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Research article

Almost periodic solutions of neutral-type differential system on time scales and applications to population models

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Abstract: We first study almost periodic solutions of neutral-type differential system on time scales and establish some basic results for the considered system. Furthermore, based on these results, the dynamic behaviors of two classes of neutral-type biological population models including host-macroparasite model and Lasota–Wazewska model are obtained. It is worth mentioning that we study almost periodic solutions for neutral-type differential system on time scales. Furthermore, using the above study and exponential dichotomy method, we investigate two types of biological population models.

Keywords: almost periodic solution; time scales; global exponential stability; exponential dichotomy
Mathematics Subject Classification: 34K14, 34K20

1. Introduction

The study of almost periodic solutions of differential equations has a long history, and many scholars have made important contributions in this area, see [1–5]. Almost periodic behaviors have been known to extensively exist in the ecological systems, electronic circuits, neural networks, and so forth. The study of almost periodic solutions has helped us understand the properties and laws of a system's motion development. More recent results for nonlinear system, see [6–11].

In 1988, Stefan Hilger [12] first introduced the theory and applications of calculus on time scales. After that, a large number of research results of dynamic equations on time scales can be found in [13–17]. However, there are few results for the study of almost periodic solutions of differential equations on time scales. In 2011, Li and Wang [18] first proposed the concept of almost periodic time scales and the definition of almost periodic functions on almost periodic time scales. Using the above results, they considered the existence and dynamic properties of the almost periodic solution for a Hopfield neural networks with time-varying delays. In [19], the authors further studied definitions of almost periodic time scales and gave some new applications on Nicholson's blowflies system on time scales. For more results about neutral-type differential system on time scales, see [20, 21] and related references. We found that there are few results for almost periodic solutions of neutral-type systems on time scales. In order to fill this gap, we will study almost periodic solutions of neutral-type systems on time scales, and we also will give its applications on neutral-type biological population models on time scales.

The main innovations of this paper are given as follows:

(1) We first consider the general theory of almost periodic solutions for neutral-type differential system on time scales and obtain some basic results for the considered system. Using the above theory and exponential dichotomy method, we study the almost periodic solutions of host-macroparasite model and Lasota-Wazewska model on time scales, which generalize the existing results in [30, 31, 33, 34].

(2) We develop the research of almost periodic solutions for neutral-type differential systems on time scales. Particularly, using the property of neutral-type operator, we obtain some new results for neutral-type differential system on time scales.

(3) Due to the fact that the system on time scales includes both discrete and continuous cases, the results obtained in this paper are applicable to both discrete and continuous systems.

The remaining setup of the paper are organized as follows: We give some preliminary results in Section 2. Section 3 gives the general theory of almost periodic solutions for neutral-type differential system on time scales. In Section 4, we study the positive almost periodic solutions of host-macroparasite model on time scales. In Section 5, we study the positive almost periodic solutions of Lasota–Wazewska model on time scales. Section 6 gives two examples for verifying our results. We draw some conclusions in Section 7.

2. Preliminary results

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. The backward jump operator ρ and the forward jump operator σ, respectively, defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \ \sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

the backward graininess $\mu = t - \rho(t)$ and the forward graininess $\mu(t) = \sigma(t) - t$. A function $g : \mathbb{T} \to \mathbb{R}$ is regressive if $1 + \mu(t)g(t) \neq 0$ for all $t \in \mathbb{T}^k$ holds. The set of regressive and rd-continuous functions g is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R})$. A function $g : \mathbb{T} \to \mathbb{R}$ is positive regressive if $1 + \mu(t)g(t) > 0$ for all $t \in \mathbb{T}^k$ holds. The set of positive regressive and rd-continuous functions g is denoted by $\mathcal{R}^+(\mathbb{T}, \mathbb{R})$. The interval $[p, q]_{\mathbb{T}}$ means $[p, q] \cap \mathbb{T}$. The intervals $[p, q)_{\mathbb{T}}$, $(p, q)_{\mathbb{T}}$ and $(p, q]_{\mathbb{T}}$ are defined similarly. $C_{rd}(\mathbb{T}, \mathbb{R})$ denotes the set of all $rd-$continuous functions on $\mathbb{T}$. For $s, t \in \mathbb{T}$ with $t > s$, the exponential function $e_\gamma(t, s)$ is defined by

$$e_\gamma(t, s) = \exp\Big(\int_s^t \zeta_{\mu(\tau)}(\gamma(\tau))\Delta\tau\Big),$$

where

$$\zeta_{\mu(\tau)}(\gamma(\tau)) = \begin{cases} \frac{1}{\mu(\tau)} Log(1 + \mu(\tau)\gamma(\tau)), \ \mu(\tau) > 0, \\ \gamma(\tau), \ \ \mu(\tau) = 0. \end{cases}$$

Lemma 2.1. [27] Let $\xi, \eta \in \mathcal{R}$. Then,

1) $e_0(t,s) \equiv 1$ and $e_\xi(t,t) \equiv 1$;
2) $e_\xi(\rho(t),s) = (1-\mu(t)\xi(t))e_\xi(t,s)$;
3) $e_\xi(t,s)e_\eta(t,s) = e_{\xi\oplus\eta}(t,s)$;
4) $e_\xi(t,s) = \frac{1}{e_\xi(s,t)} = e_{\ominus\xi}(s,t)$;
5) $e_\xi(t,s)e_\xi(s,r) = e_\xi(t,r)$.

Definition 2.1. [27] A function $\mathcal{M}:\mathbb{T}\to\mathbb{R}$ is called a delta-antiderivative of $m:\mathbb{T}\to\mathbb{R}$ if $\mathcal{M}^\Delta(t) = m(t)$ holds for all $t\in\mathbb{T}^k$. For this case, define the integral of m by

$$\int_a^t m(s)\Delta s = \mathcal{M}(t) - \mathcal{M}(a).$$

Definition 2.2. [18] Let Θ be a collection of sets which is constructed by subsets of $\mathbb{R}$. We call a time scale $\mathbb{T}$ as a almost periodic time scale, if

$$\Theta^* = \Big\{ \pm\nu \in \bigcap_{\gamma\in\Theta}\gamma : t\pm\nu\in\mathbb{T},\ \forall t\in\mathbb{T}\Big\} \neq \emptyset,$$

where Θ^* is the smallest almost periodic set of $\mathbb{T}$.
Definition 2.3. [18] Let $\mathbb{T}$ be an almost periodic time scale with respect to Θ. A function $\phi\in C(\mathbb{T},\mathbb{R}^n)$ is called almost periodic if for any $\varepsilon>0$, the set

$$\Omega(\phi,\varepsilon) = \{\nu\in\Theta^* : \|\phi(t+\nu)-\phi(t)\| < \varepsilon,\ \forall t\in\mathbb{T}\}$$

is relatively dense, i.e., for all $\varepsilon>0$, there is $m=m(\varepsilon)>0$ such that each interval of length m contains at least one $\nu\in\Omega(\phi,\varepsilon)$ satisfying $\|\phi(t+\nu)-\phi(t)\|<\varepsilon,\ \forall t\in\mathbb{T}$.
Definition 2.4. [18] Let $B(t)$ be an $n\times n$ $rd-$continuous matrix function on $\mathbb{T}$. The linear system

$$y^\Delta(t) = B(t)y(t) \tag{2.1}$$

admits an exponential dichotomy if there are constants $a_1,a_2>0$, projection P, and the fundamental solution matrix $Y(t)$ of system (2.1) satisfying:

$$\||Y(t)PY^{-1}(\sigma(s))\| \le a_1 e_{\ominus a_2}(t,\sigma(s)) \text{ for } t\ge\sigma(s),\ \ s,t\in\mathbb{T},$$

$$\||Y(t)(I-P)Y^{-1}(\sigma(s))\| \le a_1 e_{\ominus a_2}(\sigma(s),t) \text{ for } t\le\sigma(s),\ \ s,t\in\mathbb{T}.$$

