Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(2): 3797–3818.
DOI: 10.3934/math.2025176
Received: 20 December 2024
Revised: 08 February 2025
Accepted: 18 February 2025
Published: 25 February 2025

Research article

Existence and uniqueness of solutions for a class of fractional differential equation with lower-order derivative dependence

Yujun Cui, Chunyu Liang and Yumei Zou*

College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China

* **Correspondence:** Email: sdzouym@126.com.

Abstract: This study investigates boundary value problems for nonlinear fractional-order differential equations. The differential operator is interpreted in the Riemann-Liouville sense and is coupled with a non-linearrrrr term that involves the fractional derivative of the unknown function. Using the Schauder fixed point theorem, the Banach fixed point theorem, and the Leray-Schauder continuation theorem, we establish results regarding the existence and uniqueness of solutions within suitable function spaces. Additionally, we provide concrete examples of various boundary value problems involving fractional-order differential equations to demonstrate the applicability of the theory developed.

Keywords: Leray-Schauder continuity theorem; the existence and uniqueness of solutions; Schauder fixed point theorem; Banach fixed point theorem; Riemann–Liouville fractional derivatives
Mathematics Subject Classification: 34B10, 34B15

1. Introduction

The present study investigates the boundary value problem for the following fractional order differential equation

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t), D_{0^+}^{\alpha-1} u(t)), & t \in (0,1),\ 1 < \alpha \le 2, \\ u(0) = 0, \quad u(1) = \displaystyle\int_0^1 u(s)dg(s), \end{cases} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. The operator $D_{0^+}^{\alpha}$ represents the standard fractional derivative in the Riemann–Liouville sense. This type of derivative is used to model systems where memory and hereditary properties are significant, such as in viscoelastic materials, anomalous diffusion, and complex systems in physics, engineering, and biology. The order of the derivative, α, typically takes values in the range $(0, 2]$, which corresponds to non-integer derivatives that generalize classical integer-order derivatives. The boundary condition $u(1) = \int_0^1 u(s)dg(s)$ is nonlocal, as it involves an integral over the entire interval rather than a value at a specific point. Such boundary conditions are common in models where the solution at a boundary depends on the values of the solution over a region, such as in physical systems exhibiting memory effects or nonlocal interactions. The function $g(t)$ has bounded variation and satisfies condition $\displaystyle\int_0^1 u^{\alpha-1}dg(u) \neq 1$. The theory of boundary value problems for ordinary differential equations with integral boundary conditions involving a function $g(t)$ of bounded variation are widely applied in various fields of applied mathematics and physics. For instance, problems in heat conduction, groundwater flow, thermoelasticity, chemical engineering, and plasma physics can all be formulated as nonlocal problems with integral boundary conditions [1–3]. For a detailed discussion on such boundary value problems and their significance, readers are referred to [4] and the references therein.

Fractional order calculus is a theoretical framework that extends traditional calculus to encompass non-integer orders [5]. Fractional order differential equations have significant applications in various fields, effectively modeling complex phenomena in science and engineering, including control theory, viscoelastic materials, and electromagnetism, and so on, see [6, 7] and the references therein. Fractional order calculus allows for a more precise description of natural phenomena and their associated mathematical models, see [8]. Consequently, the analysis of solutions to nonlinear fractional-order differential equations has emerged as a crucial area of research. In modern academic studies, substantial progress has been achieved in both theoretical developments and practical applications. In recent years, considerable research has focused on the existence and uniqueness of solutions to boundary value problems involving fractional-order differential equations by using nonlinear analysis methods, such as fixed-point theorems on cone [9, 10], Leray–Schauder nonlinear alternative [10, 11], Schauder fixed-point theorem [12], mixed monotone method [10, 13], and Banach contraction principle [14, 15]. The study in [9] specifically addressed a Dirichlet-type boundary value problem for nonlinear fractional-order differential equations, which is described as follows

$$\begin{cases} D_{0^+}^{\alpha}u(t) = -f\left(t, u(t)\right), \ t \in [0, 1], \ 1 < \alpha \leq 2, \\ u(0) = u(1) = 0, \end{cases} \tag{1.2}$$

where $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, and $D_{0^+}^{\alpha}$ denotes the standard Riemann–Liouville fractional derivative. The authors concentrated on investigating the conditions under which positive solutions exist and their potential multiplicity. In their study, the authors first transformed the problem into an equivalent Fredholm integral equation. They then applied the fixed point theorem in a cone to successfully demonstrate the existence and multiplicity of several positive solutions.

The work in [16] focuses on analyzing a boundary value problem for a nonlinear fractional differential equation, formulated as

$$\begin{cases} D_{0^+}^{\alpha}u(t) = f\left(t, u(t)\right), \ t \in [0, 1], \ 1 < \alpha \leq 2, \\ u(0) + u'(0) = 0, \ u(1) + u'(1) = 0. \end{cases} \tag{1.3}$$

Using fixed point theory in cone spaces, the study obtained important results on the existence and multiplicity of positive solutions.

In [17], the author explored resonance multipoint boundary value problems involving nonlinear fractional-order differential equations. The problem is formulated as

$$\begin{cases} D_{0^+}^{\alpha}u(t) = -f\left(t, u(t), D_{0^+}^{\alpha-1}u(t)\right),\ t \in [0,1],\ 1 < \alpha \leq 2, \\ u(0) = 0,\ u(1) = \sum_{i=1}^{m-2} \eta_i u(\xi_i). \end{cases} \tag{1.4}$$

Through the application of spectral theory for linear operators, the study established a necessary condition for positive solutions. Furthermore, when the nonlinearity was assumed to possess a fixed sign, a necessary and sufficient condition for the existence of positive solutions was determined. Unlike the approach used in reference [17], where the necessary conditions for the existence of positive solutions to Eq (1.4) were obtained through the spectral theory of linear operators, this article establishes novel results regarding both the existence and uniqueness of solutions by employing the Leray-Schauder continuity theorem, the Schauder fixed-point theorem, and the Banach contraction principle.

The contributions of this paper are as follows. First, the boundary conditions in Eq (1.1) are quite general, since they encompass multipoint boundary value problems and nonlocal boundary value problems as special cases. Such boundary conditions introduces a bounded variation function $g(t)$ within the framework of the boundary value problem (1.1). This function is widely utilized in various fields of mathematical physics due to its significant theoretical and practical implications. Its inclusion not only complicates the computation of the Green function but also extends the problem of structural diversity and applicability, thereby enriching its analytical depth and broadening its potential applications. Second, when proving the existence of solutions for Eq (1.1) using the Leray-Schauder continuity theorem, we employ two different approaches to obtain a priori bounds for the solutions of Eq (1.1). As a result, we derive two sufficient conditions for the existence of solutions and further provide examples demonstrating that these two conditions are independent of each other. Third, in the discussion on the existence and uniqueness of solutions to Eq (1.1), we establish three sufficient conditions for the unique solvability of Eq (1.1) by imposing Lipschitz conditions on the nonlinear term, either over the entire space or within a bounded domain.

The paper is structured as follows: Section 2 defines key concepts and lemmas required for the main results. Section 3 presents the primary conclusions regarding the existence and uniqueness of solutions to Eq (1.1). In Section 4, illustrative examples are provided to illustrate the practical application of the main findings.

2. Preparatory work

In this section, we provide some essential background information for the convenience of the reader.

Definition 2.1. *[6,7] The Riemann-Liouville fractional integral of order $\alpha > 0$ is given by the function $u : (0,\infty) \to \mathbb{R}$ as*

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t u(s)(t-s)^{\alpha-1} ds,$$

provided that the integral is well-defined.

