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Research article

Novel fixed point results for a class of enriched nonspreading mappings in real Banach spaces

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Abstract: A modified Halpern-type iterative technique, having weak and strong convergence results for approximating invariant points of a new class of enriched nonspreading operators subject to some standard mild conditions in the setting of real Banach spaces, was presented in this work. It was demonstrated with an example that the class of enriched nonspreading mappings includes the class of nonspreading mappings. Again, it was demonstrated with nontrivial examples that the class of enriched nonspreading mappings and the class of enriched nonexpansive mappings are independent. Some basic properties of the class of enriched nonspreading mappings were established. The results obtained solve the open question raised in Nonlinear Analysis 73 (2010): 1562–1568 for nonspreading-type mappings incorporating an averaged mapping. Moreover, we studied the estimation of common invariant points of this new class of mappings and the class of enriched nonexpansive operators and provided a strong convergence theorem for these mappings.

Keywords: variational inequality; enriched nonexpansive mapping; pseudocontractive; monotone mapping; Hilbert space; Lipschitizian; Banach space
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1. Introduction

Throughout this research work, X will denote a real Banach space which is endowed with norm $\|.\|$ and the dual space $X^\star$; $\emptyset \neq C \subset \Omega$, $\mathbb{N}$, and $\mathbb{R}$ will represent a closed convex subset of X with at least one member, the set of positive integers, and the set of real numbers, respectively. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence in C and $T : C \longrightarrow C$ be a given mapping. We use $b_n \rightharpoonup b$ and $b_n \rightarrow b$ to denote that the sequence $\{b_n\}_{n=1}^{\infty}$ converges weakly and strongly to a point b, respectively. The generalized duality map is the operator $J_\phi : C \longrightarrow 2^{C^\star}$ associated with the gauge function ϕ given by the mapping. Then, the following identity holds:

$$J_\phi(a) = \{a^\star \in C : \langle a, a^\star \rangle = \|a\|\|a^\star\| \quad \text{and} \quad \|a^\star\| = \phi(\|a\|)\}, \tag{1.1}$$

where $\phi(\ell) = \ell^{q-1}$ for all $\ell \geq 0$ and $1 < q < \infty$. Specifically, if $q = 2$, $J_\phi = J_2$ is called the normalized duality map (NDM) represented as J which is defined by

$$J(a) = \{a^\star \in C : \langle a, a^\star \rangle = \|a\|^2 \quad \text{and} \quad \|a^\star\| = \|a\|\}.$$

It is known (see [1]) that if Ω is a real Hilbert space $\mathcal{H}$, the NDM becomes an identity, i.e., $\mathcal{J}\omega = \{\omega\}$.

Let $T : C \longrightarrow C$ be a nonlinear map. The fixed point problem is to search for a point $a \in C$ that assures

$$Ta = a. \tag{1.2}$$

We represent with $F(T)$ the set of fixed points of T, i.e., $F(T) = \{b \in C : Tb = b\}$. We use $\omega_\omega(b_n) = \{b : \exists b_n \rightharpoonup b\}$ to represent the weak ω-limit set of the sequence $\{b_n\}_{n=1}^{\infty}$. A nonlinear map $T : C \longrightarrow C$ is known as nonexpansive if it satisfies the inequality

$$\|Tb - Ta\| \leq \|a - b\|, \forall b, a \in C. \tag{1.3}$$

The mapping T is known as quasi-nonexpansive (QN), if $F(T) \neq \emptyset$ and (1.3) holds for all $b \in C$ and $a \in F(T)$.

The notion of nonexpansive operators (NM) stands as an indispensable part of the investigation of the Mann-type iterative technique for evaluating invariant points of an operator $T : C \longrightarrow C$, where C is as described above. Recall that the Mann-type iterative technique [2], developed from an arbitrary $b_1 \in C$, is given as follows:

$$b_{n+1} = (1 - \delta_n)b_n + \delta_n T b_n, \tag{1.4}$$

where $\{\delta_n\}_{n=1}^{\infty} \subset [0, 1]$ satisfies some mild conditions.

The problem of investigating fixed points of NM with respect to strong convergence has been widely studied by several authors. In this regard, Halpern [3] gave the following general iterative technique:

$$\begin{cases} u \in \Omega, \quad b_1 \in C \\ \vartheta_n = (1 - \delta_n)b_n + \delta_n T b_n \\ b_{n+1} = (1 - \alpha_n)u + \alpha_n T\vartheta_n, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\delta_n\}_{n=1}^{\infty} \subset [0,1]$ satisfy appropriate conditions and $u \in C$ is fixed. In particular, if $\alpha_n = 0$, (1.5) reduces to the standard Mann iteration (1.4). Moreover, Halpern [3] proved the strong convergence result of (1.5) when $\delta_n = 0$ and for appropriate conditions on $\{\alpha_n\}_{n=1}^{\infty}$. He further established that the control parameters

$$\mathcal{D}_1 : \lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \mathcal{D}_2 : \sum_{n=1}^{\infty} \alpha_n = \infty$$

are necessary for convergence of (1.5) to the fixed point of $\mathfrak{I}$. Thereafter, several investigations have been done to ascertain the implications of conditions $\mathcal{D}_1$ and $\mathcal{D}_2$ on the convergence of (1.5); see [4,5] for further reading.

In [6], Osilike and Isogugu studied the Halpern-type fixed point algorithm for k-strictly pseudononspreading mappings T, which includes the class of nonspreading mappings (NSM) as a special case. To achieve strong convergence results, they substituted an averaged-type mapping T_δ for the mapping T, where

$$T_\delta = (1-\delta)I + \delta T, \quad \delta \in (0,1). \tag{1.6}$$

Recently, Kohasaka and Takahashi [7, 8] studied an important class of nonlinear operators which they referred to as NMS. Let Ω be a real, smooth, strictly convex (SC) and reflexive Banach space (RBS) and denote by $j : x \longrightarrow 2^{x^\star}$ the duality mapping of x.

Let $\emptyset \neq C \subset X$ be closed and convex. A mapping $T : C \longrightarrow C$ is called nonspreading if

$$\phi(Tb, Ta) + \phi(Ta, Tb) \leq \phi(Tb, a) + \phi(Ta, b), \tag{1.7}$$

for all $b, a \in C$, where

$$\phi(b, a) = ||b||^2 - 2\langle b, j(a)\rangle + ||a||^2, \tag{1.8}$$

for all $b, a \in X$.

Kohasaka and Takahashi considered the class of NSM to study the resolvent of a maximum monotone operator in real, smooth, SC, and RBS. These mappings originate from another group of operators called firmly nonexpansive mappings (see, for example, [7, 9]). In a real Hilbert space ($\mathcal{H}$), (1.8) reduces to the following identity:

$$\phi(b, a) = ||b||^2 - 2\langle b, a\rangle + ||a||^2.$$

Consequently, if $\mathcal{H}$ and C are as described above, then T is nonspreading if

$$||Tb - Ta||^2 \leq ||Tb - a||^2 + ||Ta - b||^2, \quad \forall b, a \in C. \tag{1.9}$$

It is established in [10] that (1.9) is equivalent to the inequality

$$||Tb - Ta||^2 \leq ||b - a||^2 + \langle b - Tb, a - Ta\rangle, \quad \forall b, a \in C. \tag{1.10}$$

Remark. If T is nonspreading (resp. nonexpansive) and $F(T) \neq \emptyset$, then T is QN.

In [10], the authors studied the iterative estimation of common invariant points of NM ($\eth$) and NSM ($\mathfrak{I}$) of Λ into itself in $\mathcal{H}$. They studied a technique akin to the one employed by Moudafi in [11]. To be precise, they established the following result:

Theorem 1.1 ([10], **Theorem 4.1**). *Let $\mathcal{H}$ and C be as described above. Let $T, S : C \longrightarrow C$ be as described in Remark 1 with the property that $F(T) \cap F(S) \neq \emptyset$. Let $b_n\}_{n=1}^{\infty}$ be a real sequence generated by*

$$\begin{cases} b_1 \in C \\ b_{n+1} = (1-\alpha_n)b_n + \alpha_n[\mu_n b_n + (1-\mu_n)Tb_n], \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty} \subset [0,1]$. *Then, we have:*

(1) If $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ *and* $\sum_{n=1}^{\infty}(1-\mu_n) < \infty$, *then* $b_n \rightharpoonup \nu \in F(T)$.

(2) If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\sum_{n=1}^{\infty} \mu_n < \infty$, *then* $b_n \rightharpoonup \nu \in F(T)$.

(3) If $\liminf_{n\to\infty 1} \alpha_n(1-\alpha_n) > 0$ *and* $\liminf_{n\to\infty 1} \mu_n(1-\mu_n) > 0$, *then* $b_n \rightharpoonup \nu \in F(T) \cap F(S)$.

Finding the fixed points of NM, NSM and some other related mappings have remained invaluable topics in fixed point theory, and have been shown to be fundamental in the applied areas of signal processing [12], the split feasibility problems [13], and convex feasibility problem [14]. In subsequent works, Berinde [15, 16] came up with the notion of enriched nonlinear mappings as a generalization of the class of NM in the setup of $\mathcal{H}$. This concept was later studied in a more general Banach space by Saleem, Agwu and Igbokwe [17, 18].

Definition 1.2. A mapping $T : C \longrightarrow C$ is referred to as ψ_T-enriched Lipschitzian (or (σ, ψ_T)-enriched Lipshitzian) (see [17, 18]) (shortly, (σ, ψ_T)-ELM) if for all $b, a \in Ca$, there exist $\sigma \in [0, +\infty)$ and a continuous nondecreasing function $\psi_T : \mathcal{R}^+ \longrightarrow \mathcal{R}^+$, with $\psi_T(0) = 0$, such that

$$\|\sigma(b-a) + Tb - Ta\| \leq (\sigma+1)\psi_T(\|b-a\|). \tag{1.11}$$

Observe from inequality (1.11) that:

(a) If $\sigma = 0$, then the class of mappings called ψ_T-Lipschitzian emerges.
(b) If $\sigma = 0$ and $\psi(t) = Lt$, for $L > 0$, then (1.11) reduces to a class of mappings known as L-Lipschitzian, L represents the Lipschitz constant. In particular, if $\sigma = 0, \Psi_{\mathfrak{J}}(t) = Lt$, and $L = 1$, then (σ, ψ_T)-ELM immediately reduces to the class of NM on C.
(c) If $\psi_T(t) = t$, then inequality (1.11) becomes

$$\|\sigma(b-a) + Tb - Ta\| \leq (\sigma+1)\|b-a\|, \tag{1.12}$$

and it is called a σ-enriched nonexpansive mapping. The class of mappings defined by (1.12) was first studied by Berinde [15, 16] as a generalization of a well-known class of nonlinear mappings called NM.

