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Research article

Existence of multiple solutions for a singular $p(\cdot)$-biharmonic problem with variable exponents

Ramzi Alsaedi*

Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

* **Correspondence:** Email: ralsaedi@kau.edu.sa; ramzialsaedi@yahoo.co.uk.

Abstract: In this work, we study a multiplicity result related to a $p(\tau)$-Biharmonic equation involving singular and Hardy nonlinearities. More precisely, we use the variational method to develop the analysis of the fibering map in the Nehari manifold sets to prove the existence of two nontrivial solutions for such a problem.

Keywords: $p(\tau)$-biharmonic problem; variational method; singular equation; variable exponents
Mathematics Subject Classification: 35G30, 35J60, 35J20, 35J35

1. Introduction

The biharmonic problem with variable exponents usually arises in the context of partial differential equations where the biharmonic operator is applied to a function, but the coefficients or exponents of the problem are not constant. This type of problem has important applications in several fields, such as mathematical physics, engineering, and differential geometry, especially in modeling phenomena in which the underlying material or medium has non-uniform properties. In particular, the biharmonic equation describes the deflection of thin plates subjected to forces [1]. Su et al. [2] established the solution of the thin film epitaxy equation, and the biharmonic equation can include variable exponents to account for this heterogeneity. This is particularly useful for modeling composite materials where the material properties change gradually with position. For example, a composite plate may have different stiffnesses in different regions, and the biharmonic problem with variable exponents can model the deformation of the plate under external loads. Also, the biharmonic equation appears in the study of flow around objects [3–5]. When the viscosity of the fluid varies in space, the biharmonic operator with variable exponents can be used to model the diffusion of momentum in such media. This would allow a more realistic description of flow around objects with variable surface characteristics. The biharmonic equation was used for smoothing, denoising, and segmentation tasks [1]. When applied to images with nonuniform texture or intensity, a biharmonic operator with variable exponents can better model local image properties. For example, areas of an image with low contrast may have different smoothing behavior than areas with high contrast, leading to a variable exponent formulation in partial differential equations. A biharmonic operator with variable exponents can also be applied in the analysis of electromagnetic wave propagation in media with varying permeability, which is particularly useful in the design of optical and microwave devices with spatially graded materials.

In this paper, we fix a bounded domain $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ and consider the following system

$$P_{\lambda,\mu} \begin{cases} \Delta^2_{p(\tau)}\varphi = \lambda \ \frac{|\varphi|^{p(\tau)-2}\varphi}{\delta(\tau)^{2p(\tau)}} + b(\tau)|\varphi|^{r(\tau)-2}\varphi + \mu \ \ a(\tau)\varphi^{-\theta(\tau)} & \text{in } \Omega, \\ \varphi = 0, \quad \text{on } \partial\Omega, \end{cases}$$

where $\delta(\tau)$ is the distance between τ and the boundary of Ω. The operator $\Delta^2_{p(.)}$ is defined by:

$$\Delta^2_{p(\tau)}\varphi = \Delta(|\Delta\varphi|^{p(\tau)-2}\Delta\varphi).$$

The functions a, b, and θ are non-negative and satisfy suitable hypotheses that are fixed later, and the functions p and r are continuous on $\overline{\Omega}$, such that for any $\tau \in \overline{\Omega}$, we have

$$1 < p(\tau) < r(\tau) < p^*(\tau).$$

For a given function Z, and for any τ in Ω, we denote

$$Z^- = \inf_{x\in\overline{\Omega}} Z(x), Z^+ = \sup_{x\in\overline{\Omega}} Z(x), \text{and } Z^*(\tau) = \begin{cases} \frac{NZ(\tau)}{N-2Z(\tau)}, & if\ z(\tau) < N, \\ \infty, & if\ Z(\tau) \geq N. \end{cases} \tag{1.1}$$

Hereafter, we assume that

$$1 < p^- \leq p^+ < \frac{N}{2}, \tag{1.2}$$

and

$$0 < \lambda < C_H := \min(\frac{p^-}{p^+}\gamma_-^{p^-}, \frac{p^-}{p^+}\gamma_+^{p^+}),$$

where

$$\gamma_\pm = \frac{N(p^\pm - 1)(N - 2p^\pm)}{(p^\pm)^2}.$$

We recall that for each $\varphi \in W_0^{2,p(\tau)}(\Omega)$, we have

$$\int_\Omega \frac{|\Delta\varphi(\tau)|^{p(\tau)}}{p(\tau)} d\tau \geq C_H \int_\Omega \frac{|\varphi(\tau)|^{p(\tau)}}{p(\tau)\delta(\tau)^{2p(\tau)}} d\tau. \tag{1.3}$$

The last inequality was introduced in the reference [6] for the case $N = 1$, and in the reference [7] for the case $N \geq 2$.

Recently, problems related to biharmonic operators have been extensively studied by several authors, and by using several methods, we cite for instance the papers of Su and Chen [8], who used a combination of the bipolar Rellich inequality with Gigliardo-Nirenberg inequality and Ekeland variational principle; Su and Shi [9], who used a combination of the non-vanishing and the structure of a Palais-Smale sequence to prove ground state solutions to prove the existence of nontrivial solutions; Su and Feng [10], who used the generalized versions of Lions-type theorem to prove ground state solutions; Alsedi [11], who used a fountain theorem to prove the existence of infinitely many solutions; Dhifli and Alsedi [12], who used the Nehari manifold to prove a multiplicity result; Ji and Wang [13], who used the Nehari manifold to prove the existence of two nontrivial solutions.

Very recently, singular biharmonic equations have attracted the attention of several authors, we cite for example the works of Ghanmi and Sahbani [14], who used the mountain pass and the symmetric mountain pass theorems to prove their results; Alsaedi et al. [15], who used the Nehari manifold to prove some multiplicity results; Drissi et al. [16], who used the Nehari manifold to study biharmonic problems with Hardy nonlinearities, and Rădulescu and Repovš [17], who used variational and topological arguments to study singular biharmonic problem including the capillarity equation and the mean curvature problem. In particular, Ghanmi and Sahbani [14] considered the following singular problem:

$$\begin{cases} \Delta^2_{p(\tau)}\varphi = \lambda \ \frac{|\varphi|^{p(\tau)-2}\varphi}{\delta(\tau)^{2p(\tau)}} + b(\tau)|\varphi|^{r(\tau)-2}\varphi + a(\tau)\varphi^{-\theta(\tau)} \quad \text{in } \Omega, \\ \\ \varphi = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where a, b, and θ are three non-negative functions, $\theta \in (0,1)$. Using the condition $1 < r(\tau) < p(\tau)$, for all $\tau \in \overline{\Omega}$, the authors proved that the functional associated with the problem (1.4), is coercive and bounded below in its domain, and based on the min-max theorem, they proved that, for all $\lambda \in (0, C_H)$, the problem (1.4) admits a nontrivial weak solution. For interested readers, we cite [18–20] for other interesting related works.

