L_{\theta}e^{\overline{\lambda}|\zeta+\varphi(\mathcal{B})+\theta|}e^{-\zeta\lambda_1(\frac{q}{\delta}-\varepsilon)(\iota-Q)},
\end{aligned}$$

in which $L=L_{\theta}e^{\overline{\lambda}|\zeta+\varphi(\mathcal{B})+\theta|}\geq 1$ and $\nu=\zeta\lambda_1(\frac{q}{\delta}-\varepsilon)>0$.

Hence, (1.1) is exponentially stable. □

4. Linear nonhomogeneous problem

This section considers the linear nonhomogeneous problem.

We present the following condition: (G_3) $a(\iota)=a(\iota+\delta)$ for $\iota\in\bigcup_{l=0}^{\infty}(\varsigma_l,\iota_{l+1}]$.

Following this, we present the solution of (1.2).

Theorem 4.1. *The solution of (1.2) with $\gamma(Q)=\gamma_Q$ is*

$$\begin{aligned}
\gamma(\iota) &= \Xi(\iota,Q)\gamma_Q+\sum_{l=0}^{\phi(Q,\iota)-1}\int_{\varsigma_l}^{\iota_{l+1}}\Xi(\iota,\varsigma)a(\varsigma)(\varsigma-\varsigma_l)^{\kappa-1}d\varsigma \\
&\quad+\int_{\varsigma_{\phi(Q,\iota)}}^{\iota}\Xi(\iota,\varsigma)a(\varsigma)(\varsigma-\varsigma_{\phi(Q,\iota)})^{\kappa-1}d\varsigma+\sum_{l=1}^{\phi(Q,\iota)}\Xi(\iota,\varsigma_l)b_l.
\end{aligned}\tag{4.1}$$

Proof. For $\iota\in[\varsigma_0,\iota_1]$,

$$\gamma(\iota)=\Xi(\iota,Q)\gamma_Q+\int_{\varsigma_0}^{\iota}\Xi(\iota,\varsigma)a(\varsigma)(\varsigma-\varsigma_0)^{\kappa-1}d\varsigma.$$

If (4.1) holds for $\iota\in(\varsigma_{\phi(Q,\iota)-1},\iota_{\phi(Q,\iota)}]$, then

$$\begin{aligned}
\gamma(\iota) &= \Xi(\iota,Q)\gamma_Q+\sum_{l=0}^{\phi(Q,\iota)-2}\int_{\varsigma_l}^{\iota_{l+1}}\Xi(\iota,\varsigma)a(\varsigma)(\varsigma-\varsigma_l)^{\kappa-1}d\varsigma \\
&\quad+\int_{\varsigma_{\phi(Q,\iota)-1}}^{\iota}\Xi(\iota,\varsigma)a(\varsigma)(\varsigma-\varsigma_{\phi(Q,\iota)-1})^{\kappa-1}d\varsigma+\sum_{l=1}^{\phi(Q,\iota)-1}\Xi(\iota,\varsigma_l)b_l,
\end{aligned}$$

and for $\iota\in(\iota_{\phi(Q,\iota)},\varsigma_{\phi(Q,\iota)}]$,

$$\gamma(\iota) = (\mathcal{I}+C_{\phi(Q,\iota)})\gamma(\iota^{-}_{\phi(Q,\iota)})+b_{\phi(Q,\iota)}$$

$$= (\mathcal{I} + C_{\phi(Q,\iota)})\Big[\Xi(\iota^-_{\phi(Q,\iota)}, Q)\gamma_Q + \sum_{l=0}^{\phi(Q,\iota)-1}\int_{\varsigma_l}^{\iota_{l+1}} \Xi(\iota^-_{\phi(Q,\iota)}, \varsigma)a(\varsigma)(\varsigma - \varsigma_l)^{\kappa-1}d\varsigma$$
$$+ \sum_{l=1}^{\phi(Q,\iota)-1} \Xi(\iota^-_{\phi(Q,\iota)}, \varsigma_l)b_l\Big] + b_{\phi(Q,\iota)}.$$

Next, for $\iota \in (\varsigma_{\phi(Q,\iota)}, \iota_{\phi(Q,\iota)+1}]$,

$$\begin{aligned}
\gamma(\iota) &= \Xi(\iota, \varsigma_{\phi(Q,\iota)})\gamma(\varsigma_{\phi(Q,\iota)}) + \int_{\varsigma_{\phi(Q,\iota)}}^{\iota} \Xi(\iota, \varsigma)a(\varsigma)(\varsigma - \varsigma_{\phi(Q,\iota)})^{\kappa-1}d\varsigma \\
&= \Xi(\iota, \varsigma_{\phi(Q,\iota)})(\mathcal{I} + C_{\phi(Q,\iota)})\Xi(\iota^-_{\phi(Q,\iota)}, Q)\gamma_Q \\
&\quad + \sum_{l=0}^{\phi(Q,\iota)-1}\int_{\varsigma_l}^{\iota_{l+1}} \Xi(\iota, \varsigma_{\phi(Q,\iota)})(\mathcal{I} + C_{\phi(Q,\iota)})\Xi(\iota^-_{\phi(Q,\iota)}, \varsigma)a(\varsigma)(\varsigma - \varsigma_l)^{\kappa-1}d\varsigma \\
&\quad + \sum_{l=1}^{\phi(Q,\iota)-1} \Xi(\iota, \varsigma_{\phi(Q,\iota)})(\mathcal{I} + C_{\phi(Q,\iota)})\Xi(\iota^-_{\phi(Q,\iota)}, \varsigma_l)b_l + \Xi(\iota, \varsigma_{\phi(Q,\iota)})b_{\phi(Q,\iota)} \\
&\quad + \int_{\varsigma_{\phi(Q,\iota)}}^{\iota} \Xi(\iota, \varsigma)a(\varsigma)(\varsigma - \varsigma_{\phi(Q,\iota)})^{\kappa-1}d\varsigma \\
&= \Xi(\iota, Q)\gamma_Q + \sum_{l=0}^{\phi(Q,\iota)-1}\int_{\varsigma_l}^{\iota_{l+1}} \Xi(\iota, \varsigma)a(\varsigma)(\varsigma - \varsigma_l)^{\kappa-1}d\varsigma \\
&\quad + \int_{\varsigma_{\phi(Q,\iota)}}^{\iota} \Xi(\iota, \varsigma)a(\varsigma)(\varsigma - \varsigma_{\phi(Q,\iota)})^{\kappa-1}d\varsigma + \sum_{l=1}^{\phi(Q,\iota)} \Xi(\iota, \varsigma_l)b_l.
\end{aligned}$$

With the mathematical induction method, we obtain

$$\begin{aligned}
\gamma(\iota) &= \Xi(\iota, Q)\gamma_Q + \sum_{l=0}^{\phi(Q,\iota)-1}\int_{\varsigma_l}^{\iota_{l+1}} \Xi(\iota, \varsigma)a(\varsigma)(\varsigma - \varsigma_l)^{\kappa-1}d\varsigma \\
&\quad + \int_{\varsigma_{\phi(Q,\iota)}}^{\iota} \Xi(\iota, \varsigma)a(\varsigma)(\varsigma - \varsigma_{\phi(Q,\iota)})^{\kappa-1}d\varsigma + \sum_{l=1}^{\phi(Q,\iota)} \Xi(\iota, \varsigma_l)b_l.
\end{aligned}$$

□

Theorem 4.2. *Suppose that (G_1), (G_2), and (G_3) hold. If the solution of (1.2) is bounded, then it is $\delta-$periodic.*