Definition 2.2. *[6, 7] The fractional derivative of order $\alpha > 0$ of Riemann-Liouville type is given by the function u as*

$$D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{(n)}\int_0^t \frac{u(s)}{(t-s)^{\alpha+1-n}}ds, \quad n = [\alpha]+1,$$

where $[\alpha]$ denotes the integer part of the number α, assuming that the right-hand side is pointwise defined on $(0, +\infty)$. Here, we present a collection of commonly encountered examples for the computation of Riemann-Liouville fractional derivatives. Fractional derivative of a power function: $D_{0^+}^{\alpha}t^p = \dfrac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}t^{p-\alpha}$, $p > \alpha - 1$, specifically providing $D_{0^+}^{\alpha}t^{\alpha-m} = 0$, $m = 1, 2, 3, \ldots, N$, where N is the smallest integer greater than or equal to α.

Lemma 2.1. *[6, 7] Let $\alpha > 0$ and $u \in C(0,1) \cap L^1(0,1)$. Then we have the following assertions:*

(i) For $0 < \beta < \alpha$, $D_{0+}^{\beta}I_{0+}^{\alpha}u = I_{0+}^{\alpha-\beta}u$ and $D_{0+}^{\alpha}I_{0+}^{\alpha}u = u$,

(ii) $D_{0+}^{\alpha}u = 0$ if and only if $u(t) = c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n}$, $c_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$, where n is the smallest integer greater than or equal to α,

(iii) Suppose that $D_{0+}^{\alpha}u \in C(0,1) \cap L^1(0,1)$, and then

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) = u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n},$$

$c_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$, where n is the smallest integer greater than or equal to α.

Theorem 2.1. *[18] (Leray-Schauder Continuity Theorem)*

Let $\left(\tilde{X}, \|\cdot\|\right)$ be a Banach space, and let $T : \tilde{X} \to \tilde{X}$ be a completely continuous operator. Suppose there exists a constant $M > 0$ such that if $y = \tilde{\lambda}Ty$ for some $\tilde{\lambda} \in (0,1)$, then $\|y\| \leq M$. Under these conditions, T has a fixed point in $\tilde{X}$.

Theorem 2.2. *[19] (Banach fixed point theorem)*

Let $(\tilde{X}, d)$ be a nonempty complete metric space, and let $T : \tilde{X} \to \tilde{X}$ be a contraction mapping on $\tilde{X}$. That is, there exists a constant $q \in [0,1)$ such that for all $\tilde{u}, \tilde{v} \in \tilde{X}$,

$$d(T(\tilde{u}), T(\tilde{v})) \leq q \cdot d(\tilde{u}, \tilde{v}). \tag{2.1}$$

Then T has a unique fixed point in $\tilde{X}$.

Theorem 2.3. *[19] (Schauder fixed point theorem)*

Let $\tilde{\Omega}$ be a bounded closed convex subset of a Banach space $\tilde{X}$. If $T : \tilde{\Omega} \to \tilde{\Omega}$ is a completely continuous operator, in that case T has fixed point in $\tilde{\Omega}$.

Lemma 2.2. *Suppose $h : [0,1] \to \mathbb{R}$ represents a continuous function, and it holds that $\int_0^1 u^{\alpha-1}dg(u) \neq 1$, then fractional boundary value problem*

$$\begin{cases} D_{0+}^{\alpha}u(t) = h(t), \quad t \in (0,1), \ 1 < \alpha \leq 2, \\ u(0) = 0, \quad u(1) = \displaystyle\int_0^1 u(s)dg(s), \end{cases} \tag{2.2}$$

has a unique solution

$$u(t)=\int_0^1 G_0(t,s)h(s)ds,\ t\in[0,1], \tag{2.3}$$

where

$$G_0(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}(t-s)^{\alpha-1}+H_0(t,s), & 0\le s\le t\le 1,\\ H_0(t,s), & 0\le t\le s\le 1,\end{cases} \tag{2.4}$$

$$H_0(t,s)=\frac{-t^{\alpha-1}\displaystyle\int_s^1 (u-s)^{\alpha-1}dg(u)+(1-s)^{\alpha-1}\,t^{\alpha-1}}{\displaystyle\int_0^1 u^{\alpha-1}dg(u)-1}.$$

Proof. As stated in Lemma 2.1, there are constants $c_i\in\mathbb{R}$ for $i=1,2$ such that

$$u(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+I_{0+}^{\alpha}h(t).$$

By applying the boundary condition $u(0)=0$, we find that $c_2=0$. We obtain

$$u(t)=I_{0+}^{\alpha}h(t)+c_1t^{\alpha-1}.$$

By separately computing $u(1)$ and $\int_0^1 u(s)dg(s)$, we are able to obtain

$$u(1)=\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}h(s)ds+c_1,$$

$$\begin{aligned}\int_0^1 u(s)dg(s) &= \frac{1}{\Gamma(\alpha)}\int_0^1\int_0^t(t-s)^{\alpha-1}h(s)dsdg(t)+c_1\int_0^1 t^{\alpha-1}dg(t)\\ &= \frac{1}{\Gamma(\alpha)}\int_0^1\int_s^1(t-s)^{\alpha-1}h(s)dg(t)ds+c_1\int_0^1 t^{\alpha-1}dg(t).\end{aligned}$$

Given the condition $u(1)=\displaystyle\int_0^1 u(s)dg(s)$, consider

$$c_1=\frac{\displaystyle\int_0^1(1-s)^{\alpha-1}\,h(s)ds-\int_0^1\int_s^1(u-s)^{\alpha-1}h(s)dg(u)ds}{\Gamma(\alpha)\left[\displaystyle\int_0^1 u^{\alpha-1}dg(u)-1\right]}.$$

Therefore, the unique solution of (2.2) is given by

$$\begin{aligned}u(t) &= I_{0+}^{\alpha}h(t)+\frac{t^{\alpha-1}\displaystyle\int_0^1(1-s)^{\alpha-1}h(s)ds}{\Gamma(\alpha)\left[\displaystyle\int_0^1 u^{\alpha-1}dg(u)-1\right]}-\frac{t^{\alpha-1}\displaystyle\int_0^1\int_s^1(u-s)^{\alpha-1}h(s)dg(u)ds}{\Gamma(\alpha)\left[\displaystyle\int_0^1 u^{\alpha-1}dg(u)-1\right]}\\ &= \int_0^1 G_0(t,s)h(s)ds.\end{aligned}$$

This concludes the proof. □

Let $E=\left\{u(t)\mid u\in C[0,1] \text{ and } D_{0^+}^{\alpha-1}u\in C[0,1]\right\}$ provided with the norm

$$\|u\|=\|u\|_\infty+\left\|D_{0^+}^{\alpha-1}u\right\|_\infty,$$

where $\|u\|_\infty=\max_{t\in[0,1]}|u(t)|$. According to [12] , $(E,\|\cdot\|)$ is a Banach space.