Closer observation reveals that if ψ_T is not necessarily nondecreasing and guarantees the condition

$$\psi_T(t) < t, \quad \forall t > 0,$$

then we have the class of σ-enriched contraction mappings.

In view of the papers studied, particularly, the results obtained by Lemoto and Takahashi [10], Berinde [16], and other related results in this direction, we consider the following questions:

Question 1.3. (1) Could there be a nonlinear mapping that contains the class of mappings defined by (1.9) for which we would obtain the results in [10] as special cases?

(2) Could it be possible to obtain a strong convergence result for an averaged mapping in a more general Banach space?

Lemoto and Takahashi considered the class of nonspreading mappings and proved the weak convergence theorem as their main result in [10] in the setup of $\mathcal{H}$. Their results together with those of Kohasaka and Takahashi [8] opened a new direction in metric fixed point theory. In the current paper, we shall consider a new class of nonlinear mapping called σ-enriched nonspreading mappings ((σ)-ENSM) in the setup of Ω. Further, we present some nontrivial examples to demonstrate its existence (and its independency on the class of σ-enriched nonexpansive mappings ((σ)-ENEM)). By modifying the iterative method studied in [10], we established strong convergence theorems which solve the problems raise in Question 1.3.

The rest of the paper is organized as follows: Section 2 will consider preliminary results which will be needed in establishing our main results. Proposition 3.7, and Theorems 3.6, 3.8, 3.9, and 3.11, which will serve as our main results (including some of their consequences) and the conclusion of the results obtained in this paper, will be considered in Section 3.

2. Preliminaries

The convexity of a Banach space X is characterized by the function $\delta(\epsilon) : (0, 2] \longrightarrow [0, 1]$, known as the modulus of convex of X, defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|b + a\|}{2} : \|b\| \leq 1, \|a\| \leq 1, \|b - a\| \geq \epsilon\right\}.$$

The space C is regarded as uniformly convex if and only if $\delta(\epsilon) > 0$ for every ϵ such that $0 \leq \epsilon \leq 2$. Let $S_X = \{b \in X : \|b\| = 1\}$. For each $b, a \in S_X$, the norm of X is referred to as Gateaux differentiable provided the limit

$$\lim_{n\to 0} \frac{\|b + ta\| - \|b\|}{t} \tag{2.1}$$

exists. For the above case, X is called smooth. It is known as uniformly smooth (US) if the limit described by (2.1) is achieved uniformly in X; whereas it is called strictly convex (SC) if

$$\frac{\|b + a\|}{2} < 1,$$

whenever $b, a \in S_X$ and $b \neq a$. It is an established fact that X is uniformly convex (UC) if and only if $X^\star$ is US; for further details, see [19].

The smoothness of a Banach space X is characterized by the function $\rho : [0, \infty) \longrightarrow [0, \infty)$, known as the modulus of convexity of X, defined by

$$\rho(t) = \sup\left\{\frac{1}{2}(\|b + a\| + \|b - a\|) - 1 : b, a \in X, \|b\| = 1, \|a\| = t\right\}.$$

It is known that X is US provided

$$\lim_{n\to 0}\frac{\rho(t)}{t}=0.$$

Set $1<q\leq 2$. Then, X is called q-uniformly smooth if we can find a constant $c_q>0$ with the property that $\rho(t)\leq c_q t^q$ for all $t>0$. It is worth noting that X assumes smoothness if the mapping J (called a sequentially continuous duality mapping) dwells in X with its domain in the weak topology and the range in the weak-star topology. In this case, $b_n\rightharpoonup b\in X\Rightarrow Jb_n\overset{\star}{\rightharpoonup}Jb$; see, for example, [20] for further details. X enjoys the Opial property [21] if for any sequence $\{b_n\}_{n=1}^{\infty}$ which converges weakly in X with a weak limit $\wp$,

$$\limsup_{n\to\infty}\|b_n-b\|<\limsup_{n\to\infty}\|b_n-a\|$$

for all $a\in X$ with $b\neq a$. It is a known fact that all X with finite dimension, all Hilbert spaces and all spaces accredited to $\ell^p(1\leq p<\infty)$ admit the Opial property; see [20,21] for more details. It is also on record that if X recognizes J, then it is smooth and also assures the employment of the Opial property; see [20].

Let $X,C,\rightharpoonup$ and $\to$ be as described in section one. Let $\emptyset\neq M$ and $\emptyset\neq N$ be two subsets of X with $M\subset N$. An operator $Q_N:M\longrightarrow N$ is called sunny if

$$Q_N(Q_Nb+\xi(b-Q_Nb))=Q_Nb$$

for each $b\in X$ and $\xi\geq 0$. A mapping $Q_N:m\longrightarrow N$ is said to be a retraction if $Q_Nb=b$ for each $b\in C$.

Lemma 2.1. *[22] Let Ω be as described above and $\emptyset\neq M,N\subset\Omega$ be such that $M\subset N$. Let $Q_N:M\longrightarrow N$ be a retraction of M onto N. Then Q_N is sunny and nonexpansive if and only if*

$$\langle b-Q_N(b),j(a-Q_N(b))\rangle\leq 0,$$

for all $b\in M$ and $a\in N$, where $j(a-Q_N(b))\in J(a-Q_N(b))$ retains its usual meaning in X.

Lemma 2.2. *[22] Let Ω and J be as described above. Then,*

$$\|b+a\|^2\leq\|b\|^2+2\langle a,j(b+a)\rangle$$

for all $b,a\in X$ and for all $j(b+a)\in J(b+\ a)$.

Proposition 2.3. *Let $\mathcal{H}$ be a real Hilbert space, $\emptyset\neq C\subset\mathcal{H}$ and $T:C\longrightarrow C$ be a σ-enriched nonspreading mapping. Then, $F(T)$ is closed and convex.*

Proof. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence in $F(T)$ which converges to b. We want to show that $b\in F(T)$. Now, since

$$\begin{aligned}\|T_\beta b-b\|=\beta\|Tb-b\| &\leq \beta\|Tbi-Tb_n\|+\beta\|b_n-b\|\\ &= \beta\|\sigma(b-b_n)+Tb-Tb_n-\sigma(b-b_n)\|+\beta\|b_n-b\|\\ &\leq \beta\|\sigma(b-b_n)+Tb-Tb_n\|+\beta(\sigma+1)\|b_n-b\|,\end{aligned}\tag{2.2}$$

and since T is a σ-enriched nonspreading mapping, we have

$$\|\sigma(b-b_n)+Tb-Tb_n\|^2\ \leq\ (\sigma+1)^2\|b_n-b\|^2+2\langle b-Tb,b_n-Tb_n\rangle$$

$$= \quad (\sigma+1)^2\|b_n - b\|^2. \tag{2.3}$$

Equations (2.2) and (2.3) imply that

$$0 \leq \|Tb - b\|(\sigma+1)\|b_n - b\| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.4}$$

Hence, $b \in F(T)$.

Next, let $\vartheta_1, \vartheta_2 \in F(T)$ and $\lambda \in [0,1]$. We prove that $\lambda\vartheta_1 + (1-\lambda)\vartheta_2 \in F(\Im)$. Let $\psi = \lambda\vartheta_1 + (1-\lambda)\vartheta_2$. Then, $\vartheta_1 b = (1-\lambda)(\vartheta_1 - \vartheta_2)$ and $\vartheta_2 - b = \lambda(\vartheta_2 - \vartheta_1)$. Since

$$\begin{aligned}
\beta^2\|Tb - b\|^2 = \|b - T_\beta b\|^2 &= \|\lambda\vartheta_1 + (1-\lambda)\vartheta_2 - \Im_\beta b\|^2 \\
&= \|\lambda(\vartheta_1 - T_\beta b) + (1-\lambda)(\vartheta_2 - T_\beta b)\|^2 \\
&= \lambda\|\vartheta_1 - T_\beta b\|^2 + (1-\lambda)\|\vartheta_2 - T_\beta b\|^2 - \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2 \\
&= \lambda\|(1-\beta)\vartheta_1 + \beta T\vartheta_1 - [(1-\beta)b + \beta Tb]\|^2 \\
&+ (1-\lambda)\|(1-\beta)\vartheta_2 + \beta T\vartheta_2 - [(1-\beta)b + \beta Tb]\|^2 \\
&- \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2 \\
&= \lambda\|(1-\beta)(\vartheta_1 - b) + \beta(T\vartheta_1 - Tb)\|^2 \\
&+ (1-\lambda)\|(1-\beta)(\vartheta_2 - b) + \beta(T\vartheta_2 - Tb)\|^2 \\
&- \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2 \\
&= \frac{\lambda}{(\sigma+1)^2}\|\sigma(\vartheta_1 - b) + T\vartheta_1 - Tb\|^2 \\
&+ \frac{1-\lambda}{(\sigma+1)^2}\|\sigma(\vartheta_2 - b) + T\vartheta_2 - \Im\psi\|^2 - \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2 \\
&\leq \frac{\lambda}{(\sigma+1)^2}[(\sigma+1)^2\|\vartheta_1 - b\|^2 + 2\langle\vartheta_1 - \Im\vartheta_1, b - Tb\rangle] \\
&+ \frac{1-\lambda}{(\sigma+1)^2}[(\sigma+1)^2\|\vartheta_2 - b\|^2 + 2\langle\vartheta_2 - T\vartheta_2, b - Tb]\rangle \\
&- \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2 \\
&= \lambda\|\vartheta_1 - b\|^2 + (1-\lambda)\|\vartheta_2 - b\|^2 - \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2 \\
&= \lambda(1-\lambda)[1 - \lambda + \lambda]\|\vartheta_1 - \vartheta_2\|^2 - \lambda(1-\lambda)\|\vartheta_1 - \vartheta_2\|^2,
\end{aligned}$$

it follows that

$$\beta^2\|Tb - b\|^2 \leq 0.$$

Therefore, $b = Tb$ implies that $b \in F(T)$ as required. □

Definition 2.4. Let X be as described above, $\emptyset \neq C \subset X$ be closed and convex, and $\{b_n\}_{n=1}^\infty$ be a bounded sequence in X. For any $b \in C$, we set

$$r(b, \{b_n\}_{n=1}^\infty) = \limsup_{n\to\infty}\|b - b_n\|.$$

The asymptotic radius of $\{b_n\}_{n=1}^\infty$ with respect to C is given as

$$r(C, \{b_n\}_{n=1}^\infty) = \inf\{r(b, \{b_n\}_{n=1}^\infty) : \wp \in C\}.$$

The asymptotic center of $\{b_n\}_{n=1}^{\infty}$ with respect to C is the set

$$A(C, \{b_n\}_{n=1}^{\infty}) = \{b \in C : r(b, \{b_n\}_{n=1}^{\infty}) = r(C, \{b_n\}_{n=1}^{\infty})\}.$$

It is an established fact that if X is UC, then $A(C, \{b_n\}_{n=1}^{\infty})$ is fixed at a point (see, for instance, [3, 22]).