Proof. Since the solution of (1.2) is bounded, one set $\widetilde{\gamma}(Q + n\delta)$ is bounded. Using Theorems 2.11 and 2.12, one obtains

$$\widetilde{\gamma}(Q + (n+1)\delta) = \Xi(Q + (n+1)\delta, Q)\gamma_Q + \sum_{l=0}^{(n+1)q-1}\int_{\varsigma_l}^{\iota_{l+1}} \Xi(Q + (n+1)\delta, \varsigma)a(\varsigma)(\varsigma - \varsigma_l)^{\kappa-1}d\varsigma$$
$$+ \sum_{l=1}^{(n+1)q} \Xi(Q + (n+1)\delta, \varsigma_l)b_l$$

$$= \Xi(Q+(n+1)\delta, Q+n\delta)\Big[\Xi(Q+n\delta, Q)\gamma_Q + \sum_{l=0}^{nq-1}\int_{\varsigma_l}^{\iota_{l+1}} \Xi(Q+n\delta,\varsigma)a(\varsigma)(\varsigma-\varsigma_l)^{\kappa-1}d\varsigma$$

$$+ \sum_{l=1}^{nq}\Xi(Q+n\delta,\varsigma_l)b_l\Big] + \sum_{l=nq}^{(n+1)q-1}\int_{\varsigma_l}^{\iota_{l+1}} \Xi(Q+(n+1)\delta,\varsigma)a(\varsigma)(\varsigma-\varsigma_l)^{\kappa-1}d\varsigma$$

$$+ \sum_{l=nq+1}^{(n+1)q}\Xi(Q+(n+1)\delta,\varsigma_l)b_l$$

$$= \Xi(Q+\delta, Q)\widetilde{\gamma}(Q+n\delta) + \sum_{l=0}^{q-1}\int_{\varsigma_l}^{\iota_{l+1}} \Xi(Q+(n+1)\delta,\varsigma+n\delta)a(\varsigma+n\delta)(\varsigma-\varsigma_l)^{\kappa-1}d\varsigma$$

$$+ \sum_{l=1}^{q}\Xi(Q+(n+1)\delta,\varsigma_l+n\delta)b_{l+nq}$$

$$= \Xi(Q+\delta, Q)\widetilde{\gamma}(Q+n\delta) + \sum_{l=0}^{q-1}\int_{\varsigma_l}^{\iota_{l+1}} \Xi(Q+\delta,\varsigma)a(\varsigma)(\varsigma-\varsigma_l)^{\kappa-1}d\varsigma + \sum_{l=1}^{q}\Xi(Q+\delta,\varsigma_l)b_l$$

$$= \Xi(Q+\delta, Q)\widetilde{\gamma}(Q+n\delta) + \Gamma_q,$$

where

$$\Gamma_q = \sum_{l=0}^{q-1}\int_{\varsigma_l}^{\iota_{l+1}} \Xi(Q+\delta,\varsigma)a(\varsigma)(\varsigma-\varsigma_l)^{\kappa-1}d\varsigma + \sum_{l=1}^{q}\Xi(Q+\delta,\varsigma_l)b_l.$$

Hence,

$$\widetilde{\gamma}(Q+n\delta) = \Xi^n(Q+\delta, Q)\widetilde{\gamma}(Q) + \sum_{l=0}^{n-1}\Xi^l(Q+\delta,Q)\Gamma_q.$$

With the proof by contradiction, one supposes that $\widetilde{\gamma}(\iota)$ is not the $\delta-$periodic solution of (1.2). So there is not a $\gamma_Q \in \mathbb{R}^n$ such that

$$(\mathcal{I} - \Xi(Q+\delta, Q))\gamma_Q = \Gamma_q.$$

Fredholm alternative Theorem implies that there is a $\mathcal{Z} \in \mathbb{R}^n$ satisfying

$$(\mathcal{I} - \Xi^\top(Q+\delta, Q))\mathcal{Z} = 0, \quad <\Gamma_q, \mathcal{Z}> \neq 0.$$

Since $(\mathcal{I} - \Xi^\top(Q+\delta, Q))\mathcal{Z} = 0$, with any $n \in \mathbb{N}$, we have $[\Xi^n(Q+\delta,Q)]^\top\mathcal{Z} = \mathcal{Z}$. Also,

$$\begin{aligned} <\widetilde{\gamma}(Q+n\delta), \mathcal{Z}> &= <\Xi^n(Q+\delta,Q)\widetilde{\gamma}(Q) + \sum_{l=0}^{n-1}\Xi^l(Q+\delta,Q)\Gamma_q, \mathcal{Z}> \\ &= <\widetilde{\gamma}(Q), [\Xi^n(Q+\delta,Q)]^\top\mathcal{Z}> + \sum_{l=0}^{n-1}<\Gamma_q, [\Xi^l(Q+\delta,Q)]^\top\mathcal{Z}> \\ &= <\widetilde{\gamma}(Q), \mathcal{Z}> + n <\Gamma_q, \mathcal{Z}> \to \infty, \text{ as } n \to \infty, \end{aligned}$$

which is contradictory to the boundedness of $\widetilde{\gamma}(\iota)$. So $\widetilde{\gamma}(\iota)$ is a $\delta-$periodic solution of (1.2).

□

We analyze the existence of periodic solutions for (1.2) in two different situations.

Theorem 4.3. *Suppose* (G_1) *and* (G_2) *hold. If* $\det(\mathcal{I}-\Xi(Q+\delta,Q))\neq 0$*, (1.2) has a* $\delta-$*periodic solution with*

$$\gamma_Q=(\mathcal{I}-\Xi(Q+\delta,Q))^{-1}\Gamma_q.$$

Next, if $\det(\mathcal{I}-\Xi(Q+\delta,Q))=0$, we present

$$\begin{cases} \mathfrak{D}_{\kappa}^{\varsigma_l}\beta(\iota)=-\mathcal{B}^{\top}\beta(\iota),\ \iota\in(\varsigma_l,\iota_{l+1}],\ l\in\mathbb{N}_0,\ 0<\kappa<1,\\ \beta(\iota_l^+)=(\mathcal{I}+C_l^{\top})^{-1}\beta(\iota_l^-),\ l\in\mathbb{N},\\ \beta(\iota)=(\mathcal{I}+C_l^{\top})^{-1}\beta(\iota_l^-),\ \iota\in(\iota_l,\varsigma_l],\ l\in\mathbb{N},\\ \beta(\varsigma_l^+)=\beta(\varsigma_l^-),\ l\in\mathbb{N}. \end{cases} \tag{4.2}$$

Theorem 4.4. *Let* (G_1)*,* (G_2)*, and* (G_3) *be satisfied. (1.2) has a* $\delta-$*periodic solution iff* $<\beta_Q,\Gamma_q>=0$*, where* β_Q *is the initial value of (4.2).*

Proof. (1.2) has a $\delta-$periodic solution iff there exists γ_Q such that

$$(\mathcal{I}-\Xi(Q+\delta,Q))\gamma_Q=\Gamma_q.$$

Then,

$$\begin{aligned} <\beta_Q,\Gamma_q> &= <\beta_Q,(\mathcal{I}-\Xi(Q+\delta,Q))\gamma_Q>\\ &= <(\mathcal{I}-\Xi(Q+\delta,Q))^{\top}\beta_Q,\gamma_Q>\\ &= <(\mathcal{I}-\Xi^{\top}(Q+\delta,Q))\beta_Q,\gamma_Q>\\ &= <0,\gamma_Q>=0. \end{aligned}$$

□

5. Nonlinear problem

This section studies the $\delta-$periodic solution of (1.3).

One presents the conditions:

(G_4) For $\gamma\in\mathbb{R}^n$ and $\iota\in\bigcup_{l=0}^{\infty}(\varsigma_l,\iota_{l+1}]$, $\mathcal{A}(\iota+\delta,\gamma)=\mathcal{A}(\iota,\gamma)$.

