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Extremals for a weighted Morrey's inequality and a weighted p-Laplace equation

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Abstract: We establish a weighted Morrey's inequality. Furthermore, the existence of extremals for this weighted Morrey's inequality is studied. As an application, we prove that extremals are the weak solutions of a related weighted p-Laplace equation.

Keywords: weighted Morrey's inequality; extremal functions; p-Laplace operator
Mathematics Subject Classification: 35A15, 35A23

1. Introduction

From this article, we focus on the extremals for a weighted Morrey's inequality [2, 3, 8–12] with exponential weight and apply our findings to build the existence of the solutions for a nonlinear partial differential equation involving a weighted p-Laplace operator [20,21]. We know that partial differential equations with Laplace operator [7, 16] play an important role in calculus of variations [18], partial differential equations [4, 6, 19], potential theory [13], function theory, physics and calculus of probability. The p-Laplace operator generalizes the Laplace operator to better model nonlinear phenomena in various fields, such as non-Newtonian fluid dynamics, image processing, nonlinear elasticity, population dynamics, nonlinear heat conduction, and electromagnetic fields. Therefore, by studying the extremals of Morrey's inequality with exponential weight, we discuss whether the solutions of corresponding nonlinear partial differential equations with the p-Laplace operator exist. The purposes of this article are to construct a Morrey's inequality with exponential weight and to explore the properties of solutions for the corresponding nonlinear partial differential equation.

Morrey's inequality is the case of Sobolev's inequality under the condition $n < p < \infty$. It asserts that there is a constant $c_{q,n}$ which depends on q and n, thus

$$||u||_{C^{0,\tau}(\mathbb{R}^n)} \leq c_{q,n}||u||_{W^{1,q}(\mathbb{R}^n)}$$

for any $u \in C^1(\mathbb{R}^n)$ and $\tau = 1 - \frac{n}{q}$, $||u||_{C^{0,\tau}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} |u(x)| + \sup_{\substack{x,y\in\mathbb{R}^n \\ x\neq y}} \left\{\frac{|u(x)-u(y)|}{|x-y|^\tau}\right\}$, $||u||_{W^{1,q}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u|^q dx + \int_{\mathbb{R}^n} |\nabla u|^q dx\right)^{\frac{1}{q}}$.

In paper [10], the existence of extremals for Morrey's inequality was studied by R. Hynd and F. Seuffert through certain invariances. They obtained that there exists a nonconstant $u \in \mathscr{W}^{1,q}(\mathbb{R}^n)$ such that

$$[u]_{C^{1-\frac{n}{q}}(\mathbb{R}^n)} = C_*||\nabla u||_{L^q(\mathbb{R}^n)}$$

where $\mathscr{W}^{1,q}(\mathbb{R}^n) = \{u \in L^1_{loc}(\mathbb{R}^n) : u_{xi} \in L^q(\mathbb{R}^n) \text{ for } i = 1, \cdots, n\}$, $C_* > 0$. Moreover, they concluded that for any $\phi \in \mathscr{W}^{1,q}(\mathbb{R}^n)$,

$$C_*^q \int_{\mathbb{R}^n} |\nabla u|^{q-2}\nabla u \nabla\phi dx = \frac{|u(x_0) - u(y_0)|^{q-2}(u(x_0) - u(y_0))}{|x_0 - y_0|^{q-n}}(\phi(x_0) - \phi(y_0))$$

if $u \in \mathscr{D}^{1,q}(\mathbb{R}^n)$ is an extremal and there are two different points $x_0, y_0 \in \mathbb{R}^n$ where the $1 - \frac{n}{q}$ Hölder ratio of this extremal reaches its maximum.

In recent years, weighted isoperimetric inequalities [5], weighted Sobolev inequality, weighted Morrey's inequality, and weighted Moser–Trudinger inequality have received too much attention. For instance, X. Cabré and X. Ros-Oton studied Morrey's inequality with monomial weight in paper [1]. In a more in-depth study of the asymptotic behavior or stability of solutions to the weighted p-Laplacian equation, the exponentially weighted Morrey inequality can be used to handle situations where the region grows rapidly, thereby providing finer control. Based on this, exponential will play an important role and it is specifically considered in the generation of the p-Laplacian. Inspired by these, we build Morrey's inequality with exponential weight after studying R. Hynd and F. Seuffert's researches on weighted inequalities [14, 17, 22] as follows:

Theorem 1.1. *Let $n < q < \infty$, a constant $C_{q,n}$ which depends on q and n exists so that*

$$[u]_{C^{0,\tau}_{e^{|z|}}(\mathbb{R}^n)} \leq C_{q,n}||u||_{W^{1,q}_{e^{|x|}}(\mathbb{R}^n)} \tag{1.1}$$

for each $u \in C^1(\mathbb{R}^n)$ and $\tau = 1 - \frac{n}{q}$.

It is impossible to prove Morrey's inequality with exponential weight by directly using the traditional method of establishing Morrey's inequality [4]. We improve the traditional method to prove Morrey's inequality with exponential weight.

Subsequently, studying the properties of the extremals of Morrey's inequality with weight is of considerable significance for exploring the properties of solutions to a nonlinear partial differential equation with a weighted p-Laplace operator. Therefore, using the approach to research the extremals of Morrey's inequality in paper [12], we derive as

Theorem 1.2. *There is a nonconstant $u \in W^{1,q}_{rad,e^{|x|}}(\mathbb{R}^n)$ which makes*

$$[u]_{C^{0,\tau}_{e^{|z|}}(\mathbb{R}^n)} = C_*||u||_{W^{1,q}_{rad,e^{|x|}}(\mathbb{R}^n)}$$

where the weighted Sobolev space of radial functions $||u||_{W^{1,q}_{rad,e^{|x|}}(\mathbb{R}^n)} = \left(\sum_{|\alpha|\leq m} \int_{\mathbb{R}^n} |D^\alpha u|^q e^{|x|} dx\right)^{\frac{1}{q}}$.
In particular, $C_* > 0$ is the sharp constant in Theorem 1.1.

How to construct a sequence of functions to discuss the existence of extremals by using the invariance of norm is not easy. By improving the construction method in [12], we prove the existence of extremals by using Arzelà–Ascoli theorem and then analyze its existence.

This is how the rest is listed. We provide some definitions and notations for subsequent proof in the second section. The demonstration of a weighted Morrey's inequality is listed in Section 3. Finally, the existence of extremals for this weighted Morrey's inequality is established in Section 4.