Lemma 2.5. *[23] Let $\{\nu_n\}_{n=1}^{\infty}$ be a sequence of non-negative real real numbers validating the following inequality:*

$$\nu_{n+1} \leq (1 - \pi_n)\nu_n + \pi_n\mu_n,$$

where $\{\pi_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ satisfy the conditions:

(i) $\{\pi_n\}_{n=1}^{\infty} \subset [0, 1]$ *and* $\sum\limits_{n=0}^{\infty} \pi_n = \infty$ *or, equivalently,* $\prod\limits_{n=1}^{\infty}(1 - \pi_n) = 0$*;*
(ii) $\limsup_{n\to\infty} \mu_n \leq 0$ *or* $\sum\limits_{n=0}^{\infty} \pi_n\mu_n < \infty$.

Then, $\lim\limits_{n\to\infty} \nu_n = 0$.

Lemma 2.6. *[19] Let $\{\gamma_n\}_{n=0}^{\infty} \subset \mathbb{R}$ be such that we can find a subsequence $\{\gamma_{n_k}\}_{k=0}^{\infty}$ such that $\gamma_k < \gamma_{k+1}$ for all $k \in \mathbb{N}$. Consider the sequence of integers $\{\tau(n)\}_{n=1}^{\infty}$ given by*

$$\tau_n = \max\{i \leq n : \gamma_i \leq \gamma_{i+1}\}. \tag{2.5}$$

Then, $\{\tau(n)\}_{n=1}^{\infty}$ is a nondecreasing sequence, for all $n \geq n_0$, validating the following requirements:

(i) $\lim\limits_{n\to\infty} \tau(n) = \infty$;
(ii) $\gamma_{\tau(n)} < \gamma_{\tau(n)}, \quad \forall n \geq n_0$*;*
(iii) $\gamma_n < \gamma_{\tau(n)}, \quad \forall n \geq n_0$.

Lemma 2.7. *[1, 22] Let X be a UC and $B_\varrho\{\wp \in X : ||\wp|| \leq \varrho\}, \varrho > 0$. Then, we can find a continuous, strictly increasing function $g : [0, \infty) \longrightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$||rb + sa + tc||^2 \leq r||b||^2 + s||a||^2 + t||c||^2 - rsg(||b - a||)$$

for all $b, a, c \in B_\varrho$ and for all $r, s, t \in [0, 1,]$ with $r + s + t = 1$.

The proposition below assures some essential properties of generalized duality mapping (J_ϕ).

Proposition 2.8. *[18, 24] Let X and $X^\star$ be as described above. For $q \in (1, \infty)$, $J_\phi : X \longrightarrow 2^{X^\star}$ has the following fundamental properties:*

(1) $J_\phi(b) \neq \emptyset \forall b \in X$ *and* $D(J_\phi)$*(: the dormain of* $J_\phi) = X$*;*
(2) $J_\phi(b) = ||b||^{\phi-1}J_2(b), \quad \forall b \in X(b \neq 0)$*;*
(3) $Jb_\phi(\alpha b) = \alpha^{\phi-1}J_\phi(b), \quad \alpha \in [0, \infty)$*;*
(4) $J_\phi(-b) = -J_\phi(b)$.

3. Results and discussion

Definition 3.1. Let X be as described above. A mapping T with domain $D(T)$ and range $R(T)$ in X is known as σ-enriched nonspreading (σ-ENSM, for short) in the sense of Kurokawa and Takahashi [25] if there exists $\sigma \in [0, \infty)$ and $j(a - Ta) \in J(a - Ta)$ such that for all $b, a \in D(T)$, the following inequality holds:

$$\|\sigma(b-a) + Tb - Ta\|^2 \leq (\sigma+1)^2\|b-a\|^2 + 2\langle b - Tb, j(a - Ta)\rangle. \tag{3.1}$$

Now, by setting $\sigma = \dfrac{1}{\beta} - 1$, for some $\beta \in (0, 1]$, it follows from Proposition 2.8 (3) and (3.1) that

$$\begin{aligned}
\|\sigma(b-a) + Tb - Ta\|^2 &\leq (\sigma+1)^2\|b-a\|^2 + 2\langle b - \mathbf{b}, j(a - Ta)\rangle \\
\Leftrightarrow \Big\|\Big(\frac{1}{\beta} - 1\Big)(b-a) + Tb - Ta\Big\|^2 &\leq \frac{1}{\beta^2}\|b-a\|^2 + 2\langle b - Tb, j(a - Ta)\rangle \\
\Leftrightarrow \Big\|\Big(\frac{1-\beta}{\beta}\Big)(b-a) + Tb - Ta\Big\|^2 &\leq \frac{1}{\beta^2}\|b-a\|^2 + 2\langle b - Tb, j(a - Ta)\rangle \\
\Leftrightarrow \|(1-\beta)(b-a) + \beta Tb - \beta Ta\|^2 &\leq \|b-a\|^2 + 2\langle \beta(b - Tb), \beta j(a - Ta)\rangle \\
\Leftrightarrow \|(1-\beta)(b-a) + \beta Tb - \beta Ta\|^2 &\leq \|b-a\|^2 + 2\langle b - [(1-\beta)b \\
&+ \beta Tb], j(a - [(1-\beta)a + \beta Ta])\rangle \\
\Leftrightarrow \|(1-\beta)b + \beta Tb - [(1-\beta)a + \beta Ta]\|^2 &\leq \|b-a\|^2 + 2\langle b - [(1-\beta)b \\
&+ \beta Tb], j(a - [(1-\beta)a + \beta Ta])\rangle.
\end{aligned} \tag{3.2}$$

Remark. Observe that if $\sigma = 0$ in (3.1) (or $\beta = 1$ in (3.2)), we obtain an important class of nonspreading mappings studied in [25]. Again, if we take $T_\beta = (I - \beta)I + \beta T$, then (3.2) reduces to the inequality

$$\|T_\beta b - T_\beta a\|^2 \leq \|b-a\|^2 + 2\langle b - T_\beta b, j(a - T_\beta a)\rangle. \tag{3.3}$$

Therefore, the averaged operator T_β is a nonspreading mapping whenever T is a σ-enriched nonspreading mapping.

Remark. Any nonspreading mapping T validating (3.1) with $\sigma = 0$ is known as 0-enriched nonspreading.

Example 3.2. Let $B_\rho\{b \in \mathcal{H} : \|b\| \leq \rho\}$ for $\rho > 0$ and $C = B_2 \subset \mathcal{H}$. Define an operator $T : C \longrightarrow C$ by

$$Tb = \begin{cases} b, & b \in B_2, \\ P_{B_1}b, & b \in \setminus B_2, \end{cases}$$

where P_A is a projection map of $\mathcal{H}$ onto $\mathcal{A}$. Then, T is an enriched nonspreading mapping which does not admit continuity. Obviously, $F(\Im) = B_2$. Let $b, a \in C$. It suffices to examine the situation for which $b \in C \setminus B_2, a \in B_2$. Now, since P_{B_1} is nonexpansive (and hence 0-enriched nonexpansive) and $b - Tb = 0$, it follows that

$$\|\sigma(b-a) + Tb - Ta\|^2 = \|\sigma(b-a) + P_{B_1}b - a\|^2$$

$$= \|\sigma(b-a) + P_{B_1}b - P_{B_1}a\|^2 \leq (\sigma+)^2\|b-a\|^2$$
$$= (\sigma+1)^2\|b-a\|^2 + 2\langle b - Tb, a - Ta\rangle.$$

Therefore, T is a σ-ENSM. Clearly, T is not continuous. In fact, for $b_0 \in \partial B_2, \omega_0 \in \partial C$, consider $c_n = (1 - \frac{1}{n})b_0 + \frac{1}{n}b_0 \in C$ for each $n \geq 1$. Then, $b_n \to b_0$ but $Tb_n = P_{B_1}b_n \nrightarrow Tb_0$ because $\|P_{B_1}b_n\| = 1$ and $\|b_0\| = 2$.

Remark. Note that T is not continuous in the last example; hence, T is not uniformly continuous. In other words, the class of σ-ENSM is generally not Lipschitzian.

The following examples demonstrate the fact that the class of σ-ENEM and the class of σ-ESNM are independent.

Example 3.3. Let $\mathbb{R} \supset C = \left[\frac{1}{2}, 2\right]$ be endowed with the usual norm and let $\Im : C \longrightarrow C$ be define by $\Im\psi = \frac{1}{\wp}$ for all $\psi \in C$. Then,

(i) T is not nonexpansive.
(ii) T is $\frac{3}{2}$-enriched nonexpansive.
(iii) $F(T) = \{1\}$.
(iv) T is not a $\frac{3}{2}$-ESNM.

To validate (i) – (iv):

(i) Assume T is NE. Then, by the definition of NE, we should have

$$|Tb - Ta| = \left|\frac{a-b}{ba}\right| \leq |b-a|, \quad \forall b, a \in C,$$

which, when $b = \frac{1}{2}$ and $a = 1$, yields a contradiction.

(ii) For all $\forall b, a \in C$,

$$|\sigma(b-a) + Tb - Ta| = \left|\sigma(b-a) + \frac{1}{b} - \frac{1}{a}\right| = \left|\sigma(b-a) + \frac{a-b}{ba}\right|$$
$$= \left(\sigma - \frac{1}{ba}\right)|b-a|.$$

Observe that for any $\sigma \geq \frac{3}{2}$, the last identity becomes

$$|\sigma(b-a) + Tb - Ta| = (\sigma+1)|b-a|, \quad \forall b, a \in C,$$

and as such validates our conclusion that T is $\frac{3}{2}$-enriched nonexpansive.

(iii) $F(T) = \{1\}$ is not difficult to see.

(iv) Since every σ-enriched nonexpansive mapping satisfies the σ-enriched Lipschitz condition (see, for instance, [17]),

$$\|\sigma(b-a) + Tb - Ta\| = (\sigma+1)L\|b-a\|, \quad \forall b, a \in C,$$

where L is the Lipschitz constant, and since every σ-enriched nonspreading mapping is generally not Lipschitzian (see Example 3.2 and Remark 3 above), it follows from (*ii*) that $\mathfrak{I}$ is not a σ-enriched nonspreading mapping.