(G_5) For $\gamma\in\mathbb{R}^n$ and $\iota\in\bigcup_{l=0}^{\infty}(\varsigma_l,\iota_{l+1}]$, there is a $\overline{\mathcal{A}}>0$ such that $\|\mathcal{A}(\iota,\gamma)\|\leq\overline{\mathcal{A}}$.

One studies

$$\mathfrak{D}_{\kappa}^{\varsigma_l}\gamma(\iota)=\mathcal{B}\gamma(\iota)+\mathcal{A}(\iota,\gamma(\iota)),\ \gamma(\varsigma_{l-1})=\gamma_{l-1},\ \iota\in(\varsigma_{l-1},\iota_l],\ 0<\kappa<1,\ l\in\mathbb{N},$$

and the solution is

$$\gamma(\iota)=\Xi(\iota,\varsigma_l)\gamma_{l-1}+\int_{\varsigma_{l-1}}^{\iota}\Xi(\iota,\varsigma)\mathcal{A}(\varsigma,\gamma(\varsigma))(\varsigma-\varsigma_{l-1})^{\kappa-1}d\varsigma. \tag{5.1}$$

We set this mapping

$$\mathcal{G}_l(\gamma_{l-1}):=(\mathcal{I}+C_l)\circ\gamma(\iota_l)+b_l. \tag{5.2}$$

Equation (5.1) implies

$$\|\gamma(\iota_l)\| \leq L_\theta e^{|\varphi(\mathcal{B})+\theta|\frac{(\iota_l-\varsigma_{l-1})^\kappa}{\kappa}}\|\gamma_{l-1}\| + \frac{L_\theta\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}\Big(e^{|\varphi(\mathcal{B})+\theta|\frac{(\iota_l-\varsigma_{l-1})^\kappa}{\kappa}} - 1\Big),$$

and (5.2) implies

$$\|\mathcal{G}_l(\gamma_{l-1})\| \leq \varrho L_\theta e^{|\varphi(\mathcal{B})+\theta|\frac{(\iota_l-\varsigma_{l-1})^\kappa}{\kappa}}\|\gamma_{l-1}\| + \frac{\varrho L_\theta\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}\Big(e^{|\varphi(\mathcal{B})+\theta|\frac{(\iota_l-\varsigma_{l-1})^\kappa}{\kappa}} - 1\Big) + \overline{b},$$

where $\overline{b} = \max_{l\in\mathbb{N}} \|b_l\|$.

Then we construct this operator

$$\mathcal{G} := \mathcal{G}_q \circ \mathcal{G}_{q-1} \circ \cdots \circ \mathcal{G}_1,$$

and set $\chi_l = e^{|\varphi(\mathcal{B})+\theta|\frac{(\iota_l-\varsigma_{l-1})^\kappa}{\kappa}}$, $\varpi = \varrho L_\theta$.

Next, one presents the norm estimation of $\mathcal{G}$.

Theorem 5.1. *If* (G_5) *holds, there is*

$$\begin{aligned}\|\mathcal{G}(\gamma_Q)\| \leq\ & \varpi^q \prod_{l=1}^{q}\chi_l\|\gamma_Q\| + \frac{\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}\sum_{l=1}^{q-1}\prod_{j=l}^{q-1}\varpi^{q-j+1}\chi_q\ldots\chi_{j+1}(\chi_j-1) \\ & + \Big(\sum_{l=2}^{q}\prod_{j=l}^{q}\varpi^{q-j+1}\chi_q\ldots\chi_j + 1\Big)\overline{b} + \frac{\overline{\mathcal{A}}\varpi}{|\varphi(\mathcal{B})+\theta|}(\chi_q-1). \end{aligned} \tag{5.3}$$

Proof. For $l = 1$, there is

$$\|\gamma_1\| \leq \varpi\chi_1\|\gamma_Q\| + \frac{\varpi\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}(\chi_1 - 1) + \overline{b}.$$

For $l = 2$, there is

$$\begin{aligned}\|\gamma_2\| &\leq \varpi\chi_2\|\gamma_1\| + \frac{\varpi\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}(\chi_2-1) + \overline{b} \\ &\leq \varpi\chi_2\Big(\varpi\chi_1\|\gamma_Q\| + \frac{\varpi\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}(\chi_1-1) + \overline{b}\Big) + \frac{\varpi\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}(\chi_2-1) + \overline{b} \\ &\leq \varpi\chi_2\varpi\chi_1\|\gamma_Q\| + \frac{\varpi\chi_2\varpi\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}(\chi_1-1) + \varpi\chi_2\overline{b} + \frac{\varpi\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}(\chi_2-1) + \overline{b}.\end{aligned}$$

For $l = 3$, there is

$$\begin{aligned}\|\gamma_3\| &\leq \varpi\chi_3\|\gamma_2\| + \frac{\varpi\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}(\chi_3-1) + \overline{b} \\ &\leq \varpi\chi_3\Big(\varpi\chi_2\varpi\chi_1\|\gamma_Q\| + \frac{\varpi\chi_2\varpi\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}(\chi_1-1) + \varpi\chi_2\overline{b} + \frac{\varpi\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}(\chi_2-1) + \overline{b}\Big)\end{aligned}$$

$$
\begin{aligned}
&+\frac{\varpi\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}(\chi_3-1)+\overline{b}\\
\leq\ & \varpi\chi_3\varpi\chi_2\varpi\chi_1\|\gamma_Q\|+\frac{\varpi\chi_3\varpi\chi_2\varpi\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}(\chi_1-1)+\varpi\chi_3\varpi\chi_2\overline{b}+\frac{\varpi\chi_3\varpi\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}(\chi_2-1)+\varpi\chi_3\overline{b}\\
&+\frac{\varpi\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}(\chi_3-1)+\overline{b}.
\end{aligned}
$$

Suppose that (5.3) holds for $l=q-1$; when $l=q$,

$$
\begin{aligned}
&\|\gamma_q\|\leq\varpi\chi_q\|\gamma_{q-1}\|+\frac{\varpi\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}(\chi_q-1)+\overline{b}\\
\leq\ & \varpi\chi_q\Big[\varpi^{q-1}\prod_{l=1}^{q-1}\chi_l\|\gamma_Q\|+\frac{\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}\sum_{l=1}^{q-2}\prod_{j=l}^{q-2}\varpi^{q-j}\chi_{q-1}\ldots\chi_{j+1}(\chi_j-1)\\
&+\frac{\varpi\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}(\chi_{q-1}-1)+\Big(\sum_{l=2}^{q-1}\prod_{j=l}^{q-1}\varpi^{q-j}\chi_{q-1}\ldots\chi_j+1\Big)\overline{b}\Big]+\frac{\varpi\overline{\mathcal{A}}}{\varpi(\mathcal{B})+\theta}(\chi_q-1)+\overline{b}\\
=\ & \varpi^q\prod_{l=1}^{q}\chi_l\|\gamma_Q\|+\frac{\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}\sum_{l=1}^{q-2}\prod_{j=l}^{q-2}\varpi^{q-j+1}\chi_q\chi_{q-1}\ldots\chi_{j+1}(\chi_j-1)+\frac{\overline{\mathcal{A}}\varpi^2}{|\varphi(\mathcal{B})+\theta|}\chi_q(\chi_{q-1}-1)\\
&+\sum_{l=2}^{q-1}\prod_{j=l}^{q-1}\varpi^{q-j+1}\chi_q\chi_{q-1}\ldots\chi_j\overline{b}+\varpi\chi_q\overline{b}+\frac{\overline{\mathcal{A}}\varpi}{|\varphi(\mathcal{B})+\theta|}(\chi_q-1)+\overline{b}\\
=\ & \varpi^q\prod_{l=1}^{q}\chi_l\|\gamma_Q\|+\frac{\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}\sum_{l=1}^{q-1}\prod_{j=l}^{q-1}\varpi^{q-j+1}\chi_q\ldots\chi_{j+1}(\chi_j-1)\\
&+\sum_{l=2}^{q}\prod_{j=l}^{q}\varpi^{q-j+1}\chi_q\ldots\chi_j\overline{b}+\frac{\overline{\mathcal{A}}\varpi}{|\varphi(\mathcal{B})+\theta|}(\chi_q-1)+\overline{b}\\
=\ & \varpi^q\prod_{l=1}^{q}\chi_l\|\gamma_Q\|+\frac{\overline{\mathcal{A}}}{|\varphi(\mathcal{B})+\theta|}\sum_{l=1}^{q-1}\prod_{j=l}^{q-1}\varpi^{q-j+1}\chi_q\ldots\chi_{j+1}(\chi_j-1)\\
&+\Big(\sum_{l=2}^{q}\prod_{j=l}^{q}\varpi^{q-j+1}\chi_q\ldots\chi_j+1\Big)\overline{b}+\frac{\overline{\mathcal{A}}\varpi}{|\varphi(\mathcal{B})+\theta|}(\chi_q-1).
\end{aligned}
$$

□

Following this, we derive the equivalence between the $\delta-$periodic solution for (1.3) and the fixed point of $\mathcal{G}$.