Motivated by the above-mentioned results, our aim in this paper is to investigate a more general problem. More precisely, problem $P_{\lambda,\mu}$ contains two types of singularities (i.e., a power singularity and a singularity of Hardy type), and this makes our study more difficult. Moreover, the associated functional energy is not of class C^1, so the direct variational method cannot be applied. In particular, the energy functional does not satisfy the mountain pass geometry, which implies also that the Ekland's variational principle cannot be applied. On the other hand, the exponent r is assumed to satisfy $r > p$, and this implies that the functional energy is not coercive in its domain. To guarantee the coercivity of the associated functional energy, we will work in some subsets called Nehari manifold sets. By these sets, we prove the multiplicity of solutions in the space $X = W_0^{2,p(\tau)}(\Omega)$. For this aim, we assume the following hypotheses:

(H_1) • $a : \Omega \to \mathbb{R}$, is such that

$$a \in L^{\frac{t(\tau)}{t(\tau)+\theta(\tau)-1}}(\Omega),$$

with

$$1 < t(\tau) < p(\tau) < r(\tau) < p^*(\tau), \ \ \forall \ \tau \in \overline{\Omega}.$$

• There exist $b_1, b_2 > 0$, such that

$$0 < b_1 \leq b(\tau) \leq b_2, \ \ \forall \ \tau \in \overline{\Omega}.$$

(H_2)

$$\frac{(1-\theta^- - p^-)r^+}{(1-\theta^- - r^+)p^-} < \frac{p^+ - (1-\theta^+)}{r^- - (1-\theta^+)}.$$

Remark 1.1. *If a is a continuous function on $\overline{\Omega}$, then, the function a satisfies the hypothesis (H_1) for any*

$$t(\tau) = \frac{p^*(\tau)}{a_1}, p(\tau) = \frac{p^*(\tau)}{a_2}, r(\tau) = \frac{p^*(\tau)}{a_3},$$

with $a_1 > a_2 > a_3 > 1$. On the other hand, there are several functions satisfying hypothesis (H_2). So, we can construct a class of functions satisfying hypotheses (H_1) and (H_2).

Definition 1.1. *A function φ in the space X is a weak solution of the problem $(P_{\lambda,\mu})$ if, for each function v in X, we have*

$$\begin{aligned}\int_\Omega |\Delta\varphi|^{p(\tau)-2}\Delta\varphi\Delta v d\tau &- \lambda\int_\Omega \frac{|\varphi(\tau)|^{p(\tau)-2}}{\delta(\tau)^{2p(\tau)}}\varphi(\tau)v(\tau)d\tau \\ &- \mu\int_\Omega a(\tau)|\varphi|^{-\theta(\tau)}v(\tau)d\tau - \int_\Omega b(\tau)|\varphi(\tau)|^{r(\tau)-2}\varphi(\tau)v(\tau)d\tau = 0.\end{aligned}$$

Our main result of this paper is the following.

Theorem 1.1. *Assume assumptions (H_1) and (H_2). Then, for each $\lambda \in (0, C_H)$, the problem $(P_{\lambda,\mu})$ admits two non-trivial solutions, provided that $\mu \in (0, \mu^*)$ for some positive constant μ^*.*

In Section 2 we introduce some results on functional spaces. In Section 3, we present and prove the main result of this work.

2. Preliminaries

In this section, we recall some preliminaries on the Lebesgue and Sobolev spaces. For interested readers, we refer to the works [21–23].

The sets $C_+(\Omega)$ will denote the set of all functions μ that are continuous on Ω, and satisfy

$$\mu(\tau) > 1, \forall\tau \in \Omega.$$

For each $\mu \in C_+(\Omega)$, we define the space $L^{\mu(\tau)}(\Omega)$ by:

$$L^{\mu(\tau)}(\Omega) = \{\varphi : \Omega \to \mathbb{R}, measurable, \int_\Omega |\varphi(\tau)|^{\mu(\tau)}d\tau < \infty\}.$$

We equip the space $L^{\mu(\tau)}(\Omega)$ with the following norm,

$$|\varphi|_{\mu(\tau)} = \inf\left\{\mu > 0 : \int_\Omega |\frac{\varphi(\tau)}{\mu}|^{\mu(\tau)}d\tau \leq 1\right\}.$$

Equipped with the last norm, the space $L^{\mu(\tau)}(\Omega)$ becomes a Banach space. Moreover, it is separable and reflexive if and only if μ satisfies

$$1 < \mu^- \leq \mu^+ < \infty.$$

Proposition 2.1. *[24, 25] For any $\varphi \in L^{\mu(\tau)}(\Omega)$ and $v \in L^{\mu'(\tau)}(\Omega)$, where $\frac{1}{\mu(\tau)} + \frac{1}{\mu'(\tau)} = 1$, we have,*

$$|\int_\Omega \varphi v d\tau| \leq (\frac{1}{\mu^-} + \frac{1}{(p')^-})|\varphi|_{\mu(\tau)}|v|_{\mu'(\tau)}.$$

Let

$$\rho_{\mu(\tau)} : L^{\mu(\tau)}(\Omega) \to \mathbb{R},$$
$$\varphi \mapsto \int_\Omega |\varphi(\tau)|^{\mu(\tau)} d\tau,$$

the modular on the space $L^{\mu(\tau)}(\Omega)$.

Proposition 2.2. *[22, 24] For all $\varphi \in L^{\mu(\tau)}(\Omega)$, we have,*

(1) $|\varphi|_{\mu(\tau)} < 1(resp = 1, > 1) \Leftrightarrow \rho_{\mu(\tau)}(\varphi) < 1(resp = 1, > 1)$*;*
(2) $|\varphi|_{\mu(\tau)} > 1 \Rightarrow |\varphi|^{\mu^-}_{\mu(\tau)} \leq \rho_{\mu(\tau)}(\varphi), \leq |\varphi|^{\mu^+}_{\mu(\tau)}$*;*

(3) $|\varphi|_{\mu(\tau)} < 1 \Rightarrow |\varphi|^{\mu^+}_{\mu(\tau)} \leq \rho_{\mu(\tau)}(\varphi) \leq |\varphi|^{\mu^-}_{\mu(\tau)}$.

Also, we get the following proposition.

Proposition 2.3. *[22] Let μ and q be measurable functions such that $q \in L^\infty(\Omega)$ and $1 \leq \mu(\tau).q(\tau) \leq \infty$ for all $\tau \in \Omega$.*

Let $\varphi \in L^{\mu(\tau)}(\Omega), \varphi \neq 0$. Then,

(1) $|\varphi|_{\mu(\tau)q(\tau)} \leq 1 \Rightarrow |\varphi|^{\mu^+}_{\mu(\tau)q(\tau)} \leq ||\varphi|^{\mu(\tau)}|_{q(\tau)} \leq |\varphi|^{\mu^-}_{\mu(\tau)q(\tau)}$,
(2) $|\varphi|_{\mu(\tau)q(\tau)} \geq 1 \Rightarrow |\varphi|^{\mu^-}_{\mu(\tau)q(\tau)} \leq ||\varphi|^{\mu(\tau)}|_{q(\tau)} \leq |\varphi|^{\mu^+}_{\mu(\tau)q(\tau)}$.

Let us define the space

$$W^{2,\mu(\tau)}(\Omega) = \{\varphi \in L^{\mu(\tau)}(\Omega) : |\nabla\varphi| \in L^{\mu(\tau)}(\Omega), |\Delta\varphi| \in L^{\mu(\tau)}(\Omega)\},$$

equipped with the norm

$$\|\varphi\| = \inf\{\mu > 0 : \int_\Omega (|\frac{\Delta\varphi(\tau)}{\mu}|^{\mu(\tau)} + a(\tau)|\frac{\varphi(\tau)}{\mu}|^{\mu(\tau)})d\tau \leq 1\}.$$

$W^{2,\mu(\tau)}(\Omega)$ is a separable and reflexive Banach space (see [21, 23]).