Define the operator $T:E\to E$, as detailed below

$$(Th)(t)=\int_0^1 G_0(t,s)h(s)ds,\forall\, h\in E.$$

Lemma 2.3. *T acts as a linear completely continuous operator defined on the space E.*

Proof. Observe that $G_0(t,s)$ is a continuous function defined on $[0,1]\times[0,1]$. Consequently, for every $h\in E$, Th is a continuous function on $[0,1]$, and the derivative of order $\alpha-1$ is given by:

$$\begin{aligned}
&D_{0^+}^{\alpha-1}(Th)(t)\\
&=D_{0^+}^{\alpha-1}\left[\int_0^1 G_0(t,s)h(s)ds\right]\\
&=D_{0^+}^{\alpha-1}\left[I_{0+}^{\alpha}h(t)+\frac{\int_0^1(1-s)^{\alpha-1}h(s)ds}{\Gamma(\alpha)\left[\int_0^1u^{\alpha-1}dg(u)-1\right]}t^{\alpha-1}-\frac{\int_0^1\int_s^1(u-s)^{\alpha-1}h(s)dg(u)ds}{\Gamma(\alpha)\left[\int_0^1u^{\alpha-1}dg(u)-1\right]}t^{\alpha-1}\right]\\
&=\int_0^t h(s)ds+\frac{\int_0^1(1-s)^{\alpha-1}h(s)ds}{\int_0^1u^{\alpha-1}dg(u)-1}-\frac{\int_0^1\int_s^1(u-s)^{\alpha-1}h(s)dg(u)ds}{\int_0^1u^{\alpha-1}dg(u)-1}\\
&=\int_0^1 G_1(t,s)h(s)ds,
\end{aligned}$$

$G_1(t,s)$ along with $H_1(t,s)$ can be represented using the following equations

$$G_1(t,s)=\begin{cases}H_1(t,s)+1, & 0\le s\le t\le 1,\\ H_1(t,s), & 0\le t<s\le 1,\end{cases}$$

$$H_1(t,s)=\frac{(1-s)^{\alpha-1}-\int_s^1(u-s)^{\alpha-1}dg(u)}{\int_0^1u^{\alpha-1}dg(u)-1}.$$

Given that $D_{0^+}^{\alpha-1}(Th)(t)=\int_0^1 G_1(t,s)h(s)ds$, because $H_1(t,s)$ is defined as a continuous function on $[0,1]\times[0,1]$, it follows that $D_{0^+}^{\alpha-1}(Th)(t)$ is continuous on $[0,1]$, implying that $D_{0^+}^{\alpha-1}(Th)(t)\in C[0,1]$. Therefore, $Th\in E$.

The following demonstrates that T acts as a compact operator on the space E.

Given any bounded set $\Omega \subseteq E$, there exists a positive constant M satisfying $\|h\| \leq M$ for every $h \in \Omega$. Considering the expressions for $G_0(t,s)$ and $H_1(t,s)$, it is evident that these functions are continuous on $[0,1]\times[0,1]$, and hence, they are bounded. Therefore, there exists a constant $N > 0$ such that

$$|G_0(t,s)| \leq N, |H_1(t,s)| \leq N,\ \forall (t,s) \in [0,1]\times[0,1].$$

Thus,

$$\begin{aligned}
\|Th\|_\infty &= \max_{t\in[0,1]} |(Th)(t)| = \max_{t\in[0,1]} \int_0^1 |G_0(t,s)|\,|h(s)|\,ds \\
&\leq \max_{t\in[0,1]} \int_0^1 |G_0(t,s)|\,ds \cdot \|h\|_\infty \leq NM,
\end{aligned}$$

$$\begin{aligned}
\left\|D_{0^+}^{\alpha-1}(Th)\right\|_\infty &= \max_{t\in[0,1]} \left|D_{0^+}^{\alpha-1}(Th)(t)\right| = \max_{t\in[0,1]} \int_0^1 |G_1(t,s)|\,|h(s)|\,ds \\
&\leq \max_{t\in[0,1]} \int_0^1 (|H_1(t,s)|+1)\,ds \cdot \|h\|_\infty \leq (N+1)M,
\end{aligned}$$

consequently, we have

$$\|Th\| = \left\|D_{0^+}^{\alpha-1}(Th)\right\|_\infty + \|Th\|_\infty \leq (2N+1)M.$$

That is, $T(\Omega)$ is a bounded set in E.

To prove that $D_{0^+}^{\alpha-1}(T(\Omega))$ and $T(\Omega)$ are equicontinuous, we note that the expressions for $G_0(t,s)$ and $H_1(t,s)$ represent continuous functions on $[0,1]\times[0,1]$. As a result, these functions are uniformly continuous on the interval $[0,1]\times[0,1]$. Thus, for any given $\varepsilon > 0$, there exists a constant $\delta \in \left(0, \dfrac{\varepsilon}{3M}\right)$ such that for all $t_1, t_2 \in [0,1]$, whenever $|t_1 - t_2| < \delta$, we have

$$|G_0(t_1,s) - G_0(t_2,s)| \leq \frac{\varepsilon}{3M},$$

$$|H_1(t_1,s) - H_1(t_2,s)| \leq \frac{\varepsilon}{3M}.$$

Thus, for any $h \in \Omega$, there exists

$$\begin{aligned}
|(Th)(t_1) - (Th)(t_2)| &= \left|\int_0^1 (G_0(t_1,s) - G_0(t_2,s))\,h(s)ds\right| \\
&\leq \int_0^1 |G_0(t_1,s) - G_0(t_2,s)|\,|h(s)|\,ds \\
&\leq M\int_0^1 |G_0(t_1,s) - G_0(t_2,s)|\,ds \\
&\leq \frac{\varepsilon}{3} < \varepsilon,
\end{aligned}$$

$$\begin{aligned}\left|D_{0^+}^{\alpha-1}(Th)(t_1) - D_{0^+}^{\alpha-1}(Th)(t_2)\right| &\leq \left|\int_0^1 (H_1(t_1,s) - H_1(t_2,s))\, h(s)ds\right| + \left|\int_{t_1}^{t_2} h(s)ds\right| \\ &\leq \int_0^1 |H_1(t_1,s) - H_1(t_2,s)|\,|h(s)|\,ds + \left|\int_{t_1}^{t_2} 1ds\right| \|h\| \\ &\leq M\int_0^1 |H_1(t_1,s) - H_1(t_2,s)|\,ds + \frac{\varepsilon}{3M}\cdot M \\ &\leq \frac{2\varepsilon}{3} < \varepsilon.\end{aligned}$$

Thus, this implies that T is a compact operator on E. □

Since the function $G_0(t,s)$ is continuous on $[0,1]\times[0,1]$, and the function $G_1(t,s)$ is bounded on $[0,1]\times[0,1]$, the following constants are well-defined, given the nature of continuous functions and bounded functions on bounded and closed sets

$$l_0 = \max_{(t,s)\in[0,1]\times[0,1]} |G_0(t,s)|,$$

$$l_1 = \sup_{(t,s)\in[0,1]\times[0,1]} |G_1(t,s)|.$$

Lemma 2.4. *Given that $h \in C[0,1]$, the solution $u(t)$ of the boundary value problem for the fractional-order differential Eq (2.2) adheres to the stated properties*

$$|u(t)| \leq l_0 \int_0^1 |h(s)|\,ds,\ t\in[0,1],$$

$$\left|D_{0^+}^{\alpha-1}u(t)\right| \leq l_1 \int_0^1 |h(s)|\,ds,\ t\in[0,1].$$

Proof. The lemma 2.2 states that the solution of the fractional order differential equation $u(t)$ to the problem (2.2) satisfies

$$u(t) = \int_0^1 G_0(t,s)h(s)ds,$$

where the Green's function $G_0(t,s)$ is given by (2.4).