Example 3.4. Let $X = \mathbb{R}$ denote the reals with the usual norm. For each $\psi \in \mathbb{R}$, let the mapping $\mathfrak{I}$ be given by

$$Tb = \begin{cases} 0, & \text{if} \quad b \in (-\infty, 2], \\ 1, & \text{if} \quad b \in (2, \infty). \end{cases}$$

Then, for all $b, a \in (-\infty, 2]$ and for all $\sigma \in [0, \infty)$, we have

$$\begin{aligned} (\sigma+1)^2|b-a|^2 + 2\langle b - Tb, j(a - Ta)\rangle &= (\sigma^2 + 2\sigma + 1)|b-a|^2 + 2ba \\ &= (\sigma^2 + 2\sigma)|b-a|^2 + b^2 + a^2 \\ &\geq \sigma^2|b-a|^2 \\ &= |\sigma(b-a) + Tb - Ta|^2. \end{aligned}$$

Also, for all $b, a \in (2, \infty)$ and for all $\sigma \in [0, \infty)$, we have

$$\begin{aligned} (\sigma+1)^2|b-a|^2 + 2\langle b - Tb, j(a - Ta)\rangle &= (\sigma+1)^2|b-a|^2 + 2(b-1)(a-1) > \sigma^2|b-a|^2 \\ &= |\sigma(b-a) + Tb - Ta|^2. \end{aligned}$$

Finally, if $b \in (-\infty, 2]$ and $a \in (2, \infty)$, then for all $\sigma \in [0, \infty)$, we get

$$\begin{aligned} (\sigma+1)^2|b-a|^2 + 2\langle b - Tb, j(a - Ta)\rangle &= (\sigma^2 + 2\sigma)|b-a|^2 + b^2 + a^2 - 2b \\ &> |\sigma(b-a) - 1|^2 \\ &= |\sigma(b-a) + Tb - Ta|^2. \end{aligned}$$

Thus, for all $b, a \in X$ and for all $\sigma \in [0, \infty)$, we obtain

$$|\sigma b - a) + Tb - Ta|^2 \leq (\sigma+1)^2|b-a|^2 + 2\langle b - Tb, j(a - Ta)\rangle.$$

Hence, T is σ-enriched nonspreading. Since every σ-enriched nonexpansive mapping T must satisfy σ-enriched Lipschitz condition (see, for instance, [17])

$$||\sigma(b-a) + Tb - Ta|| \;=\; (\sigma+1)L||b-a||, \quad \forall b, a \in C,$$

where L is the Lipschitz constant. It is not difficult to see that T is not a σ-enriched nonexpansive mapping.

The next example shows that a σ-ENSM is different from a NSM thereby leading to the conclusion that the class of a σ-ENSM properly contains the class of NSM.

Example 3.5. Let $\mathbb{R}$ be as described above with the usual norm and suppose the mapping $T : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$Tb = -b.$$

Then, T is a σ-enriched nonspreading mapping. Indeed, for all $b, a \in \mathbb{R}$ with

$$|\sigma(b-a) + Tb - Ta|^2 = (\sigma-1)^2|b-a|^2,$$

and

$$(\sigma+1)^2|b-a|^2 + 2\langle b - Tb, a - Ta\rangle = (\sigma+1)^2|b-a|^2 + 2\langle 2b, 2a\rangle = (\sigma+1)^2|b-a|^2 + 8ba,$$

there exists a $\sigma \in [1,\infty)$ such that

$$|\sigma(b-a) + Tb - Ta|^2 \le (\sigma+1)^2|b-a|^2 + 2\langle b - Tb, a - Ta\rangle.$$

However, T is not nonspreading, for if $b \neq 0$ and $a = -b$, then

$$|Tb - Tb|^2 = 4b^2 > -4b^2 = |b-a|^2 + 2\langle b - Tb, a - Ta\rangle.$$

Remark. If $F(T) \neq \emptyset$ in (3.1), then we obtain a class of mapping called σ-enriched quasi-nonexpansive mappings.

Now, we give the existence theorems of invariant points of σ-ENSM in X.

Theorem 3.6. *Let X be a UC and $\emptyset \neq C \subset X$ be closed and convex. Let $T : C \longrightarrow C$ be a σ-ENSM. Then, the statements below are equivalent:*

(i) *The invariant point set of $F(T) \neq \emptyset$.*
(ii) $\exists\{b_n\}_{n=1}^{\infty} \subset C$, *with $\{b_n\}_{n=1}^{\infty}$ bounded, such that* $\liminf_{n\to\infty}\|b_n - Tb_n\| = 0$.

Proof. We can see that (i) $\Rightarrow$ (ii) is quite obvious. To establish the opposite implication, we assume $\exists\{b_n\}_{n=1}^{\infty} \subset C$, with $\{b_n\}_{n=1}^{\infty}$ bounded, such that $\liminf_{n\to\infty}\|b_n - Tb_n\| = 0$. As a consequence, $\exists\{Tb_{n_k}\}_{k=1}^{\infty} \subset \{Tb_n\}_{n=1}^{\infty}$ for which $\liminf_{k\to\infty}\|b_{n_k} - Tb_{n_k}\| = 0$. Suppose $A(C, \{b_{n_k}\}_{k=1}^{\infty}) = \{\varrho\}$. Let $\Theta_1 = \sup\{\|b_{n_k}\|, \|Tb_{n_k}\|, \|\varrho\|, \|T\varrho\| : k \in \mathbb{N}\} < \infty$.

Since the mapping T is σ-nonspreading, it follows that

$$\begin{aligned}
\|b_{n_k} - T\varrho\|^2 &= \|b_{n_k} - Tb_{n_k} + Tb_{n_k} - T\varrho\|^2 \\
&\le \|b_{n_k} - Tb_{n_k}\|^2 + \|Tb_{n_k} - T\varrho\|^2 + 2\|b_{n_k} - Tb_{n_k}\|\|Tb_{n_k} - T\varrho\| \\
&\le \|b_{n_k} - Tb_{n_k}\|^2 + \|Tb_{n_k} - T\varrho\|^2 + 2\Theta_1\|b_{n_k} - Tb_{n_k}\| \\
&= \|b_{n_k} - Tb_{n_k}\|^2 + \|\sigma(b_{n_k} - \varrho) + Tb_{n_k} - T\varrho - \sigma(\varrho - b_{n_k})\|^2 + 2\Theta_1\|b_{n_k} - Tb_{n_k}\| \\
&\le \|b_{n_k} - Tb_{n_k}\|^2 + \|\sigma(b_{n_k} - \varrho) + Tb_{n_k} - T\varrho\|^2 + \sigma^2\|\varrho - b_{n_k}\|^2 \\
&- 2\sigma\|\sigma(b_{n_k} - \varrho) + Tb_{n_k} - T\varrho\|\|\varrho - b_{n_k}\| + 2\Theta_1\|b_{n_k} - Tb_{n_k}\| \\
&\le \|b_{n_k} - Tb_{n_k}\|^2 + (\sigma+1)^2\|b_{n_k} - \varrho\|^2 + 2\langle b_{n_k} - Tb_{n_k}, j(\varrho - T\varrho)\rangle \\
&+ \sigma^2\|\varrho - b_{n_k}\|^2 - 2\sigma\|\sigma(b_{n_k} - \varrho) + Tb_{n_k} - T\varrho\|\|\varrho - b_{n_k}\| \\
&+ 2\Theta_1\|b_{n_k} - Tb_{n_k}\| \\
&= \|b_{n_k} - Tb_{n_k}\|^2 + \|b_{n_k} - \varrho\|^2 + 2\sigma(\sigma+1)\langle \varrho - b_{n_k}, j(-(b_{n_k} - \varrho))\rangle \\
&+ 2\langle b_{n_k} - Tb_{n_k}, j(\varrho - T\varrho)\rangle - 2\sigma\|\sigma(b_{n_k} - \varrho) + Tb_{n_k} - T\varrho\|
\end{aligned}$$

$$
\begin{aligned}
&\times\ \ ||\varrho - b_{n_k}|| + 2\Theta_1||b_{n_k} - Tb_{n_k}||\\
&\leq\ \ ||b_{n_k} - Tb_{n_k}||^2 + ||b_{n_k} - \varrho||^2 - 2\sigma(\sigma+1)\langle \varrho - b_{n_k}, j(b_{n_k} - \varrho)\rangle\\
&+\ \ 2\langle b_{n_k} - Tb_{n_k}, j(\varrho - T\varrho)\rangle + 2\Theta_1||b_{n_k} - Tb_{n_k}|| \quad \text{(by Proposition 2.8(4))}\\
&=\ \ ||b_{n_k} - Tb_{n_k}||^2 + ||b_{n_k} - \varrho||^2 - 2\sigma(\sigma+1)||\varrho - b_{n_k}||||b_{n_k} - \varrho||\\
&+\ \ 2||b_{n_k} - Tb_{n_k}||||\varrho - T\varrho|| + 2\Theta_1||b_{n_k} - Tb_{n_k}||\\
&\leq\ \ ||b_{n_k} - Tb_{n_k}||^2 + ||b_{n_k} - \varrho||^2 - 2\sigma(\sigma+1)||\varrho - b_{n_k}||||b_{n_k} - \varrho||\\
&+\ \ 2||b_{n_k} - Tb_{n_k}||(||\varrho|| + ||T\varrho||) + 2\Theta_1||b_{n_k} - Tb_{n_k}||\\
&\leq\ \ ||b_{n_k} - Tb_{n_k}||^2 + ||b_{n_k} - \varrho||^2 + 6\Theta_1||b_{n_k} - Tb_{n_k}||.
\end{aligned}
$$

It, therefore, follows from the last inequality that

$$
\limsup_{k\to\infty}||b_{n_k} - T\varrho||^2 \leq \limsup_{k\to\infty}[||b_{n_k} - Tb_{n_k}||^2 + ||b_{n_k} - \varrho||^2 + 6\Theta_1||b_{n_k} - Tb_{n_k}||].
$$

As a consequence, we obtain

$$
A(T\varrho, \{b_{n_k}\}_{k=1}^{\infty}) = \limsup_{k\to\infty}||b_{n_k} - T\varrho|| = \limsup_{k\to\infty}||b_{n_k} - \varrho|| = r(\varrho, \{b_{n_k}\}_{k=1}^{\infty})\}.
$$

This, by implication, entails that $T\varrho \in A(C, \{b_n\}_{n=1}^{\infty})$. In view of the uniform convexity of C, we conclude that $T\varrho = \varrho$ as required. □

The result below is an immediate consequence of Theorem 3.6.

