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Research article

Asymptotics for fractional reaction diffusion equations in periodic media

Yu Wei, Yahan Wang and Huiqin Lu*

School of Mathematics and Statistics, Shandong Normal University, Jinan, 250014, China

* **Correspondence:** Email: lhy0625@163.com.

Abstract: In this paper, the Cauchy problem for a class of reaction diffusion equations are considered with nonlocal interactions in periodic media. First, we demonstrate the existence and uniqueness of solutions that are both positive and bounded for the stationary equation. Second, we derive results concerning the existence and uniqueness of solutions for the Cauchy problem by using the semigroup theory. Finally, we analyze the behavior of the solutions to the Cauchy problem for large times by using the comparison principle.

Keywords: generalized fractional Laplacian; convolution; semigroup theory; asymptotics
Mathematics Subject Classification: 35B40, 35K57,35R11

1. Introduction

Reaction diffusion equations in the given form that can be rephrased as

$$u_t = \Delta u + f(u), \qquad x \in \mathbb{R}^N$$

were initially presented in the prestigious papers by Fisher [1] and Kolmogorov [2]. The initial impetus behind these equations stemmed from population genetics, with the aim of illuminating the spatial dissemination of beneficial genetic traits. Over numerous generations of scholarly endeavor, this class of equations has come to be extensively employed in modeling the spatial propagation or dispersion of biological species (see [3, 4] and the cited references within).

In recent years, nonlocal population models have attracted quite a lot interest (see [5–8]). Initially, the dispersal mechanism of the population is treated as nonlocal, meaning that standard Gaussian diffusion can be substituted by one that is affected through Lévy flight processes. Nonlocal dispersion strategies have been documented in the natural world (see [9]). Moreover, the idea of a potentially nonlocal species growth rate stems from real-life situations where populations derive advantages not only from immediate local resources, but also from those within their broader "accessible range" (see [10, 11]). This nonlocal characteristic can be mathematically represented through the application of an integrable kernel in convolution. It is important to highlight, from a mathematical standpoint, that these two classes of nonlocal operators exhibit marked differences. Specifically, the fractional Laplacian has a tendency to decrease the smoothness of the function, whereas convolution generally exhibits a tendency to smooth it out. Several effective techniques have been employed to numerically address both linear and nonlinear nonlocal equations, including the spline collocation method and the finite difference method. The spline collocation method focuses on the superconvergence properties of a reliable orthogonal Gaussian collocation method technique applied to two-dimensional fourth-order subdiffusion equations (see [12–14]). On the other hand, the finite difference method involves an efficient implicit finite difference scheme with an alternating direction specifically designed for solving the three-dimensional time-fractional telegraph equation (see [15, 16]).

In 2017, L. Caffarelli, S. Dipierro, and E. Valdinoci [17] investigated the following elliptic equation:

$$(-\Delta)^s u = (\delta - \mu u)u + \tau(J * u) \text{ in } \mathbb{R}^N.$$

They obtained the existence of positive solution by the energy minimization argument.

In 2021, H. Berestycki [18] studied the Cauchy problem

$$\begin{cases} v_t + (-\Delta)^\alpha v = f(x, v), & (t, x) \in (0, +\infty) \times \mathbb{R}^N, \\ v(0, x) = v_0(x), \ x \in \mathbb{R}^N. \end{cases}$$

They established the existence and uniqueness of steady-state solutions, linking these to eigenvalues, and illustrated the long-time behavioral trends of the solutions.

Drawing on the aforementioned research, we have extended the equation introduced in [18] by examining a wider range of equations and obtained conclusions that are analogous to those presented in [18].

The specific mathematical formulation we are examining is as follows. Consider $\alpha \in (0, 1)$, let $\hat{\imath}_1, \ldots, \hat{\imath}_N$ be N given real numbers, and write $\hat{\imath} = (\hat{\imath}_1, \hat{\imath}_2, \ldots, \hat{\imath}_N)$. As stated below, a function $s : \mathbb{R}^N \to \mathbb{R}$ is said to be $\hat{\imath}$–periodic if, for all $k = 1, \cdots, N$, it satisfies

$$s(x_1, \cdots, x_k + \hat{\imath}_k, \cdots, x_N) \equiv s(x_1, \cdots, x_k, \cdots, x_N).$$

Let $C_{\hat{\imath}}$ denote the periodic cell, defined as

$$C_{\hat{\imath}} = [0, \hat{\imath}_1] \times [0, \hat{\imath}_2] \times \cdots \times [0, \hat{\imath}_N].$$

We consider the fractional elliptic operator

$$L^\alpha v(t, x) := \int_{\mathbb{R}^N} (v(t, x) - v(t, x + y))K(x, y)dy,$$

where K is positive, $\hat{\imath}$–periodic, C^2 in x, singular at $y = 0$, and symmetic in y. A positive, finite constant denoted as $C_K > 1$ exists, such that $\forall x, y \in \mathbb{R}^N$

$$C_K^{-1} \leq K(x, y)|y|^{N+2\alpha} \leq C_K. \tag{1.1}$$

These operators offer a broader perspective on the fractional Laplacian $(-\Delta)^\alpha$ (see [19–21]).

In this paper, our focus is on the Cauchy problem outlined below

$$\begin{cases} v_t + L^{\alpha} v = f(x, v) + \tau(J * v - v), & (t, x) \in (0, +\infty) \times \mathbb{R}^N, \\ v(0, x) = v_0(x), \ x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $v_0 \in C(\mathbb{R}^N)$ has compact support and is non-negative. It is well-established that stationary solutions are furnished by

$$L^{\alpha} v = f(x, v) + \tau(J * v - v), \ x \in \mathbb{R}^N. \tag{1.3}$$

In the context above, $\tau \geq 0$ represents a constant. Furthermore, $J(\cdot)$ denotes a non-negative convolution kernel that is C^1, and this holds for all h in $\mathbb{R}^N$

$$J(-h) = J(h), \ \int_{\mathbb{R}^N} J(h) dh = 1 \text{ and } J(h) \geq 0.$$

Conventionally, $J * v$ signifies the convolution of two functions. This implies that $\forall x \in \mathbb{R}^N, t \in \mathbb{R}_+$

$$(J * v)(t, x) = \int_{\mathbb{R}^N} J(h) v(t, x - h) dh = \int_{\mathbb{R}^N} J(x - h) v(t, h) dh.$$

The function $f : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ possesses local $C^{0,\alpha}$ smoothness in x and exhibits Lipschitz continuity with respect to v. Additionally, it is presumed that $f(x, 0) = 0$ holds true for all $x \in \mathbb{R}^N$, and that f is C^1- differentiable in $\mathbb{R}^N \times [0, \beta]$.

For several subsequent findings, let us additionally suppose that f fulfills the condition for all $x \in \mathbb{R}^N$

$$\begin{aligned} & k \to \frac{f(x, k)}{k} \text{ is decreasing in } (0, \infty) \\ & \text{and } \exists \bar{M} > 0, \text{ such that } f(x, k) \leq 0, \text{ for all } k \geq \bar{M}. \end{aligned} \tag{1.4}$$

A typical instance of a function like f is in the following form:

$$f(x, v) = (\mu(x) - u(x) v) v \ ,$$

where both $\mu(x)$ and $u(x)$ are periodic in terms of x (see [22, 23]).

Equation (1.2) represents a model that encapsulates the dynamics of species' growth and invasion in a media environment, incorporating nonlocal dispersion. This model is suitable for situations where individuals can swiftly relocate from one place to another due to various factors, such as wind-dispersed seeds or human-assisted transportation of animals. In this context, v signifies the population density at a specific position x and time t. The diffusion term, represented by the operator L^{α} and the convolution $J * v - v$, accounts for the movement patterns of individuals. The "logistic term", denoted by $f(x, v)$, outlines the rate of population growth.