For any $t\in[0,1]$, there exists

$$\begin{aligned}|u(t)| &= \left|\int_0^1 G_0(t,s)h(s)ds\right| \leq \int_0^1 |G_0(t,s)|\,|h(s)|\,ds \\ &\leq l_0\int_0^1 |h(s)|\,ds, \\ \left|D_{0^+}^{\alpha-1}u(t)\right| &= \left|\int_0^1 G_1(t,s)h(s)ds\right| \leq \int_0^1 |G_1(t,s)|\,|h(s)|\,ds \\ &\leq l_1\int_0^1 |h(s)|\,ds.\end{aligned}$$

This proof is complete. □

Lemma 2.5. *If $h \in C[0,1]$, then the solution $u(t)$ of the boundary value problem for the fractional order differential Eq (2.2) satisfies*

$$|u(t)| \le d_0 \|h\|_\infty ,$$
$$\left|D_{0^+}^{\alpha-1} u(t)\right| \le d_1 \|h\|_\infty ,$$

where

$$d_0 = \max_{t\in[0,1]} \int_0^1 |G_0(t,s)|\, ds,$$
$$d_1 = \max_{t\in[0,1]} \int_0^1 |G_1(t,s)|\, ds.$$

Proof. For every $t \in [0,1]$, it holds that

$$\begin{aligned} |u(t)| &= \left|\int_0^1 G_0(t,s)h(s)ds\right| \le \int_0^1 |G_0(t,s)|\, \|h\|_\infty\, ds \\ &\le \max_{t\in[0,1]} \int_0^1 |G_0(t,s)|\, ds \cdot \|h\|_\infty , \\ \left|D_{0^+}^{\alpha-1} u(t)\right| &= \left|\int_0^1 G_1(t,s)h(s)ds\right| \le \int_0^1 |G_1(t,s)|\, \|h\|_\infty\, ds \\ &\le \max_{t\in[0,1]} \int_0^1 |G_1(t,s)|\, ds \cdot \|h\|_\infty . \end{aligned}$$

The evidence presented here is conclusive. □

Given the constants l_0 and l_1, as well as the expressions for d_0 and d_1, it is evident that

$$d_0 < l_0,\ d_1 < l_1.$$

However, the determination of the constants l_0, l_1, d_0 and d_1 depends on the reality that $g(t)$ is a function of bounded variation within the context of Green's function. When $g(t)$ is a complex function, calculating l_0, l_1, d_0 and d_1 becomes more challenging, making it necessary to seek range estimates for these constants.

For each pair $(t,s) \in [0,1]\times[0,1]$, the formulas for $G_0(t,s)$ and $G_1(t,s)$ yield the following results:

$$\begin{aligned} |G_0(t,s)| &\le \frac{1}{\Gamma(\alpha)}\left(|H_0(t,s)| + \left|(t-s)^{\alpha-1}\right|\right) \\ &\le \frac{1}{\Gamma(\alpha)}\left(\left|\frac{(1-s)^{\alpha-1} t^{\alpha-1} - t^{\alpha-1}\int_s^1 (u-s)^{\alpha-1} dg(u)}{\int_0^1 u^{\alpha-1} dg(u) - 1}\right| + \left|(t-s)^{\alpha-1}\right|\right) \\ &\le \frac{1}{\Gamma(\alpha)} \frac{1 + \int_0^1 u^{\alpha-1} d \bigvee_0^u (g)}{\left|\int_0^1 u^{\alpha-1} dg(u) - 1\right|} + \frac{1}{\Gamma(\alpha)}, \end{aligned}$$

$$|G_1(t,s)| \leq |H_1(t,s)| + 1 \leq \left| \frac{(1-s)^{\alpha-1} - \int_s^1 (u-s)^{\alpha-1} dg(u)}{\int_0^1 u^{\alpha-1} dg(u) - 1} \right| + 1$$

$$\leq \frac{1 + \int_0^1 u^{\alpha-1} d \bigvee_0^u (g)}{\left| \int_0^1 u^{\alpha-1} dg(u) - 1 \right|} + 1.$$

where $\bigvee_0^t (g)$ denotes the entire variation of a function g of bounded variation on the interval $[0,t]$. It is defined as the supremum of the sum of the absolute differences for all possible partitions $\{t_0, t_1 \ldots t_n\}$ of the interval $[0,t]$. Mathematically, this can be expressed as:

$$\bigvee_0^t (g) = \sup \sum_{i=1}^n \left| g(t_i) - g_(t_{i-1}) \right|.$$

This quantity, $\bigvee_0^t (g)$, provides an upper bound on the sum of the absolute variations over all possible partitions of the interval $[0,t]$.

Thus, we can obtain

$$d_0 < l_0 \leq k_0, \; d_1 < l_1 \leq k_1, \tag{2.5}$$

where

$$k_0 = \frac{1}{\Gamma(\alpha)} \frac{1 + \int_0^1 u^{\alpha-1} d \bigvee_0^u (g)}{\left| \int_0^1 u^{\alpha-1} dg(u) - 1 \right|} + \frac{1}{\Gamma(\alpha)}, \tag{2.6}$$

$$k_1 = \frac{1 + \int_0^1 u^{\alpha-1} d \bigvee_0^u (g)}{\left| \int_0^1 u^{\alpha-1} dg(u) - 1 \right|} + 1. \tag{2.7}$$

The inequalities $d_0 < l_0 < k_0$ provide an upper bound estimate for d_0(or l_0), and similarly, $d_1 < l_1 < k_1$ gives an upper bound estimate for d_1(or l_1). Theoretically, d_0(or l_0) and d_1(or l_1) exist and depend on the bounded variation function g. However, in practical applications, they may not be easily computable. Therefore, we provide a more convenient upper bound estimate.

3. Main results

In the following, we will carry out a detailed analysis of the existence of the solution.

Theorem 3.1. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Suppose there exists non-negative continuous functions α, β and γ defined on the interval* $[0,1]$ *such that*

$$|f(t,u_1,u_2)| \le \alpha(t)|u_1| + \beta(t)|u_2| + \gamma(t),\ t \in [0,1],\ u_1,u_2 \in \mathbb{R}, \tag{3.1}$$

$$l_0 \int_0^1 \alpha(s)ds + l_1 \int_0^1 \beta(s)ds < 1, \tag{3.2}$$

where $l_0 = \max_{(t,s)\in[0,1]\times[0,1]} |G_0(t,s)|$ *and* $l_1 = \sup_{(t,s)\in[0,1]\times[0,1]} |G_1(t,s)|$. *The boundary value problem associated with the fractional-order differential Eq (1.1) admiting at least one solution.*

Proof. Consider the space $E = \left\{u(t) \mid u \in C[0,1] \text{ and } D_{0^+}^{\alpha-1}u \in C[0,1]\right\}$, we define the following operator $S : E \to E$

$$(Su)(t) = \int_0^1 G_0(t,s) f\left(s, u(s), D_{0^+}^{\alpha-1}u(s)\right) ds.$$

It is straightforward to prove that S is a completely continuous operator by utilizing the continuity of the function f and Lemma 2.3. The solution of the boundary value problem for the fractional order differential Eq (1.1) in E is equivalent to solving the corresponding integral equation

$$Su = u,$$

have a fixed points in E.

Due to $(E, \|\cdot\|)$ bing a Banach space, and T is a linear completely continuous operator on E. If there exists a constant $M > 0$ such that for any $\lambda \in (0,1)$, the equation $y = \lambda Ty$ implies $\|y\| \le M$, then T has a fixed point in E. Consider $\lambda \in [0,1]$ and observe that any solution $u(t)$ to the boundary value problem for the fractional order differential Eq (1.1) in the space E possesses an a priori estimate that is independent of λ. Furthermore, the solution of this equation satisfies the following nonlinear fractional-order differential equation:

$$\begin{cases} D_{0+}^{\alpha}u(t) = \lambda f\left(t, u(t), D_{0^+}^{\alpha-1}u(t)\right), & t \in (0,1),\ 1 < \alpha \le 2, \\ u(1) = \int_0^1 u(s)dg(s), & u(0) = 0. \end{cases} \tag{3.3}$$