2. Preliminaries

Some notations, concepts, and definitions related to this article are given in this section for the convenience of understanding.

Definition 2.1. For $f, g \in L^1_{loc}(\Omega)$ and multiindex β, if

$$\int_\Omega f D^\beta \omega dx = (-1)^{|\beta|} \int_\Omega g\omega dx$$

for any test formula $\omega \in C_c^\infty(U)$, then g is the α^{th}-weak partial derivative of f and consider it as $D^\alpha f = g$.

Definition 2.2. Sobolev space $W^{k,p}(U)$ is a set that has all local summable formulas $u : U \to \mathbb{R}$ satisfies the existence of $D^\alpha u$ and belongs to $L^p(U)$.

In this paper, weight means a locally integrable formula w on $\mathbb{R}^n$; thus, $w(x) > 0$ for almost everywhere $x \in \mathbb{R}^n$.

Definition 2.3. For $0 < p < \infty$, the space $L^p_\omega(\Omega)$ is composed of all measurable formulas g on Ω as

$$||g||_{L^p_\omega(\Omega)} = \left(\int_\Omega |g|^p \omega dx \right)^{\frac{1}{p}} < \infty$$

where ω is the weight and $\Omega \subset \mathbb{R}^n$ is an open set.

Definition 2.4. (Weighted Sobolev space $W^{m,p}_w(U)$) The weighted Sobolev space $W^{m,p}_w(U)$ can be the functions $u \in L^p_w(U)$ with $D^\beta u \in L^p_w(U)$ for $|\beta| \leq l$. Thus, the norm of u in $W^{l,p}_w(U)$ is

$$||u||_{W^{1,p}_w(U)} = \left(\sum_{|\alpha| \leq l} \int_U |D^\beta u|^p w dx \right)^{\frac{1}{p}}.$$

Consider $u : \Omega \to \mathbb{R}$ as a bounded and continuous function, then

$$||u||_{C_{e^{|x|}}(\bar{\Omega})} = \sup_{x \in \Omega} \left| e^{|x|} u(x) \right|.$$

The seminorm of u: $U \to \mathbb{R}$ with τ^{th}-exponential weight Hölder is

$$[u]_{C^{0,\tau}_{e^{|z|}}(\bar{\Omega})} = \sup_{\substack{x,y \in \Omega \\ x \neq y}} \left\{ \frac{|e^{\frac{|x|}{p}} u(x) - e^{\frac{|y|}{p}} u(y)|}{|x-y|^\tau} \right\}.$$

The norm with τ^{th}-exponential weight Hölder is

$$||u||_{C^{0,\tau}_{e^{|z|}}(\bar{U})} = ||u||_{C_{e^{|x|}}(\bar{U})} + [u]_{C^{0,\tau}_{e^{|z|}}(\bar{U})}.$$

Definition 2.5. The space with τ^{th}-exponential weight Hölder $C^{0,\tau}(\bar{U})$ is composed of all functions u as

$$||u||_{C^{0,\tau}_{e^{|z|}}(\bar{U})} = ||u||_{C_{e^{|x|}}(\bar{U})} + [u]_{C^{0,\tau}_{e^{|z|}}(\bar{U})}$$

is finite.

The subspace consisting of radially symmetric functions in $W^{m,q}_{w}(\Omega)$ is written as $W^{m,q}_{rad,w}(\Omega)$.

Maximized Hölder ratio. An extremal u has two different points $a, b \in \mathbb{R}^n$ where the $1 - \frac{n}{q}$ Hölder ratio of this extremal reaches its maximum. That is

$$\sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \frac{|e^{\frac{|x|}{q}} u(x) - e^{\frac{|y|}{q}} u(y)|}{|x-y|^{\tau}} \right\} = \frac{|e^{\frac{|a|}{q}} u(a) - e^{\frac{|b|}{q}} u(b)|}{|a-b|^{\tau}}. \tag{2.1}$$

The type of weighted p − Laplace equations. Assume u is an extremal that satisfies (2.1), and δ denote the Dirac function. Then, PDE

$$-div\left(e^{|x|}|\nabla u|^{q-2}\nabla u\right) + e^{|x|}u^{q-1} = \frac{\vartheta}{C_*^q}\left(e^{\frac{|a|}{q}}\delta_a - e^{\frac{|b|}{q}}\delta_b\right)$$

is true in $\mathbb{R}^n$ where $\vartheta = \frac{\left|e^{\frac{|a|}{q}} u(a) - e^{\frac{|b|}{q}} u(b)\right|^{q-2}\left(e^{\frac{|a|}{q}} u(a) - e^{\frac{|b|}{q}} u(b)\right)}{|a-b|^{q-n}}$ and C_* is the sharp constant for this weighted Morrey's inequality.

3. Proof of Morrey's inequality with exponential weight

Proof of Theorem 1.1. First,

$$\int_{B^n(x,r)} \left|e^{\frac{|y|}{q}} u(y) - e^{\frac{|x|}{q}} u(x)\right| dy \le \frac{r^n}{n}\int_{B^n(x,r)} \frac{e^{\frac{|y|}{q}}\nabla u(y)}{|x-y|^{n-1}}dy + \frac{r^n}{qn}\int_{B^n(x,r)}\frac{\left|e^{\frac{|y|}{q}} u(y)\right|}{|x-y|^{n-1}}dy. \tag{3.1}$$

For the point $\nu \in \partial B(0,1)$ and $0 < l < r$, it can be derived that

$$\begin{aligned}
&\left|e^{\frac{|x+l\nu|}{q}} u(x+l\nu) - e^{\frac{|x|}{q}} u(x)\right| \\
&= \left|\int_0^l \frac{d}{dh} e^{\frac{|x+h\nu|}{q}} u(x+h\nu)dh\right| \\
&= \left|\int_0^l \frac{1}{q}\frac{(x+h\nu)\nu}{|x+h\nu|} e^{\frac{|x+h\nu|}{q}} u(x+h\nu)dh + \int_0^l e^{\frac{|x+h\nu|}{q}}\nabla u(x+h\nu)dh\right| \\
&\le \int_0^l \frac{1}{q}\left|e^{\frac{|x+h\nu|}{q}} u(x+h\nu)dh\right| + \int_0^l \left|e^{\frac{|x+h\nu|}{q}}\nabla u(x+h\nu)\right| dh.
\end{aligned}$$