The paper is structured as follows: Section 2 introduces the key findings of this article. In Section 3, we demonstrate the existence and uniqueness of a positive, bounded, and stationary solution to Eq (1.2) using monotone iterative techniques and the sub- and super-solution methods. Section 4 focuses on proving the existence of a global solution to (1.2) through the semigroup theory, while also presenting the comparison principle for (1.2). Additionally, we examine the long-time behavior of the solutions to (1.2) by applying the comparison principle. Conclusions are provided in Section 5.

2. The main results

First, the existence and uniqueness of the solutions to (1.3) are linked to the principal eigenvalue λ associated with the operator B, defined as outlined below

$$B\hat{\phi} = L^{\alpha}\hat{\phi} - f_v(x,0)\hat{\phi} - \tau(J * \hat{\phi} - \hat{\phi})$$

with conditions of periodicity, where $f_v(x, v)$ represents the first-order partial derivative of f with respect to v. Namely, the lemma states the following.

Lemma 2.1. *The operator B possesses a principal eigenpair $(\hat{\phi}, \lambda)$, which means that*

$$\begin{cases} B\hat{\phi}(x) = \lambda\hat{\phi}(x), \ x \in \mathbb{R}^N, \\ \hat{\phi} \text{ is positive, } \hat{\imath} - \text{periodic, and } \|\hat{\phi}\|_\infty = 1. \end{cases} \tag{2.1}$$

The eigenvalue λ signifies the smallest value in the spectrum of B.

We are now prepared to state the findings regarding the existence and uniqueness of solutions for the problem (1.3).

Theorem 2.2. *Suppose that f fulfills the condition given by (1.4) . In this case*

1) If $\lambda < 0$, there is a unique positive and bounded solution to (1.3), which is also $\hat{\imath}$-periodic.

2) If $\lambda \geq 0$, there is no non-egative bounded solution of (1.3) other than 0.

Next, we explore the existence and long-time behavior of solutions to Eq (1.2), leading us to derive the following theorem.

Theorem 2.3. *Suppose that f fulfills the condition given by (1.4). Then (1.2) has a unique solution $v(t, x)$. Moreover*

1) If $\lambda < 0$, then $v(t, x) \to q$ in $C^1_{loc}(\mathbb{R}^N)$ as $t \to \infty$, where q represents the unique positive and bounded solution to Eq (1.3).

2) If $\lambda \geq 0$, then $v(t, x)$ converges to 0 uniformly in $\mathbb{R}^N$ as $t \to \infty$.

3. Existence and uniquness of a stationary solution

First, we proceed to prove Lemma 2.1. We mainly show that the operator B has a principal eigenpair $(\hat{\phi}, \lambda)$. We establish this through the application of the Krein–Rutman theorem.

Proof of Lemma 2.1. To deal with the periodicity, we have introduced an eigenfunction within the context of the torus $C_{\hat{\imath}}$. A smooth function v with a period of $\hat{\imath}$ in $\mathbb{R}^N$ can be equivalently expressed as a smooth function defined on $C_{\hat{\imath}}$. Additionally,

$$L^{\alpha} v = \int_{\mathbb{R}^N} (v(x) - v(x + y))K(x, y)dy = \int_{C_{\hat{\imath}}} (v(t, x) - v(t, x + y))\check{K}(x, y)dy,$$

where

$$\check{K}(x, y) = \sum_{k=0}^{\infty} K(x, y + k\hat{\imath}).$$

According to (1.1), the aforementioned sum converges for all $x \in C_{\hat{\imath}}$ and $y \in C_{\hat{\imath}} \setminus \{0\}$.

It also results from (1.1) and the characterization of K that a constant $\hat{C} > 1$ exists and the statement holds for all $x, y \in \mathbb{R}^N$

$\check{K}(x, y)$ has a lower bound, symmetric with respect to y and satisfies $\hat{C}^{-1} \le \check{K}(x, y)|y|^{N+2\alpha} \le \hat{C}$.

Let $H^s(C_{\hat{i}}) = \{ v \,|v \in H^s(\mathbb{R}^N)$ and $v \equiv 0$ on $\mathbb{R}^N \setminus C_{\hat{i}}\}$. For $v \in H^s(C_{\hat{i}})$, we define

$$\|v\|_{\dot{H}^s(C_{\hat{i}})} = \left(\iint_{C_{\hat{i}}} (v(x+y) - v(x))^2 \tilde{K}(x, y) dxdy\right)^{\frac{1}{2}},$$

where $\|v\|_{\dot{H}^s(C_{\hat{i}})}$ and $\|v\|_{H^s(C_{\hat{i}})}$ represent, respectively, the homogeneous and inhomogeneous Sobolev norms on $C_{\hat{i}}$, specifically,

$$\|v\|_{H^s(C_{\hat{i}})} = \|v\|_{\dot{H}^s(C_{\hat{i}})} + \|v\|_{L^2(C_{\hat{i}})} \quad \text{and} \quad \|v\|_{L^2(C_{\hat{i}})} = (\int_{C_{\hat{i}}} |v(x)|^2 dx)^{\frac{1}{2}}.$$

For $v, \hat{v} \in H^s(C_{\hat{i}})$, we have the associated scalar product

$$\begin{aligned}\langle Bv, \hat{v}\rangle = &\iint_{C_{\hat{i}}} (v(x) - v(x+y))(\hat{v}(x) - \hat{v}(x+y))\check{K}(x, y)dxdy - \tau \int_{C_{\hat{i}}} (J * v)(x)\hat{v}(x)dx \\ &+ \tau \int_{C_{\hat{i}}} v(x)\hat{v}(x)dx - \int_{C_{\hat{i}}} f_v(x, 0)v(x)\hat{v}(x)dx.\end{aligned}$$

For $\forall v, \hat{v} \in H^s(C_{\hat{i}})$ and $\mu > 0$, we find

$$\begin{aligned}|\langle (B+\mu)v, \hat{v}\rangle| \le &\iint_{C_{\hat{i}}} (v(x) - v(x+y))(\hat{v}(x) - \hat{v}(x+y))\check{K}(x, y)dxdy + \tau \int_{C_{\hat{i}}} (J * v)(x)\hat{v}(x)dx \\ &+ (\tau+\mu) \int_{C_{\hat{i}}} v(x)\hat{v}(x)dx + \int_{C_{\hat{i}}} f_v(x, 0)v(x)\hat{v}(x)dx \\ \le &\|v\|_{\dot{H}^s} \|\hat{v}\|_{\dot{H}^s} + (2\tau + \mu + k) \|v\|_{L^2} \|\hat{v}\|_{L^2} \\ \le &\bar{k} \|v\|_{H^s} \|\hat{v}\|_{H^s},\end{aligned}$$

where $k = \|f_v(x, 0)\|_\infty$ and $\bar{k} > 0$ is a constant.

