Mathematics

AIMS Mathematics, 10(2): 3734–3758.
DOI: 10.3934/math.2025173
Received: 03 November 2024
Revised: 18 January 2025
Accepted: 06 February 2025
Published: 25 February 2025

http://www.aimspress.com/journal/Math

Research article

A soft relation approach to approximate the spherical fuzzy ideals of semigroups

Rabia Mazhar[1], Shahida Bashir[1], Muhammad Shabir[2] and Mohammed Al-Shamiri[3,*]

[1] Department of Mathematics, University of Gujrat, Gujrat 50700, Pakistan
[2] Department of Mathematics, Quaid-i-Azam University, Islamabad 45320, Pakistan
[3] Department of Mathematics, Faculty of Science and Arts, King Khalid University, Abha, Saudi Arabia

* **Correspondence:** Email: mal-shamiri@kku.edu.sa.

Abstract: Our main objective of this work was to study rough approximations of spherical fuzzy ideals by using soft relations that are free from all the complexities that are faced by many scientists. Furthermore, lower and upper approximations of spherical fuzzy subsemigroups, spherical fuzzy left (right) ideals, spherical fuzzy interior ideals, and spherical fuzzy bi-ideals of semigroups were studied using soft relations. Mainly, we proved, for a spherical fuzzy ideal of the universe, that upper and lower approximations are spherical fuzzy soft ideals but the converse may not hold, as shown by examples. Compatible relations and complete relations are needed for upper approximations and lower approximations, respectively. Also, using examples, we showed that the conditions of complete relations were necessary for lower approximations. Last, a comparison study and conclusions of the introduced technique are given, demonstrating how our work is superior and efficient in contrast to other techniques.

Keywords: spherical fuzzy set; soft set; spherical fuzzy subsemigroup; spherical fuzzy ideal; soft relation
Mathematics Subject Classification: 03E72, 18B40

---

1. Introduction

1.1. Motivations

The Fuzzy set theory was introduced by Zadeh in 1965 [1]. It was the first positive approach to tackle uncertainty in daily life problems. For understanding, the notion "young" is uncertain, because we cannot uniquely categorize people into two groups: Young people and old people. Thus, the term "young" is not an exact but a vague notion. Therefore, uncertainty and vagueness are significant to logicians, mathematicians, and scientists. After many efforts, tools are introduced to deal with such problems, which have their individual advantages and disadvantages. In a fuzzy set, uncertainty is tackled by using a membership degree, which is controlled in a closed interval [0,1]. This theory elaborates on only positive aspects of any situation, and for negative and neutral aspects, this theory fails. An extension of fuzzy set theory is intuitionistic fuzzy set theory, which also uses nonmembership degrees [2]. In intuitionistic fuzzy sets, we discuss the degree of membership and the degree of non-membership. Here, the sum of membership degree and non-membership degree lies in interval [0,1]. This theory gives information for only agree and disagree elements for a situation that is incomplete because human nature has some refusal issues, too.

The Spherical fuzzy set theory [3] is another extension of fuzzy set theory, which is used for the condition that does not rely on yes or no, but there is a rejection. This theory has three functions of degree: Membership degree, non-membership degree, and neutral degree. Here, the sum of the square of membership degree, non-membership degree, and neutral degree is a number between [0,1]. This advanced model gives more choice to the decision makers for the selection of all three degrees. In this paper, we work on the most extended form of the fuzzy set, which is the spherical fuzzy set. This generalized theory has negligible drawbacks. A good example of a spherical fuzzy set would be a decision like voting where there are four types of voters who vote for, against, or decline to vote or abstain from voting. By using this theory, an appropriate model can be generated for the voting system. This theory is also a generalization of the intuitionistic fuzzy set theory. Moreover, scientists are working on the spherical fuzzy set [4–7].

1.2. Related work

A proper semigroup was initiated by a Russian mathematician, Anton Kazimirovich Suschkewitsch [8], to provide a label for some structures that were not groups but were formed through the expansion of consequences. Semigroups are very useful in many fields such as control problems, dynamical systems, partial differential equations, sociology, stochastic differential equations, and biology. Some examples of semigroups are the collection of all mappings of a set under the composition of functions, the set of natural numbers, and set of whole numbers under either addition or multiplication.

By introducing parameterization tools, Molodtsov initiated the fundamental idea of soft set theory in 1999 [9]. This concept has a number of parameters that make it practical in many areas, such as game theory, operational research, and Riemann integration. The ideas behind parametric reduction for soft sets have been studied by many authors. Ali et al. studied some properties of soft sets, rough sets, and fuzzy soft sets [10]. The study of soft matrix and roughness of soft sets is given in [11–13].