Therefore,

$$\int_0^l \int_{S^{n-1}(0,1)} \left|e^{\frac{|x+l\nu|}{q}} u(x+l\nu) - e^{\frac{|x|}{q}} u(x)\right| dS(\nu)$$

$$\leq \int_0^l \int_{S^{n-1}(0,1)} \left| e^{\frac{|x+hv|}{q}} \nabla u(x+hv) \right| dS(v)dh + \int_0^l \int_{S^{n-1}(0,1)} \frac{1}{q} e^{\frac{|x+hv|}{q}} \left| u(x+hv) \right| dS(v)dh.$$

Following the first inequality on the right-hand side above, it must be obtained that

$$\begin{aligned}
&\int_0^l \int_{S^{n-1}(0,1)} \left| e^{\frac{|x+hv|}{q}} \nabla u(x+hv) \right| dS(v)dh \\
&= \int_0^l \int_{S^{n-1}(x,t)} \frac{\left| e^{\frac{|x+hv|}{q}} \nabla u(x+hv) \right|}{h^{n-1}} dS(y)dh \\
&= \int_{B^n(x,l)} \frac{\left| e^{\frac{|x+hv|}{q}} | \nabla u(x+hv) \right|}{|x-y|^{n-1}} dy \\
&\leq \int_{B^n(x,r)} \frac{\left| e^{\frac{|y|}{q}} \nabla u(y) \right|}{|x-y|^{n-1}} dy.
\end{aligned}$$

Following the second inequality on the right-hand side above, it implies that

$$\begin{aligned}
&\int_0^l \int_{S^{n-1}(0,1)} \frac{1}{q} e^{\frac{|x+hv|}{q}} |u(x+hv)| dS(v)dh \\
&= \int_0^l \int_{S^{n-1}(x,t)} \frac{1}{q} \frac{e^{\frac{|x+hv|}{q}} |u(x+hv)|}{h^{n-1}} dS(y)dh \\
&= \int_{B^n(x,l)} \frac{1}{q} \frac{e^{\frac{|x+hv|}{q}} |u(x+hv)|}{|x-y|^{n-1}} dy \\
&\leq \int_{B^n(x,r)} \frac{1}{q} \frac{\left| e^{\frac{|y|}{q}} u(y) \right|}{|x-y|^{n-1}} dy.
\end{aligned}$$

Then

$$\int_{S^{n-1}(0,1)} \left| e^{\frac{|x+lv|}{q}} u(x+lv) - e^{\frac{|x|}{q}} u(x) \right| dS(v) = \frac{1}{l^{n-1}} \int_{S^{n-1}(x,l)} \left| e^{\frac{|z|}{q}} u(z) - e^{\frac{|x|}{q}} u(x) \right| dS(z)$$

and

$$\int_{S^{n-1}(x,s)} \left| e^{\frac{|z|}{q}} u(z) - e^{\frac{|x|}{q}} u(x) \right| dS(z) \leq l^{n-1} \int_{B^n(x,r)} \frac{e^{\frac{|y|}{q}} \nabla u(y)}{|x-y|^{n-1}} dy + \frac{l^{n-1}}{q} \int_{B^n(x,r)} \frac{\left| e^{\frac{|y|}{q}} u(y) \right|}{|x-y|^{n-1}} dy.$$

Therefore, one can derive that

$$\int_{S^{n-1}(x,r)} \left| e^{\frac{|y|}{q}} u(y) - e^{\frac{|x|}{q}} u(x) \right| dy \leq \frac{r^n}{n} \int_{B^n(x,r)} \frac{e^{\frac{|y|}{q}} \nabla u(y)}{|x-y|^{n-1}} dy + \frac{r^n}{qn} \int_{B^n(x,r)} \frac{\left| e^{\frac{|y|}{q}} u(y) \right|}{|x-y|^{n-1}} dy.$$

By using (3.1),

$$e^{\frac{|x|}{q}} |u(x)|$$

$$\leq \fint_{B^n(x,1)} e^{\frac{|x|}{q}}|u(x)| - e^{\frac{|y|}{q}}|u(y)|dy + \fint_{B^n(x,1)} e^{\frac{|y|}{q}}|u(y)|dy$$
$$\leq \fint_{B^n(x,1)} \left|e^{\frac{|y|}{q}}u(y) - e^{\frac{|x|}{q}}u(x)\right| dy + \fint_{B^n(x,1)} e^{\frac{|y|}{q}}|u(y)|dy$$
$$\leq C_1\left(\int_{B^n(x,1)} \frac{\left|e^{\frac{|y|}{q}}\nabla u(y)\right|}{|x-y|^{n-1}}dy + \int_{B^n(x,1)} \frac{\left|e^{\frac{|y|}{q}}u(y)\right|}{|x-y|^{n-1}}dy\right) + \fint_{B^n(x,1)} e^{\frac{|y|}{q}}|u(y)|dy$$

for each $x \in \mathbb{R}^n$. Therefore,

$$e^{\frac{|x|}{q}}|u(x)| \leq C_1 \int_{B^n(x,1)} \frac{\left|e^{\frac{|y|}{q}}\nabla u(y)\right|}{|x-y|^{n-1}}dy + C_1 \int_{B^n(x,1)} \frac{\left|e^{\frac{|y|}{q}}u(y)\right|}{|x-y|^{n-1}}dy + \fint_{B^n(x,1)} \left|e^{\frac{|y|}{q}}u(y)\right| dy. \tag{3.2}$$

Following the first integral on the right-hand side of (3.2), according to the Hölder inequality,

$$\int_{B^n(x,1)} \frac{\left|e^{\frac{|y|}{q}}\nabla u(y)\right|}{|x-y|^{n-1}}dy$$
$$\leq \left(\int_{\mathbb{R}^n} \left|e^{\frac{|y|}{q}}\nabla u(y)\right|^q dy\right)^{\frac{1}{q}}\left(\int_{B^n(x,1)} \left(\frac{1}{|x-y|}\right)^{\frac{(n-1)q}{q-1}} dy\right)^{\frac{q-1}{q}}$$
$$\leq C_2\|u\|_{D^{1,q}_{e^{|y|}}(\mathbb{R}^n)}.$$

The last inequality holds because $(n-1)\frac{q}{q-1} < n$ by using $q > n$, so that

$$\int_{B^n(x,1)} \left(\frac{1}{|x-y|}\right)^{\frac{(n-1)q}{q-1}} dy < \infty.$$