Theorem 5.2. *If* (G_1)*,* (G_2)*, and* (G_4) *hold, (1.3) has a* $\delta-$*periodic solution iff* $\mathcal{G}$ *has a fixed point.*

Proof. Sufficiency.

According to the definition of $\mathcal{G}$, one has

$$\gamma_Q\ =\ \mathcal{G}(\gamma_Q)=\mathcal{G}_q\circ\mathcal{G}_{q-1}\circ\cdots\circ\mathcal{G}_1(\gamma_Q)$$

$$= \Xi(Q+\delta,Q)\gamma_Q + \sum_{l=0}^{q-1}\int_{\varsigma_l}^{\iota_{l+1}} \Xi(Q+\delta,\varsigma)\mathcal{A}(\varsigma,\gamma(\varsigma))(\varsigma-\varsigma_l)^{\kappa-1}d\varsigma + \sum_{l=1}^{q}\Xi(Q+\delta,\varsigma_l)b_l.$$

For $\iota = \widetilde{\iota} + N\delta$, Theorems 2.11–2.13 derive

$$\gamma(\iota) = \gamma(\widetilde{\iota}+N\delta) = \Xi(\widetilde{\iota}+N\delta,Q)\gamma(Q) = [\Xi(\widetilde{\iota}+\delta,\widetilde{\iota})]^N\Xi(\widetilde{\iota},Q)\gamma(Q),$$

and

$$\begin{aligned}
\gamma(\iota+\delta) &= \gamma(\widetilde{\iota}+(N+1)\delta)\\
&= \Xi(\widetilde{\iota}+(N+1)\delta,Q)\gamma_Q\\
&= [\Xi(\widetilde{\iota}+\delta,\widetilde{\iota})]^{N+1}\Xi(\widetilde{\iota},Q)\gamma_Q\\
&= [\Xi(\widetilde{\iota}+\delta,\widetilde{\iota})]^{N}\Xi(\widetilde{\iota}+\delta,Q)\gamma_Q\\
&= [\Xi(\widetilde{\iota}+\delta,\widetilde{\iota})]^{N}\gamma(\widetilde{\iota}+\delta)\\
&= [\Xi(\widetilde{\iota}+\delta,\widetilde{\iota})]^{N}\Xi(\widetilde{\iota}+\delta,Q+\delta)\gamma(Q+\delta)\\
&= [\Xi(\widetilde{\iota}+\delta,\widetilde{\iota})]^{N}\Xi(\widetilde{\iota},Q)\gamma_Q,
\end{aligned}$$

then $\gamma(\iota+\delta) = \gamma(\iota)$.

Necessity.

If $\gamma(\iota)$ is a $\delta-$periodic solution of (1.3), then $\mathcal{G}(\gamma_Q) = \gamma_Q$ and γ_Q is a fixed point of $\mathcal{G}$. □

Next, we present this condition:

(G_6) For $\gamma \in \mathbb{R}^n$ and $\iota \in \bigcup_{l=0}^{\infty}(\varsigma_l, \iota_{l+1}]$, there is a $L_{\mathcal{A}} > 0$ such that $\|\mathcal{A}(\iota,\gamma) - \mathcal{A}(\iota,\overline{\gamma})\| \le L_{\mathcal{A}}\|\gamma-\overline{\gamma}\|$.

Theorem 5.3. *Let (G_1), (G_2), (G_4), and (G_6) be satisfied. If*

$$\varpi^q e^{(L_\theta L_{\mathcal{A}}+|\varphi(\mathcal{B})+\theta|)\sum_{l=1}^{q}\frac{(\iota_l-\varsigma_{l-1})^\kappa}{\kappa}} < 1,$$

then (1.3) has a unique $\delta-$periodic solution.

Proof. Set $\gamma(\iota)$ and $\overline{\gamma}(\iota)$ respectively as the solutions of (1.3) with initial values γ_Q and $\overline{\gamma}_Q$.

By calculation, for $\iota \in [\varsigma_0, \iota_1]$, there is

$$\begin{aligned}
\|\gamma(\iota)-\overline{\gamma}(\iota)\| &\le \|\Xi(\iota,\varsigma_0)\|\|\gamma_Q-\overline{\gamma}_Q\| + \int_{\varsigma_0}^{\iota}\|\Xi(\iota,\varsigma)\|\|\mathcal{A}(\varsigma,\gamma(\varsigma))-\mathcal{A}(\varsigma,\overline{\gamma}(\varsigma))\|(\varsigma-\varsigma_0)^{\kappa-1}d\varsigma\\
&\le L_\theta e^{|\varphi(\mathcal{B})+\theta|\frac{(\iota-\varsigma_0)^\kappa}{\kappa}}\|\gamma_Q-\overline{\gamma}_Q\| + \int_{\varsigma_0}^{\iota} L_\theta e^{|\varphi(\mathcal{B})+\theta|\left(\frac{(\iota-\varsigma_0)^\kappa}{\kappa}-\frac{(\varsigma-\varsigma_0)^\kappa}{\kappa}\right)}L_{\mathcal{A}}\|\gamma(\varsigma)-\overline{\gamma}(\varsigma)\|(\varsigma-\varsigma_0)^{\kappa-1}d\varsigma,
\end{aligned}$$

then

$$e^{-|\varphi(\mathcal{B})+\theta|\frac{(\iota-\varsigma_0)^\kappa}{\kappa}}\|\gamma(\iota)-\overline{\gamma}(\iota)\| \le L_\theta\|\gamma_Q-\overline{\gamma}_Q\| + \int_{\varsigma_0}^{\iota} L_\theta e^{-|\varphi(\mathcal{B})+\theta|\frac{(\varsigma-\varsigma_0)^\kappa}{\kappa}}L_{\mathcal{A}}\|\gamma(\varsigma)-\overline{\gamma}(\varsigma)\|(\varsigma-\varsigma_0)^{\kappa-1}d\varsigma.$$

Using Lemma 2.9, one has

$$e^{-|\varphi(\mathcal{B})+\theta|\frac{(\iota-\varsigma_0)^\kappa}{\kappa}}\|\gamma(\iota)-\overline{\gamma}(\iota)\| \le L_\theta\|\gamma_Q-\overline{\gamma}_Q\|e^{L_\theta L_{\mathcal{A}}\frac{(\iota-\varsigma_0)^\kappa}{\kappa}},$$

and

$$\|\gamma(\iota)-\overline{\gamma}(\iota)\| \leq L_{\theta}\|\gamma_{Q}-\overline{\gamma}_{Q}\|e^{(L_{\theta}L_{\mathcal{A}}+|\varphi(\mathcal{B})+\theta|)\frac{(\iota-\varsigma_0)^{\kappa}}{\kappa}}.$$