Let $W^{2,\mu(\tau)}_0(\Omega)$ be the closure of $C^\infty_0(\Omega)$ in $W^{2,\mu(\tau)}(\Omega)$. Then, $W^{2,\mu(\tau)}_0(\Omega)$ is a Banach and reflexive space with the norm

$$\|\varphi\| = |\Delta\varphi|_{\mu(\tau)}.$$

Theorem 2.1. *[26] If $q \in C_+(\overline{\Omega})$ with $q(\tau) < \mu^*(\tau)$, for any $\tau \in \overline{\Omega}$, then the embedding from $W^{2,\mu(\tau)}(\Omega)$ into $L^{q(\tau)}(\Omega)$ is compact and continuous.*

Let

$$\Theta(\varphi) = \int_\Omega |\Delta\varphi|^{\mu(\tau)} d\tau.$$

Then, we have the following result.

Proposition 2.4. *[21, 23]*

(1) If $\Theta(\varphi) \geq 1$*, then* $\|\varphi\|^{\mu^-} \leq \Theta(\varphi) \leq \|\varphi\|^{\mu^+}$,
(2) If $\Theta(\varphi) \leq 1$*, then* $\|\varphi\|^{\mu^+} \leq \Theta(\varphi) \leq \|\varphi\|^{\mu^-}$,
(3) $\Theta(\varphi) \geq 1(= 1, \leq 1) \Leftrightarrow \|\varphi\| \geq 1(= 1, \leq 1)$.

3. Main results

In this section, we prove the main result of this paper. We begin by defining the functional $\Phi_{\lambda,\mu}$: $X \to \mathbb{R}$, associated with the problem $(P_{\lambda,\mu})$, as follows:

$$\begin{aligned}\Phi_{\lambda,\mu}(\varphi) &= \int_\Omega \frac{|\Delta\varphi(\tau)|^{p(\tau)}}{p(\tau)}d\tau - \lambda\int_\Omega \frac{|\varphi(\tau)|^{p(\tau)}}{p(\tau)\delta(\tau)^{2p(\tau)}}d\tau \\ &- \mu\int_\Omega \frac{a(\tau)}{1-\theta(\tau)}|\varphi|^{1-\theta(\tau)}d\tau - \int_\Omega b(\tau)\frac{|\varphi(\tau)|^{r(\tau)}}{r(\tau)}d\tau.\end{aligned}$$

We note that, due to the singular term, the functional $\Phi_{\lambda,\mu}$ is not of class C^1. Moreover, from the fact that $0 < 1-\theta(\tau) < p(\tau) < r(\tau)$, we see that $\Phi_{\lambda,\mu}$ is not bounded below in X. So, we cannot use the direct variational method to prove the existence of solutions. Throughout this paper, we consider for $\varphi \in X$, the function $J_\varphi : [0,\infty[\to \mathbb{R}$, defined by

$$J_\varphi(t) = \Phi_{\lambda,\mu}(tz).$$

We note that, by the fact that $0 < 1-\theta(\tau) < p(\tau) < r(\tau)$, the functional $\Phi_{\lambda,\mu}$ is not bounded below in X, so we will prove that $\Phi_{\lambda,\mu}$ is bounded on the Nehari manifold $\aleph$ which is defined as:

$$\aleph = \{\varphi \in X \setminus \{0\} : J'_\varphi(1) = 0\}.$$

Now, we split $\aleph$ into the following parts:

$$\aleph^+ = \{\varphi \in N : \ J''_\varphi(1) > 0\}.$$

$$\aleph^- = \{\varphi \in N : \ J''_\varphi(1) < 0\}.$$

$$\aleph^0 = \{\varphi \in N : \ J''_\varphi(1) = 0\}.$$

It is clear that:

$$J'_\varphi(1) = \int_\Omega (|\Delta\varphi(\tau)|^{p(\tau)} - \frac{|\varphi(\tau)|^{p(\tau)}}{\delta(\tau)^{2p(\tau)}})d\tau - \mu\int_\Omega a(\tau)\varphi^{1-\theta(\tau)}d\tau - \int_\Omega b(\tau)|\varphi(\tau)|^{r(\tau)}d\tau,$$

and

$$J''_\varphi(1) = \int_\Omega p(\tau)(|\Delta\varphi(\tau)|^{p(\tau)} - \frac{|\varphi(\tau)|^{p(\tau)}}{\delta(\tau)^{2p(\tau)}})d\tau - \mu\int_\Omega a(\tau)(1-\theta(\tau))\varphi^{1-\theta(\tau)}d\tau - \int_\Omega b(\tau)r(\tau)|\varphi(\tau)|^{r(\tau)}d\tau,$$

Moreover, we have

$$\varphi \in \aleph \Leftrightarrow A(\varphi) - B(\varphi) - C(\varphi) = 0, \tag{3.1}$$

where

$$A(\varphi) = \int_\Omega (|\Delta\varphi(\tau)|^{p(\tau)} - \lambda\frac{|\varphi(\tau)|^{p(\tau)}}{\delta(\tau)^{2p(\tau)}})d\tau,$$

$$B(\varphi) = \mu\int_\Omega a(\tau)|\varphi|^{1-\theta(\tau)}d\tau,$$

and

$$C(\varphi) = \int_\Omega b(\tau)|\varphi(\tau)|^{r(\tau)}d\tau.$$

Next, we will prove the following lemma.

Lemma 3.1. *Assume assumption* (H_1) *holds. Let* $\varphi \in X$, *for* $\lambda \in (0, C_H)$. *Then, we have the following:*

(1) There exists a constant $C'_H = \frac{p^-}{p^+}(1 - \frac{\lambda}{C_H}) > 0$ *such that,*

$$C'_H \ \|\varphi\|^{p^-} \leq A(\varphi) \leq \|\varphi\|^{p^+} \ \ if \ \ \|\varphi\| \geq 1.$$

$$C'_H \ \|\varphi\|^{p^+} \leq A(\varphi) \leq \|\varphi\|^{p^-} \ \ if \ \ \|\varphi\| \leq 1.$$

(2) There exists a constant $C_1 > 0$ *such that,*

$$B(\varphi) \leq C_1 \mu \max(\|\varphi\|^{1-\theta^+}, \|\varphi\|^{1-\theta^-}).$$

(3) There exists a constant $C_2 > (C'_H)^2$ *such that,*

$$C(\varphi) \leq C_2 \max(\|\varphi\|^{r^+}, \|\varphi\|^{r^-}).$$

Proof. (1) Let $\varphi \in X$ and $0 < \lambda < C_H$. By (1.3), we have

$$\frac{\lambda}{C_H} \int_\Omega \frac{|\Delta\varphi(\tau)|^{p(\tau)}}{p(\tau)} d\tau \geq \lambda \int_\Omega \frac{|\varphi(\tau)|^{p(\tau)}}{p(\tau)\delta(\tau)^{2p(\tau)}} d\tau.$$