The rough set was introduced by Pawlak [14]. It attracts many researchers as it handles vagueness, uncertainty, and complexities in problems. It reduces the redundant data, which helps us make easy decisions in daily life problems. For a crisp set, two major approximations are made. The lower approximation includes objectives that positively belong to the target set, whereas the upper approximation is a group that contains elements that may belong to a target set. In this theory, rough approximations are given by using binary relations, whereas rough approximations, in terms of soft binary relations, provide various binary relations. To fabricate approximations, the equivalence relations are vital in Pawlak rough sets. The rough set approach is an authoritative approach in case we come across imperfect data [15].

1.3. Literature review and research gap

Scientists have added spherical fuzzy sets, covering relations, such as soft relations, to the Pawlak rough set model [16, 17]. Shahzaib et al. studied spherical fuzzy sets and their representation [18]. Many scientists worked on the applications of the spherical fuzzy set [19–21]. Ajay et al. worked on Einstein exponential operation laws of spherical fuzzy sets [22]. Kanwal and Shabir introduced roughness of the fuzzy subsemigroup in terms of soft relations [23]. Shabir et al. studied approximations of bipolar soft sets by soft relations [24]. Anwar et al. approximated the intuitionistic fuzzy sets and ideals based on soft relations [25]. Bilal and Shabir worked on approximations of pythagorean fuzzy sets over dual universes by soft binary relations [26]. Prasertpong worked on roughness of soft sets based on set-valued picture hesitant fuzzy relations [27]. Mazhar et al. worked on the approximation of spherical fuzzy sets using soft relations [28]. Approximations of spherical fuzzy ideals in terms of soft relations have not been studied, which is a generalization of all the theories. In this paper, we work on this new technique having negligible drawbacks.

1.4. Contributions

By extending previous studies, we work on the roughness of spherical fuzzy ideals of semigroups based on soft relations, which gives information about membership degree, non-membership degree, and neutral membership degree.

Ideals in semigroups are vital for learning structures and properties of such algebraic theories. As in any algebraic system, ideals can also be used to build quotient semigroups. Quotient structures are generally easier to manage and convert tedious problems to nontrivial ones. In automation, approximation of spherical fuzzy ideals contributes to defining control parameters under inexact input data. Ideals in semigroups help to explain the congruence relations. In fact, congruences in semigroups are defined in terms of ideals. Thus, ideals in semigroups give a fundamental concept with which to study, simplify, and classify the structure of semigroup analogs with what happens in ring theory or any other part of algebra. In this paper, we extend the approximation of spherical fuzzy sets to approximation of spherical fuzzy ideals using soft relations. By combining the rough set theory, soft binary relations, and spherical fuzzy set theory on semigroup structure, we approximate spherical fuzzy ideals using soft binary relation and obtain lower and upper approximations of the spherical fuzzy set by using afterset and foreset. The main theorem of this study is that "If X is a spherical fuzzy ideal of S_2, then $(\overline{C}^X, U)$ and $(\underline{C}^X, U)$ are spherical fuzzy soft ideals of S_1" for upper approximations, (C, U) is soft compatible relation and for lower approximations, and (C, U) is soft complete relation. The converse of this theorem may not hold, as shown by examples.

For convenience, we demonstrate the proposed work by a frame diagram given in Figure 1.

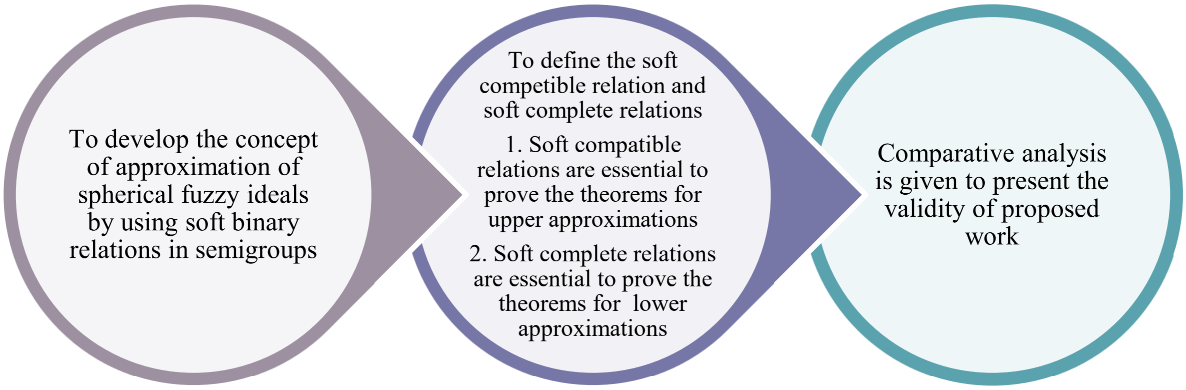

Figure 1. Frame diagram of the proposed work.