For $v \in H^s(C_{\hat{i}})$ and $\mu > 0$, v fulfills a Garding inequality, that is

$$\begin{aligned}\langle (B+\mu)v, v\rangle \ge &\iint_{C_{\hat{i}}} (v(x) - v(x+y))^2 \check{K}(x, y)dxdy - \tau \int_{C_{\hat{i}}} (J * v)(x)v(x)dx \\ &+ \tau \int_{C_{\hat{i}}} v(x)^2 dx - \int_{C_{\hat{i}}} f_v(x, 0)v(x)^2 dx + \mu \int_{C_{\hat{i}}} v(x)^2 dx \\ \ge &\|v\|^2_{\dot{H}^s} - \tau \|v\|^2_{L^2} + \tau \|v\|^2_{L^2} - \|f_v(x, 0)\|_\infty \|v\|^2_{L^2} + \mu \|v\|^2_{L^2} \\ \ge &\|v\|^2_{\dot{H}^s} + (\mu - \|f_v(x, 0)\|_\infty) \|v\|^2_{L^2}.\end{aligned}$$

Therefore, for a sufficiently high value of the constant $\mu > \|f_v(x, 0)\|_\infty$, it is apparent that the operator $\langle (B+\mu)\cdot, \cdot\rangle$ is bijective. Hence, due to the Lax–Milgram theorem, the operator $(B+\mu)$ admits a globally defined inverse that is compact

$$\check{A} = (B+\mu)^{-1} : H^s(\mathbb{R}^N) \to \mathrm{Dom}(L^\alpha).$$

In accordance with the Krein–Rutman theorem, a positive eigenfunction $\hat{\phi}$ exists, corresponding to the operator $\check{A}$ that has the eigenvalue $r(\check{A})$. Hence,

$$L^{\alpha}\hat{\phi} - f_v(x,0)\hat{\phi} - \tau(J * \hat{\phi} - \hat{\phi}) = (\frac{1}{r(\check{A})} - \mu)\hat{\phi}.$$

This aligns precisely with Eq (2.1), where $\lambda = r(\check{A})^{-1} - \mu$.

Next, we give an important strong elliptic maximum theorem that is needed for the following proof. Throughout the article, we frequently employ the strong elliptic maximum theorem in our discussions.

Lemma 3.1. *Assume that* $m \in C(\mathbb{R}^N, \mathbb{R}^+)$ *, f satisfies (1.4), and*

$$L^{\alpha}m(x) \geq f(x,m) + \tau(J * m(x) - m(x)), \ x \in \mathbb{R}^N, \tag{3.1}$$

where $\tau \geq 0$*. Then either m is identically zero in* $\mathbb{R}^N$ *or* $m > 0$ *in* $\mathbb{R}^N$*.*

Proof. Because f exhibits local continuity in relation to m, we can obtain $\forall x \in \mathbb{R}^N$

$$\begin{aligned} &|f(x,m) - f(x,0)| \leq c(x)|m(x)| \\ &\Longrightarrow f(x,m) \geq -c(x)m(x), \end{aligned} \tag{3,2}$$

where $c(x) > 0$. From (3.1) and (3.2), we have

$$L^{\alpha}m + c(x)m - \tau(J * m - m) \geq f(x,m) + c(x)m \geq 0. \tag{3.3}$$

Next, suppose that $m \not\equiv 0$ in $\mathbb{R}^N$. By continuity, $x_1 \in \mathbb{R}^N$ and $\varepsilon_1 > 0$ exist such that $m(\hat{y}) \geq \varepsilon_1$ for all $\hat{y} \in B_1 = B(x_1, \varepsilon_1) \in \mathbb{R}^N$. If $m(\hat{x}) = 0$ at some $\hat{x} \in \mathbb{R}^N \setminus B_1$, we have

$$L^{\alpha}m(\hat{x}) \leq \int_{\mathbb{R}^N \setminus B_1} -m(\hat{x}+y)K(\hat{x},y)dy + \int_{B_1} -\varepsilon_1 K(\hat{x},y)dy.$$

Again, the first integral is non-positive and the second one is strictly negative; hence $L^{\alpha}m(\hat{x}) < 0$. Since $c(\hat{x})m(\hat{x}) = 0$ and $\tau J * m(\hat{x}) \geq 0$, a contradiction with (3.3) is reached. Thus, we must have $m > 0$ in $\mathbb{R}^N$.

Next, we begin with the aspect of existence, which is relatively straightforward in this context.

Proof of Theorem 2.2. (1) (i) We first establish the existence of a positive and finite solution to Eq (1.3). Consider $\hat{\phi}$ as the unique positive solution to

$$\begin{cases} B\hat{\phi}(x) = \lambda\hat{\phi}(x), \ x \in \mathbb{R}^N, \\ \hat{\phi} \text{ is positive, } \hat{\imath} - \text{periodic and } \|\hat{\phi}\|_{\infty} = 1, \end{cases} \tag{3.4}$$

where $\lambda < 0$. Given that $f(x,v)$ is continuously differentiable in $\mathbb{R}^N \times [0,\beta]$, for sufficiently small values of $\varepsilon' > 0$,

$$f(x,\varepsilon'\hat{\phi}) \geq \varepsilon' f_v(x,0)\hat{\phi} + \frac{\lambda}{2}\varepsilon'\hat{\phi} \ \text{ in } \ \mathbb{R}^N. \tag{3.5}$$

Therefore, it is possible to deduce that

$$\varepsilon' L^{\alpha}\hat{\phi} - \tau\varepsilon'(J * \hat{\phi} - \hat{\phi}) - f(x,\varepsilon'\phi) \leq \frac{\lambda}{2}\varepsilon'\hat{\phi} \leq 0 \text{ in } \mathbb{R}^N.$$

Futhermore, $\varepsilon'\hat{\phi}$ acts as a sub-solution to Eq (1.3) under periodicity constraints. In addition, for any given $\hat{M} \geq M_0 =: (\tau + \mathrm{lip}_v(f) + M) > 0$, subsequently, the constant $\hat{M}$ serves as a super-solution to Eq (1.3) when subjected to periodicity conditions, particularly for values of ε' that are sufficiently small and it holds that $\varepsilon'\hat{\phi} \leq \hat{M}$ across $\mathbb{R}^N$. As a result, through the application of an iterative approach, it is demonstrated that there is a positive, bounded solution ψ to (1.3) with $\varepsilon'\hat{\phi} \leq \psi \leq \hat{M}$ in $\mathbb{R}^N$. In fact, we define the order interval

$$[\underline{\psi}, \overline{\psi}] = \{\psi \in C^1(\mathbb{R}^N) : \varepsilon\hat{\phi} := \underline{\psi}(x) \leq \psi(x) \leq \overline{\psi}(x) =: \hat{M}, \forall x \in \mathbb{R}^N\}.$$

Let $\tilde{g}(x, \psi) = f(x, \psi) + \tau(J * \psi - \psi)$. For any specified $\bar{u} \in [\underline{\psi}, \overline{\psi}]$, by using the variational method, we find that the linear equation

$$L^\alpha \psi + \hat{M}\psi = \tilde{g}(x, \bar{u}) + \hat{M}\bar{u} \quad x \in \mathbb{R}^N,$$

has a unique solution ψ that belongs to the space $C^{1+\alpha}(\mathbb{R}^N)$, and this solution is represented as $\psi = T(\bar{u})$.