Next, (5.2) implies

$$\begin{aligned}\|\gamma_1-\overline{\gamma}_1\| &= \|\mathcal{G}_1(\gamma_Q)-\mathcal{G}_1(\overline{\gamma}_Q)\|\\ &\leq \varrho\|\gamma(\iota_1)-\overline{\gamma}(\iota_1)\|\\ &\leq \varrho L_{\theta}\|\gamma_Q-\overline{\gamma}_Q\|e^{(L_{\theta}L_{\mathcal{A}}+|\varphi(\mathcal{B})+\theta|)\frac{(\iota_1-\varsigma_0)^{\kappa}}{\kappa}}.\end{aligned}$$

For $\iota\in(\varsigma_1,\iota_2]$, there is

$$\begin{aligned}&\|\gamma(\iota)-\overline{\gamma}(\iota)\| \leq \|\Xi(\iota,\varsigma_1)\|\|\gamma_1-\overline{\gamma}_1\|+\int_{\varsigma_1}^{\iota}\|\Xi(\iota,\varsigma)\|\|\mathcal{A}(\varsigma,\gamma(\varsigma))-\mathcal{A}(\varsigma,\overline{\gamma}(\varsigma))\|(\varsigma-\varsigma_1)^{\kappa-1}d\varsigma\\ \leq\ & L_{\theta}e^{|\varphi(\mathcal{B})+\theta|\frac{(\iota-\varsigma_1)^{\kappa}}{\kappa}}\|\gamma_1-\overline{\gamma}_1\|+\int_{\varsigma_1}^{\iota}L_{\theta}e^{|\varphi(\mathcal{B})+\theta|\left(\frac{(\iota-\varsigma_1)^{\kappa}}{\kappa}-\frac{(\varsigma-\varsigma_1)^{\kappa}}{\kappa}\right)}L_{\mathcal{A}}\|\gamma(\varsigma)-\overline{\gamma}(\varsigma)\|(\varsigma-\varsigma_1)^{\kappa-1}d\varsigma,\end{aligned}$$

then

$$e^{-|\varphi(\mathcal{B})+\theta|\frac{(\iota-\varsigma_1)^{\kappa}}{\kappa}}\|\gamma(\iota)-\overline{\gamma}(\iota)\| \leq L_{\theta}\|\gamma_1-\overline{\gamma}_1\|+\int_{\varsigma_1}^{\iota}L_{\theta}e^{-|\varphi(\mathcal{B})+\theta|\frac{(\varsigma-\varsigma_1)^{\kappa}}{\kappa}}L_{\mathcal{A}}\|\gamma(\varsigma)-\overline{\gamma}(\varsigma)\|(\varsigma-\varsigma_1)^{\kappa-1}d\varsigma.$$

Using Lemma 2.9, one has

$$e^{-|\varphi(\mathcal{B})+\theta|\frac{(\iota-\varsigma_1)^{\kappa}}{\kappa}}\|\gamma(\iota)-\overline{\gamma}(\iota)\| \leq L_{\theta}\|\gamma_1-\overline{\gamma}_1\|e^{L_{\theta}L_{\mathcal{A}}\frac{(\iota-\varsigma_1)^{\kappa}}{\kappa}},$$

and

$$\begin{aligned}\|\gamma(\iota)-\overline{\gamma}(\iota)\| &\leq L_{\theta}\|\gamma_1-\overline{\gamma}_1\|e^{(L_{\theta}L_{\mathcal{A}}+|\varphi(\mathcal{B})+\theta|)\frac{(\iota-\varsigma_1)^{\kappa}}{\kappa}}\\ &\leq L_{\theta}\varrho L_{\theta}\|\gamma_Q-\overline{\gamma}_Q\|e^{(L_{\theta}L_{\mathcal{A}}+|\varphi(\mathcal{B})+\theta|)\left(\frac{(\iota_1-\varsigma_0)^{\kappa}}{\kappa}+\frac{(\iota-\varsigma_1)^{\kappa}}{\kappa}\right)}.\end{aligned}$$

Next, (5.2) implies

$$\begin{aligned}\|\gamma_2-\overline{\gamma}_2\| &= \|\mathcal{G}_2(\gamma_1)-\mathcal{G}_1(\overline{\gamma}_1)\|\\ &\leq \varrho\|\gamma(\iota_2)-\overline{\gamma}(\iota_2)\|\\ &\leq (\varrho L_{\theta})^2\|\gamma_Q-\overline{\gamma}_Q\|e^{(L_{\theta}L_{\mathcal{A}}+|\varphi(\mathcal{B})+\theta|)\left(\frac{(\iota_1-\varsigma_0)^{\kappa}}{\kappa}+\frac{(\iota_2-\varsigma_1)^{\kappa}}{\kappa}\right)}.\end{aligned}$$

According to the above calculation, there is

$$\begin{aligned}\|\gamma_{q-1}-\overline{\gamma}_{q-1}\| &= \|\mathcal{G}_{q-1}(\gamma_{q-2})-\mathcal{G}_{q-1}(\overline{\gamma}_{q-2})\|\\ &\leq \varrho\|\gamma(\iota_{q-1})-\overline{\gamma}(\iota_{q-1})\|\\ &\leq (\varrho L_{\theta})^{q-1}\|\gamma_Q-\overline{\gamma}_Q\|e^{(L_{\theta}L_{\mathcal{A}}+|\varphi(\mathcal{B})+\theta|)\sum_{l=1}^{q-1}\frac{(\iota_l-\varsigma_{l-1})^{\kappa}}{\kappa}}.\end{aligned}$$

For $\iota\in(\varsigma_{q-1},\iota_q]$, there is

$$\|\gamma(\iota)-\overline{\gamma}(\iota)\| \leq \|\Xi(\iota,\varsigma_{q-1})\|\|\gamma_{q-1}-\overline{\gamma}_{q-1}\|+\int_{\varsigma_{q-1}}^{\iota}\|\Xi(\iota,\varsigma)\|\|\mathcal{A}(\varsigma,\gamma(\varsigma))-\mathcal{A}(\varsigma,\overline{\gamma}(\varsigma))\|(\varsigma-\varsigma_{q-1})^{\kappa-1}d\varsigma$$

$$\leq L_\theta e^{|\varphi(\mathcal{B})+\theta|\frac{(\iota-\varsigma_{q-1})^\kappa}{\kappa}}\|\gamma_{q-1}-\overline{\gamma}_{q-1}\|+\int_{\varsigma_{q-1}}^{\iota} L_\theta e^{|\varphi(\mathcal{B})+\theta|\left(\frac{(\iota-\varsigma_{q-1})^\kappa}{\kappa}-\frac{(\varsigma-\varsigma_{q-1})^\kappa}{\kappa}\right)}L_{\mathcal{A}}\|\gamma(\varsigma)-\overline{\gamma}(\varsigma)\|(\varsigma-\varsigma_{q-1})^{\kappa-1}d\varsigma,$$

then

$$e^{-|\varphi(\mathcal{B})+\theta|\frac{(\iota-\varsigma_{q-1})^\kappa}{\kappa}}\|\gamma(\iota)-\overline{\gamma}(\iota)\|\leq L_\theta\|\gamma_{q-1}-\overline{\gamma}_{q-1}\|+\int_{\varsigma_{q-1}}^{\iota} L_\theta e^{-|\varphi(\mathcal{B})+\theta|\frac{(\varsigma-\varsigma_{q-1})^\kappa}{\kappa}}L_{\mathcal{A}}\|\gamma(\varsigma)-\overline{\gamma}(\varsigma)\|(\varsigma-\varsigma_{q-1})^{\kappa-1}d\varsigma.$$