We shall demonstrate that all possible solutions $u(t)$ of Eq (3.3) in E satisfy a priori estimates that are independent of λ. To establish this, we will apply Lemma 2.4 and we examine

$$\begin{aligned}
&\int_0^1 \left|D_{0^+}^{\alpha} u(s)\right| ds \\
&= \lambda \int_0^1 \left|f\left(s, u(s), D_{0^+}^{\alpha-1} u(s)\right)\right| ds \\
&\leq \int_0^1 \left(\alpha(s)\left|u(s)\right| + \beta(s)\left|D_{0^+}^{\alpha-1} u(s)\right| + \gamma(s)\right) ds \\
&= \int_0^1 \alpha(s)\left|u(s)\right| ds + \int_0^1 \beta(s)\left|D_{0^+}^{\alpha-1} u(s)\right| ds + \int_0^1 \gamma(s) ds \\
&\leq l_0 \int_0^1 \alpha(s) ds \cdot \int_0^1 \left|D_{0^+}^{\alpha} u(s)\right| ds + l_1 \int_0^1 \beta(s) ds \cdot \int_0^1 \left|D_{0^+}^{\alpha} u(s)\right| ds + \int_0^1 \gamma(s) ds.
\end{aligned}$$

Thus, we have

$$\int_0^1 \left|D_{0^+}^{\alpha} u(s)\right| ds \leq \sigma,$$

where

$$\sigma = \frac{\int_0^1 \gamma(s) ds}{1 - l_0 \int_0^1 \alpha(s) ds - l_1 \int_0^1 \beta(s) ds}.$$

Using Lemma 2.3 in conjunction with Eq (3.3), it can be deduced that

$$\begin{aligned}
\left|u(t)\right| &\leq l_0 \int_0^1 \left|\lambda f\left(s, u(s), D_{0^+}^{\alpha-1} u(s)\right)\right| ds \\
&= l_0 \int_0^1 \left|D_{0^+}^{\alpha} u(s)\right| ds \leq l_0 \sigma,
\end{aligned}$$

$$\begin{aligned}
\left|D_{0^+}^{\alpha-1} u(t)\right| &\leq l_1 \int_0^1 \left|\lambda f\left(s, u(s), D_{0^+}^{\alpha-1} u(s)\right)\right| ds \\
&= l_1 \int_0^1 \left|D_{0^+}^{\alpha} u(s)\right| ds \leq l_1 \sigma.
\end{aligned}$$

From the equation estimated above, we demonstrate that all possible solutions $u(t)$ of Eq (3.3) in E have a priori estimates that are independent of λ, such that

$$\|u\| = \max_{t\in[0,1]} \left|u(t)\right| + \max_{t\in[0,1]} \left|D_{0^+}^{\alpha-1} u(t)\right| \leq (l_0 + l_1)\,\sigma.$$

Therefore, according to Theorem 2.1, a fixed point of S exists, which leads to the conclusion that the nonlinear boundary value problem for the fractional-order differential Eq (1.1) admits at least one solution. □

Corollary 3.1. *Suppose that $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ is continuous. Assume that there are non-negative continuous functions α, β, and γ defined on the interval $[0,1]$ such that*

$$|f(t,u_1,u_2)| \le \alpha(t)|u_1| + \beta(t)|u_2| + \gamma(t),\ t\in[0,1],\ u_1,u_2\in\mathbb{R},$$

$$k_0\int_0^1 \alpha(s)ds + k_1\int_0^1 \beta(s)ds < 1,$$

where k_0 and k_1 are given by (2.6) and (2.7). The boundary value problem associated with the fractional-order differential Eq (1.1) admits at least one solution.

Corollary 3.2. *Assume that $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ is continuous. Suppose there exists non-negative continuous function γ defined on the interval $[0,1]$ such that*

$$|f(t,u_1,u_2)| \le \gamma(t),\ t\in[0,1],\ u_1,u_2\in\mathbb{R}.$$

The boundary value problem associated with the fractional-order differential Eq (1.1) admits at least one solution.

Theorem 3.2. *Suppose that $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ is continuous. Assume that there are non-negative continuous functions α, β, and γ defined on the interval $[0,1]$ such that*

$$|f(t,u_1,u_2)| \le \alpha(t)|u_1| + \beta(t)|u_2| + \gamma(t),\ t\in[0,1],\ u_1,u_2\in\mathbb{R},$$

$$\begin{aligned}\max\Big\{&\max_{t\in[0,1]}\int_0^1 \alpha(s)\left(|G_0(t,s)|+|G_1(t,s)|\right)ds,\\ &\max_{t\in[0,1]}\int_0^1 \beta(s)\left(|G_0(t,s)|+|G_1(t,s)|\right)ds\Big\} < 1.\end{aligned} \tag{3.4}$$

The boundary value problem associated with the fractional-order differential Eq (1.1) admits at least one solution.

Proof. According to the proof of Theorem 3.1, it suffices to demonstrate the existence of fixed points of S. Consequently, for $\lambda\in[0,1]$, any solutions $u(t)$ must satisfy this equality

$$u = \lambda S u.$$

If condition (3.4) is satisfied, then the following statement holds:

$$\begin{aligned}|u(t)| &= |\lambda(Su)(t)| = \left|\lambda\int_0^1 G_0(t,s)f\left(s,u(s),D_{0^+}^{\alpha-1}u(s)\right)ds\right|\\ &\le \int_0^1 |G_0(t,s)|\left|f\left(s,u(s),D_{0^+}^{\alpha-1}u(s)\right)\right|ds\\ &\le \int_0^1 \alpha(s)|G_0(t,s)||u(s)|ds + \int_0^1 \beta(s)|G_0(t,s)|\left|D_{0^+}^{\alpha-1}u(s)\right|ds + \int_0^1 \gamma(s)|G_0(t,s)|ds\\ &\le \int_0^1 \alpha(s)|G_0(t,s)|ds\cdot\|u\|_\infty + \int_0^1 \beta(s)|G_0(t,s)|ds\cdot\left\|D_{0^+}^{\alpha-1}u\right\|_\infty + \int_0^1 \gamma(s)|G_0(t,s)|ds,\end{aligned}$$

$$
\begin{aligned}
&\left|D_{0^+}^{\alpha-1}u(t)\right| \\
&= \left|\lambda D_{0^+}^{\alpha-1}(Su)(t)\right| = \left|\lambda \int_0^1 G_1(t,s) f\left(s,u(s),D_{0^+}^{\alpha-1}u(s)\right) ds\right| \\
&\le \int_0^1 |G_1(t,s)| \left|f\left(s,u(s),D_{0^+}^{\alpha-1}u(s)\right)\right| ds \\
&\le \int_0^1 \alpha(s)\,|G_1(t,s)|\,|u(s)|\,ds + \int_0^1 \beta(s)\,|G_1(t,s)| \left|D_{0^+}^{\alpha-1}u(s)\right| ds + \int_0^1 \gamma(s)\,|G_1(t,s)|\,ds \\
&\le \int_0^1 \alpha(s)\,|G_1(t,s)|\,ds \cdot \|u\|_\infty + \int_0^1 \beta(s)\,|G_1(t,s)|\,ds \cdot \left\|D_{0^+}^{\alpha-1}u\right\|_\infty + \int_0^1 \gamma(s)\,|G_1(t,s)|\,ds.
\end{aligned}
$$