We assert that T exhibits monotonicity within the order interval $[\underline{\psi}, \overline{\psi}]$, namely, $T(\bar{u}_1) \leq T(\bar{u}_2)$ if $\bar{u}_1, \bar{u}_2 \in [\underline{\psi}, \overline{\psi}]$, and $\bar{u}_1 \leq \bar{u}_2$. Given that $\bar{u}_1 \leq \bar{u}_2$, and considering that $f(x, \psi)$ fulfills a Lipschitz condition in terms of ψ, we can deduce that

$$\tilde{g}(x, \bar{u}_2) - \tilde{g}(x, \bar{u}_1) \geq -\hat{M}(\bar{u}_2 - \bar{u}_1).$$

Actually, by if we write $v_1 = T(\bar{u}_1), v_2 = T(\bar{u}_2)$, then $h = v_2 - v_1$ satisfies

$$L^\alpha h + \hat{M}h = \tilde{g}(x, \bar{u}_2) - \tilde{g}(x, \bar{u}_1) + \hat{M}(\bar{u}_2 - \bar{u}_1) \geq 0, \quad x \in \mathbb{R}^N.$$

Utilizing the maximum principle (see [24]), we conclude that $h \geq 0$, namely, $T(\bar{u}_1) = v_1 \leq v_2 = T(\bar{u}_2)$. Define

$$\underline{\psi}_1 = T(\underline{\psi}), \quad \underline{\psi}_{i+1} = T(\underline{\psi}_i), \quad \overline{\psi}_1 = T(\overline{\psi}), \quad \overline{\psi}_{i+1} = T(\overline{\psi}_i),$$

by the monotonicity of T, $\underline{\psi}_1 \leq \overline{\psi}_1$. We assert that the sequences $\{\underline{\psi}_i\}$ and $\{\overline{\psi}_i\}$ conform to the following properties:

$$\underline{\psi} \leq \underline{\psi}_i \leq \underline{\psi}_{i+1} \leq \overline{\psi}_{i+1} \leq \overline{\psi}_i \leq \overline{\psi}, \quad \forall i \geq 1. \tag{3.6}$$

Let $\bar{u} = \overline{\psi} - \overline{\psi}_1$. Since $\overline{\psi}$ satisfies

$$L^\alpha \overline{\psi} + \hat{M}\overline{\psi} \geq \tilde{g}(x, \overline{\psi}) + \hat{M}\overline{\psi} \quad x \in \mathbb{R}^N,$$

we see that $\bar{u}$ satisfies

$$L^\alpha \bar{u} + \hat{M}\bar{u} \geq 0 \quad x \in \mathbb{R}^N.$$

The application of the maximum principle results in $\bar{u} \geq 0$, which implies that $\overline{\psi}_1 \leq \overline{\psi}$. Similarly, it can be derived that $\underline{\psi} \leq \underline{\psi}_1$. Consequently, we have the inequality chain $\underline{\psi} \leq \underline{\psi}_1 \leq \overline{\psi}_1 \leq \overline{\psi}$. Leveraging the monotonicity property of T, we can inductively deduce Eq (3.6).

As $\underline{\psi}_i, \overline{\psi}_i \in [\underline{\psi}, \overline{\psi}]$, we have $\left\| f(\cdot, \underline{\psi}_i(\cdot)), f(\cdot, \overline{\psi}_i(\cdot)) \right\|_\infty \leq C$. If we assume that $q > N$, then the L^q theory asserts that $\left\| \underline{\psi}_i, \overline{\psi}_i \right\|_{W^2_q(\mathbb{R}^N)} \leq C_1$. This leads to $\tilde{g}(\cdot, \underline{\psi}_i(\cdot)), \tilde{g}(\cdot, \overline{\psi}_i(\cdot)) \in C^\alpha(\mathbb{R}^N)$, and $\|\tilde{g}(\cdot, \underline{\psi}_i(\cdot)), \tilde{g}(\cdot, \overline{\psi}_i(\cdot))\|_\alpha \leq C_2$. Therefore

$$\|\underline{\psi}_i, \overline{\psi}_i\|_{C^{1+\alpha}} \leq C_3,$$

according to the Schauder theory. In the preceding context, the positive constants C, C_1, C_2, and C_3 are all independent of i. Noting that $C^{1+\alpha}(\mathbb{R}^N) \hookrightarrow C^{1+\beta}(\mathbb{R}^N)$ is compact for $\beta \in (0, \alpha)$, there are subsequences of $\{\underline{\psi}_i\}$ and $\{\overline{\psi}_i\}$ converging to $\tilde{\psi}$ and $\hat{\psi}$ in $C^1(\mathbb{R}^N)$, respectively. Given that these sequences exhibit monotonicity and are bounded, by the uniqueness of limits, $\underline{\psi}_i \to \tilde{\psi}$ and $\overline{\psi}_i \to \hat{\psi}$ in $C(\mathbb{R}^N)$. Taking $i \to \infty$ in $\underline{\psi}_{i+1} = T(\underline{\psi}_i)$ and $\overline{\psi}_{i+1} = T(\overline{\psi}_i)$, we observe that $\tilde{\psi}$ and $\hat{\psi}$ are both positive and bounded solutions to Eq (1.3).

(ii) Next, we prove that the positive and bounded solutions to Eq (1.3) are unique.

Assume that $\psi_1(x), \psi_2(x)$ are any two solutions to Eq (1.3) that are both positive and bounded. In that case, there is an $\breve{M} > M_0$ such that $\max\{\psi_1(x), \psi_2(x)\} \leq \breve{M}$. Hence, $\psi_1 \leq \breve{M}$ and $\psi_2 \leq \breve{M}$. Similar to the proof of (i) above, we easily know that

$$\psi_1 = T(\psi_1) \leq T(\breve{M}) =: \bar{u}_1,$$

$$\psi_1 \leq T(\bar{u}_1) = \bar{u}_2.$$

In turn, we have

$$\psi_1 \leq \bar{u}_k,\ k = 1, 2, 3, \ldots.$$

Since this sequence is monotone and bounded, and due to the uniqueness of the limits, as $k \to \infty, \bar{u}_k \to \breve{\psi}$ and $\psi_1 \leq \breve{M}$ in $\mathbb{R}^N$. On the basis of symmetry and Eq (1.3), we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \psi_1(x) f(x, \breve{\psi}) dx &= \int_{\mathbb{R}^N} L^\alpha \breve{\psi}(x) \psi_1 dx - \tau \int_{\mathbb{R}^N} (J * \breve{\psi} - \breve{\psi})(x) \psi_1(x) dx \\
&= \int_{\mathbb{R}^N} L^\alpha \psi_1(x) \breve{\psi}(x) dx - \tau \int_{\mathbb{R}^N} (J * \psi_1 - \psi_1)(x) \breve{\psi}(x) dx \\
&= \int_{\mathbb{R}^N} \breve{\psi}(x) f(x, \psi_1) dx;
\end{aligned}$$

hence

$$\int_{\mathbb{R}^N} \psi_1(x) \breve{\psi}(x) \Big(\frac{f(x, \breve{\psi})}{\breve{\psi}} - \frac{f(x, \psi_1)}{\psi_1}\Big) dx = 0. \tag{3.7}$$

Because $\frac{f(x,v)}{v}$ is monotonically decreasing, if $\psi_1 \not\equiv \breve{\psi}$ in $\mathbb{R}^N$, then $\int_{\mathbb{R}^N} v(x) \breve{\psi}(x) (\frac{f(x,\breve{\psi})}{\breve{\psi}} - \frac{f(x,\psi_1)}{\psi_1}) dx < 0$, which contradicts (3.7). Hence, $\psi_1(x) \equiv \breve{\psi}(x)$ in $\mathbb{R}^N$. Analogously, it can be demonstrated that $\psi_2(x) \equiv \breve{\psi}(x)$ in $\mathbb{R}^N$.

If p is a positive and bounded solution to Eq (1.3), then for each $1 \leq j \leq N$, the function defined by $x \to p(x_1, x_2, \cdots, x_j + \hat{l}_j, x_N)$ is also a positive solution. Given the uniqueness of the positive and bounded solutions, it is straightforward to conclude that p is $\hat{l}$-periodic. This establishes Part 1) of Theorem 2.2.