Consider the following nonlinear system

$$y^\Delta(t) = B(t)y(t) + \phi(t),\ t\in\mathbb{T}, \tag{2.2}$$

where $B(t)$ is defined by (2.1), $\phi(t)$ is almost periodic vector value function.
Lemma 2.2. [18] If system (2.1) admits an exponential dichotomy, then system (2.2) has a unique almost periodic solution $y(t)$ as follows:

$$y(t) = \int_{-\infty}^t Y(t)PY^{-1}(\sigma(s))\phi(s)\Delta s - \int_t^{+\infty} Y(t)(I-P)Y^{-1}(\sigma(s))\phi(s)\Delta s.$$

Consider the following linear system

$$y^{\Delta}(t) = \mathrm{diag}(-b_1(t), -b_2(t), \cdots, -b_n(t))y(t),\ t \in \mathbb{T}. \tag{2.3}$$

Lemma 2.3. [18] For $i = 1, 2, \cdots, n$, assume that $b_i(t)$ is almost periodic function on $\mathbb{T}$ with $b_i(t) > 0$, $-b_i(t) \in \mathcal{R}^+$, and $\inf_{t\in\mathbb{T}} b_i(t) > 0$. Then, the linear system (2.3) admits an exponential dichotomy on $\mathbb{T}$.

Remark 2.1. Let $B(t) = \mathrm{diag}(-b_1(t), -b_2(t), \cdots, -b_n(t))$. Then, $Y(t) = e_B(t, t_0)$ is a fundamental solution matrix of the linear system (2.3).

Remark 2.2. Exponential dichotomy method has wide applications in non linear differential equations, for example, in [22], Sasu provided a new approach concerning the characterization of exponential dichotomy of difference equations by means of admissible pair of sequence spaces; Jiang [23] extended Hartman's theorem to the systems with generalized exponential dichotomy; the study of the exponential dichotomy of evolution equations using input-output techniques, see [24–26].

In this paper, we use the notations:

$$\hat{f}(t) = \sup_{t\in\mathbb{T}} |f(t)|,\quad \check{f}(t) = \inf_{t\in\mathbb{T}} |f(t)|,$$

where f is a bounded rd-continuous function.

3. Almost periodic solution of neutral-type system on time scales

Consider the following neutral-type system on time scales:

$$(y(t) - Cy(t-\tau))^{\Delta} = B(t)y(t) + \phi(t),\ t \in \mathbb{T}, \tag{3.1}$$

where $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^T$ and $\phi(t) = (\phi_1(t), \phi_2(t), \cdots, \phi_n(t))^T$ are rd-continuous vector functions, $\tau > 0$ is a constant, $C = \mathrm{diag}(c_1, c_2, \cdots, c_n)$, c_i is constant, $i = 1, 2, \cdots, n$, and $B(t)$ is an $n \times n$ $rd-$continuous matrix function. Define the operator $\mathcal{A}$ by

$$\mathcal{A} : \Omega \to \Omega,\ (\mathcal{A}y)(t) = y(t) - Cy(t-\tau),\ t \in \mathbb{T}, \tag{3.2}$$

where $\Omega = \{\omega : \omega(t) \in C_{rd}(\mathbb{T}, \mathbb{R}^n), \omega(t)$ is almost periodic vector function$\}$.

Lemma 3.1 [28] If $\|C\| < 1$, then the operator $\mathcal{A}$ has a unique rd-continuous bounded inverse $\mathcal{A}^{-1}$ satisfying

$$\|\mathcal{A}^{-1}y\| \le \frac{\|y\|}{1 - \|C\|},$$

where $\mathcal{A}$ is defined by (3.2).

From (3.1) and (3.2), we can change system (3.1) into the following system:

$$(\mathcal{A}y)^{\Delta}(t) = B(t)(\mathcal{A}y)(t) + B(t)Cy(t-\tau) + \phi(t),\ t \in \mathbb{T}. \tag{3.3}$$

Consider the following line system:

$$(\mathcal{A}y)^{\Delta}(t) = B(t)(\mathcal{A}y)(t), \tag{3.4}$$

where $B(t) = \text{diag}(-b_1(t), -b_2(t), \cdots, -b_n(t))$, and $b_i(t)$ is rd-continuous function. By Lemma 2.3, we have the following lemma:

Lemma 3.2, For $i = 1, 2, \cdots, n$, assume that $b_i(t)$ is almost periodic function on $\mathbb{T}$ with $b_i(t) > 0$, and $-b_i(t) \in \mathcal{R}^+$ and $\inf_{t\in\mathbb{T}} b_i(t) > 0$. Then, the linear system (3.4) admits an exponential dichotomy on $\mathbb{T}$.

By Lemma 2.2 we have the following lemma:

Lemma 3.3. [18] If system (3.4) admits an exponential dichotomy, then system (3.3) has a unique almost periodic solution $(\mathcal{A}y)(t)$ as follows:

$$(\mathcal{A}y)(t) = \int_{-\infty}^{t} Y(t)PY^{-1}(\sigma(s))\tilde{\phi}(s)\Delta s - \int_{t}^{+\infty} Y(t)(I-P)Y^{-1}(\sigma(s))\tilde{\phi}(s)\Delta s,$$

where $\tilde{\phi}(s) = B(s)Cy(s-\tau) + \phi(s)$, and $Y(t)$ is the fundamental solution matrix of system (3.3).

4. Applications in host-macroparasite model

In this section, we will study the dynamic properties of almost periodic solution to host-macroparasite model by using the theory results of Section 3.