Using Lemma 2.9, one has

$$e^{-|\varphi(\mathcal{B})+\theta|\frac{(\iota-\varsigma_{q-1})^\kappa}{\kappa}}\|\gamma(\iota)-\overline{\gamma}(\iota)\|\leq L_\theta\|\gamma_{q-1}-\overline{\gamma}_{q-1}\|e^{L_\theta L_{\mathcal{A}}\frac{(\iota-\varsigma_{q-1})^\kappa}{\kappa}},$$

and

$$\begin{aligned}\|\gamma(\iota)-\overline{\gamma}(\iota)\| &\leq L_\theta\|\gamma_{q-1}-\overline{\gamma}_{q-1}\|e^{(L_\theta L_{\mathcal{A}}+|\varphi(\mathcal{B})+\theta|)\frac{(\iota-\varsigma_{q-1})^\kappa}{\kappa}}\\ &\leq L_\theta(\varrho L_\theta)^{q-1}\|\gamma_Q-\overline{\gamma}_Q\|e^{(L_\theta L_{\mathcal{A}}+|\varphi(\mathcal{B})+\theta|)\sum_{l=1}^{q-1}\frac{(\iota_l-\varsigma_{l-1})^\kappa}{\kappa}}e^{(L_\theta L_{\mathcal{A}}+|\varphi(\mathcal{B})+\theta|)\frac{(\iota-\varsigma_{q-1})^\kappa}{\kappa}}.\end{aligned}$$

Next, (5.2) implies

$$\begin{aligned}\|\gamma_q-\overline{\gamma}_q\| &= \|\mathcal{G}_q(\gamma_{q-1})-\mathcal{G}_1(\overline{\gamma}_{q-1})\|\\ &\leq \varrho\|\gamma(\iota_q)-\overline{\gamma}(\iota_q)\|\\ &\leq (\varrho L_\theta)^q\|\gamma_Q-\overline{\gamma}_Q\|e^{(L_\theta L_{\mathcal{A}}+|\varphi(\mathcal{B})+\theta|)\sum_{l=1}^{q}\frac{(\iota_l-\varsigma_{l-1})^\kappa}{\kappa}}.\end{aligned}$$

Hence,

$$\|\mathcal{G}\gamma_Q-\mathcal{G}\overline{\gamma}_Q\|\leq\varpi^q\|\gamma_Q-\overline{\gamma}_Q\|e^{(L_\theta L_{\mathcal{A}}+|\varphi(\mathcal{B})+\theta|)\sum_{l=1}^{q}\frac{(\iota_l-\varsigma_{l-1})^\kappa}{\kappa}}.$$

Since

$$\varpi^q e^{(L_\theta L_{\mathcal{A}}+|\varphi(\mathcal{B})+\theta|)\sum_{l=1}^{q}\frac{(\iota_l-\varsigma_{l-1})^\kappa}{\kappa}}<1,$$

$\mathcal{G}$ is a contraction mapping. Then, $\mathcal{G}$ has a unique fixed point such that $\mathcal{G}\gamma_Q=\gamma_Q$. □

Theorem 5.4. *Let* (G_1)*,* (G_2)*,* (G_4)*, and* (G_5) *be satisfied. If*

$$\rho:=\varpi^q\prod_{l=1}^{q}\chi_l<1,$$

the Eq (1.3) has at least one $\delta-$*periodic solution and* $\|\gamma_Q\|\leq\psi:=\frac{\widetilde{\rho}}{1-\rho}$*, where*

$$\begin{aligned}\widetilde{\rho} &= \frac{\overline{\mathcal{A}}}{|\psi(\mathcal{B})+\theta|}\sum_{l=1}^{q-1}\prod_{j=l}^{q-1}\varpi^{q-j+1}\chi_q\ldots\chi_{j+1}(\chi_j-1)\\ &\quad+\Big(\sum_{l=2}^{q}\prod_{j=l}^{q}\varpi^{q-j+1}\chi_q\ldots\chi_j+1\Big)\overline{b}+\frac{\overline{\mathcal{A}}\varpi}{|\psi(\mathcal{B})+\theta|}(\chi_q-1).\end{aligned}$$

Proof. $\|\gamma_Q\|\leq\psi$ and (5.3) imply

$$\|\mathcal{G}(\gamma_Q)\|\leq\rho\|\gamma_Q\|+\widetilde{\rho}\leq\psi.$$

Then $\mathcal{G}:\overline{B(0,\psi)}\to\overline{B(0,\psi)}$. Brouwers fixed-point theorem implies that $\mathcal{G}$ has fixed points, and Theorem 5.2 obtains that (1.3) has at least one $\delta-$periodic solution. □

6. Examples

Example 6.1. *Consider (1.1). Let* $\varsigma_0 = \frac{1}{2},\ \kappa = \frac{1}{2},\ \varsigma_l = l + \frac{1}{2},\ \iota_l = l,\ q = 1,\ \delta = 1,\ l \in \mathbb{N}$, *and*

$$\mathcal{B} = \begin{pmatrix} -10 & 0 \\ 0 & -5 \end{pmatrix} \in \mathbb{R}^{2\times 2},\ C_l = \begin{pmatrix} e^2 - 1 & 0 \\ 0 & e^2 - 1 \end{pmatrix} \in \mathbb{R}^{2\times 2},$$
$$\gamma_Q = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{R}^2.$$

And we can obtain

$$e^{\mathcal{B}\iota} = \begin{pmatrix} e^{-10\iota} & 0 \\ 0 & e^{-5\iota} \end{pmatrix},\ \mathcal{I} + C_l = \begin{pmatrix} e^2 & 0 \\ 0 & e^2 \end{pmatrix},$$

so $\|\mathcal{I} + C_l\| = e^2$, $\ln \varrho = 2$ *and*

$$\begin{aligned} \Xi(\iota, Q) &= \prod_{l=1}^{\phi(Q,\iota)} (\mathcal{I} + C_l) e^{\mathcal{B}\left[\left(\frac{(\iota - \varsigma_{\phi(Q,\iota)})^\kappa}{\kappa}\right)_+ + \sum_{l=0}^{\phi(Q,\iota)-1} \frac{(\iota_{l+1} - \varsigma_l)^\kappa}{\kappa}\right]} \\ &= \begin{pmatrix} e^{2\phi(Q,\iota)} & 0 \\ 0 & e^{2\phi(Q,\iota)} \end{pmatrix} \times \begin{pmatrix} e^{-10(2(\iota - (\phi(Q,\iota) + \frac{1}{2}))_+^{\frac{1}{2}} + \phi(Q,\iota)\sqrt{2})} & 0 \\ 0 & e^{-5(2(\iota - (\phi(Q,\iota) + \frac{1}{2}))_+^{\frac{1}{2}} + \phi(Q,\iota)\sqrt{2})} \end{pmatrix} \end{aligned}$$

Then, we set $\varepsilon = \theta = 0.5$ *and obtain*

$$\begin{aligned} \|\Xi(\iota, Q)\| &= e^{-5(2(\iota - (\phi(Q,\iota) + \frac{1}{2}))_+^{\frac{1}{2}} + \phi(Q,\iota)\sqrt{2})} e^{2\phi(Q,\iota)} \\ &\le e^{4.5\sqrt{2}} e^{\phi(Q,\iota)(2 - 4.5\sqrt{2})} \\ &\le e^{4.5\sqrt{2}} e^{-(0.5-1)(2-4.5\sqrt{2})(\iota - \frac{1}{2})}, \end{aligned}$$

in which $L = e^{4.5\sqrt{2}} > 1$ *and* $v = 0.5(4.5\sqrt{2} - 2) > 0$.

Thus, (1.1) is exponentially stable with $L = e^{4.5\sqrt{2}}$ *and* $v = 0.5(4.5\sqrt{2} - 2)$.