Following the second integral on the right-hand side of (3.2), according to the Hölder inequality, it can be derived that

$$\int_{B^n(x,1)} \frac{|e^{\frac{|y|}{q}}u(y)|}{|x-y|^{n-1}}dy \leq \int_{B^n(x,1)} \left|e^{\frac{|y|}{q}}u(y)\right|^q dy^{\frac{1}{q}}\left(\int_{B^n(x,1)} \left(\frac{1}{|x-y|}\right)^{\frac{(n-1)q}{q-1}} dy\right)^{\frac{q-1}{q}} = C_3\|u\|_{L^q_{e^{|y|}}(\mathbb{R}^n)}.$$

Following the third integral on the right-hand side of (3.2), according to the Hölder inequality,

$$\fint_{B^n(x,1)} \left|e^{\frac{|y|}{q}}u(y)\right| dy \leq \left(\int_{B^n(x,1)} \left|e^{|y|}u(y)^q\right| dy\right)^{\frac{1}{q}} = C_4\|u\|_{L^q_{e^{|y|}}(\mathbb{R}^n)}.$$

Therefore,

$$e^{\frac{|x|}{q}}|u(x)| \leq C_2\|u\|_{D^{1,q}_{e^{|y|}}(\mathbb{R}^n)} + C_5\|u\|_{L^q_{e^{|y|}}(\mathbb{R}^n)}.$$

Next, it will be proved that

$$\frac{\left|e^{\frac{|x|}{q}}u(x) - e^{\frac{|y|}{q}}u(y)\right|}{|x-y|^{1-\frac{n}{q}}} \leq C_6\|u\|_{W^{1,q}_{e^{|x|}}(\mathbb{R}^n)}.$$

We take any $x, y \in \mathbb{R}^n$. $M = B^n(x,r) \cap B^n(y,r)$. Then,

$$\begin{aligned}
&\left|e^{\frac{|x|}{q}} u(x) - e^{\frac{|y|}{q}} u(y)\right| \\
&= \fint_M \left|e^{\frac{|x|}{q}} u(x) - e^{\frac{|y|}{q}} u(y)\right| dz \\
&= \fint_M \left|\left(e^{\frac{|x|}{q}} u(x) - e^{\frac{|z|}{q}} u(z)\right) - \left(e^{\frac{|y|}{q}} u(y) - e^{\frac{|z|}{q}} u(z)\right)\right| dz \\
&\leq \fint_M \left|e^{\frac{|x|}{q}} u(x) - e^{\frac{|z|}{q}} u(z)\right| dz + \fint_M \left|e^{\frac{|y|}{q}} u(y) - e^{\frac{|z|}{q}} u(z)\right| dz.
\end{aligned} \tag{3.3}$$

Following the first integral on the right-hand side of (3.3), according to inequality (3.1), one can obtain that

$$\begin{aligned}
&\fint_W \left|e^{\frac{|x|}{q}} u(x) - e^{\frac{|z|}{q}} u(z)\right| dz \\
&\leq C_7 \fint_{B^n(x,r)} \left|e^{\frac{|x|}{q}} u(x) - e^{\frac{|z|}{q}} u(z)\right| dz \\
&\leq C_7 \left(\int_{B^n(x,r)} \frac{e^{\frac{|z|}{q}} \nabla u(z)}{|x-z|^{n-1}} dz + \int_{B^n(x,r)} \frac{e^{\frac{|z|}{q}} u(z)}{|x-z|^{n-1}} dz \right).
\end{aligned} \tag{3.4}$$

Following the first integral on the right-hand side of (3.4), according to the Hölder inequality,

$$\begin{aligned}
&\int_{B^n(x,r)} \frac{e^{\frac{|z|}{q}} \nabla u(z)}{|x-z|^{n-1}} dz \\
&\leq \left(\int_{B^n(x,r)} e^{\frac{|z|}{q} q} |\nabla u(z)|^q dz \right)^{\frac{1}{q}} \left(\int_{B^n(x,r)} \left(\frac{1}{|x-z|^{n-1}} \right)^{\frac{q}{q-1}} dz \right)^{\frac{q-1}{q}} \\
&\leq \|u\|_{D^{1,q}_{e^{|z|}}(\mathbb{R}^n)} \left(r^{n-(n-1)\frac{q}{q-1}} \right)^{\frac{q-1}{q}} \\
&= r^{1-\frac{n}{q}} \|u\|_{D^{1,q}_{e^{|z|}}(\mathbb{R}^n)}.
\end{aligned}$$

Following the second integral on the right-hand side of (3.4), according to the Hölder inequality, one can derive that

$$\begin{aligned}
&\int_{B^n(x,r)} \frac{e^{\frac{|z|}{q}} u(z)}{|x-z|^{n-1}} dz \\
&\leq \left(\int_{B^n(x,r)} \left|e^{\frac{|z|}{q}} u(z)\right|^q dz \right)^{\frac{1}{q}} r^{1-\frac{n}{q}} \\
&= \left(\int_{B^n(x,r)} e^{|z|} |u^q(z)| dz \right)^{\frac{1}{q}} r^{1-\frac{n}{q}} \\
&= r^{1-\frac{n}{q}} \|u\|_{L^q_{e^{|z|}}(\mathbb{R}^n)}.
\end{aligned}$$

Likewise, for the second integral on the right-hand side of (3.3), then

$$\left|e^{\frac{|x|}{q}} u(x) - e^{\frac{|y|}{q}} u(y)\right| \leq C_7 r^{1-\frac{n}{q}} \|u\|_{L^q_{e^{|z|}}(\mathbb{R}^n)} + C_7 r^{1-\frac{n}{q}} \|u\|_{D^{1,q}_{e^{|z|}}(\mathbb{R}^n)}.$$

Therefore, we have

$$[u]_{C^{0,\tau}_{e^{|z|}}(\mathbb{R}^n)} \le C_{q,n}\|u\|_{W^{1,q}_{e^{|z|}}(\mathbb{R}^n)}.$$

□

4. Proof of the existence of extremals

We want to use a variation of the classical Arzelà-Ascoli theorem for getting a subsequence $(v_{j_m})_{m\in\mathbb{N}}$ which converge uniformly locally on a continual function $v : \mathbb{R}^n \to \mathbb{R}$. Moreover, v can be the nonconstant extremal of a weighted Morrey's inequality.