In 1995, May and Anderson [29] first introduced the host-macroparasite model

$$x'(t) = -ax(t) + \frac{bx(t)}{[1 + cx(t-\tau)]^{N+1}}, \tag{4.1}$$

where $x(t)$ denotes the number of sexually mature worms in the human community. The means of other parameters, see [29]. After that, the authors [30] investigated the oscillation of system (4.1). In 2015, Yao [31] studied a class of host-macroparasite model with multiply delays and variable coefficients on time scales:

$$x^{\Delta}(t) = -a(t)x(t) + \sum_{i=1}^{n} \frac{b_i(t)x(t-\tau_i(t))}{[1 + x_i(t-\tau_i(t))]^{N_i+1}}. \tag{4.2}$$

Using the contraction mapping fixed point theorem and exponential dichotomy, the author obtained the existence and global exponential stability of positive almost periodic solution for Eq (4.2). But, there exist few results for positive almost periodic solution of neutral-type host-macroparasite model. Hence, this paper will study the following neutral-type host-macroparasite model on time scales:

$$(x(t) - c_0x(t-\gamma))^{\Delta} = -a(t)x(t) + \frac{b(t)x(t-\tau(t))}{[1 + c(t)x(t-\tau(t))]^{N+1}}, \ t \in \mathbb{T}, \tag{4.3}$$

where $\mathbb{T}$ is an almost periodic time scale, c_0 is a constant with $0 < c_0 < 1$, $a(t)$, $b(t)$, $c(t)$, and $\tau(t)$ are all positive almost periodic functions with $-a \in \mathcal{R}^+$, γ and $N > 0$ are constants. Let

$$(\mathcal{A}x)(t) = x(t) - c_0x(t-\gamma).$$

Then, Eq (4.3) can be rewritten by

$$(\mathcal{A}x)^{\Delta}(t) = -a(t)(\mathcal{A}x)(t) - a(t)c_0x(t-\gamma) + \frac{b(t)x(t-\tau(t))}{[1 + c(t)x(t-\tau(t))]^{N+1}}, \ t \in \mathbb{T}. \tag{4.4}$$

Considering biological significance, we only focus on positive almost periodic solutions of equation (4.3). Let $\Xi = \{u : u \in C_{rd}(\mathbb{T}, \mathbb{R}), u(t)$ is almost periodic function$\}$ with the norm $\|u\| = \sup_{t\in\mathbb{T}} |u(t)|$, then Ξ is a Banach space. In view of (4.4), for $u \in \Xi$, consider the following auxiliary equation:

$$(\mathcal{A}x)^{\Delta}(t) = -a(t)(\mathcal{A}x)(t) - a(t)c_0u(t-\gamma) + \frac{b(t)u(t-\tau(t))}{[1 + c(t)u(t-\tau(t))]^{N+1}},\ t \in \mathbb{T}. \tag{4.5}$$

Since $\check{a} > 0$, it follows by Lemma 3.2 that the linear equation $(\mathcal{A}x)^{\Delta}(t) = -a(t)(\mathcal{A}x)(t)$ admits an exponential dichotomy on $\mathbb{T}$. By Lemma 3.3, Eq (4.5) has a unique almost periodic solution

$$(\mathcal{A}x)(t) = \int_{-\infty}^{t} e_{-a}(t, \sigma(s))\Big[- a(s)c_0u(s-\gamma) + \frac{b(s)u(s-\tau(s))}{[1 + c(s)u(s-\tau(s))]^{N+1}}\Big]\Delta s.$$

For $u \in \Xi$, define the operator $\Gamma : \Xi \to \Xi$ by

$$\Gamma[(\mathcal{A}u)](t) = \int_{-\infty}^{t} e_{-a}(t, \sigma(s))\Big[- a(s)c_0u(s-\gamma) + \frac{b(s)u(s-\tau(s))}{[1 + c(s)u(s-\tau(s))]^{N+1}}\Big]\Delta s.$$

Obviously, $u(t)$ is the almost periodic solution of Eq (4.3) if and only if $\mathcal{A}u$ is the fixed point of the operator Γ. In this section, we need the following assumptions:
(H_1) $N + 1 - c(t) > 0$ for all $t \in \mathbb{T}$;
(H_2) there exist $\lambda_1, \lambda_2 > 0$ with $\frac{1}{N+1-\hat{c}} \le \lambda_1 \le \lambda_2$ such that

$$\frac{1}{\check{a}}\Big(M_1 - \frac{\check{a}c_0\lambda_1}{1-c_0}\Big) \le \lambda_2,$$

$$\frac{1}{\hat{a}}\Big(\frac{\check{b}\lambda_2}{[1 + \hat{c}\lambda_2]^{N+1}} - \frac{\hat{a}c_0\lambda_2}{1-c_0}\Big) \ge \lambda_1,$$

where M_1 is defined by (4.8).
(H_3) $\frac{\hat{a}c_0+\hat{b}}{\check{a}(1-c_0)} < 1$.
Theorem 4.1. Suppose that assumptions (H_1)–(H_3) are satisfied, then Eq (4.3) has a unique almost periodic positive solution.
Proof: Let $\Omega = \{u : u \in \Xi,\ \lambda_1 \le (\mathcal{A}u)(t) \le \lambda_2,\ t \in \mathbb{T}\}$, where λ_1 and λ_2 are defined by assumption (H_2). We first show that $\Gamma(\mathcal{A}\Omega) \subset \mathcal{A}\Omega$. Since $\mathcal{A}'(u) = 1 - c_0 > 0$, then $\mathcal{A}^{-1}$ is increasing on $\mathbb{R}$. Hence, for each $u \in \Omega$, by Lemma 3.1, we have

$$\frac{\lambda_1}{1-c_0} \le u(t) \le \frac{\lambda_2}{1-c_0}. \tag{4.6}$$

For each $u \in \Omega$, by (4.6) we have

$$\begin{aligned}\Gamma[(\mathcal{A}u)](t) &= \int_{-\infty}^{t} e_{-a}(t, \sigma(s))\Big[- a(s)c_0u(s-\gamma) + \frac{b(s)u(s-\tau(s))}{[1 + c(s)u(s-\tau(s))]^{N+1}}\Big]\Delta s \\ &\le \int_{-\infty}^{t} e_{-a}(t, \sigma(s))\Big[- \check{a}c_0\frac{\lambda_1}{1-c_0} + \frac{\hat{b}u(s-\tau(s))}{[1 + \check{c}u(s-\tau(s))]^{N+1}}\Big]\Delta s.\end{aligned} \tag{4.7}$$

Consider the function $f_1(x) = \frac{x}{(1+\check{c}x)^{N+1}}$, $x \in \mathbb{R}$. Since $f_1'(x) = \frac{1-(-\check{c}+N+1)x}{(1+\check{c}x)^{N+2}}$, in view of assumption (H_1), $f_1(x)$ is increasing on $[0, \frac{1}{N+1-\check{c}}]$ and decreasing on $[\frac{1}{N+1-\check{c}}, +\infty)$. Hence,

$$\frac{\hat{b}u(s-\tau(s))}{[1+\check{c}u(s-\tau(s))]^{N+1}} \le \frac{\hat{b}\frac{1}{N+1-\check{c}}}{[1+\check{c}\frac{1}{N+1-\check{c}}]^{N+1}} := M_1. \tag{4.8}$$

It follows by (4.7), (4.8), and assumption (H_2) that

$$\begin{aligned}\Gamma[(\mathcal{A}u)](t) &\le \left(M_1 - \frac{\check{a}c_0\lambda_1}{1-c_0}\right)\int_{-\infty}^{t} e_{-\check{a}}(t,\sigma(s))\Delta s\\ &= \frac{1}{\check{a}}\left(M_1 - \frac{\check{a}c_0\lambda_1}{1-c_0}\right)\\ &\le \lambda_2.\end{aligned} \tag{4.9}$$

On the other hand, for each $u \in \Omega$, by (4.6) we have

$$\Gamma[(\mathcal{A}u)](t) \ge \int_{-\infty}^{t} e_{-a}(t,\sigma(s))\Big[-\hat{a}c_0\frac{\lambda_2}{1-c_0} + \frac{\check{b}u(s-\tau(s))}{[1+\hat{c}u(s-\tau(s))]^{N+1}}\Big]\Delta s. \tag{4.10}$$