1.5. Organization of the paper

This paper is organized as follow: Some foundational definitions and concepts are given in Section 2. In Section 3, many theorems and examples based on roughness of spherical fuzzy ideals in semigroups using soft binary relations are given. In the last section, conclusions and future work are given.

2. Preliminaries

In this section, some basic notions about binary relations, spherical fuzzy sets, soft sets, and rough sets are given. Throughout the paper $(S,\cdot)$, $(S_1,\cdot)$, and $(S_2,\cdot)$ are semigroups.

Definition 1. A binary relation $\mathcal{R}$ from S_1 to S_2 is a subset of $S_1 \times S_2$. A binary relation $\mathcal{R}$ on S is a subset of $S \times S$.

Definition 2. [14] A binary relation $\mathcal{R}$ on S is said to be reflexive if $(r,r) \in \mathcal{R}$, symmetric if $(r,s) \in \mathcal{R}$ implies $(s,r) \in \mathcal{R}$ and transitive if $(r,s) \in \mathcal{R}$, and $(s,t) \in \mathcal{R}$ implies $(r,t) \in \mathcal{R}$ for all $r,s,t \in \mathcal{R}$. A binary relation $\mathcal{R}$ is said to be an equivalence relation if $\mathcal{R}$ is reflexive, symmetric, and transitive. Equivalence relation partitions S into equivalence classes.

If $U(\neq \phi) \subseteq S$ is a union of some equivalence classes, then U is definable. Otherwise, it is not definable. Then, we approximate U into two definable subsets called lower and upper approximations of U as $\underline{\mathcal{R}}(U) = \cup\{[s]_{\mathcal{R}} : [s]_{\mathcal{R}} \subseteq U\}$ and $\overline{\mathcal{R}}(U) = \cup\{[s]_{\mathcal{R}} : [s]_{\mathcal{R}} \cap U \neq \phi\}$. The pair $\left(\underline{\mathcal{R}}(U), \overline{\mathcal{R}}(U)\right)$ is called a rough set; if $\underline{\mathcal{R}}(U) \neq \overline{\mathcal{R}}(U)$ and if $\underline{\mathcal{R}}(U) = \overline{\mathcal{R}}(U)$ then U is definable. $\underline{\mathcal{R}}(U) - \overline{\mathcal{R}}(U)$ represents a boundary region.

Definition 3. [9] For a subset U of E (set of parameters), a pair (C,U) is called a soft set over S if C is a mapping $C: U \rightarrow P(S)$ where $P(S)$ is the power set of S. Thus, $C(u)$ is a subset of S for all $u \in U$. Hence, a soft set over S is a parameterized collection of subsets of S.

Definition 4. [3] A spherical fuzzy set U of S is a set of the form $U = \{\langle s, \mathcal{P}_U(s), \mathcal{I}_U(s), \mathcal{N}_U(s) \rangle : s$

$\in$ S}; where $\mathcal{P}_U: S \to [0,1]$, $\mathcal{I}_U: S \to [0,1]$ and $\mathcal{N}_U: S \to [0,1]$ are called functions of a positive membership, function of a neutral membership, and function of a negative membership degree, respectively, satisfying $0 \le \mathcal{P}_U^2(s) + \mathcal{I}_U^2(s) + \mathcal{N}_U^2(s) \le 1$, for all $s \in S$. For spherical fuzzy set U $\delta_U(s) = \sqrt{1 - (\mathcal{P}_U^2(s) + \mathcal{I}_U^2(s) + \mathcal{N}_U^2(s))}$ is said to be refusal degree for $s \in S$. A triplet $\langle \mathcal{P}_U(s), \mathcal{I}_U(s), \mathcal{N}_U(s) \rangle$ is called spherical fuzzy number and it is denoted by $n = \langle \mathcal{P}_n, \mathcal{I}_n, \mathcal{N}_n \rangle$, where $\mathcal{P}_n, \mathcal{I}_n, \mathcal{N}_n \in [0,1]$, with condition $0 \le \mathcal{P}_n^2 + \mathcal{I}_n^2 + \mathcal{N}_n^2 \le 1$.