This implies that

$$\int_\Omega \frac{|\Delta\varphi(\tau)|^{p(\tau)}}{p(\tau)} d\tau - \lambda \int_\Omega \frac{|\varphi(\tau)|^{p(\tau)}}{p(\tau)\delta(\tau)^{2p(\tau)}} d\tau \geq (1 - \frac{\lambda}{C_H}) \int_\Omega \frac{|\Delta\varphi(\tau)|^{p(\tau)}}{p(\tau)} d\tau. \tag{3.2}$$

Moreover, we have

$$p^- \left(\int_\Omega \frac{|\Delta\varphi(\tau)|^{p(\tau)}}{p(\tau)} d\tau - \lambda \int_\Omega \frac{|\varphi(\tau)|^{p(\tau)}}{p(\tau)\delta(\tau)^{2p(\tau)}} d\tau \right) \leq A(\varphi) \leq \Theta(\varphi). \tag{3.3}$$

From (3.2) and (3.3), we get

$$p^-(1 - \frac{\lambda}{C_H}) \int_\Omega \frac{|\Delta\varphi(\tau)|^{p(\tau)}}{p(\tau)} d\tau \leq A(\varphi) \leq \Theta(\varphi).$$

So,

$$\frac{p^-}{p^+}(1 - \frac{\lambda}{C_H})\Theta(\varphi) \leq A(\varphi) \leq \Theta(\varphi).$$

Then, by Proposition 2.4, we deduce the assertion (1).

(2) By (H_1) and Proposition 2.1, we get

$$\int_\Omega a(\tau)\varphi^{1-\theta(\tau)} d\tau \ \ \leq \ \ |a|_{\frac{t(\tau)}{t(\tau)+\theta(\tau)-1}} \| |\varphi|^{1-\theta(\tau)} |_{\frac{t(\tau)}{1-\theta(\tau)}}.$$

Using, $1 < t(\tau) < p^*(\tau)$, Proposition 2.2, and Theorem 2.1, we have

$$\begin{aligned} B(\varphi) \ &\leq \ \mu \ |a|_{\frac{t(\tau)}{t(\tau)+\theta(\tau)-1}} \max(|\varphi|_{t(\tau)}^{1-\theta^+}, |\varphi|_{t(\tau)}^{1-\theta^-}) \\ &\leq \ \mu \ |a|_{\frac{t(\tau)}{t(\tau)+\theta(\tau)-1}} \max(\|\varphi\|^{1-\theta^+}, \|\varphi\|^{1-\theta^-}). \end{aligned}$$

(3) By (H_1), Proposition 2.2, and Theorem 2.1, we get

$$\begin{aligned} C(\varphi) &= \int_\Omega b(\tau)|\varphi(\tau)|^{r(\tau)}\tau \le b_2|\varphi|^{r^+}_{r(\tau)} + b_2|\varphi|^{r^-}_{r(\tau)} \\ &\le C[\|\varphi\|^{r^+} + \|\varphi\|^{r^-}] \le C_2 \max(\|\varphi\|^{r^+}, \|\varphi\|^{r^-}), \end{aligned}$$

where, $C_2 = 2C + (C'_H)^2$.

□

Lemma 3.2. *Assume that assumption (H_1) holds. For $\lambda \in (0, C_H)$, $\Phi_{\lambda,\mu}$ is coercive and bounded below on $\aleph$.*

Proof. Let $\varphi \in N$, for $\|\varphi\| > 1$, from (3.1) and Lemma 3.1, we have

$$\begin{aligned} \Phi_{\lambda,\mu}(\varphi) &\ge \frac{1}{p^+}A(\varphi) - \frac{1}{1-\theta^+}B(\varphi) - \frac{1}{r^-}C(\varphi) \\ &\ge \frac{1}{p^+}A(\varphi) - \frac{1}{1-\theta^+}B(\varphi) - \frac{1}{r^-}[A(\varphi) - B(\varphi)] \\ &\ge (\frac{1}{p^+} - \frac{1}{r^-})A(\varphi) - (\frac{1}{1-\theta^+} - \frac{1}{r^-})B(\varphi) \\ &\ge (\frac{1}{p^+} - \frac{1}{r^-})C'_H\|\varphi\|^{p^-} - \mu C_1(\frac{1}{1-\theta^+} - \frac{1}{r^-})\|\varphi\|^{1-\theta^-}. \end{aligned}$$

Since $0 < 1-\theta^+ < 1-\theta^- < p^- < p^+ < r^- < r^+$, we get,

$$\Phi_{\lambda,\mu}(\varphi) \to +\infty, \ \ as \ \ \|\varphi\| \to +\infty.$$

Hence, the lemma is proved. □

Lemma 3.3. *Under assumption (H_1), there exists $\mu_0 > 0$, given by*

$$\mu_0 = \frac{1}{C_1}(\frac{r^- - p^+}{r^- - (1-\theta^+)})(\frac{[p^- - (1-\theta^-)](C'_H)^2}{[r^+ - (1-\theta^-)]C_2})^{\frac{p^+-(1-\theta^+)}{r^--p^+}},$$

such that for any $\lambda \in (0, C_H)$, and $\mu \in (0, \mu_0)$, we have $\aleph^0 = \emptyset$. The positive constants C_H, C_1, C_2, and C'_H are given in (1.3) *and in Lemma 3.1.*

Proof. We suppose that there exists $0 < \mu < \mu_0$ such that $\aleph^0 \neq \emptyset$. Then, there exists $\varphi \in \aleph^0$. So, if $\|\varphi\| < 1$, then by (3.1), we have

$$\begin{aligned} 0 = J''_\varphi(1) &\le p^+A(\varphi) - (1-\theta^+)B(\varphi) - r^-C(\varphi) \\ &\le p^+A(\varphi) - (1-\theta^+)B(\varphi) - r^-(A(\varphi) - B(\varphi)) \\ &\le (p^+ - r^-)A(\varphi) + C_1[r^- - (1-\theta^+)]B(\varphi). \end{aligned}$$

Then

$$(r^- - p^+)A(\varphi) \le C_1[r^- - (1-\theta^+)]B(\varphi).$$

By Lemma 3.1, we have

$$(r^- - p^+) \ \ C'_H\|\varphi\|^{p^+} \le \mu C_1[r^- - (1-\theta^+)]\|\varphi\|^{1-\theta^+}.$$

Then,

$$\frac{(r^- - p^+)\ C'_H}{C_1[r^- - (1-\theta^+)]}\|\varphi\|^{p^+-(1-\theta^+)} \le \mu. \tag{3.4}$$

Again from (3.1), we have

$$\begin{aligned} 0 = J''_\varphi(1) &\ge p^- A(\varphi) - (1-\theta^-)B(\varphi) - r^+ C(\varphi) \\ &\ge p^- A(\varphi) - (1-\theta^-)(A(\varphi) - C(\varphi)) - r^+ C(\varphi) \\ &\ge [p^- - (1-\theta^-)]A(\varphi) - [r^+ - (1-\theta^-)]C(\varphi). \end{aligned}$$

Then, using Lemma 3.1, we obtain

$$C_2\|\varphi\|^{r^-}[r^+ - (1-\theta^-)] \ge [p^- - (1-\theta^-)]\ C'_H\|\varphi\|^{p^+}.$$