Consider the function $f_2(x) = \frac{x}{(1+\hat{c}x)^{N+1}}$, $x \in \mathbb{R}$. Since $f_2(x)$ is decreasing on $[\frac{1}{N+1-\hat{c}}, +\infty)$ and $\frac{1}{N+1-\hat{c}} \le \lambda_1 \le u \le \lambda_2$, then,

$$\frac{\check{b}u(s-\tau(s))}{[1+\hat{c}u(s-\tau(s))]^{N+1}} \ge \frac{\check{b}\lambda_2}{[1+\hat{c}\lambda_2]^{N+1}}. \tag{4.11}$$

It follows by (4.10), (4.11), and assumption (H_2) that

$$\begin{aligned}\Gamma[(\mathcal{A}u)](t) &\ge \left(\frac{\check{b}\lambda_2}{[1+\hat{c}\lambda_2]^{N+1}} - \frac{\hat{a}c_0\lambda_2}{1-c_0}\right)\int_{-\infty}^{t} e_{-\hat{a}}(t,\sigma(s))\Delta s\\ &= \frac{1}{\hat{a}}\left(\frac{\check{b}\lambda_2}{[1+\hat{c}\lambda_2]^{N+1}} - \frac{\hat{a}c_0\lambda_2}{1-c_0}\right)\\ &\ge \lambda_1.\end{aligned} \tag{4.12}$$

Based on (4.9) and (4.12), we have $\Gamma(\mathcal{A}\Omega) \subset \mathcal{A}\Omega$. Next, we show that Γ is a contraction mapping on Ω. For $u_1, u_2 \in \Omega$, we have

$$\begin{aligned}&|\Gamma[(\mathcal{A}u_1)](t) - \Gamma[(\mathcal{A}u_2)](t)|\\ &= \Big|\int_{-\infty}^{t} e_{-a}(t,\sigma(s))\Big[-a(s)c_0(u_1(s-\gamma)-u_2(s-\gamma))\\ &\quad + \frac{b(s)u_1(s-\tau(s))}{[1+c(s)u_1(s-\tau(s))]^{N+1}} - \frac{b(s)u_2(s-\tau(s))}{[1+c(s)u_2(s-\tau(s))]^{N+1}}\Big]\Delta s\Big|\\ &\le \frac{1}{\check{a}}\hat{a}c_0\|u_1-u_2\| + \hat{b}\int_{-\infty}^{t} e_{-a}(t,\sigma(s))\left|\frac{u_1(s-\tau(s))}{[1+c(s)u_1(s-\tau(s))]^{N+1}} - \frac{u_2(s-\tau(s))}{[1+c(s)u_2(s-\tau(s))]^{N+1}}\right|\Delta s.\end{aligned} \tag{4.13}$$

Let $g(x) = \frac{x}{[1+c(s)x]^{N+1}}$, then $g'(x) = \frac{1-(-c(s)+N+1)x}{(1+c(s)x)^{N+2}}$. Thus,

$$\begin{aligned}&\left|\frac{u_1(s-\tau(s))}{[1+c(s)u_1(s-\tau(s))]^{N+1}} - \frac{u_2(s-\tau(s))}{[1+c(s)u_2(s-\tau(s))]^{N+1}}\right|\\ &= |g'(\xi)||u_1(s-\tau(s)) - u_2(s-\tau(s))|\\ &= \left|\frac{1-(-c(s)+N+1)\xi}{(1+c(s)\xi)^{N+2}}\right||u_1(s-\tau(s)) - u_2(s-\tau(s))|,\end{aligned} \tag{4.14}$$

where ξ lies between $u_1(s-\tau(s))$ and $u_2(s-\tau(s))$. Obviously,

$$\left|\frac{1-(-c(s)+N+1)\xi}{(1+c(s)\xi)^{N+2}}\right| \le 1. \tag{4.15}$$

Thus, from (4.14) and (4.15), we have

$$\left|\frac{u_1(s-\tau(s))}{[1+c(s)u_1(s-\tau(s))]^{N+1}} - \frac{u_2(s-\tau(s))}{[1+c(s)u_2(s-\tau(s))]^{N+1}}\right| \le \|u_1-u_2\|. \tag{4.16}$$

It follows by (4.13), (4.16), and Lemma 3.1 that

$$\begin{aligned}\|\Gamma[(\mathcal{A}u_1)](t)-\Gamma[(\mathcal{A}u_2)](t)\| &\le \left(\frac{\hat{a}c_0}{\check{a}}+\frac{\hat{b}}{\check{a}}\right)\|u_1-u_2\| \\ &\le \frac{\hat{a}c_0+\hat{b}}{\check{a}(1-c_0)}\|\mathcal{A}u_1-\mathcal{A}u_2\|.\end{aligned}$$

From $\frac{\hat{a}c_0+\hat{b}}{\check{a}(1-c_0)}<1$, the operator Γ is a contraction mapping. Therefore, the operator Γ has a unique fixed point $\mathcal{A}u$ in Ω. This means that Eq (4.3) has a unique positive almost periodic solution $u(t)$.

Theorem 4.2. Suppose that assumptions (H_1)–(H_3) are satisfied. Then, Eq (4.3) has a unique globally exponentially stable positive almost periodic solution.

Proof: Since assumptions (H_1)–(H_3) hold, it follows by Theorem 4.1 that Eq (4.3) has a unique positive almost periodic solution $u^*(t)$ with $\frac{\lambda_1}{1-c_0} \le u^*(t) \le \frac{\lambda_2}{1-c_0}$. For $\tilde{\tau}=\max\{\gamma, \sup_{t\in\mathbb{T}}\tau(t)\}$, let $\phi_1(t)$ be the initial function of $u^*(t)$, i.e., $u^*(t,\phi_1)=\phi_1(t)$ for $t\in[-\tilde{\tau},0]_{\mathbb{T}}$. Suppose that $u(t)$ is an arbitrary positive solution of Eq (4.3) with the initial function $u(t,\phi_2)=\phi_2(t)$ for $t\in[-\tilde{\tau},t_0]_{\mathbb{T}}$. Let $v(t)=u(t)-u^*(t)$. By (4.4), we have

$$\begin{aligned}(\mathcal{A}v)^{\Delta}(t) &= ((\mathcal{A}u)(t)-(\mathcal{A}u^*)(t))^{\Delta} \\ &= -a(t)(\mathcal{A}v)(t)-a(t)c_0(u(t-\gamma)-u^*(t-\gamma)) \\ &\quad + \frac{b(t)u(t-\tau(t))}{[1+c(t)u(s-\tau(t))]^{N+1}} - \frac{b(t)u^*(t-\tau(t))}{[1+c(t)u^*(t-\tau(t))]^{N+1}} \\ &= -a(t)(\mathcal{A}v)(t)+f(t),\end{aligned} \tag{4.17}$$

where

$$f(t) = -a(s)c_0(u(t-\gamma)-u^*(t-\gamma)) + \frac{b(t)u(t-\tau(t))}{[1+c(t)u(t-\tau(t))]^{N+1}} - \frac{b(t)u^*(t-\tau(t))}{[1+c(t)u^*(t-\tau(t))]^{N+1}}.$$