(2) Assume that $\lambda \geq 0$ and q is a non-negative bounded solution of Eq (1.3). Let $\bar{\phi}$ denote the first eigenfunction corresponding to (3.4). According to the hypothesis of (1.4), it is valid that $f(x, \eta\bar{\phi}(x)) < f_v(x, 0)\eta\bar{\phi}(x)$ for every $x \in \mathbb{R}^N$ and for all $\eta > 0$. Consequently

$$L^\alpha \eta\bar{\phi}(x) - f(x, \eta\bar{\phi}) - \tau\eta(J * \bar{\phi} - \bar{\phi}) > \lambda_1 \eta\bar{\phi} \geq 0 \text{ in } \mathbb{R}^N \tag{3.8}$$

for all $\eta > 0$.

Since $\bar{\phi}$ is bounded away from zero from below and q is non-negative and bounded, it is possible to define

$$\eta' = \inf\{\eta > 0, \eta\bar{\phi} > q \text{ in } \mathbb{R}^N\} \geq 0.$$

Assuming that $\eta' > 0$, let $\bar{s} := \eta'\bar{\phi} - q$; hence, $\bar{s} \geq 0$, and there is a sequence of points $\{x_{\bar{n}}\} \in \mathbb{R}^N$ such that as $\bar{n} \to +\infty$, $\bar{s}\,(x_{\bar{n}}) \to 0$.

Assuming initially that a certain subsequence can be extracted, $x_{\bar{n}} \to \bar{x} \in \mathbb{R}^N$ as $\bar{n} \to +\infty$. By virtue of continuity, it is established that $\bar{s} \geq 0$ in $\mathbb{R}^N$ and $\bar{s}(\bar{x}) = 0$. Given that $\eta'\bar{\phi}$ satisfies the conditions of (3.8), making it a super-solution of Eq (1.3) with periodic boundary conditions, it is straightforward to deduce from Lemma 3.1 that $\bar{s}$ must be identically equal to 0. Thus $q \equiv \eta'\bar{\phi}$. Noticing that $\lambda \geq 0$, it is deduced from (1.3) and (3.8) that

$$0 < L^{\alpha}q(x) - f(x, q(x)) - \tau(J * q(x) - q(x)) = 0.$$

This results in a contradiction.

In typical circumstances, let $\{\bar{x}_{\bar{n}}\} \subset C_{\hat{l}}$ be a sequence such that $x_{\bar{n}} - \bar{x}_{\bar{n}} \in \prod_{i=1}^{N} l_i\mathbb{Z}$. Then, by possibly extracting a subsequence, we can assume the existence of $\bar{x}_{\infty} \in C_{\hat{l}}$ with the property that $\bar{x}_{\bar{n}} \to \bar{x}_{\infty}$ as $\bar{n} \to \infty$. Subsequently, define $\bar{\phi}_{\bar{n}}(x) = \bar{\phi}\,(x + x_{\bar{n}})$ and $q_{\bar{n}}(x) = q\,(x + x_{\bar{n}})$. Considering the periodicity of both L^{α} and f in relation to x, the functions $\eta'\bar{\phi}_{\bar{n}}$ and $q_{\bar{n}}$ comply with the necessary properties

$$L^{\alpha}(\gamma'\bar{\phi}_{\bar{n}})(x + \overline{x}_{\bar{n}}) - f(x + \overline{x}_{\bar{n}}, \gamma'\bar{\phi}_{\bar{n}}) - \tau(J * \gamma'\bar{\phi}_{\bar{n}} - \gamma'\bar{\phi}_{\bar{n}}) > 0,$$
$$L^{\alpha}(q_{\bar{n}})(x + \overline{x}_{\bar{n}}) - f(x + \overline{x}_{\bar{n}}, q_{\bar{n}}) - \tau(J * q_{\bar{n}} - q_{\bar{n}}) = 0$$

for any $x \in \mathbb{R}^N$. According to standard elliptic estimates, it can be deduced that (after possibly extracting a subsequence) $q_{\bar{n}}$ converges in C^1_{loc} to a function q_{∞} satisfying

$$L^{\alpha}(q_{\infty})(x + \overline{x}_{\infty}) - f(x + \overline{x}_{\infty}, q_{\infty}) - \tau(J * q_{\infty} - q_{\infty}) = 0,\ x \in \mathbb{R}^N.$$

Meanwhile, the sequence $(\eta'\bar{\phi}_{\bar{n}})$ converges to $\eta'\bar{\phi}_{\infty} := \eta'\phi\,(\cdot + \bar{x}_{\infty})$, and

$$L^{\alpha}(\eta'\bar{\phi}_{\infty})(x + \overline{x}_{\infty}) - f(x + \overline{x}_{\infty}, \eta'\bar{\phi}_{\infty}) - \tau\eta'(J * \bar{\phi}_{\infty} - \bar{\phi}_{\infty}) > 0 \text{ in } \mathbb{R}^N.$$

Let us define $\bar{s}_{\infty}(x) := \eta'\bar{\phi}_{\infty}(x) - q_{\infty}(x)$. We can obtain

$$\bar{s}_{\infty}(x) = \lim_{\bar{n}\to+\infty}\left[\eta'\bar{\phi}\,(x + x_{\bar{n}}) - q\,(x + x_{\bar{n}})\right];$$

hence, $\bar{s}_{\infty}(x) = \lim_{\bar{n}\to+\infty} \bar{s}\,(x + x_{\bar{n}})$. Therefore, $\bar{s}_{\infty} \geq 0$ and satisfies $\bar{s}_{\infty}(0) = 0$. Consequently, according to Lemma 3.1, we have $\bar{s}_{\infty} = 0$, ao by repeating the above steps, we can get a contradiction.

In conclusion, we only have $\eta' = 0$, thus $q \equiv 0$, and the second part of Theorem 2.2 is proved to be valid..

4. Long-time behavior of solutions

In this section, we focus on proving Theorem 2.3. At the outset, we introduce a well-known principle related to the fractional heat kernel.

Proposition 4.1. *A positive constant $\tilde{C}_1$, greater than 1, exists such that the heat kernel $q_\alpha(x-y,t)$ linked to the operator $\partial t + L^\alpha$ fulfills the inequalities stated below for all $t > 0$*

$$\tilde{C}_1^{-1} \times \min(t^{-\frac{N}{2\alpha}}, \frac{t}{|x-h|^{N+2\alpha}}) \le q_\alpha(x-h,t) \le \tilde{C}_1 \times \min(t^{-\frac{N}{2\alpha}}, \frac{t}{|x-h|^{N+2\alpha}}).$$

The proof for this proposition can be found in [25]. Next, we establish the existence, uniqueness, and comparison principle for the solution to Eq (1.3).

In this section, we first re-examine the notion of a mild solution associated with the linear problem

$$\begin{cases} v_t + L^\alpha v = \bar{w}(t) \quad \text{in} \quad [0,\tilde{T}], \\ v(0) = v_0, \end{cases} \tag{4.1}$$

where $\tilde{T} > 0, v_0 \in D$, and $\bar{w} \in (C([0,\tilde{T}]); D)$ are provided, and

$$D = \{\ v \mid v : \mathbb{R}^N \to \mathbb{R} \text{ is bounded and has uniform continuity in } \mathbb{R}^N\}. \tag{4.2}$$

The mild solution of (4.1) means that

$$v(t) = T_t v_0 + \int_0^t T_{t-s}\bar{w}(s)ds \tag{4.3}$$

$\forall t \in [0,\tilde{T}]$ and $T_t v_0(x) = \int_{\mathbb{R}^N} p_\alpha(t, x-h)v_0(h)dh$ (p_α is given in Proposition 4.1). It is easy to confirm that $v \in C([0,\tilde{T}]; D)$.