Then,

$$\|\varphi\| \ge (\frac{[p^- - (1-\theta^-)]C'_H}{C_2[r^+ - (1-\theta^-)]})^{\frac{1}{r^- - p^+}}. \tag{3.5}$$

Combining (3.4) and (3.5), we get

$$\mu \ge \frac{1}{C_1}(\frac{r^- - p^+}{r^- - (1-\theta^+)})(\frac{[p^- - (1-\theta^-)]\ (C'_H)^2}{[r^+ - (1-\theta^-)]C_2})^{\frac{p^+-(1-\theta^+)}{r^- - p^+}}. \tag{3.6}$$

Now, if $\|\varphi\| \ge 1$, we have

$$\|\varphi\| \ge (\frac{[p^- - (1-\theta^-)]C'_H}{C_2[r^+ - (1-\theta^-)]})^{\frac{1}{r^+ - p^-}}, \tag{3.7}$$

and

$$\mu \ge \frac{1}{C_1}(\frac{r^- - p^+}{r^- - (1-\theta^+)})(\frac{[p^- - (1-\theta^-)]\ (C'_H)^2}{[r^+ - (1-\theta^-)]C_2})^{\frac{p^--(1-\theta^-)}{r^+ - p^-}}. \tag{3.8}$$

Using $C_2 > (C'_H)^2$ and $0 < 1-\theta^+ < 1-\theta^- < 1 < q^- < q^+ < p^- < p^+ < r^- < r^+$, we have,

$$0 < \frac{[p^- - (1-\theta^-)](C'_H)^2}{[r^+ - (1-\theta^-)]C_2} < 1, \tag{3.9}$$

$$\frac{q^- - (1-\theta^-)}{r^+ - p^-} < \frac{p^+ - (1-\theta^+)}{r^- - p^+}. \tag{3.10}$$

Combining, (3.6), (3.8), (3.9), and (3.10), we get

$$\mu \ge \frac{1}{C_1}(\frac{r^- - p^+}{r^- - (1-\theta^+)})(\frac{[p^- - (1-\theta^-)]\ (C'_H)^2}{[r^+ - (1-\theta^-)]C_2})^{\frac{p^+-(1-\theta^+)}{r^- - p^+}},$$

which is a contradiction. □

Remark 3.1. *For any $0 < \lambda < C_H$, $0 < \mu < \mu_0$ and from Lemmas 3.2 and 3.3, we obtain that*

(1) $N = \aleph^+ \cup \aleph^-$.
(2) $\Phi_{\lambda,\mu}$ *is coercive and bounded below on* $\aleph^+$ *and* $\aleph^-$.

Then, we note,

$$m = \inf_{\varphi\in\aleph}(\Phi_{\lambda,\mu}(\varphi));\;\; m^+ = \inf_{\varphi\in\aleph^+}(\Phi_{\lambda,\mu}(\varphi));\;\; m^- = \inf_{\varphi\in\aleph^-}(\Phi_{\lambda,\mu}(\varphi)).$$

Lemma 3.4. *If* (H_1) *and* (H_2) *hold,then for any* $0 < \lambda < C_H$ *and* $0 < \mu < \mu_0$*, we have*

$$m \le m^+ < 0.$$

Proof. If $\varphi \in \aleph^+$, from (3.1) we have,

$$\begin{aligned} 0 &< J''_{\varphi}(1) \le p^+A(\varphi) - (1-\theta^+)B(\varphi) - r^-C(\varphi)\\ &\le p^+A(\varphi) - (1-\theta^+)(A(\varphi) - C(\varphi)) - r^-C(\varphi)\\ &\le [p^+ - (1-\theta^+)]A(\varphi) - [r^- - (1-\theta^+)]C(\varphi). \end{aligned}$$

Then,

$$C(\varphi) < \frac{(p^+ - (1-\theta^+))}{r^- - (1-\theta^+)}A(\varphi). \tag{3.11}$$

On the other hand, we also have from (3.1) that:

$$\begin{aligned} \Phi_{\lambda,\mu}(\varphi) &\le \frac{1}{p^-}A(\varphi) - \frac{1}{1-\theta^-}B(\varphi) - \frac{1}{r^+}C(\varphi)\\ &\le \frac{1}{p^-}A(\varphi) - \frac{1}{1-\theta^-}[A(\varphi) - C(\varphi)] - \frac{1}{r^+}C(\varphi)\\ &\le (\frac{1}{p^-} - \frac{1}{1-\theta^-})A(\varphi) + (\frac{1}{1-\theta^-} - \frac{1}{r^+})C(\varphi). \end{aligned}$$

By to (3.11), we get,

$$\Phi_{\lambda,\mu}(\varphi) < (\frac{1}{p^-} - \frac{1}{1-\theta^-})A(\varphi) + (\frac{1}{1-\theta^-} - \frac{1}{r^+})\frac{p^+ - (1-\theta^+)}{r^- - (1-\theta^+)}A(\varphi).$$

Then,

$$\Phi_{\lambda,\mu}(\varphi) \le \frac{(1-\theta^- - p^-)r^+ - p^-(1-\theta^- - r^+) \quad \frac{p^+-(1-\theta^+)}{r^- -(1-\theta^+)}}{p^-r^+(1-\theta^-)}A(\varphi).$$

From (H_2), we have,

$$\frac{(1-\theta^- - p^-)r^+ - p^-(1-\theta^- - r^+) \quad \frac{p^+-(1-\theta^+)}{r^- -(1-\theta^+)}}{p^-r^+(1-\theta^-)} < 0.$$

So, $\Phi_{\lambda,\mu}(\varphi) < 0, \forall\; \varphi \in \aleph^+$. Thus $m \le m^+ < 0$. □

Lemma 3.5. *If* (H_1) *and* (H_2) *hold, then for* $0 < \lambda < C_H$*,* $0 < \mu < \frac{1-\theta^+}{p^+}\mu_0$*, there exists* $k_0 > 0$*, such that*

$$m^- \ge k_0 > 0.$$

Proof. Let $\varphi \in \aleph^-$. Then we know that $\varphi''_{\varphi}(1) < 0$. Moreover, we have two cases:
Case 1. $\|\varphi\| < 1$. From Lemma 3.2 and (3.5), we have

$$\begin{aligned}\Phi_{\lambda,\mu}(\varphi) &\geq (\frac{1}{p^+} - \frac{1}{r^-})C'_H\|\varphi\|^{p^+} - \mu C_1(\frac{1}{1-\theta^+} - \frac{1}{r^-})\|\varphi\|^{1-\theta^+} \\ &\geq \|\varphi\|^{1-\theta^+}[(\frac{1}{p^+} - \frac{1}{r^-})C'_H\|\varphi\|^{p^+-(1-\theta^+)} - \mu C_1(\frac{1}{1-\theta^+} - \frac{1}{r^-})] \\ &\geq (\frac{p^- - (1-\theta^-)}{C_2[r^+ - (1-\theta^-)]})^{\frac{1-\theta^+}{r^- - p^+}}[(\frac{1}{p^+} - \frac{1}{r^-})([\frac{p^- - (1-\theta^-)](C'_H)^2}{C_2[r^+ - (1-\theta^-)]})^{\frac{p^+-(1-\theta^+)}{r^- - p^+}} \\ &- \mu C_1(\frac{1}{1-\theta^+} - \frac{1}{r^-})] = d_1.\end{aligned}$$