By (4.17), we get

$$(\mathcal{A}v)(t) = e_{-a}(t,t_0)(\mathcal{A}v)(t_0)+\int_{t_0}^{t} e_{-a}(t,t_0)f(s)\Delta s, \ t_0\in[-\tilde{\tau},0]_{\mathbb{T}}, \tag{4.18}$$

where $(\mathcal{A}v)(t_0)=(\mathcal{A}\phi_1)(t_0)-(\mathcal{A}\phi_2)(t_0)=\mathcal{A}(\phi_1(t_0)-\phi_2(t_0))$. Note that

$$\begin{aligned}|f(s)| &= \left|-a(s)c_0(u(s-\gamma)-u^*(s-\gamma)) + \frac{b(s)u(s-\tau(s))}{[1+c(s)u(s-\tau(s))]^{N+1}} - \frac{b(s)u^*(s-\tau(s))}{[1+c(s)u^*(s-\tau(s))]^{N+1}}\right| \\ &\le \hat{a}c_0\|v\| + \hat{b}\left|\frac{u(s-\tau(s))}{[1+c(s)u(s-\tau(s))]^{N+1}} - \frac{u^*(s-\tau(s))}{[1+c(s)u^*(s-\tau(s))]^{N+1}}\right| \\ &\le (\hat{a}c_0+\hat{b})\|v\| \\ &\le (\hat{a}c_0+\hat{b})\frac{1}{1-c_0}\|\mathcal{A}v\|.\end{aligned} \tag{4.19}$$

The proof of (4.19) is similar to one of (4.16). From (4.18) and (4.19), we have

$$\|\mathcal{A}v\| \leq e_{-a}(t,t_0)\|\mathcal{A}(\phi_1-\phi_2)\| + \int_{t_0}^{t} e_{-a}(t,s)(\hat{a}c_0+\hat{b})\frac{1}{1-c_0}\|\mathcal{A}v\|\Delta s,$$

and

$$\frac{\|\mathcal{A}v\|}{e_{-a}(t,t_0)} \leq \|\mathcal{A}(\phi_1-\phi_2)\| + \int_{t_0}^{t} \frac{1}{e_{-a}(s,t_0)}(\hat{a}c_0+\hat{b})\frac{1}{1-c_0}\|\mathcal{A}v\|\Delta s.$$

Using the Gronwall inequality on time scales, we have

$$\frac{\|\mathcal{A}v\|}{e_{-a}(t,t_0)} \leq \|\mathcal{A}(\phi_1-\phi_2)\|e_{\mu}(t,t_0),$$

and

$$\begin{aligned}\|\mathcal{A}v\| &\leq \|\mathcal{A}(\phi_1-\phi_2)\|e_{\mu}(t,t_0)e_{-a}(t,t_0)\\ &\leq \|\mathcal{A}(\phi_1-\phi_2)\|e_{\mu}(t,t_0)e_{-\check{a}}(t,t_0)\\ &\leq \|\mathcal{A}(\phi_1-\phi_2)\|e_{-(\check{a}-\mu)}(t,t_0),\end{aligned} \tag{4.20}$$

where $\mu = \frac{\hat{a}c_0+\hat{b}}{1-c_0}$. By assumption ($H_3$), we have $\check{a}-\mu > 0$. Using Lemma 3.1 and (4.20), we arrive at

$$\|v\| \leq \frac{1}{1-c_0}\|\mathcal{A}\|\|\phi_1-\phi_2\|e_{-(\check{a}-\mu)}(t,t_0) \leq \frac{1}{(1-c_0)^2}\|\phi_1-\phi_2\|e_{-(\check{a}-\mu)}(t,t_0),$$

i.e.,

$$\|u(t)-u^*(t)\| \leq \frac{1}{(1-c_0)^2}\|\phi_1-\phi_2\|e_{-(\check{a}-\mu)}(t,t_0),$$

which implies that $u^*(t)$ is globally exponentially stable.

Remark 4.1. Due to the similar research methods and results between Eq (4.3) with single time-varying delay and Eq (4.3) with multiply time-varying delays, we only study Eq (4.3) with single time-varying delay in this paper.

5. Applications in Lasota–Wazewska model

In this section, we will study the dynamic properties of Lasota–Wazewska model by using the theory results of Section 3. In 1988, Wazewska–Czyzewska and Lasota [32] fist introduced a model for the survival of red blood cells in an animal which is called Lasota-Wazewska model. After that, Gopalsamy and Trofimchuk [33] investigated the existence of almost periodic solutions for Lasota-Wazewska model with delay as follows:

$$x'(t) = -a(t)x(t) + b(t)e^{-\alpha x(t-\tau)},$$

where $x(t)$ denotes numbers of red blood cells, $a(t), b(t) > 0$ are almost periodic functions. α and τ are positive constants. Stamov [34] studied almost periodic solutions for Lasota-Wazewska model with impulse. The authors [35] studied almost periodic solutions for Lasota–Wazewska model with multiple time-varying delays. However, there are few results for almost periodic solutions of neutral-type Lasota–Wazewska model on time scales. Therefore, this paper is aim to study the following neutral-type Lasota–Wazewska model on time scales:

$$(x(t)-cx(t-\gamma))^{\Delta} = -a(t)x(t) + b(t)e^{-d(t)x(t-\tau(t))}, \quad t\in\mathbb{T}, \tag{5.1}$$

where $x(t)$ denotes numbers of red blood cells, $a(t), b(t), d(t) > 0$ are almost periodic functions with $-a \in \mathcal{R}^+$. c and γ are positive constants with $0 < c < 1$. The time-varying delay $\tau(t) > 0$ is a almost periodic function. Let

$$(\mathcal{A}x)(t) = x - cx(t-\gamma).$$

Then, Eq (5.1) can be rewritten by

$$(\mathcal{A}x)^\Delta(t) = -a(t)(\mathcal{A}x)(t) - a(t)cx(t-\gamma) + b(t)e^{-d(t)x(t-\tau(t))},\ t \in \mathbb{T}. \tag{5.2}$$

Let $\mathbb{B} = \{u : u \in C_{rd}(\mathbb{T}, \mathbb{R}), u(t)$ is almost periodic function$\}$ with the norm $\|u\| = \sup_{t\in\mathbb{T}} |u(t)|$, then $\mathbb{B}$ is a Banach space. In view of (5.2), for $u \in \mathbb{B}$, consider the following auxiliary equation:

$$(\mathcal{A}x)^\Delta(t) = -a(t)(\mathcal{A}x)(t) - a(t)cu(t-\gamma) + b(t)e^{-d(t)u(t-\tau(t))},\ t \in \mathbb{T}. \tag{5.3}$$

From Lemmas 3.2 and 3.3, Eq (5.3) has a unique almost periodic solution

$$(\mathcal{A}x)(t) = \int_{-\infty}^{t} e_{-a}(t, \sigma(s))\Big[- a(s)cu(s-\gamma) + b(s)e^{-d(s)u(s-\tau(s))}\Big]\Delta s.$$

For $u \in \mathbb{B}$, define the operator $\Gamma : \mathbb{B} \to \mathbb{B}$ by

$$\Gamma[(\mathcal{A}u)](t) = \int_{-\infty}^{t} e_{-a}(t, \sigma(s))\Big[- a(s)cu(s-\gamma) + b(s)e^{-d(s)u(s-\tau(s))}\Big]\Delta s.$$

Obviously, $u(t)$ is the almost periodic solution of Eq (5.1) if and only if $\mathcal{A}u$ is the fixed point of the operator Γ. In this section, we need the following assumptions:
(A_1) There exist constants $\mu_1, \mu_2 > 0$ with $\frac{1}{\check{d}} \le \mu_1 \le \mu_2$ such that

$$0 < \frac{1}{\check{a}}\Big(\frac{\hat{b}}{\mu_1\check{d}e} - \frac{\check{a}c\mu_1}{1-c}\Big) \le \mu_2,$$

$$\frac{1}{\hat{a}}\Big((1-c)\check{b}e^{-\hat{d}\mu_2} - \frac{\hat{a}c\mu_2}{1-c}\Big) \ge \mu_1.$$

(A_2) $\frac{\hat{a}c+\hat{b}}{\check{a}(1-c)} < 1$.