Further,

$$\Xi(Q + \delta, Q) = (\mathcal{I} + C_1) e^{\mathcal{B}\frac{(\iota_1 - \varsigma_0)^\kappa}{\kappa}} = \begin{pmatrix} e^2 & 0 \\ 0 & e^2 \end{pmatrix} \begin{pmatrix} e^{-10\sqrt{2}} & 0 \\ 0 & e^{-5\sqrt{2}} \end{pmatrix} = \begin{pmatrix} e^{2-10\sqrt{2}} & 0 \\ 0 & e^{2-5\sqrt{2}} \end{pmatrix}.$$

$$rank(\mathcal{I} - \Xi(Q + \delta, Q)) = n.$$

Then (1.1) only has the trivial $1-$*periodic solution.*

Example 6.2. *Consider (1.2) and* $\kappa = \frac{1}{2}, \varsigma_0 = 0, \varsigma_l = l, \iota_l = l - \frac{1}{2}, \delta = 1, q = 1, l \in N$,

$$a(\iota) = \begin{pmatrix} \iota - \varsigma_l \\ 0 \end{pmatrix},\ \iota \in (\varsigma_l, \iota_{l+1}],\ l \in \mathbb{N}.$$

Set

$$\mathcal{B} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, C_l = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, b_l = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

It is easily obtain

$$\Xi(\iota,\varsigma)=\prod_{l=\phi(Q,\varsigma)+1}^{\phi(Q,\iota)}(\mathcal{I}+C_l)e^{\mathcal{B}\left[\left(\frac{(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa}}{\kappa}\right)_+-\left(\frac{(\varsigma-\varsigma_{\phi(Q,\varsigma)})^{\kappa}}{\kappa}\right)_++\sum_{l=\phi(Q,\varsigma)}^{\phi(Q,\iota)-1}\frac{(\iota_{l+1}-\varsigma_l)^{\kappa}}{\kappa}\right]}$$

$$=\begin{pmatrix}1 & \phi(Q,\iota)-\phi(Q,\varsigma)\\ 0 & 1\end{pmatrix}e^{\left(2(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa}\right)_+-\left(2(\varsigma-\varsigma_{\phi(Q,\varsigma)})^{\kappa}\right)_++\sum_{l=\phi(Q,\varsigma)}^{\phi(Q,\iota)-1}2(\iota_{l+1}-\varsigma_l)^{\kappa}}$$

$$\times\begin{pmatrix}1 & (4(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa})_+-(4(\varsigma-\varsigma_{\phi(Q,\varsigma)})^{\kappa})_++\sum_{l=\phi(Q,\varsigma)}^{\phi(Q,\iota)-1}4(\iota_{l+1}-\varsigma_l)^{\kappa}\\ 0 & 1\end{pmatrix}$$

$$=e^{\left(2(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa}\right)_+-\left(2(\varsigma-\varsigma_{\phi(Q,\varsigma)})^{\kappa}\right)_++\sum_{l=\phi(Q,\varsigma)}^{\phi(Q,\iota)-1}2(\iota_{l+1}-\varsigma_l)^{\kappa}}$$

$$\times\begin{pmatrix}1 & (4(\iota-\varsigma_{\phi(Q,\iota)})^{\kappa})_+-(4(\varsigma-\varsigma_{\phi(Q,\varsigma)})^{\kappa})_++\sum_{l=\phi(Q,\varsigma)}^{\phi(Q,\iota)-1}4(\iota_{l+1}-\varsigma_l)^{\kappa}+\phi(Q,\iota)-\phi(Q,\varsigma)\\ 0 & 1\end{pmatrix}.$$

Next,

$$\begin{aligned}\Gamma_q &= \int_{\varsigma_0}^{\iota_1}\Xi(1,\varsigma)a(\varsigma)(\varsigma-\varsigma_0)^{\kappa-1}d\varsigma+\sum_{l=1}^{\phi(Q,1)}\Xi(1,\varsigma_l)b_l\\ &= \begin{pmatrix}\frac{e^{\sqrt{2}}}{2}-\frac{\sqrt{2}}{2}\\ 0\end{pmatrix}.\end{aligned}$$

By

$$\Xi(1,\varsigma_0)=\begin{pmatrix}e^{\sqrt{2}} & e^{\sqrt{2}}+2^{1.5}e^{\sqrt{2}}\\ 0 & e^{\sqrt{2}}\end{pmatrix},$$

and

$$(\mathcal{I}-\Xi(1,\varsigma_0))^{-1}=\begin{pmatrix}\frac{1}{1-e^{\sqrt{2}}} & -\frac{e^{\sqrt{2}}+2^{1.5}e^{\sqrt{2}}}{1-2e^{\sqrt{2}}+e^{2\sqrt{2}}}\\ 0 & \frac{1}{1-e^{\sqrt{2}}}\end{pmatrix},$$

so

$$\gamma_Q=(\mathcal{I}-\Xi(1,\varsigma_0))^{-1}\Gamma_q=\begin{pmatrix}\frac{\frac{e^{\sqrt{2}}}{2}-\frac{\sqrt{2}}{2}}{1-e^{\sqrt{2}}}\\ 0\end{pmatrix}.$$

Thus, for $t\in(\varsigma_{\phi(Q,\iota)},\iota_{\phi(Q,\iota)+1}]$,

$$\begin{aligned}\gamma(\iota,Q,\gamma_Q)=\Xi(\iota,Q)\gamma_Q&+\sum_{l=0}^{\phi(Q,\iota)-1}\int_{\varsigma_l}^{\iota_{l+1}}\Xi(\iota,\varsigma)a(\varsigma)(\varsigma-\varsigma_l)^{\kappa-1}d\varsigma\\ &+\int_{\varsigma_{\phi(Q,\iota)}}^{\iota}\Xi(\iota,\varsigma)a(\varsigma)(\varsigma-\varsigma_{\phi(Q,\iota)})^{\kappa-1}d\varsigma+\sum_{l=1}^{\phi(Q,\iota)}\Xi(\iota,\varsigma_l)b_l\end{aligned}$$