According to the definition of the space of

$$
E = \left\{u(t) \mid u(t) \in C[0,1] \text{ and } D_{0^+}^{\alpha-1}u(t) \in C[0,1]\right\},
$$

it follows that

$$
\begin{aligned}
\|u\| &= \|u\|_\infty + \left\|D_{0^+}^{\alpha-1}u\right\|_\infty \\
&\le \max_{t\in[0,1]} \int_0^1 \alpha(s)\left(|G_0(t,s)| + |G_1(t,s)|\right) ds \cdot \|u\|_\infty \\
&\quad + \max_{t\in[0,1]} \int_0^1 \beta(s)\left(|G_0(t,s)| + |G_1(t,s)|\right) ds \cdot \left\|D_{0^+}^{\alpha-1}u\right\|_\infty \\
&\quad + \max_{t\in[0,1]} \int_0^1 \gamma(s)\left(|G_0(t,s)| + |G_1(t,s)|\right) ds \\
&\le \max\left\{\max_{t\in[0,1]} \int_0^1 \alpha(s)\left(|G_0(t,s)| + |G_1(t,s)|\right) ds,\right. \\
&\qquad \left.\max_{t\in[0,1]} \int_0^1 \beta(s)\left(|G_0(t,s)| + |G_1(t,s)|\right) ds\right\} \|u\|_\infty \\
&\quad + \max\left\{\max_{t\in[0,1]} \int_0^1 \alpha(s)\left(|G_0(t,s)| + |G_1(t,s)|\right) ds,\right. \\
&\qquad \left.\max_{t\in[0,1]} \int_0^1 \beta(s)\left(|G_0(t,s)| + |G_1(t,s)|\right) ds\right\} \left\|D_{0^+}^{\alpha-1}u\right\|_\infty \\
&\quad + \max_{t\in[0,1]} \int_0^1 \gamma(s)\left(|G_0(t,s)| + |G_1(t,s)|\right) ds \\
&= \max\left\{\max_{t\in[0,1]} \int_0^1 \alpha(s)\left(|G_0(t,s)| + |G_1(t,s)|\right) ds,\right. \\
&\qquad \left.\max_{t\in[0,1]} \int_0^1 \beta(s)\left(|G_0(t,s)| + |G_1(t,s)|\right) ds\right\} \|u\| \\
&\quad + \max_{t\in[0,1]} \int_0^1 \gamma(s)\left(|G_0(t,s)| + |G_1(t,s)|\right) ds.
\end{aligned}
$$

By condition (3.4), the Eq (1.1) admits an a priori estimate in E for all possible solutions $u(t)$ that are independent of λ. Consequently, by Theorem 2.1, there exists a fixed point for S, which implies that the boundary value problem associated with the fractional-order differential Eq (1.1) admits at least one solution. □

Corollary 3.3. *Suppose that* $f : [0, 1]\times\mathbb{R}^2 \to \mathbb{R}$ *is continuous. Assume that there exists a non-negative continuous function* γ *defined on the interval* $[0, 1]$, *and two positive constants* α_0 *and* β_0 *are given such that*

$$|f(t, u_1, u_2)| \le \alpha_0 |u_1| + \beta_0 |u_2| + \gamma(t),\ t \in [0, 1],\ u_1, u_2 \in \mathbb{R}, \tag{3.5}$$

and

$$l_0\alpha_0 + l_1\beta_0 < 1, \tag{3.6}$$

or

$$(d_0 + d_1) \max\{\alpha_0, \beta_0\} < 1, \tag{3.7}$$

where $l_0 = \max_{(t,s)\in[0,1]\times[0,1]} |G_0(t, s)|$, $l_1 = \sup_{(t,s)\in[0,1]\times[0,1]} |G_1(t, s)|$, $d_0 = \max_{t\in[0,1]} \int_0^1 |G_0(t, s)|\, ds$, *and* $d_1 = \max_{t\in[0,1]} \int_0^1 |G_1(t, s)|\, ds$. *Given these conditions, the boundary value problem associated with the fractional-order differential Eq (1.1) has at least one solution.*

In the subsequent analysis, we focus on examining the uniqueness of solutions through a detailed study of relevant theories and the integration of existing theorems and lemmas, providing a comprehensive discussion of the conditions ensuring uniqueness.

Theorem 3.3. *Suppose that* $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous. Assume that there are non-negative continuous functions* α, β, *and* γ *defined on the interval* $[0, 1]$ *such that*

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \le \alpha(t) |u_1 - v_1| + \beta(t) |u_2 - v_2|, \tag{3.8}$$

for all $t \in [0, 1]$ *and* $u_1, u_2, v_1, v_2 \in \mathbb{R}$. *Under condition (3.4), the boundary value problem associated with the fractional-order differential Eq (1.1) admits at least one solution.*

Proof. In the subsequent discussion, we prove that the operator S is a contraction mapping on E.

Let $u, v \in E$, with

$$\begin{aligned}
&\|Su - Sv\|_\infty \\
&= \max_{t\in[0,1]} |(Su)(t) - (Sv)(t)| \\
&= \max_{t\in[0,1]} \left| \int_0^1 G_0(t,s) \left(f\left(s, u(s), D_{0^+}^{\alpha-1}u(s)\right) - f\left(s, v(s), D_{0^+}^{\alpha-1}v(s)\right)\right) ds \right| \\
&\le \max_{t\in[0,1]} \int_0^1 |G_0(t,s)| \left(\alpha(s) |u(s) - v(s)| + \beta(s) \left| D_{0^+}^{\alpha-1} (u(s) - v(s))\right| \right) ds \\
&\le \max_{t\in[0,1]} \int_0^1 |G_0(t,s)| \left(\alpha(s) \|u - v\|_\infty + \beta(s) \left\| D_{0^+}^{\alpha-1}u - D_{0^+}^{\alpha-1}v \right\|_\infty \right) ds \\
&\le \max_{t\in[0,1]} \int_0^1 |G_0(t,s)| \, \alpha(s) ds \cdot \|u - v\|_\infty + \max_{t\in[0,1]} \int_0^1 |G_0(t,s)| \beta(s) ds \cdot \left\| D_{0^+}^{\alpha-1}u - D_{0^+}^{\alpha-1}v \right\|_\infty,
\end{aligned}$$

$$\begin{aligned}
&\left\| D_{0^+}^{\alpha-1}(Su) - D_{0^+}^{\alpha-1}(Sv) \right\|_\infty \\
&= \max_{t\in[0,1]} \left| D_{0^+}^{\alpha-1}(Su)(t) - D_{0^+}^{\alpha-1}(Sv)(t) \right| \\
&= \max_{t\in[0,1]} \left| \int_0^1 G_1(t,s) \left(f\left(s, u(s), D_{0^+}^{\alpha-1}u(s)\right) - f\left(s, v(s), D_{0^+}^{\alpha-1}v(s)\right)\right) ds \right| \\
&\le \max_{t\in[0,1]} \int_0^1 |G_1(t,s)| \left(\alpha(s) |u(s) - v(s)| + \beta(s) \left| D_{0^+}^{\alpha-1} (u(s) - v(s))\right| \right) ds \\
&\le \max_{t\in[0,1]} \int_0^1 |G_1(t,s)| \left(\alpha(s) \|u - v\|_\infty + \beta(s) \left\| D_{0^+}^{\alpha-1}u - D_{0^+}^{\alpha-1}v \right\|_\infty \right) ds \\
&\le \max_{t\in[0,1]} \int_0^1 |G_1(t,s)| \beta(s) ds \cdot \left\| D_{0^+}^{\alpha-1}u - D_{0^+}^{\alpha-1}v \right\|_\infty + \max_{t\in[0,1]} \int_0^1 |G_1(t,s)| \, \alpha(s) ds \cdot \|u - v\|_\infty,
\end{aligned}$$

we successfully obtained

$$\begin{aligned}
\|Su - Sv\| &= \|Su - Sv\|_\infty + \left\| D_{0^+}^{\alpha-1} (Su) - D_{0^+}^{\alpha-1} (Sv) \right\|_\infty \\
&\le \max_{t\in[0,1]} \int_0^1 \alpha(s) |G_0(t,s)| \, ds \cdot \|u - v\|_\infty \\
&\quad + \max_{t\in[0,1]} \int_0^1 \beta(s) |G_0(t,s)| \, ds \cdot \left\| D_{0^+}^{\alpha-1}u - D_{0^+}^{\alpha-1}v \right\|_\infty \\
&\quad + \max_{t\in[0,1]} \int_0^1 \alpha(s) |G_1(t,s)| \, ds \cdot \|u - v\|_\infty \\
&\quad + \max_{t\in[0,1]} \int_0^1 \beta(s) |G_1(t,s)| \, ds \cdot \left\| D_{0^+}^{\alpha-1}u - D_{0^+}^{\alpha-1}v \right\|_\infty
\end{aligned}$$

$$\leq \max\left\{\max_{t\in[0,1]}\int_0^1 \alpha(s)\left(|G_0(t,s)|+|G_1(t,s)|\right)ds,\right.$$
$$\left.\max_{t\in[0,1]}\int_0^1 \beta(s)\left(|G_0(t,s)|+|G_1(t,s)|\right)ds\right\}\|u-v\|.$$

The fulfillment of condition (3.4) ensures that S is a contraction operator on E. As a result, S has a unique fixed point. □

Following the approach used in the proof of Theorem 3.1, the uniqueness of the solution can also be established under the condition (3.8).