Theorem 5.1. Suppose that assumptions (A_1) and (A_2) are satisfied, then Eq (5.1) has a unique positive almost periodic solution.

Proof: Let $\Omega = \{u : u \in \mathbb{B},\ \mu_1 \le (\mathcal{A}u)(t) \le \mu_2,\ t \in \mathbb{T}\}$, where μ_1 and μ_2 are defined by assumption (A_1). We first show that $\Gamma(\mathcal{A}\Omega) \subset \mathcal{A}\Omega$. For each $u \in \Omega$, by Lemma 3.1, we have

$$\frac{\mu_1}{1-c} \le u(t) \le \frac{\mu_2}{1-c}. \tag{5.4}$$

For each $u \in \Omega$, by (5.4), we have

$$\begin{aligned}\Gamma[(\mathcal{A}u)](t) &= \int_{-\infty}^{t} e_{-a}(t, \sigma(s))\Big[- a(s)cu(s-\gamma) + b(s)e^{-d(s)u(s-\tau(s))}\Big]\Delta s \\ &\le \int_{-\infty}^{t} e_{-a}(t, \sigma(s))\Big[- \check{a}c\frac{\mu_1}{1-c} + \frac{\hat{b}}{\mu_1}u(s-\tau(s))e^{-\check{d}u(s-\tau(s))}\Big]\Delta s.\end{aligned} \tag{5.5}$$

Since $f(x) = xe^{-\check{d}x}$ is increasing on $(0, \frac{1}{\check{d}}]$ and decreasing on $[\frac{1}{\check{d}}, +\infty)$, then,

$$u(s-\tau(s))e^{-\check{d}u(s-\tau(s))} \leq \frac{1}{\check{d}e}. \tag{5.6}$$

From (5.5), (5.6), and assumption (A_1), we have

$$\begin{aligned}\Gamma[(\mathcal{A}u)](t) &\leq \Big(\frac{\hat{b}}{\mu_1\check{d}e} - \frac{\check{a}c\mu_1}{1-c}\Big)\int_{-\infty}^{t} e_{-\check{a}}(t,\sigma(s))\Delta s \\ &= \frac{1}{\check{a}}\Big(\frac{\hat{b}}{\mu_1\check{d}e} - \frac{\check{a}c\mu_1}{1-c}\Big) \\ &\leq \mu_2.\end{aligned} \tag{5.7}$$

On the other hand, for each $u \in \Omega$, by (5.4), we have

$$\Gamma[(\mathcal{A}u)](t) \geq \int_{-\infty}^{t} e_{-\hat{a}}(t,\sigma(s))\Big[-\hat{a}c\frac{\mu_2}{1-c} + \frac{(1-c)\check{b}}{\mu_2}u(s-\tau(s))e^{-\hat{d}u(s-\tau(s))}\Big]\Delta s. \tag{5.8}$$

Since $g(x) = xe^{-\hat{d}x}$ is decreasing on $[\frac{1}{\hat{d}}, +\infty)$ and $\frac{1}{\hat{d}} \leq \mu_1 \leq u \leq \mu_2$, then,

$$u(s-\tau(s))e^{-\hat{d}u(s-\tau(s))} \geq \mu_2 e^{-\hat{d}\mu_2}. \tag{5.9}$$

From (5.8), (5.9), and assumption (A_1), we have

$$\begin{aligned}\Gamma[(\mathcal{A}u)](t) &\geq \Big((1-c)\check{b}e^{-\hat{d}\mu_2} - \frac{\hat{a}c\mu_2}{1-c}\Big)\int_{-\infty}^{t} e_{-\hat{a}}(t,\sigma(s))\Delta s \\ &= \frac{1}{\hat{a}}\Big((1-c)\check{b}e^{-\hat{d}\mu_2} - \frac{\hat{a}c\mu_2}{1-c}\Big) \\ &\geq \mu_1.\end{aligned} \tag{5.10}$$

From (5.7) and (5.10), we have $\Gamma(\mathcal{A}\Omega) \subset \mathcal{A}\Omega$. Next, we show that Γ is a contraction mapping on Ω. For $u_1, u_2 \in \Omega$, we have

$$\begin{aligned}&|\Gamma[(\mathcal{A}u_1)](t) - \Gamma[(\mathcal{A}u_2)](t)| \\ &= \Big|\int_{-\infty}^{t} e_{-a}(t,\sigma(s))\Big[-a(s)c\big(u_1(s-\gamma) - u_2(s-\gamma)\big) \\ &+ b(s)e^{-d(s)u_1(s-\tau(s))} - b(s)e^{-d(s)u_2(s-\tau(s))}\Big]\Delta s\Big| \\ &\leq \frac{\hat{a}c}{\check{a}}\|u_1-u_2\| + \hat{b}\int_{-\infty}^{t} e_{-\check{a}}(t,\sigma(s))\Big|e^{-d(s)u_1(s-\tau(s))} - e^{-d(s)u_2(s-\tau(s))}\Big|\Delta s \\ &\leq \frac{\hat{a}c}{\check{a}}\|u_1-u_2\| + \hat{b}\|u_1-u_2\|\int_{-\infty}^{t} e_{-\check{a}}(t,\sigma(s))\frac{d(s)}{e^{d(s)\xi}}\Delta s \\ &\leq \Big(\frac{\hat{a}c}{\check{a}} + \hat{b}\Big)\|u_1-u_2\|,\end{aligned}$$

i.e.,

$$\|\Gamma[(\mathcal{A}u_1)](t) - \Gamma[(\mathcal{A}u_2)](t)\| \le \left(\frac{\hat{a}c}{\check{a}} + \frac{\hat{b}}{\check{a}}\right)\frac{1}{1-c}\|(\mathcal{A}u_1 - (\mathcal{A}u_2\|, \tag{5.11}$$

where ξ lies between $u_1(s-\tau(s))$ and $u_2(s-\tau(s))$ with $\frac{d(s)}{e^{d(s)\xi}} \le 1$. From $\frac{\hat{a}c+\hat{b}}{\check{a}(1-c)} < 1$, the operator Γ is a contraction mapping. Therefore, the operator Γ has a unique fixed point $\mathcal{A}u$ in Ω. This means that Eq (5.1) has a unique positive almost periodic solution $u(t)$.