Proposition 3.7. *Let X and C be as described in Theorem 3.6. Let $T : C \longrightarrow X$ be a σ-ESNM with $F(T) \neq \emptyset$. If $b_n \rightharpoonup \varrho \in C$ and $(I - T)b_n \to 0$, then $\varrho \in F(T)$.*

Theorem 3.8. *Let T and C be as in Theorem 3.6 with X admitting the Opial property. Let $T : C \longrightarrow C$ be a σ-ESNM such that $F(T) \neq \emptyset$. If $\{\gamma_n\}_{n=1}^{\infty}$ is a sequence in $(0, 1)$ with $0 < \alpha \leq \gamma_n \leq 1 - \alpha < 1$, and $\{b_n\}_{n=1}^{\infty}$ is a sequence in C developed from*

$$
b_{n+1} = (1 - \gamma_n)b_n + \gamma_n T_\beta b_n, \quad \forall n \in \mathbb{N}, \tag{3.4}
$$

where $T_\beta = (I - \beta)I + \beta T$, then (3.4) converges weakly to an element of F.

Proof. Let $\varrho \in F(T) = F(T_\beta)$ be arbitrarily chosen. Then, by Lemma 2.7, we can find a strictly increasing function $g : [0, \infty) \longrightarrow [0, \infty)$, characterized by convexity and the continuity property, with $g(0) = 0$ such that

$$
\begin{aligned}
||b_{n+1} - \varrho||^2 &= ||(1 - \gamma_n)(b_n - \varrho) + \gamma_n(T_\beta b_n - \varrho)||^2\\
&\leq (1 - \gamma_n)||b_n - \varrho||^2 + \gamma_n||T_\beta b_n - \varrho||^2 - \gamma_n(1 - \gamma_n)g(||b_n - T_\beta b_n||)\\
&= (1 - \gamma_n)||b_n - \varrho||^2 + \frac{\gamma_n}{(\sigma+1)^2}||(1 - \sigma)(b_n - \varrho) + Tb_n - T\varrho||^2\\
&- \gamma_n(1 - \gamma_n)g\Big(\frac{1}{\sigma+1}||b_n - Tb_n||\Big)\\
&\leq (1 - \gamma_n)||\wp_n - \varrho||^2 + \gamma_n||b_n - \varrho||^2 + \frac{\gamma_n}{(\sigma+1)^2}\langle b_n - Tb_n, j(\varrho - T\varrho)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&- \quad \gamma_n(1-\gamma_n)g\Big(\frac{1}{\sigma+1}\|b_n - Tb_n\|\Big) \\
&= \quad (1-\gamma_n)\|b_n - \varrho\|^2 + \gamma_n\|b_n - \varrho\|^2 - \gamma_n(1-\gamma_n)g\Big(\frac{1}{\sigma+1}\|b_n - Tb_n\|\Big) \\
&\leq \quad \|b_n - \varrho\|^2 - \alpha^2 g\Big(\frac{1}{\sigma+1}\|b_n - Tb_n\|\Big).
\end{aligned}
\tag{3.5}
$$

Since $\alpha > 0$ and $\sigma \in [0, \infty)$, it follows from (3.5) that

$$\|b_{n+1} - \varrho\| \leq \|b_n - \varrho\|.$$

This implies that $\lim_{n\to\infty}\|b_n - \varrho\|$ exists. Therefore, $\{b_n\}_{n=1}^{\infty}$ is bounded. By setting

$$\lim_{n\to\infty}\|b_n - \varrho\| = \delta,$$

we obtain from (3.5) that

$$\alpha^2 g\Big(\frac{1}{\sigma+1}\|b_n - Tb_n\|\Big) \leq \|b_n - \varrho\| - \|b_{n+1} - \varrho\|,$$

which yields that

$$\lim_{n\to\infty}\|b_n - Tb_n\| = 0.$$

But, $\{b_n\}_{n=1}^{\infty}$ is bounded. Therefore, $\exists\{b_{n_k}\}_{k=1}^{\infty} \subset \{b_n\}_{n=1}^{\infty}$ such that $b_{n_k} \rightharpoonup \varrho$. Also, $\lim_{n\to\infty}\|b_n - Tb_n\| = 0$ implies that $\lim_{k\to\infty}\|b_{n_k} - Tb_{n_k}\| = 0$. From Proposition 3.7, $(I - T)b_n \to 0$ as $n \to \infty$. Consequently, $\varrho \in F(\mathfrak{T})$. To conclude, it suffices to establish the fact that for another subsequence $\{b_{n_i}\}_{i=1}^{\infty} \subseteq \{b_n\}_{n=1}^{\infty}$ which is characterized by the weak convergence property (i.e., $b_{n_i} \rightharpoonup \nu$ as $n \to \infty$), we have $\varrho = \nu$. Suppose otherwise and let $\varrho \neq \nu$. Then, we get from Opial's theorem that

$$
\begin{aligned}
\lim_{n\to\infty}\|b_n - \varrho\| &= \lim_{k\to\infty}\|b_{n_k} - \varrho\| < \lim_{k\to\infty}\|b_{n_k} - \nu\| = \lim_{n\to\infty}\|b_n - \nu\| \\
&= \lim_{i\to\infty}\|b_{n_i} - \nu\| < \lim_{i\to\infty}\|b_{n_i} - \varrho\| = \lim_{n\to\infty}\|b_n - \varrho\|.
\end{aligned}
$$

This is a contradiction. Consequently, $\{b_n\}_{n=1}^{\infty}$ converges weakly to $\varrho \in F(T)$. □

Theorem 3.9. *Let X be a UC which admits a weakly sequentially continuous duality mapping J, $\emptyset \neq C \subset X$ be closed and convex, and $T : C \longrightarrow C$ be a σ-enriched nonspreading mapping such that $F(T) \neq \emptyset$. Let $\{\gamma_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be two sequences in* $(0, 1)$ *such that the following requirements are validated:*

(a) $\lim_{n\to\infty}\gamma_n = 0$;
(b) $\sum_{n=1}^{\infty}\gamma_n = \infty$;
(c) $\liminf_{k\to\infty}\delta_n(1-\delta_n) > 0$.

Let the sequence $\{b_n\}_{n=1}^{\infty}$ be developed from

$$
\begin{cases}
u \in C, \quad b_1 \in C \quad \text{chosen arbitrarily}, \\
\vartheta_n = (1-\delta_n)b_n + \delta_n T_\beta b_n, \\
b_{n+1} = \gamma_n u + (1-\gamma_n)\vartheta_n, \quad \forall n \in \mathbb{N},
\end{cases}
\tag{3.6}
$$

where $T_\beta = (I-\beta)I + \beta T$. Then, $\{b_n\}_{n=1}^{\infty}$ given (3.6) *admits strong convergence to a point $\varrho \in Q_F u$, where Q_F denotes a sunny nonexpansive retraction from X onto $F(T)$.*

Proof. Since T_β is quasi-nonexpansive, we have that $F(T)$ is closed and convex. Set

$$\xi = Q_F.$$

We shall divide the rest of the proof into several steps.

Step 1. We demonstrate that $\{b_n\}_{n=1}^{\infty}$, $\{\vartheta_n\}_{n=1}^{\infty}$, and $\{Tb_n\}_{n=1}^{\infty}$ are bounded. First, we establish that $\{b_n\}_{n=1}^{\infty}$ admits boundedness.

Fix $\varrho \in F(T_\beta) = F(T)$. Using Lemma 2.7, we can find a strictly increasing function $g : [0,\infty) \longrightarrow [0,\infty)$ which is characterized by continuity, convexity, and $g(0) = 0$ properties such that the following estimates hold:

$$\begin{aligned}
\|\vartheta_n - \varrho\|^2 &= \|(1-\delta_n)b_n + \delta_n T_\beta b_n - \varrho\|^2 \\
&\le (1-\delta_n)\|b_n - \varrho\|^2 + \delta_n\|T_\beta b_n - \varrho\|^2 - \delta_n(1-\delta_n)g(\|b_n - T_\beta b_n\|) \\
&\le (1-\delta_n)\|b_n - \varrho\|^2 + \frac{\delta_n}{(\sigma+1)^2}\|\sigma(b_n - \varrho) + Tb_n - T\varrho\|^2 \\
&- \delta_n(1-\delta_n)g\Big(\frac{1}{\sigma+1}\|b_n - Tb_n\|\Big) \\
&\le (1-\delta_n)\|b_n - \varrho\|^2 + \frac{\delta_n}{(\sigma+1)^2}[(\sigma+1)^2\|b_n - \varrho\|^2 + 2\langle b_n - Tb_n, j(\varrho - T\varrho)] \\
&- \delta_n(1-\delta_n)g\Big(\frac{1}{\sigma+1}\|b_n - Tb_n\|\Big) \\
&= \|b_n - \varrho\|^2 + \frac{2\delta_n}{(\sigma+1)^2}\langle b_n - Tb_n, j(\varrho - T\varrho) - \delta_n(1-\delta_n)g\Big(\frac{1}{\sigma+1}\|b_n - Tb_n\|\Big) \\
&\le \|b_n - \varrho\|^2. \qquad (3.7)
\end{aligned}$$

Again, from (3.6), we have

$$\begin{aligned}
\|b_{n+1} - \varrho\| &= \|\gamma_n u + (1-\gamma_n)\vartheta_n - \varrho\| \\
&\le \gamma_n\|u - \varrho\| + (1-\gamma_n)\|\vartheta_n - \varrho\| \\
&\le \gamma_n\|u - \varrho\| + (1-\gamma_n)\|b_n - \varrho\| \quad \text{(by (3.7))} \\
&\le \max\{\|u - \varrho\|, \|b_n - \varrho\|\}.
\end{aligned}$$

Using induction, we get

$$\|b_{n+1} - \varrho\| \le \max\{\|u - \varrho\|, \|b_1 - \varrho\|\}, \quad \forall n \in \mathbb{N}.$$

The last inequality yields that $\{\|b_n - \varrho\|\}_{n=1}^{\infty}$ is bounded and as a consequence, $\{b_n\}_{n=1}^{\infty}$ is bounded. The boundedness of $\{\vartheta_n\}_{n=1}^{\infty}$ and $\{Tb_n\}_{n=1}^{\infty}$ follows from the above result and (3.6).