Lemma 4.2. *Suppose that f fulfills the condition given by (1.4). Then, (1.2) has a unique global solution v.*

Proof. Let $G(t, v(t,x)) = \tau(J * v(t,x) - v(t,x)) + f(x, v(t,x))$, for all $\tilde{T} > 0$. The mild solution of (1.2) is equivalent to

$$v(t) = T_t v_0 + \int_0^t T_{t-s}G(s, v(s))ds \tag{4.4}$$

$\forall t \in [0,\tilde{T}]$.

Please note that the map $W_{v_0} : C([0,\tilde{T}]; D) \longrightarrow C([0,\tilde{T}]; D)$ defined by

$$W_{v_0}(v)(t) = T_t v_0 + \int_0^t T_{t-s}G(s, v(s))ds$$

has a Lipschitz property in $C([0,\tilde{T}]; D)$ with a constant of

$$||W_{v_0}||_{\text{Lip}} \le TM'(\text{Lip}_v(f) + \tau). \tag{4.5}$$

In this context, $\text{Lip}_v(f)$ refers to the Lipschitz constant of f with respect to v and $M' := sup_{t\in[0,T]}||T_t||$. For any semigroup that is strongly continuous, it holds that that $||T_t|| \le Ce^{wt}$ for certain positive constants C and w. Utilizing (4.5) and the expression (4.4), it can be deduced by induction that $W^n_{v_0}$ is Lipschitz– continuous in $C([0,\tilde{T}]; D)$ with a Lipschitz constant of $\{TM'(\text{Lip}_v(f) + \tau\}^n/n!$, where n represents an arbitrary positive integer. If we select n to be sufficiently large, this constant will be smaller than 1. Consequently, through a direct extension of the contraction mapping theorem, not just $W_{v_0}^n$ but also W_{v_0} possesses a unique fixed point. Therefore, for every $\tilde{T} > 0$, there is a unique mild solution v to Eq (4.3). It is also straightforward to observe that this solution is represented by taking the limit of the iterative sequence $N_{v_0}^i(u), i \in \mathbb{Z}^+$, of any given element $u \in C([0, \tilde{T}]; D)$. Specifically, by setting $u = u(t) \equiv v_0$, we have

$$v = \lim_{i \to +\infty} (N_{v_0})^i(v_0).$$

Considering $0 < \tilde{T} < T''$, the mild solution within the interval $(0, T'')$ must agree with the mild solution on the subinterval $(0, \tilde{T})$ due to the principle of uniqueness. Hence, under the assumption (4.2), the unique mild solution of Eq (1.2) can be extended to all $t \in [0, \infty)$, i.e., it persists for all time.

We now introduce the comparison principle.

Lemma 4.3. *Let $\alpha \in (0, 1), \tau \geq 0$. The function $\tilde{f} : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ belongs to the class $C^{0,\alpha}$ and is Lipschitz-continuous with respect to the variable v. If $\bar{u}, \bar{v} \in C\left([0, \infty); C_{u,b}\left(\mathbb{R}^N\right)\right)$ are such that*

$$\forall (t, x) \in [0, \infty) \times \mathbb{R}^N, \quad \bar{v}_t + L^\alpha \bar{v} \leq \tilde{f}(x, \bar{v}) + \tau(J * \bar{v}), \qquad \bar{u}_t + L^\alpha \bar{u} \geq \tilde{f}(x, \bar{u}) + \tau(J * \bar{u}),$$

and satisfy $\bar{v}(0, \cdot) \leq \bar{u}(0, \cdot)$ on $\mathbb{R}^N$, then $\bar{v} \leq \bar{u}$ on $[0, \infty) \times \mathbb{R}^N$.

Proof. It is sufficient to demonstrate that $\bar{v} \leq \bar{u}$ holds on $\left[0, \bar{T}\right] \times \mathbb{R}^N$ for any $\bar{T} < \infty$.

Define $m := \max \{\text{lip}_v(f), \|\bar{v}\|_\infty, \|\bar{u}\|_\infty\} + \tau$, and let

$$\tilde{v}(t, x) := e^{mt}\bar{v}(t, x), \quad \tilde{u}(t, x) := e^{mt}\bar{u}(t, x), \quad g(t, x, \bar{v}) := m\bar{u} + e^{mt}\tilde{f}\left(x, e^{-mt}\bar{v}\right).$$

Then, $\tilde{v}$ is a sub-solution and $\tilde{u}$ is a super-solution to the following equation:

$$\omega_t + L^\alpha \omega = \tau(J * \omega) + m\omega + e^{mt}\tilde{f}(x, e^{-mt}\omega) = \tau(J * \omega) + g(t, x, \omega).$$

Moreover, $\max \{\|\tilde{u}\|_\infty, \|\tilde{v}\|_\infty\} \leq me^{m\bar{T}}$, and if we let $S := [0, \bar{T}] \times \left[-me^{m\bar{T}}, me^{m\bar{T}}\right]$, hence $\forall (t, \bar{v}) \in S$, we can find that

$$0 \leq g_{\bar{v}}(t, x, \bar{v}) \leq 2m. \tag{4.6}$$

Since $\tilde{v}(0, \cdot) \leq \tilde{u}(0, \cdot)$, proving that $\bar{v} \leq \bar{u}$ amounts to proving that $\tilde{v} \leq \tilde{u}$. For any small ε' in the interval (0,1), $\omega_{\varepsilon'} \geq 0$ and $\lim_{\varepsilon' \to 0^+} \omega_{\varepsilon'} = 0$ exist such that

$$\sup_{\substack{(\tau,y),(t,x) \in [0,T] \times \mathbb{R}^N \\ \max\{|\tau - t|, |y - x|\} \leq \varepsilon'}} \max\{|\tilde{v}(\tau, y) - \tilde{v}(t, x)|, |\tilde{u}(\tau, y) - \tilde{u}(t, x)|\} \leq \omega'_\varepsilon. \tag{4.7}$$

Now, consider any arbitrary point (t, x) belonging to the set $[0, \bar{T} - 2\varepsilon'] \times \mathbb{R}^N$. If there is a point $(\bar{t}, \bar{x}) \in B_{\varepsilon'}(t + \varepsilon', x)$ such that $\tilde{v}\,(\bar{t}, \bar{x}) \leq \tilde{u}\,(\bar{t}, \bar{x})$, then we can deduce from (4.6) and (4.7) that

$$\begin{aligned} g(t, x, \tilde{v}) - g(t, x, \tilde{u}) &= m(\tilde{v} - \tilde{u}) + e^{mt}f(x, e^{-mt}\tilde{v}) - e^{mt}f(x, e^{-mt}\tilde{u}) \\ &= m(\tilde{v}(t, x) - \tilde{v}(\bar{t}, \bar{x}) + \tilde{v}(\bar{t}, \bar{x}) - \tilde{u}(\bar{t}, \bar{x}) + \tilde{u}(\bar{t}, \bar{x}) - \tilde{u}(t, x)) + lip(f)|\tilde{v}(t, x) - \tilde{u}(t, x)| \\ &\leq 2m\omega_{\varepsilon'} + 2m\omega_{\varepsilon'} = 4m\omega_{\varepsilon'}. \end{aligned}$$

If $\tilde{u} \geq \tilde{v}$ holds on $B_{\varepsilon'}(t + \varepsilon', x)$ instead, then it follows from the inequality $g_v \leq 2M$ that

$$[g(t, x, \tilde{u}) - g(t, x, \tilde{v})](t, x) \leq 2m(\tilde{u} - \tilde{v}).$$