Theorem 5.2. Suppose that assumptions (A_1) and (A_2) are satisfied. Then Eq (5.1) has a unique globally exponentially stable positive almost periodic solution.

Proof: Since assumptions (A_1) and (A_2) hold, it follows by Theorem 5.1 that Eq (5.1) has a unique almost periodic positive solution $u^*(t)$ with $\frac{\mu_1}{1-c} \le u^*(t) \le \frac{\mu_2}{1-c}$. For $\tilde{\tau} = \max\{\gamma, \sup_{t\in\mathbb{T}} \tau(t)\}$, let $\phi_1(t)$ be the initial function of $u^*(t)$, i.e., $u^*(t,\phi_1) = \phi_1(t)$ for $t \in [-\tilde{\tau}, 0]_{\mathbb{T}}$ and $u(t)$ be an arbitrary positive solution of Eq (5.1) with the initial function $u(t,\phi_2) = \phi_2(t)$ for $t \in [-\tilde{\tau}, t_0]_{\mathbb{T}}$. Let $y(t) = u(t) - u^*(t)$. By (5.2), we have

$$\begin{aligned}(\mathcal{A}y)^{\Delta}(t) &= \big((\mathcal{A}u)(t) - (\mathcal{A}u^*)(t)\big)^{\Delta}\\ &= -a(t)(\mathcal{A}y)(t) - a(t)c\big(u(t-\gamma) - u^*(t-\gamma)\big)\\ &\quad + b(t)e^{-d(t)u(t-\tau(t))} - b(t)e^{-d(t)u^*(t-\tau(t))}\\ &= -a(t)(\mathcal{A}y)(t) + f(t),\end{aligned} \tag{5.12}$$

where

$$f(t) = -a(t)c\big(u(t-\gamma) - u^*(t-\gamma)\big) + b(t)e^{-d(t)u(t-\tau(t))} - b(t)e^{-d(t)u^*(t-\tau(t))}.$$

By (5.12), we get

$$(\mathcal{A}y)(t) = e_{-a}(t,t_0)(\mathcal{A}y)(t_0) + \int_{t_0}^{t} e_{-a}(t,t_0)f(s)\Delta s,\ t_0 \in [-\tilde{\tau}, 0]_{\mathbb{T}}, \tag{5.13}$$

where $(\mathcal{A}y)(t_0) = (\mathcal{A}\phi_1)(t_0) - (\mathcal{A}\phi_2)(t_0) = \mathcal{A}(\phi_1(t_0) - \phi_2(t_0))$. Note that

$$\begin{aligned}|f(s)| &= \left| -a(s)c\big(u(s-\gamma) - u^*(s-\gamma)\big) + b(s)e^{-d(s)u(s-\tau(s))} - b(s)e^{-d(s)u^*(s-\tau(s))}\right|\\ &\le \hat{a}c\|y\| + \hat{b}\left|e^{-d(s)u(s-\tau(s))} - e^{-d(s)u^*(s-\tau(s))}\right|\\ &\le (\hat{a}c + \hat{b})\|y\|\\ &\le (\hat{a}c + \hat{b})\frac{1}{1-c}\|\mathcal{A}y\|.\end{aligned} \tag{5.14}$$

The proof of (5.14) is similar to one of (5.11). From (5.13) and (5.14), we have

$$\|\mathcal{A}y\| \le e_{-a}(t,t_0)\|\mathcal{A}(\phi_1 - \phi_2)\| + \int_{t_0}^{t} e_{-a}(t,s)(\hat{a}c + \hat{b})\frac{1}{1-c}\|\mathcal{A}y\|\Delta s,$$

and

$$\frac{\|\mathcal{A}y\|}{e_{-a}(t,t_0)} \le \|\mathcal{A}(\phi_1 - \phi_2)\| + \int_{t_0}^{t} \frac{1}{e_{-a}(s,t_0)}(\hat{a}c + \hat{b})\frac{1}{1-c}\|\mathcal{A}y\|\Delta s.$$

Using the Gronwall inequality on time scales, we have

$$\frac{\|\mathcal{A}y\|}{e_{-a}(t,t_0)} \le \|\mathcal{A}(\phi_1 - \phi_2)\|e_{\lambda}(t,t_0),$$

and

$$\begin{aligned}\|\mathcal{A}y\| &\leq \|\mathcal{A}(\phi_1-\phi_2)\|e_{\mu}(t,t_0)e_{-a}(t,t_0)\\ &\leq \|\mathcal{A}(\phi_1-\phi_2)\|e_{\lambda}(t,t_0)e_{-\check{a}}(t,t_0)\\ &\leq \|\mathcal{A}(\phi_1-\phi_2)\|e_{-(\check{a}-\lambda)}(t,t_0),\end{aligned} \tag{5.15}$$

where $\lambda = \frac{\hat{a}c+\hat{b}}{1-c}$. It follows by assumption (A_2) that $\check{a} - \lambda > 0$. Using Lemma 3.1 and (5.15), we arrive at

$$\|y\| \leq \frac{1}{1-c}\|\mathcal{A}\|\|\phi_1-\phi_2\|e_{-(\check{a}-\lambda)}(t,t_0) \leq \frac{1}{(1-c)^2}\|\phi_1-\phi_2\|e_{-(\check{a}-\lambda)}(t,t_0),$$

i.e.,

$$\|u(t)-u^*(t)\| \leq \frac{1}{(1-c)^2}\|\phi_1-\phi_2\|e_{-(\check{a}-\lambda)}(t,t_0),$$

which implies that $u^*(t)$ is globally exponentially stable.

Remark 5.1. The neutral-type equation encompasses a wider range of mathematical models and has important applications in many aspects. Due to the complexity of neutral-type equations compared to general functional differential equations, in this paper, we utilized the properties of neutral-type operators, fixed point theorems and inequalities on time scales to study the dynamic behavior of two types of mathematical models.

6. Two numerical examples

Since host-macroparasite model and Lasota–Wazewska model on time scale $\mathbb{T} = \mathbb{R}$ have been studied extensively, we focus on the above two classes of models on time scale $\mathbb{T} = \mathbb{Z}$.

Example 6.1. Consider the following neutral-type host-macroparasite model on $\mathbb{T} = \mathbb{Z}$:

$$\Delta(x(k)-c_0x(k-\gamma)) = -a(k)x(k) + \frac{b(k)x(k-\tau(k))}{[1+c(k)x(k-\tau(k))]^{N+1}},\ k\in\mathbb{Z}, \tag{6.1}$$

where

$$\Delta(x(k)) = x(k+1)-x(k),\ c_0 = 1\times 10^{-4},\ a(k) = 0.05+0.01\sin\sqrt{3}k,$$
$$b(k) = 0.02+0.01\sin\sqrt{2}k,\ c(k) = 0.03+0.01\sin\pi k,\ \gamma = 0.5,\ \tau(k) = e^{0.2\sin k}.$$

We have

$$\hat{a} = 0.06,\ \check{a} = 0.04,\ \hat{b} = 0.03,\ \check{b} = 0.01,\ \hat{c} = 0.04,\ \check{c} = 0.02.$$

Choose $N = 0.01,\ \lambda_1 = 1.1,\ \lambda_2 = 10$, then,

$$N+1-c(k) = 0.98-0.01\sin\pi k > 0,\ M_1 \approx 2.95\times 10^{-2},$$

$$\frac{1}{\check{a}}\Big(M_1 - \frac{\check{a}c_0\lambda_1}{1-c_0}\Big) \approx 0.74 \leq \lambda_2,$$

$$\frac{1}{\hat{a}}\Big(\frac{\check{b}\lambda_2}{[1+\hat{c}\lambda_2]^{N+1}} - \frac{\hat{a}c_0\lambda_2}{1-c_0}\Big) \approx 1.19 \geq \lambda_1,$$

$$\frac{\hat{a}c_0+\hat{b}}{\check{a}(1-c_0)} \approx 0.73 < 1.$$

Hence, all conditions of Theorem 4.2 hold, Eq (6.1) has a unique globally exponentially stable positive almost periodic solution. The trajectory of Eq (6.1) is shown in Figure 1.