$$
\begin{aligned}
&= e^{2(\iota-\varsigma_{\phi(Q,\iota)})^{\frac{1}{2}}+\sqrt{2}\phi(Q,\iota)} \begin{pmatrix} \frac{\frac{e^{\sqrt{2}}}{2}-\frac{\sqrt{2}}{2}}{1-e^{\sqrt{2}}} \\ 0 \end{pmatrix} \\
&+ \sum_{l=0}^{\phi(Q,\iota)-1} \int_{\varsigma_l}^{\iota_{l+1}} \begin{pmatrix} e^{2(\iota-\varsigma_{\phi(Q,\iota)})^{\frac{1}{2}}-2(\varsigma-\varsigma_{\phi(Q,\varsigma)})^{\frac{1}{2}}+\sum\limits_{l=\phi(Q,\varsigma)}^{\phi(Q,\iota)-1} 2(\iota_{l+1}-\varsigma_l)^{\frac{1}{2}}} (\varsigma-\varsigma_l)^{\frac{1}{2}} \\ 0 \end{pmatrix} d\varsigma \\
&+ \int_{\varsigma_{\phi(Q,\iota)}}^{\iota} \begin{pmatrix} e^{2(\iota-\varsigma_{\phi(Q,\iota)})^{\frac{1}{2}}-2(\varsigma-\varsigma_{\phi(Q,\varsigma)})^{\frac{1}{2}}} (\varsigma-\varsigma_{\phi(Q,\iota)})^{\frac{1}{2}} \\ 0 \end{pmatrix} d\varsigma \\
&+ \sum_{l=1}^{\phi(Q,\iota)} \begin{pmatrix} e^{2(\iota-\varsigma_{\phi(Q,\iota)})^{\frac{1}{2}}+\sum\limits_{l=\phi(Q,\varsigma_l)}^{\phi(Q,\iota)-1} 2(\iota_{l+1}-\varsigma_l)^{\frac{1}{2}}} \\ 0 \end{pmatrix} \\
&= e^{2(\iota-\varsigma_{\phi(Q,\iota)})^{\frac{1}{2}}} \Bigg[\begin{pmatrix} e^{\sqrt{2}\phi(Q,\iota)} \frac{\frac{e^{\sqrt{2}}}{2}-\frac{\sqrt{2}}{2}}{1-e^{\sqrt{2}}} \\ 0 \end{pmatrix} \\
&+ \sum_{l=0}^{\phi(Q,\iota)-1} \int_{\varsigma_l}^{\iota_{l+1}} \begin{pmatrix} e^{-2(\varsigma-\varsigma_{\phi(Q,\varsigma)})^{\frac{1}{2}}+\sum\limits_{l=\phi(Q,\varsigma)}^{\phi(Q,\iota)-1} 2(\iota_{l+1}-\varsigma_l)^{\frac{1}{2}}} (\varsigma-\varsigma_l)^{\frac{1}{2}} \\ 0 \end{pmatrix} d\varsigma \\
&+ \int_{\varsigma_{\phi(Q,\iota)}}^{\iota} \begin{pmatrix} e^{-2(\varsigma-\varsigma_{\phi(Q,\varsigma)})^{\frac{1}{2}}} (\varsigma-\varsigma_{\phi(Q,\iota)})^{\frac{1}{2}} \\ 0 \end{pmatrix} d\varsigma + \sum_{l=1}^{\phi(Q,\iota)} \begin{pmatrix} e^{\sum\limits_{l=\phi(Q,\varsigma_l)}^{\phi(Q,\iota)-1} 2(\iota_{l+1}-\varsigma_l)^{\frac{1}{2}}} \\ 0 \end{pmatrix} \Bigg] \\
&= e^{2(\iota-\varsigma_{\phi(Q,\iota)})^{\frac{1}{2}}} \\
&\times \begin{pmatrix} e^{\sqrt{2}\phi(Q,\iota)} \frac{\frac{e^{\sqrt{2}}}{2}-\frac{\sqrt{2}}{2}}{1-e^{\sqrt{2}}} + (\frac{1}{2}-\frac{\sqrt{2}}{2}e^{-\sqrt{2}}-e^{-\sqrt{2}}) \times \frac{e^{\sqrt{2}}(1-e^{\sqrt{2}\phi(Q,\iota)})}{1-e^{\sqrt{2}}} \\ 0 \end{pmatrix} \\
&+ e^{2(\iota-\varsigma_{\phi(Q,\iota)})^{\frac{1}{2}}} \\
&\times \begin{pmatrix} -(\iota-\varsigma_{\phi(Q,\iota)}) e^{-2(\iota-\varsigma_{\phi(Q,\iota)})^{\frac{1}{2}}} - (\iota-\varsigma_{\phi(Q,\iota)})^{\frac{1}{2}} e^{-2(\iota-\varsigma_{\phi(Q,\iota)})^{\frac{1}{2}}} - \frac{1}{2} e^{-2(\iota-\varsigma_{\phi(Q,\iota)})^{\frac{1}{2}}} + \frac{1}{2} \\ 0 \end{pmatrix} \\
&+ e^{2(\iota-\varsigma_{\phi(Q,\iota)})^{\frac{1}{2}}} \begin{pmatrix} \frac{1-e^{\sqrt{2}\phi(Q,\iota)}}{1-e^{\sqrt{2}}} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\frac{e^{\sqrt{2}}}{2}-\frac{\sqrt{2}}{2}}{1-e^{\sqrt{2}}} e^{2(\iota-\varsigma_{\phi(Q,\iota)})^{\frac{1}{2}}} - (\iota-\varsigma_{\phi(Q,\iota)}) - (\iota-\varsigma_{\phi(Q,\iota)})^{\frac{1}{2}} - \frac{1}{2} + \frac{1}{2} e^{2(\iota-\varsigma_{\phi(Q,\iota)})^{\frac{1}{2}}} \\ 0 \end{pmatrix}.
\end{aligned}
$$

Then

$$\gamma(\iota+1, 0, \gamma_Q) = \gamma(\iota, 0, \gamma_Q),$$

so there is a 1*-periodic solution. The component of the solution is in Figure 1. Further,*

$$\|\gamma(\iota)\| \leq \frac{\frac{e^{\sqrt{2}}}{2}-\frac{\sqrt{2}}{2}}{1-e^{\sqrt{2}}} e^{\sqrt{2}} + \frac{1}{2} e^{\sqrt{2}},$$

and Theorem 4.2 is verified.

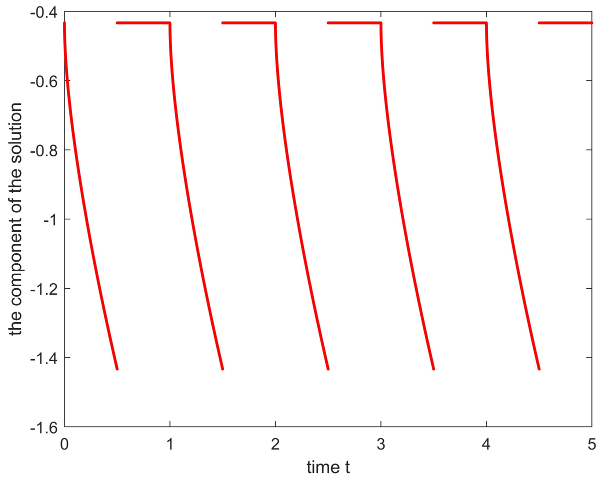

Figure 1. The component of the solution for Example 6.2.

Example 6.3. *Consider (1.3) and* $\kappa = \frac{1}{2}, \varsigma_0 = 0, \varsigma_l = l, \iota_l = l - \frac{1}{2}, \delta = 1, q = 1, l \in N,$

$$\mathcal{A}(\iota, \gamma(\iota)) = \begin{pmatrix} (\iota - \varsigma_l)\cos\gamma \\ 0 \end{pmatrix}, \ \iota \in (\varsigma_l, \iota_{l+1}], \ l \in \mathbb{N}.$$

Set

$$\mathcal{B} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, C_l = \begin{pmatrix} -\frac{9}{10} & 0 \\ 0 & -\frac{9}{10} \end{pmatrix}, b_l = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \gamma_Q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Next, $\overline{\mathcal{A}} = \frac{1}{2}$, $\chi_1 = e^{|\varphi(\mathcal{B})+\theta|\frac{(\iota_1-\varsigma_0)^\kappa}{\kappa}} = e^{\sqrt{2}}$, $\varpi = \varrho L_\theta = \frac{1}{10}$, $\rho := \varpi^q \prod_{l=1}^{q} \chi_l = \frac{e^{\sqrt{2}}}{10} < 1$ *and*

$$\widetilde{\rho} = \frac{\overline{\mathcal{A}}\varpi}{|\varphi(\mathcal{B}) + \theta|}(\chi_1 - 1) + \overline{b} = \frac{e^{\sqrt{2}} - 1}{20} + 1.$$

Thus, (1.3) has at least one $1-$*periodic solution and* $1 = \|\gamma_Q\| < \psi = 1.96$.

7. Conclusions

In this paper, we investigate the existence and stability of solutions for periodic conformable systems with non-instantaneous impulses. A key focus is the introduction and analysis of the conformable Cauchy matrix and its properties. We conduct separate studies on linear homogeneous, linear nonhomogeneous, and nonlinear systems. For the linear nonhomogeneous system, we employ the constant variation method to derive the solution expression. Moreover, we discuss the existence of periodic solutions for the linear nonhomogeneous system under two conditions. Regarding the nonlinear system, the existence and uniqueness of periodic solutions are transformed into the existence and uniqueness of fixed points of a corresponding operator. This transformation enables us to utilize related fixed-point theory to analyze the existence of periodic solutions.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no conflict of interest.

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