Using these estimates, along with the fact that $\tilde{v}$ and $\tilde{u}$ are, respectively, a sub-solution and a super-solution, let $w = \tilde{v} - \tilde{u}$. We can then see

$$h := (w)_t + (L)^\alpha w \leq \tau(J * w) + g(\cdot, \tilde{u}(\cdot, \cdot)) - g(\cdot, \tilde{v}(\cdot, \cdot)) \leq \tau(J * w) + \max\{2mw, 4m\omega_{\varepsilon'}\}.$$

The principle of Duhamel, applied to smooth solutions of the linear partial differential equation

$$w_t + L^\alpha w = h(t, x),$$

now yields

$$w(t, x) \leq T_t [w(0, \cdot)] (x) + \int_0^t T_{t-\tau} [\max\{4m\omega_{\varepsilon'}(\tau, \cdot), 2mw\} + \tau(J * w)] (x) d\tau$$

$\forall (x, t) \in \mathbb{R}^N \times [0, \bar{T} - 2\varepsilon']$. Since the operator T_t preserves the order and satisfies $T_t[1] \equiv 1$, we can conclude that $T_t [\omega(0, \cdot)] \leq 0$ on $\mathbb{R}^N$ (due to the fact that $\bar{v}(0, \cdot) \leq \bar{u}(0, \cdot)$). Furthermore, if we define $\xi(t) := \max\{\sup_{x \in \mathbb{R}^N} w(x, t), 0\}$, then

$$\xi(t) \leq 4mT\omega_{\varepsilon'} + \int_0^t (2m + \tau)\xi(\tau) d\tau$$

for all $t \in [0, \bar{T} - 2\varepsilon']$. Applying Gronwall's inequality now gives us

$$\xi(t) \leq 4mT\omega_{\varepsilon'} e^{(2m+\tau)T},$$

where $t \in [0, \bar{T} - 2\varepsilon']$. Therefore

$$\limsup_{\varepsilon' \to 0} \sup_{(t,x) \in [0, \bar{T} - 2\varepsilon'] \times \mathbb{R}^N} w(t, x) = 0,$$

which shows that $\bar{u} \leq \bar{v}$ on $[0, \bar{T}] \times \mathbb{R}^N$.

Remark 4.4. *We can establish that the solution u of (1.2) is a classical solution by referring to Theorem 1.5, and its proof presented in Section 6.1 of [26]. Additionally, Definition 2.1 in Section 4.2 of [26] and the clarification given in [7] also support this conclusion.*

We now apply Proposition 4.1 and Lemma 4.3 to show that the unique solution of (1.2) has the following property.

Lemma 4.5. *Let $v(t, x)$ be the unique solution to Eq (1.2). Then a positive constant $\bar{c}$ exists that depends on v_0 and α, and holds for every $x \in \mathbb{R}^N$*

$$\frac{\bar{c}}{1 + |x|^{N+2\alpha}} \leq v(1, x).$$

Proof. Let $\underline{v}$ represent the solution to the given equation

$$\begin{cases} \partial_t \underline{v}(t, x) + L^\alpha(\underline{v})(t, x) = -2\bar{M}\underline{v}, & (t, x) \in (0, +\infty) \times \mathbb{R}^N, \\ \underline{v}(x, 0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{4.8}$$

where $\bar{M} = \max\{\tau, \mathrm{lip}_v(f)\}$. By virtue of Proposition 4.1, we are able to solve Eq (4.8) and subsequently discover that

$$\underline{v}(t,x) = e^{-2\bar{M}} \int_{\mathbb{R}^N} q_\alpha(x,h,t) v_0(h) dh.$$

Hence, for any positive t, we obtain:

$$e^{-2\bar{M}t} \int_{\mathrm{supp}(v_0)} C^{-1} \times v_0(h) \min\left(t^{-\frac{N}{2\alpha}}, \frac{t}{|x-h|^{N+2\alpha}}\right) dh \leq \underline{v}(t,x)$$
$$\Rightarrow e^{-2\bar{M}} \int_{\mathrm{supp}(v_0)} C^{-1} \times v_0(h) \min\left(1, \frac{1}{|x-h|^{N+2\alpha}}\right) dh \leq \underline{v}(1,x).$$

Thanks to the theorem of dominated convergence, we obtain :

$$\left(1 + |x|^{N+2\alpha}\right) \times e^{-2\bar{M}} \int_{\mathrm{supp}(v_0)} C^{-1} \times v_0(h) \min\left(1, \frac{1}{|x-h|^{N+2\alpha}}\right) dh \underset{|x|\to\infty}{\longrightarrow} e^{-2\bar{M}} \int_{\mathrm{supp}\ (v_0)} C^{-1} \times v_0(h) dh.$$

Hence, we deduce, through a compactness reasoning, that the following statement holds for all $x \in \mathbb{R}^N$

$$\frac{e^{-2\bar{M}} C^{-1}}{(1 + |x|^{N+2\alpha})} \leq \underline{v}(1,x),$$

where the aforementioned constant C is a new value that solely depends on v_0. Moreover thanks to Lemma 4.3, it holds that for any $t \geq 0$, $x \in \mathbb{R}^N$

$$0 < \underline{v}(t,x) \leq v(t,x),$$

and hence

$$0 < \frac{\bar{c}}{(1 + |x|^{N+2\alpha})} \leq v(1,x), \ \forall x \in \mathbb{R}^N.$$

Subsequently, we present a particular instance of Lemma 2.1. Analogous to the proof of Lemma 2.1, we can draw the following conclusion.

Lemma 4.6. *Let B_r represent the open sphere in $\mathbb{R}^N$ with center at the origin and a radius of r. Then there are a unique real number (the principal eigenvalue) $\bar{\lambda}_r$, and a unique function $\bar{\varphi}_r$ (the principal eigenfunction) with $C^{1+\alpha}\left(\overline{B_r}\right)$ that satisfy*

$$\begin{cases} L^\alpha \bar{\varphi}_r - f_v(x+y,0)\bar{\varphi}_r - \tau(J * \bar{\varphi}_r - \bar{\varphi}_r) = \bar{\lambda}_r \bar{\varphi}_r & \text{in } B_r, \\ \bar{\varphi}_r > 0 & \text{in } B_r, \\ \bar{\varphi}_r = 0 & \text{on } \partial B_r, \\ \|\bar{\varphi}_r\|_\infty = 1. \end{cases} \tag{4.9}$$

Similarly to Theorem 1.1 in [17], we can easily obtain the following Lemma.

Lemma 4.7. *When r goes to $+\infty$, then $\bar{\lambda}_r$ converges to λ.*

Finally, we use the properties given above to prove Theorem 2.3.

Proof of Theorem 2.3. (1) Let $v(t,x)$ represent the solution to

$$\begin{cases} \partial_t v + L^\alpha v = f(x,v) + \tau(J * v - v), & (t,x) \in (0,+\infty) \times \mathbb{R}^N, \\ v(x,0) = v_0(x), & x \in \mathbb{R}^N. \end{cases}$$

We first assume that $\lambda < 0$. According to Lemma 4.7, we can choose an r that is sufficiently large to ensure $\bar{\lambda}_r < 0$. Let $\bar{\varphi}_r$ be the function that fulfills (4.9). That is, $\bar{\varphi}_r \in C^{1+\alpha}\left(\overline{B_r}\right)$ and satisfies

$$\begin{cases} L^\alpha \bar{\varphi}_r - \tau(J * \bar{\varphi}_r - \bar{\varphi}_r) - f_v(x,0)\bar{\varphi}_r = \bar{\lambda}_r \bar{\varphi}_r \text{ in } B_r, \\ \bar{\varphi}_r \text{ is positive within the ball } B_r, \bar{\varphi}_r = 0 \text{ on } \partial B_r, \|\bar{\varphi}_r\|_\infty = 1. \end{cases}$$