So, if

$$\mu < (\frac{1}{p^+} - \frac{1}{r^-})([\frac{p^- - (1-\theta^-)](C'_H)^2}{C_2[r^+ - (1-\theta^-)]})^{\frac{p^+-(1-\theta^+)}{r^- - p^+}} \frac{1}{C_1(\frac{1}{1-\theta^+} - \frac{1}{r^-})} = \frac{1-\theta^+}{p^+}\mu_0,$$

we conclude that $\Phi_{\lambda,\mu}(\varphi) \geq d_1 > 0$.
Case 2. $\|\varphi\| > 1$. From Lemma 3.2 and (3.7), we have

$$\begin{aligned}\Phi_{\lambda,\mu}(\varphi) &\geq (\frac{1}{p^+} - \frac{1}{r^-})C'_H\|\varphi\|^{p^-} - \mu C_1(\frac{1}{1-\theta^+} - \frac{1}{r^-})\|\varphi\|^{1-\theta^-} \\ &\geq \|\varphi\|^{1-\theta^-}[(\frac{1}{p^+} - \frac{1}{r^-})C'_H\|\varphi\|^{p^--(1-\theta^-)} - \mu C_1(\frac{1}{1-\theta^+} - \frac{1}{r^-})] \\ &\geq (\frac{p^- - (1-\theta^-)}{C_2[r^+ - (1-\theta^-)]})^{\frac{1-\theta^-}{r^+ - p^-}}[(\frac{1}{p^+} - \frac{1}{r^-})([\frac{p^- - (1-\theta^-)](C'_H)^2}{C_2[r^+ - (1-\theta^-)]})^{\frac{p^--(1-\theta^-)}{r^+ - p^-}} \\ &- \mu C_1(\frac{1}{1-\theta^+} - \frac{1}{r^-})] = d_2.\end{aligned}$$

So, if we have

$$\mu < (\frac{1}{p^+} - \frac{1}{r^-})(\frac{p^- - (1-\theta^-)}{C_2[r^+ - (1-\theta^-)]})^{\frac{p^--(1-\theta^-)}{r^+ - p^-}} \frac{1}{C_1(\frac{1}{1-\theta^+} - \frac{1}{r^-})},$$

then we obtain $J(\varphi) \geq d_2 > 0$.
Now, a simple calculation shows that

$$\frac{1-\theta^+}{p^+}\mu_0 < (\frac{1}{p^+} - \frac{1}{r^-})(\frac{p^- - (1-\theta^-)}{C_2[r^+ - (1-\theta^-)]})^{\frac{p^--(1-\theta^-)}{r^+ - p^-}} \frac{1}{C_1(\frac{1}{1-\theta^+} - \frac{1}{r^-})}.$$

Hence, if we put $k_0 = \min(d_1, d_2)$, then from the above study, we obtain $\Phi_{\lambda,\mu}(\varphi) \geq k_0 > 0$, which implies that

$$m^- = \inf_{\varphi \in N^-}(\Phi_{\lambda,\mu}(\varphi)) \geq k_0.$$

□

Lemma 3.6. *If* (H_1) *and* (H_2) *hold, then for* $\varphi \in X\backslash\{0\}$*, there exists* $\mu' > 0$*, such that, for* $0 < \mu < \mu'$ *and* $0 < \lambda < C_H$*, there exists* $t^* > 0$ *and* $t^+ < t^-$ *such that,* $t^-\varphi \in \aleph^-$*,* $t^+\varphi \in \aleph^+$*,*

$$\Phi_{\lambda,\mu}(t^+\varphi) = \inf_{0\leq t\leq t^*}\Phi_{\lambda,\mu}(tz) \ \text{ and } \ \Phi_{\lambda,\mu}(t^-\varphi) = \sup_{t\geq 0}\Phi_{\lambda,\mu}(tz).$$

Proof. Let $\varphi \in X\backslash\{0\}$. For all $t > 1$, we have

$$A(\varphi)t^{p^- -1} - B(\varphi)t^{-\theta^+} - C(\varphi)t^{r^+ -1} \leq J'_\varphi(t) \leq A(\varphi)t^{p^+ -1} - B(\varphi)t^{-\theta^-} - C(\varphi)t^{r^- -1},$$

and for all $0 < t \leq 1$, we get

$$A(\varphi)t^{p^+ -1} - B(\varphi)t^{-\theta^-} - C(\varphi)t^{r^- -1} \leq J'_\varphi(t) \leq A(\varphi)t^{p^- -1} - B(\varphi)t^{-\theta^+} - C(\varphi)t^{r^+ -1}.$$

Now, we introduce the following function

$$h(t) = A(\varphi)t^{\alpha} - B(\varphi)t^{-\theta} - C(\varphi)t^{\beta}, \forall t > 0,$$

where, $\beta > \alpha > \theta > 0, A(\varphi), B(\varphi), C(\varphi) \geq 0$, we have

$$h(t) = 0 \Leftrightarrow t^{\theta}h(t) = 0 \Leftrightarrow A(\varphi)t^{\alpha+\theta} - C(\varphi)t^{\beta+\theta} = B(\varphi).$$

Define $\tau(t) = A(\varphi)t^{\alpha+\theta} - C(\varphi)t^{\beta+\theta}$. So, τ possesses a unique maximum point at

$$t_{\max} = \left(\frac{A(\varphi)(\alpha+\theta)}{C(\varphi)(\beta+\theta)}\right)^{\frac{1}{(\beta-\alpha)}}.$$

$B(\varphi) = \mu \int_\Omega a(\tau)|\varphi|^{1-\theta(\tau)}\tau > 0$ and if $\mu > 0$, is small enough such that $B(\varphi) < \tau(t_{\max})$, then there exist $0 < t_1 < t_{\max} < t_2 < \infty$ such that $\tau(t^+) = \tau(t^-) = B(\varphi)$, $\tau'(t^+) > 0$, and $\tau'(t^-) < 0$, that is, t^+ and t^- are two solutions of the equation $h(t) = 0$, for all $t > 0$.

The graph $J'_\varphi(t)$ is between two graphs

$$U_\varphi(t) = A(\varphi)t^{p^- -1} - B(\varphi)t^{-\theta^+} - C(\varphi)t^{r^+ -1},$$

and

$$V_\varphi(t) = A(\varphi)t^{p^+ -1} - B(\varphi)t^{-\theta^-} - C(\varphi)t^{r^- -1}.$$

Now, using the discussion as τ, there exists $\mu' > 0$ and $0 < t^+ < t^- < \infty$ such that $J'_\varphi(t^-) = J'_\varphi(t^+) = 0$, and $t^+\varphi \in \aleph^+$ $t^-\varphi \in \aleph^-$ λ for all $\mu \in (0, \mu')$. □

Nextly, we pose $\mu^* = \min\{\mu', \frac{1-\theta^+}{p^+}\mu_0\}$, where μ' and μ_0, are given in Lemmas 3.6 and 3.3, respectively.