Step 2. Now, for any $n \in \mathbb{N}$, we want to show that

$$\|b_{n+1} - \xi\|^2 \le (1-\gamma_n)\|b_n - \xi\|^2 + 2\gamma_n\langle u - \xi, j(b_{n+1} - \xi)\rangle. \qquad (3.8)$$

To do this, note that for each $n \in \mathbb{N}$, (3.7) (with $\varrho = \xi$) gives

$$\|\vartheta_n - \xi\|^2 \leq \|b_n - \xi\|^2 + \frac{2\delta_n}{(\sigma+1)^2}\langle b_n - Tb_n, j(\xi - \Im\xi)\rangle - \delta(1-\delta_n)g\Big(\frac{1}{\sigma+1}\|b_n - Tb_n\|\Big).$$

This, together with (3.6), gives

$$\begin{aligned} \|b_{n+1} - \xi\| &= \|\gamma_n u + (1-\gamma_n)\vartheta_n - \xi\| \\ &\leq \gamma_n\|u - \xi\| + (1-\gamma_n)\Big[\|b_n - \xi\|^2 \\ &- \delta_n(1-\delta_n)g\Big(\frac{1}{\sigma+1}\|b_n - Tb_n\|\Big)\Big]. \end{aligned} \tag{3.9}$$

Set $\Theta_2 = \sup\Big\{\|u-\xi\| - \|b_n-\xi\|^2 + \delta(1-\delta_n)g\Big(\frac{1}{\sigma+1}\|b_n - Tb_n\|\Big) : n \in \mathbb{N}\Big\}$. Then, we obtain from (3.9) that

$$\delta_n(1-\delta_n)g\Big(\frac{1}{\sigma+1}\|b_n - Tb_n\|\Big) \leq \|\wp_n - \xi\|^2 - \|b_{n+1} - \xi\|^2 + \gamma_n\Theta_2. \tag{3.10}$$

Now, from Lemma 2.2 and (3.6), we get

$$\begin{aligned} \|b_{n+1} - \xi\|^2 &= \|\gamma_n u + (1-\gamma_n)\vartheta_n - \xi\|^2 \\ &= \|\gamma_n(u - \xi) + (1-\gamma_n)(\vartheta_n - \xi)\|^2 \\ &\leq (1-\gamma_n)^2\|\vartheta_n - \xi)\|^2 + 2\gamma_n\langle u - \xi), j(b_{n+1} - \xi)\rangle \\ &\leq (1-\gamma_n)\|\vartheta_n - \xi)\|^2 + 2\gamma_n\langle u - \xi), j(b_{n+1} - \xi)\rangle \\ &\leq (1-\gamma_n)\|b_n - \xi)\|^2 + 2\gamma_n\langle u - \xi), j(b_{n+1} - \xi)\rangle. \end{aligned}$$

Step 3. Now, we demonstrate that $\lim_{n\to\infty} b_n = \xi$.

To do this, we consider the two cases below:

Case A. If the sequence $\{\|b_n-\xi\|\}_{n=1}^{\infty}$ is monotonically decreasing, then there exists an $n_0 \in \mathbb{N}$ for which $\{\|b_n-\xi\|\}_{n=n_0}^{\infty}$ is decreasing. Consequently, $\{\|b_n-\xi\|\}_{n=1}^{\infty}$ is convergent and as such $\lim_{n\to\infty}(\|b_n-\xi\|^2 - \|b_{n+1} - \xi\|^2) = 0$. This, in view of condition (c) and (3.10), yields

$$\lim_{n\to\infty} g\Big(\frac{1}{\sigma+1}\|b_n - Tb_n\|\Big) = 0.$$

From the property of g, we have

$$\lim_{n\to\infty}\|b_n - Tb_n\| = 0. \tag{3.11}$$

Since from (3.6)

$$b_n - \vartheta_n = \delta_n(b_n - T_\beta b_n) = \frac{\delta_n}{\sigma+1}(b_n - Tb_n) \quad \text{and} \quad b_{n+1} - \vartheta_n = \gamma_n(u - \vartheta_n),$$

it follows from (3.11) and condition (a) that

$$\lim_{n\to\infty}\|b_n - \vartheta_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|b_{n+1} - \vartheta_n\| = 0. \tag{3.12}$$

Therefore, using the triangular inequality and (3.12), we get

$$\lim_{n\to\infty}\|b_{n+1}-\wp_n\| = \lim_{n\to\infty}\|b_{n+1}-\vartheta_n+\vartheta_n-b_n\| \le \lim_{n\to\infty}[\|b_{n+1}-\vartheta_n\|+\|\vartheta_n-b_n\|] \to 0 \tag{3.13}$$

as $n\to\infty$ (by (3.12)).

Since $\{b_n\}_{n=1}^{\infty}$ is bounded, there exists a subsequence $\{b_{n_k}\}_{k=1}^{\infty}$ of $\{b_n\}_{n=1}^{\infty}$ such that $b_{n_k+1} \rightharpoonup \varpi \in \Lambda$ as $k\to\infty$. It, therefore, follows from Proposition 3.7 and (3.11) that $\varpi \in F(T)$. This, together with Lemma 2.1, implies that

$$\limsup_{n\to\infty}\langle u-\xi, j(b_{n+1}-\xi)\rangle = \lim_{n\to\infty}\langle u-\xi, j(b_{n+1}-\xi)\rangle = \langle u-\xi, j(\varpi-\xi)\rangle \le 0. \tag{3.14}$$

Thus, by Lemma 2.5, the result follows immediately.

Case B. If the sequence $\{\|b_n-\xi\|\}_{n=1}^{\infty}$ is not eventually decreasing, then there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ of $\{n\}_{n=1}^{\infty}$such that

$$\|b_{n_k}-\xi\| < \|b_{n_k+1}-\xi\|$$

for all $k\in\mathbb{N}$. Using Lemma 2.6, we can find a nondecreasing sequence $\{m_j\}_{j=1}^{\infty}\subset\mathbb{N}$ such that $m_j\to\infty$ and

$$\|b_{m_j}-\xi\| < \|b_{m_j+1}-\xi\| \quad \text{and} \quad \|b_j-\xi\| < \|b_{m_j+1}-\xi\|,$$

for all $j\in\mathbb{N}$. This, together with (3.10), yields

$$\delta_{m_j}(1-\delta_{m_j})g\Big(\frac{1}{\sigma+1}\|b_{m_j}-Tb_{m_j}\|\Big) \le \|b_{m_j}-\xi\|^2 - \|b_{m_j+1}-\xi\|^2 + \gamma_{m_j}\Theta_2.$$

From the requirements of (a) and (c) and the property of g, it follows that

$$\lim_{j\to\infty}\|b_{m_j}-Tb_{m_j}\| = 0. \tag{3.15}$$

Using the same method employed in Case A, we obtain

$$\limsup_{j\to\infty}\langle u-\xi, j(b_{m_j}-\xi)\rangle = \limsup_{j\to\infty}\langle u-\xi, j(b_{m_j+1}-\xi)\rangle \le 0.$$

Since from (3.8)

$$\|b_{m_j+1}-\xi\|^2 \le (1-\gamma_{m_j})\|b_{m_j}-\xi\|^2 + 2\gamma_{m_j}\langle u-\xi, j(b_{m_j+1}-\xi)\rangle \tag{3.16}$$

and $\|b_{m_j}-\xi\| < \|b_{m_j+1}-\xi\|$, it follows that

$$\begin{aligned}\gamma_{m_j}\|b_{m_j}-\xi\|^2 &\le \|b_{m_j}-\xi\|^2 - \|b_{m_j+1}-\xi\|^2 + 2\gamma_{m_j}\langle u-\xi, j(b_{m_j+1}-\xi)\rangle \\ &\le 2\gamma_{m_j}\langle u-\xi, j(b_{m_j+1}-\xi)\rangle.\end{aligned} \tag{3.17}$$

In particular, since $\gamma_{m_j}>0$, it follows from (3.17) that

$$\|b_{m_j}-\xi\|^2 \le 2\langle u-\xi, j(b_{m_j+1}-\xi)\rangle$$

and hence

$$\lim_{j\to\infty}\|b_{m_j}-\xi\| = 0.$$

The last identity, together with (3.16), yields

$$\lim_{j\to\infty}\|b_{m_j+1}-\xi\|=0.$$

On the other hand, we have that $\|\wp_{m_j}-\xi\|<\|b_{m_j+1}-\xi\|\ \forall j\in\mathbb{N}$, which yields $b_j\to\xi$ as $j\to\infty$. Hence, $b_n\to\xi$ as $n\to\infty$ and the proof is complete. □

Corollary 3.10. *Let $\emptyset\neq C\subset X$ be closed and convex, where X is a real Hilbert space. Let $T:C\longrightarrow C$ be a σ-enriched nonspreading mapping such that $F(T)\neq\emptyset$, and $\{\gamma_n\}_{n=1}^{\infty}$ be a sequence in $(0,1)$ which validates the requirements that:*

(a) $\lim_{n\to\infty}\gamma_n=0$;

(b) $\sum_{n=1}^{\infty}\gamma_n=\infty$.

Then $\{\wp_n\}_{n=1}^{\infty}$ developed from

$$\begin{cases}u\in C, \quad b_1\in C \quad \textit{chosen arbitrarily},\\ b_{n+1}=\gamma_n u+(1-\gamma_n)T_\beta b_n, \quad \forall n\in\mathbb{N},\end{cases}\tag{3.18}$$

where $T_\beta=(I-\beta)I+\beta T$ admits strong convergence to a point $\varrho\in Q_F u$, where P_F is the metric projection from X onto F.

Theorem 3.11. *Let X and C be as described in Theorem 3.9. Let $T_{\beta,1}:C\longrightarrow C$ be a σ-enriched nonspreading mapping and $T_{\beta,2}:C\longrightarrow C$ be a σ-enriched nonexpansive mapping such that $F(T_1)\cap F(T_2)\neq\emptyset$. Let $\{\gamma_n\}_{n=1}^{\infty},\{\delta_{n,1}\}_{n=1}^{\infty},\{\delta_{n,2}\}_{n=1}^{\infty}$, and $\{\delta_{n,3}\}_{n=1}^{\infty}$ be four sequences in $[0,1]$ which validate the requirements that:*

(a) $\lim_{n\to\infty}\gamma_n=0$;

(b) $\sum_{n=1}^{\infty}\gamma_n=\infty$;

(c) $\delta_{n,1}+\delta_{n,2}+\delta_{n,3}=1$;

(d) $\liminf_{k\to\infty}\delta_n(1-\delta_n)>0$.

Then, $\{b_n\}_{n=1}^{\infty}$ developed from

$$\begin{cases}u\in C, \quad b_1\in C \quad \textit{chosen arbitrarily},\\ \vartheta_n=\delta_{n,1}T_{\beta,1}b_n+\delta_{n,2}T_{\beta,2}b_n+\delta_{n,3}b_{n,3},\\ b_{n+1}=\gamma_n u+(1-\gamma_n)\vartheta_n, \quad \forall n\in\mathbb{N}\end{cases}\tag{3.19}$$

admits strong convergence to a point $\varrho\in Q_F u$, where $T_{\beta,1}=(I-\beta)I+\beta T_1$, $T_{\beta,2}=(I-\beta)I+\beta T_2$, and Q_F denotes a sunny nonexpansive retraction from X onto F.