Proof of Theorem 1.2. First, we can define

$$\Lambda = \frac{1}{C} = \inf\left\{\|u\|_{W^{1,p}_{rad,e^{|x|}}(\mathbb{R}^n)} : u \in W^{1,p}_{rad,e^{|x|}}(\mathbb{R}^n), [u]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)} = 1\right\}$$

so that one can choose a minimizing sequence$(u_j)_{j\in\mathbb{N}}$ and make $\Lambda = \lim_{j\to\infty}\|u_j\|_{W^{1,p}_{rad,e^{|x|}}(\mathbb{R}^n)}$. Then, by choosing $x_j, y_j \in \mathbb{R}^n$, $x_j \neq y_j$ and $\lambda = |x_j - y_j|$ then

$$1 = [u_j]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)} < \frac{\left|e^{\frac{|x_j|}{p}}u_j(x_j) - e^{\frac{|y_j|}{p}}u_j(y_j)\right|}{|x_j - y_j|^{1-\frac{n}{p}}} + \frac{1}{j} \tag{4.1}$$

and

$$\lim_{j\to\infty}\frac{1}{\lambda} = 1.$$

Let the rotation transformation O_j be: $O_je_n = \frac{y_j-x_j}{|x_j-y_j|}$ where $e_n = (0,\cdots,0,1)$. Then we can construct two function sequences

$$v_{1j}(z) = |x_j - y_j|^{\frac{n}{p}-1}\left|\left(u_j(|x_j - y_j|O_jz + x_j) - u_j(x_j)\right)\right|$$

and

$$v_{2j}(z) = |x_j - y_j|^{\frac{n}{p}}\left|\left(u_j(|x_j - y_j|O_jz + x_j) - u_j(x_j)\right)\right|.$$

Let $v_{3j}(z) = \frac{1}{2}v_{1j}(z) + \frac{1}{2\lambda}v_{2j}(z)$ for $z \in \mathbb{R}^n$ and $j \in \mathbb{N}$. Applying the invariances of the seminorms $u \to [u]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)}$, $u \to \|u\|_{D^{1,p}_{e^{|x|}}(\mathbb{R}^n)}$ and $u \to \|u\|_{L^p_{e^{|x|}}(\mathbb{R}^n)}$, one can conclude $[v_{1j}]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)} = 1$, $[v_{2j}]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)} = \lambda$,

$$\begin{aligned}
&[v_{3j}]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)}\\
&= \left[\frac{1}{2}v_{1j} + \frac{1}{2\lambda}v_{2j}\right]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)}\\
&= \sup_{\substack{x,y\in\mathbb{R}^n\\ x\neq y}}\left\{\frac{\left|e^{\frac{|x|}{p}}\left(\frac{1}{2}v_{1j}(x) + \frac{1}{2\lambda}v_{2j}(x)\right) - e^{\frac{|y|}{p}}\left(\frac{1}{2}v_{1j}(y) + \frac{1}{2\lambda}v_{2j}(y)\right)\right|}{|x-y|^{1-\frac{n}{p}}}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\substack{x,y\in\mathbb{R}^n\\ x\neq y}} \left\{ \frac{\left| \frac{1}{2}e^{\frac{|x|}{p}}v_{1j}(x) - \frac{1}{2}e^{\frac{|y|}{p}}v_{1j}(y) + \frac{1}{2\lambda}e^{\frac{|x|}{p}}v_{2j}(x) - \frac{1}{2\lambda}e^{\frac{|y|}{p}}v_{2j}(y) \right|}{|x-y|^{1-\frac{n}{p}}} \right\} \\
&= \sup_{\substack{x,y\in\mathbb{R}^n\\ x\neq y}} \left\{ \frac{\frac{1}{2}\left| e^{\frac{|x|}{p}}v_{1j}(x) - e^{\frac{|y|}{p}}v_{1j}(y) \right| + \frac{1}{2\lambda}\left| e^{\frac{|x|}{p}}v_{2j}(x) - e^{\frac{|y|}{p}}v_{2j}(y) \right|}{|x-y|^{1-\frac{n}{p}}} \right\} \\
&= \frac{1}{2}[v_{1j}]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)} + \frac{1}{2\lambda}[v_{2j}]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)} \\
&= 1.
\end{aligned}
$$

The fourth equation above holds because $(\frac{1}{2}e^{\frac{|x|}{p}}v_{1j}(x) - \frac{1}{2}e^{\frac{|y|}{p}}v_{1j}(y))\frac{|x_j-y_j|}{\lambda} = \frac{1}{2\lambda}e^{\frac{|x|}{p}}v_{2j}(x) - \frac{1}{2\lambda}e^{\frac{|y|}{p}}v_{2j}(y)$ can be obtained from $v_{1j}(z)|x_j - y_j| = v_{2j}(z)$ so that $\frac{1}{2}e^{\frac{|x|}{p}}v_{1j}(x) - \frac{1}{2}e^{\frac{|y|}{p}}v_{1j}(y)$ and $\frac{1}{2\lambda}e^{\frac{|x|}{p}}v_{2j}(x) - \frac{1}{2\lambda}e^{\frac{|y|}{p}}v_{2j}(y)$ are both positive or negative. We also obtain that

$$
\begin{aligned}
&\|v_{3j}\|_{W^{1,p}_{rad,e^{|x|}}(\mathbb{R}^n)} \\
&= \|v_{3j}\|_{L^p_{e^{|x|}}(\mathbb{R}^n)} + \|v_{3j}\|_{D^{1,p}_{e^{|x|}}(\mathbb{R}^n)} \\
&= \left\| \frac{1}{2}v_{1j} + \frac{1}{2\lambda}v_{2j} \right\|_{L^p_{e^{|x|}}(\mathbb{R}^n)} + \left\| \frac{1}{2}v_{1j} + \frac{1}{2\lambda}v_{2j} \right\|_{D^{1,p}_{e^{|x|}}(\mathbb{R}^n)} \\
&= \frac{1}{2}\|v_{1j}\|_{L^p_{e^{|x|}}(\mathbb{R}^n)} + \frac{1}{2\lambda}\|v_{2j}\|_{L^p_{e^{|x|}}(\mathbb{R}^n)} + \frac{1}{2}\|v_{1j}\|_{D^{1,p}_{e^{|x|}}(\mathbb{R}^n)} + \frac{1}{2\lambda}\|v_{2j}\|_{D^{1,p}_{e^{|x|}}(\mathbb{R}^n)} \\
&= \frac{1}{2\lambda}\|u_j\|_{L^p_{e^{|x|}}(\mathbb{R}^n)} + \frac{1}{2\lambda}\|u_j\|_{L^p_{e^{|x|}}(\mathbb{R}^n)} + \frac{1}{2}\|u_j\|_{D^{1,p}_{e^{|x|}}(\mathbb{R}^n)} + \frac{1}{2}\|u_j\|_{D^{1,p}_{e^{|x|}}(\mathbb{R}^n)} \\
&= \frac{1}{\lambda}\|u_j\|_{L^p_{e^{|x|}}(\mathbb{R}^n)} + \|u_j\|_{D^{1,p}_{e^{|x|}}(\mathbb{R}^n)}.
\end{aligned}
$$