Proposition 3.1. *If* (H_1) *and* (H_2) *hold, then for* $\lambda \in (0, C_H)$ *and* $\mu \in (0, \mu^*)$ *the functional* $\Phi_{\lambda,\mu}$ *has a minimizer* $\varphi_1 \in \aleph^+$*, such that,*

$$\Phi_{\lambda,\mu}(\varphi_1) = m^+ < 0.$$

Proof. The functional $\Phi_{\lambda,\mu}$ is bounded below in $\aleph^+$. So, there exists $\{v_n\}$, such that, $\Phi_{\lambda,\mu}(v_n) \to m^+$. By Lemma 3.2, we conclude that $\{v_n\}$ is bounded on reflexive space X, so, up to a sub-sequence, there exist $\{v_n\}$ and φ_1 in X such that

$$\begin{cases} v_n \rightharpoonup \varphi_1, & \text{weakly in } X, \\ v_n \to \varphi_1, & \text{strongly in } L^{\beta(\tau)}(\Omega), \ 1 \leq \beta(\tau) < p^*(\tau), \\ v_n \to \varphi_1, & \text{a.e in } \Omega. \end{cases}$$

Next, by Proposition 2.3, we get

$$\lim_{n\to\infty} B(v_n) = B(\varphi_1), \quad \lim_{n\to\infty} C(v_n) = C(\varphi_1). \tag{3.12}$$

$$\lim_{n\to\infty} \int_\Omega \frac{|v_n(\tau)|^{p(\tau)}}{p(\tau)\delta^{2p(\tau)}} d\tau = \int_\Omega \frac{|\varphi_1(\tau)|^{p(\tau)}}{p(\tau)\delta^{2p(\tau)}} d\tau. \tag{3.13}$$

We prove that $\varphi_1 \neq 0$ and $B(\varphi_1) > 0$. If $B(\varphi_1) = 0$, since $(v_n) \in \aleph^+$ and by (3.1) and (3.12), we get

$$\Phi_{\lambda,\mu}(v_n) \geq (\frac{1}{p^+} - \frac{1}{r^-})A(v_n) - (\frac{1}{1-\theta^+} - \frac{1}{r^-})B(v_n).$$

Then,

$$\lim_{n\to\infty} \Phi_{\lambda,\mu}(v_n) \geq (\frac{1}{p^+} - \frac{1}{r^-}) \lim_{n\to\infty} A(v_n) \geq 0.$$

From, Lemma 3.4, $\lim_{n\to\infty} \Phi_{\lambda,\mu}(v_n) = m^+ < 0$. This is a contradiction. Then, $B(\varphi_1) > 0$ and $\varphi_1 \in X\backslash\{0\}$.
Now, we will be showing that $v_n \to \varphi_1$ strongly in X.
Supposing the contrary, then $v_n \nrightarrow \varphi_1$ strongly in X. By the Brezis-Lieb Lemma (see [27]), we get

$$\int_\Omega \frac{|\Delta(\varphi_1)|^{p(\tau)}}{p(\tau)} d\tau < \liminf_{n\to\infty} \int_\Omega \frac{|\Delta(v_n)|^{p(\tau)}}{p(\tau)} d\tau. \tag{3.14}$$

Using (3.12)–(3.14), we get

$$\Phi_{\lambda,\mu}(\varphi_1) < \lim_{n\to\infty} \Phi_{\lambda,\mu}(v_n). \tag{3.15}$$

From Lemma 3.6, for $\varphi_1 \in X\backslash\{0\}$, there exists $t^+ > 0$, such that $t^+\varphi_1 \in \aleph^+$. Since, $v_n \nrightarrow \varphi_1$ in X, we concluded that,

$$A(t^+\varphi_1) < \liminf_{n\to\infty} A(t^+v_n). \tag{3.16}$$

Then, by Proposition 2.3, we get

$$B(t^+\varphi_1) = \lim_{n\to\infty} B(t^+v_n), \;\; C(t^+\varphi_1) = \lim_{n\to\infty} C(t^+v_n). \tag{3.17}$$

By (3.16) and (3.17), we obtain

$$0 = J'_{\varphi_1}(t^+) < \lim_{n\to\infty} J'_{v_n}(t^+).$$

Then for n large enough, we obtain

$$J'_{v_n}(t^+) > 0. \tag{3.18}$$

Now, since $v_n \in \aleph^+$ for all $n \in \mathbb{N}$, we have, $J'_{v_n}(1) = 0$. Thus $t^+ \neq 1$. Clearly, t^+ is a minimizer of $g(t) = \Phi_{\lambda,\mu}(tz_1)$, for $t > 0$. Then, by (3.15), we get

$$\Phi_{\lambda,\mu}(t^+\varphi_1) \leq \Phi_{\lambda,\mu}(\varphi_1) < \lim_{n\to\infty} \Phi_{\lambda,\mu}(v_n) = \inf_{\varphi\in\aleph^+} \Phi_{\lambda,\mu}(\varphi).$$

This is a contracted $t^+\varphi_1 \in \aleph^+$. So, $v_n \to \varphi_1$ strongly in $\aleph^+ \cup \aleph^-$ as $n \to +\infty$, and by Lemma 3.2, $\aleph^0 = \emptyset$. Then, $\varphi_1 \in \aleph^+$, and by Lemma 3.4,

$$\Phi_{\lambda,\mu}(\varphi_1) = \lim_{n\to\infty} \Phi_{\lambda,\mu}(v_n) = m^+ < 0.$$

□

Proposition 3.2. *If* (H_1) *and* (H_2) *hold, then for* $\lambda \in (0, C_H)$ *and* $\mu \in (0, \mu^*)$ *the functional* $\Phi_{\lambda,\mu}$ *has a minimizer* $\varphi_2 \in \aleph^-$*, such that*

$$\Phi_{\lambda,\mu}(\varphi_2) = m^- > 0.$$

Proof. Let $\{v_n\}$ in $\aleph^-$, such that, $\Phi_{\lambda,\mu}(v_n) \to m^-$. Lemma 3.2 implies that $\{v_n\}$ is bounded in reflexive space X. So, up to a sub-sequence, there exist $\{v_n\}$ and φ_2 in X such that,

$$\begin{cases} v_n \rightharpoonup \varphi_2, & \text{weakly in } \ X, \\ v_n \to \varphi_2, & \text{strongly in } \ L^{\beta(\tau)}(\Omega), \ \ 1 \le \beta(\tau) < p^*(\tau), \\ v_n \to \varphi_2, & \text{a.e in } \ \Omega, \end{cases}$$

and

$$\lim_{n\to\infty} C(v_n) = C(\varphi_2). \tag{3.19}$$

Next, we have $\varphi_2 \neq 0$. Indeed, if $\varphi_2 = 0$, from (3.19), we obtain,

$$C(v_n) \to 0, \ \text{ as } \ n \to \infty. \tag{3.20}$$

Using the fact that $\{v_n\} \in \aleph^-$, Eq (3.1), and Lemma 3.5, we have

$$0 < k_0 < \Phi_{\lambda,\mu}(v_n) \le (\frac{1}{p^-} - \frac{1}{1-\theta^-})A(v_n) + (\frac{1}{1-\theta^-} - \frac{1}{r^+})C(v_n).$$

Then, by (3.20) and the fact that $1 - \theta^- < p^-$, we obtain

$$0 < k_0 \le \lim_{n\to\infty} \Phi_{\lambda,\mu}(v_n) \le 0,$$

which is a contradiction. So, $\varphi_2 \in X\backslash\{0\}$. On the other hand, by Lemma 3.6 there exists a positive real t^- such that $t^-\varphi_2 \in \aleph^-$.