Theorem 3.4. *Let $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ be a continuous function. Suppose there exist non-negative continuous functions α and β defined on the interval $[0,1]$ such that condition (3.2) holds. In that case, the boundary value problem (1.1) for the nonlinear fractional-order differential equation has a unique solution.*

For any given positive number M, define the closed region in $[0,1]\times\mathbb{R}^2$

$$D_M=\{(t,u,v): t\in[0,1], |u|\leq d_0M, |v|\leq d_1M\},$$

where d_0, d_1 are given by Lemma 2.5.

For any $u\in E=\left\{u(t)\mid u(t)\in C[0,1] \text{ and } D_{0^+}^{\alpha-1}u(t)\in C[0,1]\right\}$, define

$$\|u\|_1=\max_{t\in[0,1]}\frac{\|u\|_\infty}{d_0}+\max_{t\in[0,1]}\frac{\left\|D_{0^+}^{\alpha-1}u\right\|_\infty}{d_1}.$$

Since the constants d_0 and d_1 are positive, the norm $\|u\|_1$ is well-defined. Consequently, the two norms $\|u\|_1$ and $\|u\|$ are equivalent. Let

$$B_M=\{u\in E:\|u\|_1\leq M\},$$

thus, B_M is a convex and closed set in $(E,\|u\|_1)$.

Theorem 3.5. *Let $M>0$ be a constant, and suppose the function $f(t,u,v)$ is continuous on the domain D_M and satisfies*

$$|f(t,u,v)|<M,$$

for all $(t,u,v)\in D_M$. If the condition (3.8) in Theorem 3.3 and the following condition

$$l_0\int_0^1\alpha(s)ds+l_1\int_0^1\beta(s)ds<1,$$

are satisfied. Thus, the fractional-order differential Eq (1.1) possesses exactly one solution.

Proof. Let the operator S be defined as

$$(Su)(t)=\int_0^1 G_0(t,s)f\left(s,u(s),D_{0^+}^{\alpha-1}u(s)\right)ds.$$

We prove below that $S(B_M) \subset B_M$. For any $u \in B_M$, examine

$$|u(t)| \le \|u\|_\infty \le d_0 \|u\|_1 \le d_0 M, \ \forall\, t \in [0,1],$$

$$\left|D_{0^+}^{\alpha-1} u(t)\right| \le \left\|D_{0^+}^{\alpha-1} u\right\|_\infty \le d_1 \|u\|_1 \le d_1 M, \ \forall\, t \in [0,1].$$

According to the definition of D_M, for any $u \in B_M$, the function $f\left(t, u(t), D_{0^+}^{\alpha-1} u(t)\right)$ is continuous on $[0,1]$ and satisfies

$$\left|f\left(t, u(t), D_{0^+}^{\alpha-1} u(t)\right)\right| \le M, \ \forall\, t \in [0,1].$$

By employing the proof technique used in Lemma 2.5, it can be deduced that

$$|(Su)(t)| \le d_0 \|h\|_\infty \le d_0 M, \ \forall\, t \in [0,1],$$

$$\left|D_{0^+}^{\alpha-1} (Su)(t)\right| \le d_1 \|h\|_\infty \le d_1 M, \ \forall\, t \in [0,1],$$

where $h(t) = f\left(t, u(t), D_{0^+}^{\alpha-1} u(t)\right)$, thus

$$\|Su\|_\infty \le d_0 M \ and \ \left\|D_{0^+}^{\alpha-1} (Su)\right\|_\infty \le d_1 M,$$

where

$$\|Su\|_1 = \max_{t \in [0,1]} \frac{\|Su\|_\infty}{d_0} + \max_{t \in [0,1]} \frac{\left\|D_{0^+}^{\alpha-1} (Su)\right\|_\infty}{d_1} \le M.$$

Hence, this establishes that $S(B_M) \subset B_M$.

Analogously to the proof of Lemma 2.3, it can be easily demonstrated that $S : B_M \to B_M$ is a completely continuous operator. By Theorem 2.3, it follows that the operator S has at least one fixed point.

The following proof establishes that S has precisely one fixed point in B_M.

Let u_1, u_2 be fixed points of the operator S in B_M. Define $w = u_1 - u_2$, then the function w satisfies

$$\begin{cases} D_{0+}^{\alpha} w(t) = h(t), \ \ t \in (0,1), \ 1 < \alpha \le 2, \\ w(0) = 0, \ \ w(1) = \int_0^1 w(s)\,dg(s), \end{cases} \tag{3.9}$$

where $h(t) = f\left(t, u_1(t), D_{0^+}^{\alpha-1} u_1(t)\right) - f\left(t, u_2(t), D_{0^+}^{\alpha-1} u_2(t)\right)$.

Thus based on Lemma 2.4, we obtain that

$$|w(t)| \le l_0 \int_0^1 |h(s)|\, ds, \ t \in [0,1],$$

$$\left|D_{0^+}^{\alpha-1} w(t)\right| \le l_1 \int_0^1 |h(s)|\, ds, \ t \in [0,1].$$

Consequently, from Eq (3.9), we obtain that

$$\begin{aligned}
&\int_0^1 \left|D_{0^+}^{\alpha} w(s)\right| ds \\
\leq &\int_0^1 \left|f\left(s, u_1(s), D_{0^+}^{\alpha-1}u_1(s)\right) - f\left(s, u_2(s), D_{0^+}^{\alpha-1}u_2(s)\right)\right| ds \\
\leq &\int_0^1 \left(\alpha(s)\left|u_1(s) - u_2(s)\right| + \beta(s)\left|D_{0^+}^{\alpha-1}u_1(s) - D_{0^+}^{\alpha-1}u_2(s)\right|\right) ds \\
= &\int_0^1 \left(\alpha(s)\left|w(s)\right| + \beta(s)\left|D_{0^+}^{\alpha-1}w(s)\right|\right) ds \\
\leq &\, l_0 \int_0^1 \left|D_{0^+}^{\alpha} w(s)\right| ds \cdot \int_0^1 \alpha(s)ds + l_1 \int_0^1 \left|D_{0^+}^{\alpha} w(s)\right| ds \cdot \int_0^1 \beta(s)ds \\
= &\int_0^1 \left|D_{0^+}^{\alpha} w(s)\right| \left(l_0 \int_0^1 \alpha(s)ds + l_1 \int_0^1 \beta(s)ds\right) ds.
\end{aligned}$$

Note that the condition (3.2) implies $\int_0^1 \left|D_{0^+}^{\alpha} w(s)\right| ds = \int_0^1 h(s)ds = 0$, and therefore

$$h(t) \equiv 0, \ \forall\, t \in [0, 1].$$

Consider the fractional order differential equation given by $w(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + I_{0^+}^{\alpha} h(t),\ c_1, c_2 \in \mathbb{R}$. Applying the condition $w(0) = 0$, it follows that $c_2 = 0$. Subsequently, using boundary condition $w(1) = \int_0^1 w(s)dg(s)$, we obtain $c_1 = c_1 \int_0^1 t^{\alpha-1}dg(t)$. Given the assumption that $\int_0^1 w(t)dg(t) \neq 1$, we conclude that $c_1 = 0$. Consequently, it follows that $w(t) = 0$. Thus, if u_1 and u_2 are fixed points of the operator S in B_M, it must follow that $u_1 = u_2$. In summary, there is precisely one fixed point associated with this operator S in B_M. □

4. Application

Now, we provide a concrete example to demonstrate the validity of the results obtained.