By Lemma 4.5, we have, for an $\varepsilon_1 > 0$ that is small enough, $\varepsilon_1 \bar{\varphi}_r < v(1,x)$ in B_r. Let us extend the function $\varepsilon_1 \bar{\varphi}_r$ to the entire space $\mathbb{R}^N$ by setting $u_0(x) := \varepsilon_1 \bar{\varphi}_r(x)$ for x in B_r, and define $u_0(x) := 0$ in $\mathbb{R}^N \backslash B_r$. Next, we introduce the function u_1 as follows:

$$\begin{cases} \partial_t u_1 + L^\alpha u_1 = \tau(J * u_1 - u_1) + f(x,u_1), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N, \\ u_1(0,x) = u_0(x), x \in \mathbb{R}^N. \end{cases} \tag{4.10}$$

For a sufficiently small $\varepsilon_1 > 0$ and a large enough r, $\varepsilon_1 \bar{\varphi}_r$ acts as a sub-solution of Eq (4.10) in B_r, and thus u_0 serves as a sub-solution of Eq (4.10) in $\mathbb{R}^N$. From Lemma 4.3, we have $u_0(x) \le u_1(\delta, x)$ for all $\delta > 0$. We claim that u_1 is nondecreasing in time t. In fact, for all $\delta > 0$, we set $\tilde{w}(t,x) = u_1(t+\delta, x)$; hence, $\tilde{w}$ serves as the solution of

$$\begin{cases} \partial_t \tilde{w} + L^\alpha \tilde{w} = \tau(J * \tilde{w} - \tilde{w}) + f(x,\tilde{w}), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N, \\ \tilde{w}(0,x) = u_1(\delta,x), \qquad x \in \mathbb{R}^N. \end{cases}$$

Since $u_1(0,x) \le u_1(\delta,x)$, by Lemma 4.3, we can obtain

$$u_1(t,x) \le u_1(t+\delta,x),$$

which implies that the function u_1 is nondecreasing with respect to time t. By Lemma 4.3, if $u_1(0,x) \le v(1,x)$ holds for all $x \in \mathbb{R}^N$, then it follows that

$$u_1(t,x) \le v(t,x) \;\text{ in } \mathbb{R}_+ \times \mathbb{R}^N.$$

Furthermore, for a sufficiently small $\varepsilon_1 > 0$, $u_0(x) \le q(x)$ in $\mathbb{R}^N$, where q represents the unique bounded and positive solution to Eq (1.3) . Given that q is a stationary solution of (1.2), this leads to the conclusion that

$$u_1(t,x) \le q(x) \text{ in } \mathbb{R}_+ \times \mathbb{R}^N.$$

Since u_1 is nondecreasing with respect to time t, standard elliptic estimates suggest that u_1 converges in $C^{1+\alpha}_{loc}\left(\mathbb{R}^N\right)$ to a bounded stationary solution $\underline{u}_\infty (\le q)$ of (1.2). Moreover, it holds that $\underline{u}_\infty(0) \ge u_1(0,0) \ge \varepsilon_1 \bar{\varphi}_r(0) > 0$. By applying Lemma 3.1, it is deduced that $\underline{u}_\infty > 0$ in $\mathbb{R}^N$. Thanks to Theorem 2.2, we can see that $\underline{u}_\infty \equiv q$.

Subsequently, according to (1.4), we have the option to select a positive constant, denoted as $\tilde{M} > 0$ such that $f(x,k) \le 0$ holds in $\mathbb{R}^N$ for all $k \ge \tilde{M}$ and for every x in $\mathbb{R}^N$. We select $\tilde{M}$ to be sufficiently sufficiently large to ensure that $\tilde{M} \ge \|v_0\|_\infty > 0$ in $\mathbb{R}^N$ and let u_2 as follow:

$$\begin{cases} \partial_t u_2 + L^\alpha u_2 = \tau(J * u_2 - u_2) + f(x,u_2), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N, \\ u_2(0,x) = \tilde{M}, \qquad x \in \mathbb{R}^N. \end{cases}$$

Because $\tilde{M}$ serves as a super-solution of (1.3), we can prove that u_2 exhibits a nonincreasing behavior with respect to time t. By Lemma 4.3, $u_2(0,x) = \tilde{M} \ge v_0(x) \ge 0$ in $\mathbb{R}^N$ implies

$$u_2(t,x) \ge v(t,x) \ge 0, \quad \text{for all} \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N.$$

Therefore, standard elliptic estimates indicate that as t approaches positive infinity, u_2 converges in $C^1_{loc}\left(\mathbb{R}^N\right)$ to a bounded and non-negative solution $\overline{u_\infty}(\leq \bar{M})$ of (1.3). According to Lemma 3.1 and Theorem 2.2, either $\overline{u_\infty} \equiv 0$ or $\overline{u_\infty} \equiv q$. In conclusion, we have

$$u_1(t,x) \leq v(1+t,x) \leq u_2(1+t,x), \quad \text{for all} \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N. \tag{4.11}$$

As $u_1(t,x) \to q(x)$ as $t \to +\infty$, it is deduced from (4.11) that $\overline{u_\infty} \equiv q$, and that $v(t,x)$ converges to $q(x)$ in $C^1_{\text{loc}}\left(\mathbb{R}^N\right)$ as $t \to +\infty$.

(2) Next, we assume that $\lambda \geq 0$. Following the same reasoning as before, a positive constant, denoted as $\grave{M} > 0$, exists such that for all $k \geq \grave{M}$ and every $x \in \mathbb{R}^N$, the inequality $f(x,k) \leq 0$ is satisfied. By choosing $\grave{M}$ to be sufficiently large to ensure that $v_0 \leq \grave{M}$, one can once again derive the desired result by defining u_2 as previously described

$$u_2(t,x) \geq v(t,x) \geq 0, \quad \text{for all} \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N. \tag{4.12}$$

However, according to the outcome of Theorem 2.2, u_2 converges in $C^1_{loc}(\mathbb{R}^N)$ to 0 as $t \to \infty$. Furthermore, this convergence is uniform with respect to x. Given that u_2 exhibits periodicty in x for all $t \geq 0$, including $t = 0$, and considering that Eq (1.3) is also periodic in x, it can be deduced from Eq (4.12) that $v(t,x)$ converges to 0 uniformly as $t \to \infty$.

5. Conclusions

In this study, we have investigated the long-time asymptotic behavior of Eq (1.2) in a periodic environment. Our main findings include the existence and uniqueness of equilibrium solutions (Theorem 2.2) and the long-term asymptotic behavior of the solutions (Theorem 2.3). The results obtained in this research have significant implications for understanding the spread and extinction of biological species. By establishing the existence and uniqueness of stationary solutions and characterizing their long-time behavior, we provide a theoretical foundation for modeling biological processes in periodic environments. While our study provides valuable insights into the long-time behavior of Eq (1.2), there are several avenues for future research. First, the extension of our results to more general classes of equations and environments remains an open problem. Second, the development of numerical methods to simulate the long-time behavior of these equations could provide further understanding of their dynamics. Finally, exploring the biological implications of our findings in specific ecological contexts could lead to practical applications in conservation and management strategies. In summary, this research contributes to the understanding of the long-term asymptotic behavior of Eq (1.2) in periodic environments and highlights its relevance to biological applications. Future work will focus on extending these results and exploring their practical implications.

Author contributions

Yu Wei: Writing–review and editing, writing–original draft, formal analysis, methodology, software; Yahan Wang: writing–review and editing, software; Huiqin Lu: writing–review and editing, methodology, investigation, funding acquisition. All authors have read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declares they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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