Proof. Since $T_{\beta,1}$ and $T_{\beta,2}$ are quasi-nonexpansive, we have that $F(T)\cap F(S)$ is closed and convex. Set

$$\xi=Q_F.$$

We shall divide the rest of the proof into several steps.

Step 1. We demonstrate that $\{b_n\}_{n=1}^{\infty}$, $\{\vartheta_n\}_{n=1}^{\infty}$, $\{Tb_n\}_{n=1}^{\infty}$, and $\{Sb_n\}_{n=1}^{\infty}$ are bounded. First, we establish that $\{b_n\}_{n=1}^{\infty}$ is bounded.

Let $\varrho \in F = F(T_1) \cap F(T_2)$ be fixed. Using Lemma 2.7, we can find a strictly increasing function $g : [0, \infty) \longrightarrow [0, \infty)$ which is characterized by continuity, convexity, and $g(0) = 0$ properties such that the following estimates hold:

$$\begin{aligned}
\|\vartheta_n - \varrho\|^2 &= \|\delta_{n,1}T_{\beta,1}b_n + \delta_{n,2}T_{\beta,2}b_n + \delta_n b_{n,3} - \varrho\|^2 \\
&\leq \delta_{n,1}\|T_{\beta,1}b_n - \varrho\|^2 + \delta_{n,2}\|T_{\beta,2}b_n - \varrho\|^2 + \delta_{n,3}\|b_n - \varrho\|^2 - \delta_{n,i}\delta_{n,3}g(\|b_n - T_{\beta,i}b_n\|) \\
&\leq \frac{\delta_{n,1}}{(\sigma+1)^2}\|\sigma(b_n - \varrho) + T_1 b_n - T\varrho\|^2 + \frac{\delta_{n,2}}{(\sigma+1)^2}\|\sigma(b_n - \varrho) + T_2 b_n - T\varrho\|^2 \\
&+ \delta_{n,3}\|b_n - \varrho\|^2 - \delta_{n,i}\delta_{n,3}g\Big(\frac{1}{\sigma+1}\|b_n - T_i b_n\|\Big) \\
&\leq \frac{\delta_{n,1}}{(\sigma+1)^2}[(\sigma+1)^2\|b_n - \varrho\|^2 + \langle b_n - T_1, j(\varrho - T_1\varrho)\rangle] \\
&+ \frac{\delta_{n,2}}{(\sigma+1)^2}[(\sigma+1)^2\|b_n - \varrho\|^2] + \delta_{n,3}\|b_n - \varrho\|^2 - \delta_{n,i}\delta_{n,3}g\Big(\frac{1}{\sigma+1}\|b_n - T_i\wp_n\|\Big) \\
&\leq \|\wp_n - \varrho\|^2 - \delta_{n,i}\delta_{n,3}g\Big(\frac{1}{\sigma+1}\|\wp_n - \Im_i b_n\|\Big) \\
&\leq \|b_n - \varrho\|^2, \quad i = 1, 2.
\end{aligned} \tag{3.20}$$

Also, from (3.19), we have

$$\begin{aligned}
\|b_{n+1} - \varrho\| &= \|\gamma_n u + (1-\gamma_n)\vartheta_n - \varrho\| \\
&\leq \gamma_n\|u - \varrho\| + (1-\gamma_n)\|\vartheta_n - \varrho\| \\
&\leq \gamma_n\|u - \varrho\| + (1-\gamma_n)\|b_n - \varrho\| \quad \text{(by (3.20))} \\
&\leq \max\{\|u - \varrho\| + \|b_n - \varrho\|\},
\end{aligned}$$

which by induction yields

$$\|b_{n+1} - \varrho\| \leq \max\{\|u - \varrho\| + \|b_1 - \varrho\|\}, \quad \forall n \in \mathbb{N},$$

and as a consequence, it follows that $\{\|\wp_n - \varrho\|\}_{n=1}^{\infty}$ is bounded. The boundedness of $\{b_n\}_{n=1}^{\infty}, \{\vartheta_n\}_{n=1}^{\infty}, \{T_1 b_n\}_{n=1}^{\infty}$, and $\{\Im_2 b_n\}_{n=1}^{\infty}$ follows directly from the boundedness of $\{b_n\}_{n=1}^{\infty}$ and (3.19).

Step 2. We establish that

$$\|b_{n+1} - \xi\|^2 \leq (1-\gamma_n)\|b_n - \xi\|^2 + 2\gamma_n\langle u - \xi, j(b_{n+1} - \xi)\rangle, \tag{3.21}$$

for any $n \in \mathbb{N}$. To do this, note that for each $n \in \mathbb{N}$ and $i = 1, 2$, (3.20) (with $\varrho = \xi$) gives

$$\|\vartheta_n - \xi\|^2 \leq \|b_n - \xi\|^2 - \delta_{n,i}\delta_{n,3}g\Big(\frac{1}{\sigma+1}\|b_n - T_i b_n\|\Big).$$

This, together with (3.19), gives

$$\|b_{n+1} - \xi\| = \|\gamma_n u + (1-\gamma_n)\vartheta_n - \xi\|$$

$$\leq \quad \gamma_n\|u-\xi\| + (1-\gamma_n)\Big[\|b_n-\xi\|^2 - \delta_{n,i}\delta_{n,3}g\Big(\frac{1}{\sigma+1}\|b_n - T_ib_n\|\Big)\Big]. \tag{3.22}$$

Set $\Theta_3 = \sup\Big\{\|u-\xi\| - \|b_n-\xi\|^2 + \delta_{n,i}\delta_{n,3}g\Big(\frac{1}{\sigma+1}\|b_n - T_ib_n\|\Big) : n \in \mathbb{N}, i = 1,2\Big\}$. Then, we obtain from (3.22) that

$$\delta_{n,i}\delta_{n,3}g\Big(\frac{1}{\sigma+1}\|b_n - T_ib_n\|\Big) \leq \|b_n-\xi\|^2 - \|b_{n+1}-\xi\|^2 + \gamma_n\Theta_3. \tag{3.23}$$

Now, from Lemma 2.2 and (3.19), we get

$$\begin{aligned} \|b_{n+1}-\xi\|^2 &= \|\gamma_n u + (1-\gamma_n)\vartheta_n - \xi\|^2 \\ &= \|\gamma_n(u-\xi) + (1-\gamma_n)(\vartheta_n-\xi)\|^2 \\ &\leq (1-\gamma_n)^2\|\vartheta_n-\xi)\|^2 + 2\gamma_n\langle u-\xi), j(b_{n+1}-\xi)\rangle \\ &\leq (1-\gamma_n)\|\vartheta_n-\xi)\|^2 + 2\gamma_n\langle u-\xi), j(b_{n+1}-\xi)\rangle \\ &\leq (1-\gamma_n)\|b_n-\xi)\|^2 + 2\gamma_n\langle u-\xi), j(b_{n+1}-\xi)\rangle. \end{aligned}$$

Step 3. We demonstrate that $\wp_n \to \xi$ as $n \to \infty$.

To show this, consider the two cases below:

Case A. If the sequence $\{\|b_n-\xi\|\}_{n=1}^{\infty}$ is monotonically decreasing, then $\exists n_0 \in \mathbb{N}$ for which $\{\|b_n-\xi\|\}_{n=n_0}^{\infty}$ is decreasing. Consequently, $\{\|b_n-\xi\|\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty}(\|b_n-\xi\|^2 - \|\wp_{n+1}-\xi\|^2) = 0$. This, in view of requirement (c) and (3.23), yields

$$\lim_{n\to\infty} g\Big(\frac{1}{\sigma+1}\|b_n - T_ib_n\|\Big) = 0, \quad i = 1,2.$$

Employing the property of g, we have

$$\lim_{n\to\infty}\|b_n - T_ib_n\| = 0, \quad i = 1,2. \tag{3.24}$$

Since from (3.19)

$$\begin{aligned} b_n - \vartheta_n &= \delta_{n,1}(T_{\beta,1}b_n - b_n) + \delta_{n,2}(T_{\beta,2}b_n - b_n) \\ &= \frac{1}{\sigma+1}[\delta_{n,1}(T_1b_n - \wp_n) + \delta_{n,2}(T_2b_n - b_n)] \end{aligned}$$

and $b_{n+1} - \vartheta_n = \gamma_n(u - \vartheta_n)$, it follows from (3.24) and condition (a) that

$$\lim_{n\to\infty}\|b_n - \vartheta_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|b_{n+1} - \vartheta_n\| = 0. \tag{3.25}$$

Therefore, using the triangular inequality and (3.25), we get

$$\lim_{n\to\infty}\|b_{n+1} - b_n\| = \lim_{n\to\infty}\|b_{n+1} - \vartheta_n + \vartheta_n - b_n\| \leq \lim_{n\to\infty}[\|b_{n+1} - \vartheta_n\| + \|\vartheta_n - b_n\|] = 0. \tag{3.26}$$

Since $\{b_n\}_{n=1}^{\infty}$ is bounded, we can find a subsequence $\{b_{n_k}\}_{k=1}^{\infty}$ of $\{b_n\}_{n=1}^{\infty}$ such that $b_{n_k+1} \rightharpoonup \varpi \in C$ as $k \to \infty$. It, therefore, follows from Proposition 3.7 and (3.24) that $\varpi \in F$. This, together with Lemma 2.1, implies that

$$\limsup_{n\to\infty}\langle u-\xi, j(b_{n+1}-\xi)\rangle = \lim_{n\to\infty}\langle u-\xi, j(b_{n+1}-\xi)\rangle = \langle u-\xi, j(\varpi-\xi)\rangle \leq 0. \tag{3.27}$$

Thus, by Lemma 2.5, the result follows immediately.

Case B. Using the approach employed in establishing Theorem 3.9, we can show that $\lim\limits_{n\to\infty} b_n = \varrho$, and the proof is complete. □

Remark. (1) The result of this research work solves the question posed by Kurokawa and Takahashi; see Remark on page 1567 in [25].

(2) Theorem 4.1 of [10] admits only a weak convergent result while our Theorem 3.11 admits a strong convergence result. However, it is worth mentioning that the technique involved in proving Theorem 3.11 is very different from the one employed in proving Theorem 4.1.

(3) In most cases, strong convergence results are better than weak convergence results in applications.

4. Rate of convergence

For a nonempty convex subset C of a space X and $T :\longrightarrow C$:

(1) The Mann (M_n) iteration method (see [2]) is defined by the following sequence $\{b_n\}$:

$$\begin{cases} b_0 \in C \\ b_{n+1} = (1-\gamma_n)b_n + \gamma_n T b_n \end{cases}, \tag{4.1}$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$.