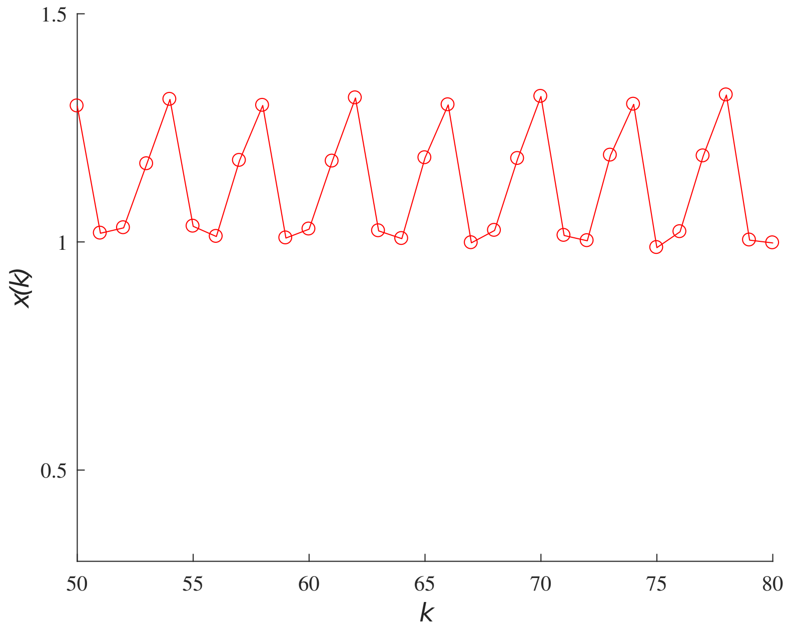

Figure 1. Positive almost periodic solution of Eq (6.1).

Example 6.2. Consider the following neutral-type Lasota–Wazewska model on time scale $\mathbb{T} = \mathbb{Z}$:

$$\Delta(x(k) - cx(k-\gamma)) = -a(k)x(k) + b(k)e^{-d(k)x(k-\tau(k))}, \quad k \in \mathbb{Z}, \tag{6.2}$$

where

$$\Delta(x(k)) = x(k+1) - x(k), \; c = 1 \times 10^{-4}, \; a(k) = 0.06 - 0.01\cos\sqrt{2}k,$$
$$b(k) = 0.03 + 0.01\sin\sqrt{3}k, \; d(k) = 20 + 10\sin\pi k, \quad \gamma = 0.3, \; \tau(k) = e^{0.3\cos k}.$$

We have

$$\hat{a} = 0.07, \; \check{a} = 0.05, \; \hat{b} = 0.04, \; \check{b} = 0.02, \; \hat{d} = 30, \; \check{d} = 10.$$

Choose $\mu_1 = 0.034, \; \mu_2 = 2$, then,

$$0 < \frac{1}{\check{a}}\left(\frac{\hat{b}}{\mu_1 \check{d} e} - \frac{\check{a}c\mu_1}{1-c}\right) \approx 0.87 \le \mu_2,$$

$$\frac{1}{\hat{a}}\left((1-c)\check{b}e^{-\hat{d}\mu_2} - \frac{\hat{a}c\mu_2}{1-c}\right) \approx 0.132 \ge \mu_1,$$

$$\left(\frac{\hat{a}c}{\check{a}} + \frac{\hat{b}}{\check{a}}\right)\frac{1}{1-c} \approx 0.826 < 1.$$

Hence, all conditions of Theorem 5.2 hold, Eq (6.2) has a unique globally exponentially stable positive almost periodic solution. The trajectory of Eq (6.2) is shown in Figure 2.

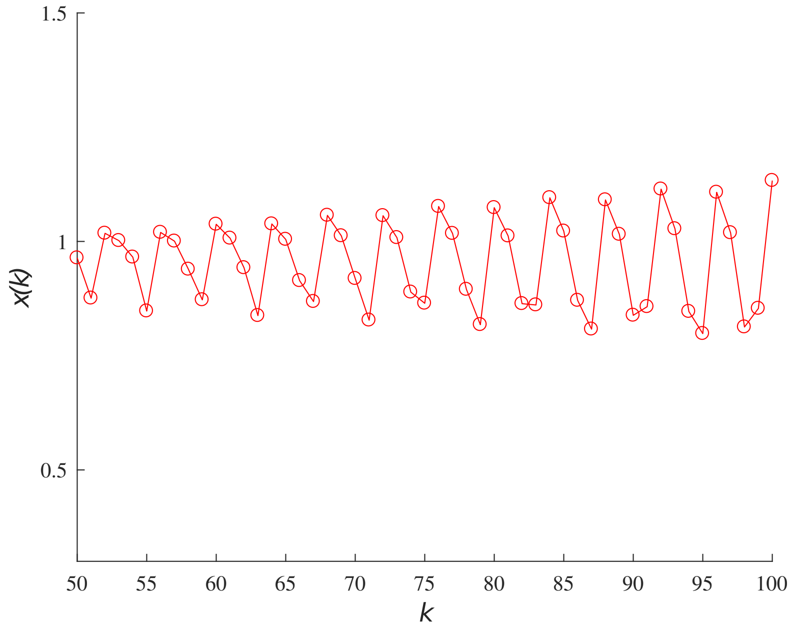

Figure 2. Positive almost periodic solution of Eq (6.2).

7. Conclusions

In practical applications, almost periodic solutions can more accurately characterize the actual development and changes than periodic solutions. In the present paper, we first study the general theory of almost periodic solutions for neutral-type differential system on time scales. Our theory generalize the corresponding one in [18]. We find that the above theory combined with the properties of neutral operators can facilitate the study of neutral biological population models on time scales. By using the above theory, we obtain the existence and exponential stability of almost periodic solutions for two classes of neutral-type biological population models including host-macroparasite model and Lasota–Wazewska model. In the future work, we will explore the dynamic behaviors of almost periodic solutions for neutral-type population models with impulsive terms on time scales and study the dynamic behaviors of almost periodic solutions for neutral-type population models with stochastic terms on time scales.

Author contributions

Jing Ge: Methodology, writing-review and editing; Xiaoliang Li: Supervision, methodology; Bo Du: Writing-original draft; Famei Zheng: Formal analysis. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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