Example 4.1. *The problem of determining boundary values for a nonlinear differential equation involving fractional-order terms is considered:*

$$\begin{cases} D_{0^+}^{\frac{3}{2}} u(t) = c(t) + a(t)\dfrac{u(t)}{1 + \left(D_{0^+}^{\frac{1}{2}} u(t)\right)^2} + b(t) D_{0^+}^{\frac{1}{2}} u(t) \sin u(t), \quad t \in [0, 1], \\ u(1) = \displaystyle\int_0^1 u(s)ds, \quad u(0) = 0, \end{cases} \tag{4.1}$$

where $a, b, c \in C[0, 1]$, $\alpha = \frac{3}{2}$ and $g(s) = s$. After performing a simple calculation, it is determined that $\int_0^1 u^{\alpha-1}dg = \dfrac{2}{3} \neq 1$. Furthermore, we have obtained the following values: $l_0 = \frac{\sqrt{101\sqrt{7}-179}}{9}$, $l_1 = \frac{3\sqrt{3}}{4}$, $d_0 = \frac{4\sqrt{15}}{25}$, $d_1 = \frac{6}{5}$.

(i) The function f is defined such that

$$f = \frac{t}{2} + \frac{7}{10}\frac{u(t)}{1+\left(D_{0^+}^{\frac{1}{2}}u(t)\right)^2} + \frac{1}{10}D_{0^+}^{\frac{1}{2}}u(t)\sin u(t),$$

where f satisfies the inequality (3.5) with $\alpha_0 = \frac{7}{10}$ and $\beta_0 = \frac{1}{10}$. After performing the calculations, we obtain condition $l_0\alpha_0 + l_1\beta_0 \approx 0.8604 < 1$, and we also obtain condition (3.7) such that $(d_0 + d_1)\max\{\alpha_0, \beta_0\} \approx 1.27 > 1$. Therefore, it can be concluded that condition (3.6) is satisfied, while condition (3.7) is not. Thus, the boundary value problem associated with the fractional-order differential Eq (4.1) admits at least one solution.

(ii) Alternatively, consider

$$f = \frac{t}{2} + \frac{27}{50}\frac{u(t)}{1+\left(D_{0^+}^{\frac{1}{2}}u(t)\right)^2} + \frac{13}{25}D_{0^+}^{\frac{1}{2}}u(t)\sin u(t),$$

it is satisfies the inequality(3.5) with $\alpha_0 = \frac{27}{50}$ and $\beta_0 = \frac{13}{25}$. After performing the calculations, we obtain important condition $l_0\alpha_0 + l_1\beta_0 \approx 1.2390 > 1$, and we also obtain at important condition such that $(d_0 + d_1)\max\{\alpha_0, \beta_0\} \approx 0.9826 < 1$. Therefore, it can be concluded that condition (3.7) is satisfied, while condition (3.6) is not. Thus, the boundary value problem associated with the fractional-order differential Eq (4.1) admits at least one solution.

5. Conclusions

In this paper, we explore boundary value problems for nonlinear fractional-order differential equations through the application of the Schauder fixed point theorem, the Leray-Schauder continuation theorem, and the Banach fixed point theorem. Prior to addressing the main content, we investigate the relationships between three sets of constants: l_i, d_i, and k_i (where $i \in 0, 1$). These constants are then employed in the core results to demonstrate the existence and uniqueness of solutions for the boundary value problems associated with fractional-order differential equations.

Author contributions

Yujun Cui: Methodology, Writing-review and editing; Chunyu Liang: Methodology, Writing-original draft; Yumei Zou: Supervision, Methodology, Writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The research is supported by NSF of China, No: 12371173,11801322, by the Shandong Provincial Natural Science Foundation, No: ZR2018MA011.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. J. M. Gallardo, Second order differential operators with integral boundary conditions and generation of semigroups, *Rocky Mountain J. Math.,* **30** (2000), 1265–1292. http://doi.org/10.1216/rmjm/1021477351
2. G. L. Karakostas, P. Ch. Tsamatos, Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary-value problems, *Electron. J. Differential Equation,* **30** (2002), 1–17. http://doi.org/10.1023/A:1013391904853
3. A. Lomtatidze, L. Malaguti, On a nonlocal boundary-value problems for second order nonlinear singular differential equations, *Georgian Math.,* **7** (2000), 133–154. http://doi.org/10.1515/GMJ.2000.133
4. C. Corduneanu, *Integral Equations and Applications*, Cambridge: Cambridge University Press, 1991. http://doi.org/10.1017/CBO9780511569395
5. H. O. Fattorini, *Second order linear differential equations in Banach spaces*, North-Holland, 1985. http://doi.org/10.2140/pjm.1990.142.175
6. A. A. Kilbas, J. J. Trujillo, Differential equations of fractional order: Methods, results and problems—I, *Appl. Anal.,* **78** (2001), 153–192. http://doi.org/10.1080/00036810108840931
7. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006. http://doi.org/10.1016/S0304-0208(06)80001-0
8. I. Bachar, H. Maagli, H. Eltayeb, Existence and iterative method for some Riemann fractional nonlinear boundary value problems, *Mathematics*, **7** (2019), 961. http://doi.org/10.3390/math7100961
9. Z. Bai, H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.,* **311** (2005), 495–505. https://doi.org/10.1016/j.jmaa.2005.02.052
10. X. Xu, D. Jiang, C. Yuan, Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation, *Nonlinear Anal. Theory Methods Appl.,* **71** (2009), 4676–4688. http://doi.org/10.1016/j.na.2009.03.030
11. N. Kosmatov, A singular boundary value problem for nonlinear differential equations of fractional order, *J. Appl. Math. Comput.,* **29** (2009), 125–135. http://doi.org/10.1007/s12190-008-0104-x
12. X. Su, Boundary value problem for a coupled system of nonlinear fractional differential equations, *Appl. Math. Lett.,* **22** (2009), 64–69. http://doi.org/10.1016/j.aml.2008.03.001
13. C. Zhai, M. Hao, Fixed point theorems for mixed monotone operators with perturbation and applications to fractional differential equation boundary value problems, *Nonlinear Anal. Theory Methods Appl.,* **75** (2012), 2542–2551. http://doi.org/10.1016/j.na.2011.10.048
14. Y. Cui, Uniqueness of solution for boundary value problems for fractional differential equations, *Appl. Math. Lett.,* **51** (2016), 48–54. http://doi.org/10.1016/j.aml.2015.07.002

15. I. Bachar, H. Maagli, H. Eltayeb, Existence and uniqueness of solutions for a class of fractional nonlinear boundary value problems under mild assumptions, *Adv. Differ. Equ.,* **2021** (2021), 22. http://doi.org/10.1186/s13662-020-03176-w

16. S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional differential equations, *Electron. J. Differ. Equations.,* **2006** (2006), 1–12.

17. Y. Wang, Necessary conditions for the existence of positive solutions to fractional boundary value problems at resonance, *Appl. Math. Lett.,* **97** (2019), 34–40. http://doi.org/10.1016/j.aml.2019.05.007

18. J. Mawhin, Leray-Schauder continuation theorems in the absence of a priori bounds, *Topol. Methods Nonlinear Anal.,* **4** (1997), 179–200. http://doi.org/10.12775/TMNA.1997.008

19. A. Granas, J. Dugundji, *Fixed point theory*, New York: Springer. http://doi.org/10.1007/978-0-387-21593-8