(2) The sequence $\{b_n\}$ given by

$$\begin{cases} b_0 \in C \\ \vartheta_n = (1-\delta_n)b_n + \delta_n T b_n \\ b_{n+1} = (1-\gamma_n)b_n + \gamma_n T \vartheta_n \end{cases}, \tag{4.2}$$

where $\{\gamma_n\}, \{\delta_n\}$ are sequences in $(0, 1)$, is called the Ishikawa (Ish_n) method (see [26]).

(3) Our method (I_n) is given by

$$\begin{cases} b_0 \in C \\ \vartheta_n = (1-\delta_n)b_n + \delta_n T_\beta b_n \\ b_{n+1} = (1-\gamma_n)u + \gamma_n \vartheta_n \end{cases}, \tag{4.3}$$

where $\{\gamma_n\}, \{\delta_n\}$ are sequences in $(0, 1)$ and $T_\beta = (1-\beta)I + \beta T$ (with $\beta \in (0, 1)$).

Definition 4.1. [27] Suppose that $\{c_n\}$ and $\{d_n\}$ are two real convergent sequences with limits c and d, respectively. Then, $\{c_n\}$ is said to converge faster than $\{d_n\}$ if

$$\lim_{n\to\infty} \left| \frac{c_n - c}{d_n - d} \right| = 0.$$

Now, using the example below, we prove that the iteration process I_n used in obtaining our main result of Theorem 3.9 is faster than the Mann M_n and Ishikawa Ish_n methods for enriched nonspreading operators.

Example 4.2. Suppose $T:\mathbb{R}\longrightarrow\mathbb{R}=-b, \gamma_n=\frac{4}{\sqrt{n}}, \beta=\frac{5}{6}, \delta_n=\frac{1}{4}$, and $u=\frac{1}{2}$. It is clear that T is an enriched nonspreading mapping with a unique fixed point of 0 (see Example 3.5 above). Also, it is not difficult to see that Example 4.2 satisfies all the conditions of Theorem 3.9.

Proof. Since $\gamma_n=\frac{4}{\sqrt{n}}, \beta=\frac{5}{6}$, and $\delta_n=\frac{1}{4}$, it follows from M_n, Ish_n, and I_m that for $b_0\neq 0$,

$$\begin{aligned} M_n &= (1-\gamma_n)b_n+\gamma_n Tb_n \\ &= \Big(1-\frac{4}{\sqrt{n}}\Big)b_n-\frac{4}{\sqrt{n}}b_n=\Big(1-\frac{8}{\sqrt{n}}\Big)b_n=\prod_{i=2}^{n}\Big(1-\frac{8}{\sqrt{i}}\Big)b_0, \end{aligned}$$

$$\begin{aligned} Ish_n &= (1-\gamma_n)b_n+\gamma_n[(1-\delta_n)b_n+\delta_n Tb_n] \\ &= \Big(1-\frac{4}{\sqrt{n}}\Big)b_n+\frac{4}{\sqrt{n}}T\Big(\frac{1}{2}b_n\Big)=\Big(1-\frac{4}{\sqrt{n}}\Big)b_n-\frac{4}{\sqrt{n}}\Big(\frac{1}{2}b_n\Big)=\Big(1-\frac{6}{\sqrt{n}}\Big)b_n=\prod_{i=2}^{n}\Big(1-\frac{6}{\sqrt{i}}\Big)b_0, \end{aligned}$$

and

$$\begin{aligned} I_n &= (1-\gamma_n)u+\gamma_n[(1-\delta_n)b_n+\delta_n((1-\beta)I+\beta T)b_n] \\ &= \frac{4}{2\sqrt{n}}+\Big(1-\frac{4}{\sqrt{n}}\Big)\Big[\Big(1-\frac{1}{4}\Big)b_n+\frac{1}{4}\Big(\Big(1-\frac{5}{6}\Big)b_n-\frac{5}{6}b_n\Big)\Big] \\ &= \frac{2}{\sqrt{n}}+\Big(1-\frac{4}{\sqrt{n}}\Big)\Big[\frac{3}{4}b_n-\frac{1}{6}b_n\Big]=\frac{2}{\sqrt{n}}+\frac{7}{12}\Big(1-\frac{4}{\sqrt{n}}\Big) \\ &= \Big(\frac{7}{12}-\frac{1}{3\sqrt{n}}\Big)b_n=\prod_{i=2}^{n}\Big(\frac{7}{12}-\frac{1}{3\sqrt{n}}\Big)b_0. \end{aligned}$$

Now, consider

$$\begin{aligned} \Big|\frac{I_n-0}{M_n-0}\Big| &= \left|\frac{\prod_{i=2}^{n}\Big(\frac{7}{12}-\frac{1}{3\sqrt{n}}\Big)b_0}{\prod_{i=2}^{n}\Big(1-\frac{8}{\sqrt{i}}\Big)b_0}\right|=\left|\frac{\prod_{i=2}^{n}\Big(\frac{7}{12}-\frac{1}{3\sqrt{n}}\Big)}{\prod_{i=2}^{n}\Big(1-\frac{8}{\sqrt{i}}\Big)}\right| \\ &= \left|\prod_{i=2}^{n}\left(1-\frac{\frac{5}{12}-\frac{8}{\sqrt{i}}+\frac{1}{3\sqrt{i}}}{\Big(1-\frac{8}{\sqrt{i}}\Big)}\right)\right|=\left|\prod_{i=2}^{n}\left(1-\frac{1}{12\sqrt{i}}\frac{5i-92\sqrt{i}}{\sqrt{i}-8}\right)\right|. \end{aligned}$$

It is not difficult to see that

$$\begin{aligned} 0 &\leq \lim_{n\to\infty}\left|\prod_{i=2}^{n}\left(1-\frac{1}{12\sqrt{i}}\frac{5i-92\sqrt{i}}{\sqrt{i}-8}\right)\right| \\ l &\leq \lim_{n\to\infty}\prod_{i=2}^{n}\left(1-\frac{1}{i}\right)=\lim_{n\to\infty}\frac{1}{n}. \end{aligned} \tag{4.4}$$

Hence,

$$\lim_{n\to\infty}\left|\frac{I_n - 0}{M_n - 0}\right| = 0.$$

Thus, our iteration scheme converges faster than Mann's iteration method to the fixed point of T.

Similarly,

$$\begin{aligned}\left|\frac{I_n - 0}{Ish_n - 0}\right| &= \left|\frac{\prod_{i=2}^{n}\left(\frac{7}{12} - \frac{1}{3\sqrt{n}}\right)b_0}{\prod_{i=2}^{n}\left(1 - \frac{6}{\sqrt{i}}\right)b_0}\right| = \left|\frac{\prod_{i=2}^{n}\left(\frac{7}{12} - \frac{1}{3\sqrt{n}}\right)}{\prod_{i=2}^{n}\left(1 - \frac{6}{\sqrt{i}}\right)}\right| \\ &= \left|\prod_{i=2}^{n}\left(1 - \frac{\frac{5}{12} - \frac{6}{\sqrt{i}} + \frac{1}{3\sqrt{i}}}{\left(1 - \frac{6}{\sqrt{i}}\right)}\right)\right| = \left|\prod_{i=2}^{n}\left(1 - \frac{1}{12\sqrt{i}}\frac{5i - 68\sqrt{i}}{\sqrt{i} - 6}\right)\right|\end{aligned}$$

with

$$\begin{aligned}0 &\le \lim_{n\to\infty}\left|\prod_{i=2}^{n}\left(1 - \frac{1}{12\sqrt{i}}\frac{5i - 68\sqrt{i}}{\sqrt{i} - 6}\right)\right| \\ &\le \lim_{n\to\infty}\prod_{i=2}^{n}\left(1 - \frac{1}{i}\right) = \lim_{n\to\infty}\frac{1}{n}. \end{aligned} \tag{4.5}$$

Therefore,

$$\lim_{n\to\infty}\left|\frac{I_n - 0}{Ish_n - 0}\right| = 0.$$

Thus, our iteration scheme converges faster than Ishikawa's iteration method to the fixed point of T. □

In general, we notice that for $x_0 = b_0 = -1, u = 0.5$, and $\gamma_n = \frac{2}{n+5}$, we can choose $\beta = \delta_n = \frac{5}{5}$. Thus, all the conditions of Theorem 3.9 are fulfilled and $\{x_n\} = \{b_n\}$ converges to $0 = P_{F(T)}u$ (see Figure 1 below). Similarly, for $x_0 = b_0 = 0.7, u = -1$, and $\gamma_n = \frac{2}{n+5}$, the sequence $\{x_n\} = \{b_n\}$ converges to $0 = P_{F(T)}u$ (see Figure 1 below). A closer observation on Figure 1 shows that the convergence of the sequence $\{x_n\} = \{b_n\}$ to the fixed point of T is independent of the numerical values of the initial point $x_0 = b_0$ and u.

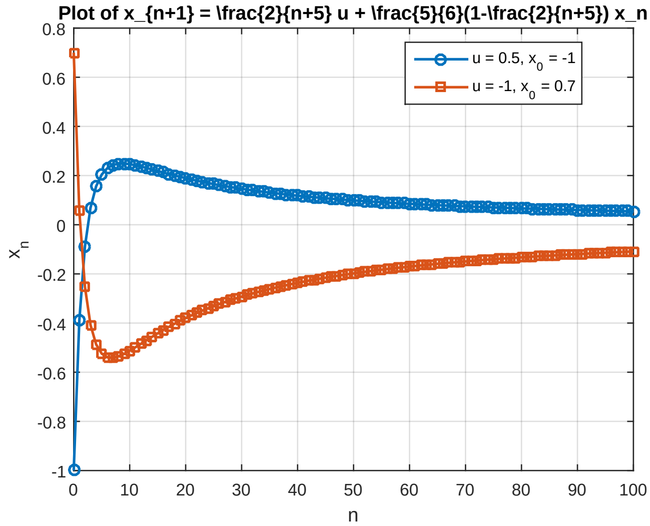

Figure 1. Figure of $\{a_n\}$ with initial values $u = 0.8, a_0 = -1$ and $u = -1, a_0 = 0.7$.

Author contributions

Asima Razzaque: Investigation, Writing review and editing; Imo Kalu Agwu: Conceptualization, Formal analysis, Investigation, Writing original draft preparation, Writing review and editing; Naeem Saleem: Conceptualization, Formal analysis, Investigation, Writing original draft preparation, Writing review and editing; Donatus Ikechi Igbokwe: Conceptualization; Maggie Aphane: Formal analysis, Writing review and editing. All authors have read and agreed to the published version of the manuscript. All the authors have read and approved the current version of this manuscript.

Use of Generative-AI tools declaration

The authors declare that AI was not involved in any manner during the writing of this manuscript.

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Conflict of interest

The authors declare that they do not have any competing interests.

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