Next, we will prove that v_n converges strongly to φ_2 in X. Assume that this is not true. Then, by the Brezis-Lieb Lemma (see [27]), we have

$$\int_\Omega \frac{|\Delta(t^-\varphi_2)|^{p(\tau)}}{p(\tau)} d\tau < \liminf_{n\to\infty} \int_\Omega \frac{|\Delta(t^-v_n)|^{p(\tau)}}{p(\tau)} d\tau. \tag{3.21}$$

On the other hand, from Eq (3.21) and Proposition 2.3, we obtain

$$\Phi_{\lambda,\mu}(t^-\varphi_2) < \lim_{n\to\infty} \Phi_{\lambda,\mu}(t^-v_n). \tag{3.22}$$

$$0 = J'_{\varphi_2}(t^-) < \lim_{n\to\infty} J'_{v_n}(t^-).$$

Thus for n large enough, we conclude that

$$J'_{v_n}(t^-) > 0. \tag{3.23}$$

Since $v_n \in \aleph^-$, then we have $J'_{v_n}(1) = 0$ and using (3.23), we get $t^- \neq 1$. Observe that the function $G(t) = \Phi_{\lambda,\mu}(tv_n)$, for $t > 0$, attains its maximum at $t = 1$ and using (3.22), we get

$$\Phi_{\lambda,\mu}(t^-\varphi_2) < \lim_{n\to\infty} \Phi_{\lambda,\mu}(t^-v_n) \le \lim_{n\to\infty} \Phi_{\lambda,\mu}(v_n) = \inf_{\varphi\in\aleph^-} \Phi_{\lambda,\mu}(\varphi).$$

This contradicts the fact that $t^{-}\varphi_2 \in \aleph^{-}$. Then, $v_n \to \varphi_2$ strongly in X as $n \to +\infty$, and thus $\varphi_2 \in \aleph$. By Lemma 3.2 we have, $\aleph^0 = \emptyset$ and by Lemma 3.4, we get,

$$\Phi_{\lambda,\mu}(\varphi_2) = \lim_{n\to\infty} \Phi_{\lambda,\mu}(v_n) = m^{-} > 0.$$

So, we conclude that $\varphi_2 \in \aleph^{-}$. □

Lemma 3.7. *If (H_1) and (H_2) hold and $\varphi \in \aleph^{\pm}$, there exist $\epsilon > 0$, and a continuous function $\alpha : B_\epsilon(0) \to (0, +\infty)$, such that*

$$\alpha(0) = 1, \;\; \alpha(v)(\varphi + v) \in N^{+}, \;\; \forall \;\; v \in B_\epsilon(0),$$

where

$$B_\epsilon(0) = \{v \in X : \|v\| < \epsilon\}.$$

Proof. Let $\varphi \in \aleph^{+}$ and let the function $f : X \times \mathbb{R} \to \mathbb{R}$ defined by:

$$\begin{aligned} f(v,t) &= J'_{\varphi+v}(t) \\ &= \int_\Omega t^{p(\tau)-1}|\Delta(v+\varphi)(\tau)|^{p(\tau)}d\tau - \int_\Omega t^{p(\tau)-1}\frac{|(v+\varphi)(\tau)|^{p(\tau)}}{\delta(\tau)^{2p(\tau)}}d\tau \\ &- \mu\int_\Omega t^{-\theta(\tau)}a(\tau)(v+\varphi)^{1-\theta(\tau)}d\tau - \int_\Omega b(\tau)t^{r(\tau)-1}|(v+\varphi)(\tau)|^{r(\tau)}d\tau, \;\; \forall \;\; v \in X. \end{aligned}$$

Since $\varphi \in \aleph^{+}$, ve have $f(0,1) = J'_{\varphi}(1) = 0$ and $f'(0,1) = J''_{\varphi}(1) > 0$.
Then, from the implicit function theorem, there exist $\epsilon > 0$ and a continuous function $\alpha : B_\epsilon(0) \to (0, +\infty)$, such that

$$f(v, \alpha(v)) = 0, \;\; \alpha(0) = 1. \tag{3.24}$$

Using (3.24), we get

$$\alpha(v)(\varphi + v) \in \aleph, \;\; \forall \;\; v \in B_\epsilon(0).$$

Taking $\epsilon > 0$ even smaller if necessary, we can also have

$$\alpha(v)(\varphi + v) \in \aleph^{+}, \;\; \forall \;\; v \in B_\epsilon(0).$$

The proof for the case $\varphi \in \aleph^{-}$ is very similar, so we omit it. □

Proof of Theorem 1.1. By Lemma 3.7, we can find $\vartheta(t) > 0, \; t \in [0, t_0]$ such that,

$$\vartheta(t)(\varphi_1 + th) \in \aleph^{+}, \;\; \vartheta(t) \to 1 \;\; as \;\; t \to 0^{+}.$$

Then, by Proposition 3.1, we have, $\forall \; t \in [0, t_0]$,

$$m^{+} = \Phi_{\lambda,\mu}(\varphi_1) \le \Phi_{\lambda,\mu}(\vartheta(t)(\varphi_1 + th)).$$

So, $\forall \; t \in [0, t_1]$ with $0 < t_1 \le t_0$, we get

$$m^{+} \le \Phi_{\lambda,\mu}(\varphi_1) \le \Phi_{\lambda,\mu}(\varphi_1 + th).$$

Then,

$$0 \leq \Phi_{\lambda,\mu}(\varphi_1 + th) - \Phi_{\lambda,\mu}(\varphi_1).$$

So, for $t > 0$, we get

$$0 \leq \lim_{t\to 0} \frac{\Phi_{\lambda,\mu}(\varphi_1 + th) - \Phi_{\lambda,\mu}(\varphi_1)}{t},$$

which yields to

$$\begin{aligned}\int_\Omega |\Delta\varphi_1|^{p(\tau)-2}\Delta\varphi_1\Delta h d\tau \quad &- \quad \lambda \int_\Omega \frac{|\varphi_1(\tau)|^{p(\tau)-2}}{\delta(\tau)^{2p(\tau)}}\varphi_1(\tau)h(\tau)d\tau \\ &- \quad \mu \int_\Omega a(\tau)|\varphi_1|^{-\theta(\tau)}h(\tau)d\tau - \int_\Omega b(\tau)|\varphi_1(\tau)|^{r(\tau)-2}\varphi_1(\tau)h(\tau)d\tau \geq 0.\end{aligned}$$

Since the function h is arbitrary, then we can change h by $-h$ in the last inequality, and we conclude that $\varphi_1 \in \aleph^+$ is a nontrivial weak solution to the problem $(P_{\lambda,\mu})$.

Now, by Lemma 3.7 and Proposition 3.2, the proof is the same for $\varphi_2 \in \aleph^-$. By this stage, the proof of Theorem 1.1 is now completed.

4. Conclusions

In this paper, we studied a $p(x)$-biharmonic problem involving two types of nonlinearities: Singular and Hardy type. More precisely, we combine a variational method with the Nehari manifold method to prove that such a problem admits two nontrivial solutions. We will generalize this study to problems involving the $p(\cdot,\cdot)$-Laplace operator, and we will extend this study to double-phase and multi-phase problems.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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