Since $\lim_{j\to\infty} \frac{1}{\lambda} = 1$, we have

$$
\lim_{j\to\infty} \|v_{3j}\|_{W^{1,p}_{rad,e^{|x|}}(\mathbb{R}^n)} = \lim_{j\to\infty} \left(\|u_j\|_{L^p_{e^{|x|}}(\mathbb{R}^n)} + \|u_j\|_{D^{1,p}_{e^{|x|}}(\mathbb{R}^n)} \right) = \|u_j\|_{W^{1,p}_{rad,e^{|x|}}(\mathbb{R}^n)} = \Lambda.
$$

Moreover, one can conclude $v_{3j}(0) = 0$ and

$$
v_{3j}(e_n) = \frac{1}{2}v_{1j}(e_n) + \frac{1}{2\lambda}v_{2j}(e_n) = \frac{1}{2}\frac{\left|u_j(y_j) - u_j(x_j)\right|}{|x_j - y_j|^{1-\frac{n}{p}}} + \frac{1}{2\lambda}\frac{\left|u_j(y_j) - u_j(x_j)\right|}{|x_j - y_j|^{\frac{n}{p}}} < 1 - \frac{1}{j}
$$

by using (4.1). Then, we use the Arzelà–Ascoli theorem in its variant form to obtain $(v_{k_m})_{m\in\mathbb{N}}$ which converge uniformly to a continuous formula locally $v : \mathbb{R}^n \to \mathbb{R}$.

Then one can derive that

$v(0) = 0$, $v(e_n) = 1$ and $[v]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)} \leq 1$. We can also conclude that

$$
1 = \frac{v(e_n) - v(0)}{|e_n - 0|^{1-\frac{n}{p}}} \leq [v]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)}.
$$

Therefore,

$$
[v]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)} = 1.
$$

Suppose that $(\nabla v_{j_m})_m$ converges weakly in $L^p_{e^{|z|}}(\mathbb{R}^n)$. This implies the weak limit of ∇v_{j_m} in $L^p_{e^{|z|}}(\mathbb{R}^n)$ can be the weak derivative of v. Therefore, we have $v \in W^{1,p}_{rad,e^{|x|}}(\mathbb{R}^n)$ and

$$\Lambda = \lim_{m\to\infty} \inf \|v_{j_m}\|_{W^{1,p}_{rad,e^{|x|}}(\mathbb{R}^n)} \geq \|v\|_{W^{1,p}_{rad,e^{|x|}}(\mathbb{R}^n)}.$$

Because

$$1 = [v]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)} \leq \frac{1}{\Lambda}\|v\|_{W^{1,p}_{rad,e^{|x|}}(\mathbb{R}^n)} \leq 1,$$

we can conclude that v is a nonconstant extremal of this weighted Morrey's inequality. □

Corollary 4.1. *Suppose $x_0, y_0 \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ are distinct and satisfy $\left(1 - e^{\frac{|x_0|}{p}}\right)\alpha = \left(1 - e^{\frac{|y_0|}{p}}\right)\beta$. Then there exists an extremal u that satisfies $u(x_0) = \alpha$, $u(y_0) = \beta$, and the $1-\frac{n}{p}$ Hölder ratio of this extremal reaches its maximum at x_0 and y_0.*

Proof of Corollary 4.1. We give a rotation transformation O, which makes $O\left(\frac{y_0-x_0}{|y_0-x_0|}\right) = e_n$ and let

$$u(x) = \left(e^{\frac{|y_0|}{p}}\beta - e^{\frac{|x_0|}{p}}\alpha\right)v\left(O\left(\frac{x-x_0}{|y_0-x_0|}\right)\right) + \alpha.$$

Since the proof of Theorem 1.2, we know that an extremal v exists that fulfills $v(0) = 0$, $v(e_n) = 1$, and the two points 0 and e_n where the $1-\frac{n}{p}$ Hölder ratio of this extremal reaches its maximum. Therefore, we can derive that $u(x_0) = \alpha$ and $u(y_0) = \beta$. Using translation invariance, rotation invariance and scaling invariance of $[u]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)}$, one can conclude that

$$\begin{aligned}
&[u(x)]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)} \\
&= \left[\left(e^{\frac{|y_0|}{p}}\beta - e^{\frac{|x_0|}{p}}\alpha\right)v\left(O\left(\frac{x-x_0}{|y_0-x_0|}\right)\right) + \alpha\right]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)} \\
&= \left[\left(e^{\frac{|y_0|}{p}}\beta - e^{\frac{|x_0|}{p}}\alpha\right)v\left(O\left(\frac{x-x_0}{|y_0-x_0|}\right)\right)\right]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)} \\
&= \left|e^{\frac{|y_0|}{p}}\beta - e^{\frac{|x_0|}{p}}\alpha\right|\left[v\left(O\left(\frac{x-x_0}{|y_0-x_0|}\right)\right)\right]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)} \\
&= \left|e^{\frac{|y_0|}{p}}\beta - e^{\frac{|x_0|}{p}}\alpha\right|\left[v\left(\frac{x-x_0}{|y_0-x_0|}\right)\right]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)} \\
&= \frac{\left|e^{\frac{|y_0|}{p}}\beta - e^{\frac{|x_0|}{p}}\alpha\right|}{|x_0-y_0|^{1-\frac{n}{p}}}[v(x)]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)} \\
&= \frac{\left|e^{\frac{|y_0|}{p}}\beta - e^{\frac{|x_0|}{p}}\alpha\right|}{|x_0-y_0|^{1-\frac{n}{p}}} \\
&= \frac{\left|e^{\frac{|x_0|}{p}}u(x_0) - e^{\frac{|y_0|}{p}}u(y_0)\right|}{|x_0-y_0|^{1-\frac{n}{p}}}.
\end{aligned}$$

□

Theorem 4.2. *Assume $u \in W^{1,q}$ is an extremal and $1-\frac{n}{p}$ Hölder ratio of this extremal reaches its maximum at x_0 and y_0. We denote $\zeta = \left(e^{\frac{x_0}{q}}\phi(x_0) - e^{\frac{|y_0|}{q}}\phi(y_0)\right)$ where $\phi \in W^{1,q}(\mathbb{R}^n)$ and $\vartheta = \frac{\left|e^{\frac{|x_0|}{q}}u(x_0)-e^{\frac{|y_0|}{q}}u(y_0)\right|^{q-2}\left(e^{\frac{|x_0|}{q}}u(x_0)-e^{\frac{|y_0|}{q}}u(y_0)\right)}{|x_0-y_0|^{q-n}}$. Then*

$$C_*^q \int_{\mathbb{R}^n} \left(e^{|x|}|\nabla u|^{q-2}\nabla u \nabla \phi + e^{|x|}u^{q-1}\phi\right) dx = \vartheta\zeta \tag{4.2}$$

for each ϕ.

Proof. Since u is an extremal and two distinct points $x_0, y_0 \in \mathbb{R}^n$ are here, where the $1-\frac{n}{q}$ Hölder ratio of this extremal reaches its maximum, we have

$$\begin{aligned}
&[u]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)}\\
&= \sup_{x,y\in U}\left\{\frac{\left|e^{\frac{|x|}{q}}u(x) - e^{\frac{|y|}{q}}u(y)\right|}{|x-y|^{1-\frac{n}{q}}}\right\}\\
&= \frac{\left|e^{\frac{|x_0|}{q}}u(x_0) - e^{\frac{|y_0|}{q}}u(y_0)\right|}{|x_0-y_0|^{1-\frac{n}{q}}}\\
&= C_*\|u\|_{W^{1,q}_{rad,e^{|x|}}(\mathbb{R}^n)}\\
&= C_*\|u\|_{L^q_{e^{|x|}}(\mathbb{R}^n)} + C_*\|u\|_{D^{1,q}_{e^{|x|}}(\mathbb{R}^n)}.
\end{aligned} \tag{4.3}$$

Moreover, one can conclude that

$$\begin{aligned}
&[u]^q_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)}\\
&= \frac{\left|e^{\frac{|x_0|}{q}}u(x_0) - e^{\frac{|y_0|}{q}}u(y_0)\right|^q}{|x_0-y_0|^{q-n}}\\
&= C_*^q\left(\|u\|^q_{L^q_{e^{|x|}}(\mathbb{R}^n)} + \|u\|^q_{D^{1,q}_{e^{|x|}}(\mathbb{R}^n)}\right)\\
&= C_*^q\left(\int_{\mathbb{R}^n} e^{|x|}u^q dx + \int_{\mathbb{R}^n} e^{|x|}|Du|^q dx\right).
\end{aligned}$$

Since Theorem 1.1, we substitute $u + t\phi$ into (1.1) so that

$$\begin{aligned}
&[u+t\phi]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)}{}^q\\
&= \frac{\left|e^{\frac{|x_0|}{q}}(u(x_0)+t\phi(x_0)) - e^{\frac{|y_0|}{q}}(u(y_0)+t\phi(y_0))\right|^q}{|x_0-y_0|^{q-n}}\\
&= \frac{\left|e^{\frac{|x_0|}{q}}u(x_0) - e^{\frac{|y_0|}{q}}u(y_0) + e^{\frac{|x_0|}{q}}\phi(x_0) - e^{\frac{|y_0|}{q}}\phi(y_0)\right|^q}{|x_0-y_0|^{q-n}}
\end{aligned}$$

$$\leq C_*^q \int_{\mathbb{R}^n} e^{|x|}(u+t\phi)^q dx + C_*^q \int_{\mathbb{R}^n} e^{|x|}|\nabla u + t\nabla\phi|^q dx \tag{4.4}$$

where $\phi \in W^{1,q}_{rad,e^{|x|}}(\mathbb{R}^n)$ and $t > 0$.

For any convex function h on interval I, we know that it has a property as follows:

$$h(x_2) \geq h(x_1) + h'(x_1)(x_2 - x_1).$$

Therefore, by using the convexity of the function $x \mapsto |x|^q$ for each $x \in \mathbb{R}$,

$$\begin{aligned}
&\frac{\left|e^{\frac{|x_0|}{q}}u(x_0) - e^{\frac{|y_0|}{q}}u(y_0) + t\left(e^{\frac{|x_0|}{q}}\phi(x_0) - e^{\frac{|y_0|}{q}}\phi(y_0)\right)\right|^p}{|x_0-y_0|^{p-n}} \\
&\geq \frac{\left|e^{\frac{|x_0|}{q}}u(x_0) - e^{\frac{|y_0|}{q}}u(y_0)\right|^q}{|x_0-y_0|^{q-n}} + \frac{tq\zeta}{|x_0-y_0|^{1-\frac{n}{q}}} \frac{\left|e^{\frac{|x_0|}{q}}u(x_0) - e^{\frac{|y_0|}{q}}u(y_0)\right|^{q-1}\left(e^{\frac{|x_0|}{q}}u(x_0) - e^{\frac{|y_0|}{q}}u(y_0)\right)}{|x_0-y_0|^{(1-\frac{n}{q})(q-1)}\left|e^{\frac{|x_0|}{q}}u(x_0) - e^{\frac{|y_0|}{q}}u(y_0)\right|} \\
&= \frac{\left|e^{\frac{|x_0|}{q}}u(x_0) - e^{\frac{|y_0|}{q}}u(y_0)\right|^q}{|x_0-y_0|^{q-n}} + tq\frac{\left|e^{\frac{|x_0|}{q}}u(x_0) - e^{\frac{|y_0|}{q}}u(y_0)\right|^{q-2}\left(e^{\frac{|x_0|}{q}}u(x_0) - e^{\frac{|y_0|}{q}}u(y_0)\right)\zeta}{|x_0-y_0|^{q-n}}.
\end{aligned} \tag{4.5}$$

We subtract (4.4) from (4.3) to obtain

$$\begin{aligned}
&\frac{\left|e^{\frac{|x_0|}{q}}u(x_0) - e^{\frac{|y_0|}{q}}u(y_0) + t\left(e^{\frac{|x_0|}{q}}\phi(x_0) - e^{\frac{|y_0|}{q}}\phi(y_0)\right)\right|^q - \left|e^{\frac{|x_0|}{q}}u(x_0) - e^{\frac{|y_0|}{q}}u(y_0)\right|^q}{|x_0-y_0|^{q-n}} \\
&\leq C_*^q \int_{\mathbb{R}^n} \left(e^{|x|}(u+t\phi)^q + e^{|x|}|\nabla u + t\nabla\phi|^q - e^{|x|}u^p - e^{|x|}|\nabla u|^q\right) dx.
\end{aligned} \tag{4.6}$$

Next, we substitute (4.5) into (4.6) to obtain

$$\begin{aligned}
&C_*^q \int_{\mathbb{R}^n} \left(\frac{e^{|x|}(u+t\phi)^q + e^{|x|}|\nabla u + t\nabla\phi|^q - e^{|x|}u^q - e^{|x|}|\nabla u|^q}{qt}\right) dx \\
&\geq \frac{\left|e^{\frac{|x_0|}{q}}u(x_0) - e^{\frac{|y_0|}{q}}u(y_0)\right|^{q-2}\left(e^{\frac{|x_0|}{q}}u(x_0) - e^{\frac{|y_0|}{p}}u(y_0)\right)\zeta}{|x_0-y_0|^{q-n}}.
\end{aligned} \tag{4.7}$$

From [12], we are aware that a constant c_q exists, then

$$0 \leq \frac{|\nabla u + t\nabla\phi|^q - |\nabla u|^q}{qt} - |\nabla u|^{q-2}\nabla u \nabla\phi \leq c_q \Theta(x)$$

for $t \in (0, 1]$ and $\Theta(x) = c_q \begin{cases} t^{q-1}|\nabla\phi|^q, & 1 < q < 2, \\ t|\nabla\phi|^2(|\nabla u| + |\nabla\phi|)^{q-2}, & 2 \leq q < \infty \end{cases}.$

Therefore,

$$0 \leq \frac{e^{|x|}|\nabla u + t\nabla\phi|^q - e^{|x|}|\nabla u|^q}{qt} - e^{|x|}|\nabla u|^{q-2}\nabla u \nabla\phi \leq c_q e^{|x|}\Theta(x). \tag{4.8}$$

By using L'Hopital's rule, one can conclude that

$$\lim_{t\to 0}\frac{e^{|x|}(u+t\phi)^q-e^{|x|}u^q}{qt}=\lim_{t\to 0}\frac{e^{|x|}q(u+t\phi)^{q-1}\phi}{q}=\lim_{t\to 0}e^{|x|}(u+t\phi)^{q-1}\phi=e^{|x|}u^{q-1}\phi.$$

Therefore, taking $t\to 0$ in (4.7), one can derive that

$$C_*^q\int_{\mathbb{R}^n}\left(e^{|x|}|\nabla u|^{q-2}\nabla u\nabla\phi+e^{|x|}u^{q-1}\phi\right)dx\geq\vartheta\zeta. \tag{4.9}$$

Moreover, choosing $-\phi$ in (4.9), we derive that

$$C_*^q\int_{\mathbb{R}^n}\left(e^{|x|}|\nabla u|^{q-2}\nabla u\nabla\phi+e^{|x|}u^{q-1}\phi\right)dx=\vartheta\zeta.$$

□

Remark 4.3. From Theorem 4.2, we can also derive that u is a weak method for PDE

$$-div\left(e^{|x|}|\nabla u|^{q-2}\nabla u\right)+e^{|x|}u^{q-1}=\frac{\vartheta}{C_*^q}\left(e^{\frac{|x_0|}{q}}\delta_{x_0}-e^{\frac{|y_0|}{q}}\delta_{y_0}\right)$$

and

$$-div\left(e^{|x|}|\nabla u|^{q-2}\nabla u\right)+e^{|x|}u^{q-1}=0\qquad in\ \mathbb{R}^n\setminus\{x_0,y_0\}.$$

Theorem 4.4. *Assume $x_0,y_0\in\mathbb{R}^n$ are two distinct points, and $u\in W^{1,q}_{rad,e^{|x|}}(\mathbb{R}^n)$ is an extremal with*

$$[u]_{C^{0,\gamma}_{e^{|z|}}(\mathbb{R}^n)}=\frac{\left|e^{\frac{|x_0|}{q}}u(x_0)-e^{\frac{|y_0|}{q}}u(y_0)\right|}{|x_0-y_0|^{1-\frac{n}{q}}}.$$

Then for any $v\in W^{1,q}_{rad,e^{|x|}}(\mathbb{R}^n)$ with $v(x_0)=u(x_0)$ and $v(y_0)=u(y_0)$,

$$\int_{\mathbb{R}^n}e^{|x|}\left(|\nabla u|^q+u^q\right)dx\leq\int_{\mathbb{R}^n}e^{|x|}\left(|\nabla v|^q+u^{q-1}v\right)dx.$$

Proof. By using (4.8), we select $\phi=v-u$ and $t=1$. Then

$$0\leq\frac{e^{|x|}|\nabla v|^q-e^{|x|}|\nabla u|^q}{q}-e^{|x|}|\nabla u|^{q-2}\nabla u\nabla(v-u). \tag{4.10}$$

Combining (4.10), (4.2), $v(x_0)=u(x_0)$ and $v(y_0)=u(y_0)$, then

$$\begin{aligned}&\int_{\mathbb{R}^n}e^{|x|}|\nabla v|^q dx\\ \geq&\int_{\mathbb{R}^n}e^{|x|}|\nabla u|^q dx+p\int_{\mathbb{R}^n}e^{|x|}|\nabla u|^{q-2}\nabla u(\nabla v-\nabla u)dx\\ =&\int_{\mathbb{R}^n}e^{|x|}|\nabla u|^q dx+pc\left(e^{\frac{|x_0|}{q}}(v-u)(x_0)-e^{\frac{|y_0|}{q}}(v-u)(y_0)\right)-\int_{\mathbb{R}^n}e^{|x|}u^{q-1}(v-u)dx\\ \geq&\int_{\mathbb{R}^n}e^{|x|}|\nabla u|^q dx-\int_{\mathbb{R}^n}e^{|x|}u^{q-1}v dx+\int_{\mathbb{R}^n}e^{|x|}u^q dx.\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^n}e^{|x|}(|\nabla u|^q+u^q)dx\leq\int_{\mathbb{R}^n}e^{|x|}\left(|\nabla v|^q+u^{q-1}v\right)dx.$$

□

5. Conclusions

This study presents three principal contributions. First, a weighted Morrey's inequality with exponential weight was established. Subsequently, the existence of extremals for this inequality was rigorously investigated. As a principal application, these theoretical advances were shown to guarantee the existence of weak solutions for a related weighted p-Laplace equation and a novel integral inequality emerging directly from the established framework.

While the establishment of a novel weighted Morrey inequality and the investigation of its existence of extremals represent significant advances, several fundamental properties-including boundedness, uniqueness, symmetry, and regularity-remain unresolved. Furthermore, the stability of this weighted Morrey's inequality with exponential weight merits systematic investigation.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interests regarding the publication of this article.

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