Definition 5. [3] Let $U = \langle \mathcal{P}_U, \mathcal{I}_U, \mathcal{N}_U \rangle$, and $V = \langle \mathcal{P}_V, \mathcal{I}_V, \mathcal{N}_V \rangle$ be two spherical fuzzy subsets of S. Then

1) $U \subseteq V$ if and only if $\mathcal{P}_U(s) \le \mathcal{P}_V(s), \mathcal{I}_U(s) \le \mathcal{I}_V(s)$ and $\mathcal{N}_U(s) \ge \mathcal{N}_V(s)$ for all $s \in S$;
2) $U = V$ if and only if $U \subseteq V$ and $V \subseteq U$;
3) $U \cup V = \langle \mathcal{P}_U(s) \vee \mathcal{P}_V(s), \mathcal{I}_U(s) \wedge \mathcal{I}_V(s), \mathcal{N}_U(s) \wedge \mathcal{N}_V(s) \rangle$;
4) $U \cap V = \langle \mathcal{P}_U(s) \wedge \mathcal{P}_V(s), \mathcal{I}_U(s) \wedge \mathcal{I}_V(s), \mathcal{N}_U(s) \vee \mathcal{N}_V(s) \rangle$;
5) $U^c = \langle \mathcal{P}_U(s), \mathcal{I}_U(s), \mathcal{N}_U(s) \rangle^c = \langle \mathcal{N}_U(s), \mathcal{I}_U(s), \mathcal{P}_U(s) \rangle$.

Definition 6. [28] The spherical fuzzy universe set $S = 1_S = \langle 1, 0, 0 \rangle$ and spherical fuzzy empty set $\phi = 0_S = \langle 0, 0, 0 \rangle$ of S, where $1(s) = 1$ and $0(s) = 0$ for all $s \in S$.

Definition 7. [28] A pair (C, U) is called a spherical fuzzy soft set over S, where $C: U \to Sp(S)$ and $C(u)$ is a spherical fuzzy set of S for all $u \in U$. Hence, a spherical fuzzy soft set over S is a parameterized collection of spherical fuzzy sets of S.

For two spherical fuzzy soft sets (C_1, U_1) and (C_2, U_2) over a common universe S, (C_1, U_1) is a spherical fuzzy soft subset of (C_2, U_2) if (1) $U_1 \subseteq U_2$ and (2) $C_1(u)$ is a spherical fuzzy soft subset of $C_2(u)$ for all $u \in U$. Two spherical fuzzy soft sets (C_1, U_1) and (C_2, U_2) over a common universe S are said to be spherical fuzzy soft equal if (C_1, U_1) is a spherical fuzzy soft subset of (C_2, U_2) and (C_2, U_2) is a spherical fuzzy soft subset of (C_1, U_1).

The union and intersection of two spherical fuzzy soft sets (C_1, U) and (C_2, U) over a common universe S are the spherical fuzzy soft sets (H, U) and (K, U) respectively, where $H(u) = C_1(u) \cup C_2(u)$ and $K(u) = C_1(u) \cap C_2(u)$ for all $u \in U$.

Definition 8. [29] A spherical fuzzy subset $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ of a semigroup S is called a spherical fuzzy subsemigroup of S, if

1) $\mathcal{P}_X(s_1 s_2) \ge \mathcal{P}_X(s_1) \wedge \mathcal{P}_X(s_2)$;
2) $\mathcal{I}_X(s_1 s_2) \ge \mathcal{I}_X(s_1) \wedge \mathcal{I}_X(s_2)$;
3) $\mathcal{N}_X(s_1 s_2) \le \mathcal{N}_X(s_1) \vee \mathcal{N}_X(s_2)$ for all $s_1, s_2 \in S$.

Definition 9. [29] A spherical fuzzy subset $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ of a semigroup S is called a spherical fuzzy left ideal (resp. right ideal) of S, if

1) $\mathcal{P}_X(s_1 s_2) \ge \mathcal{P}_X(s_2)$ (resp. $\mathcal{P}_X(s_1 s_2) \ge \mathcal{P}_X(s_1)$);
2) $\mathcal{I}_X(s_1 s_2) \ge \mathcal{I}_X(s_2)$ (resp. $\mathcal{I}_X(s_1 s_2) \ge \mathcal{I}_X(s_1)$);
3) $\mathcal{N}_X(s_1 s_2) \le \mathcal{N}_X(s_2)$ (resp. $\mathcal{N}_X(s_1 s_2) \le \mathcal{N}_X(s_1)$) for all $s_1, s_2 \in S$.

If X is both spherical fuzzy left and spherical fuzzy right ideal of S, then X is spherical fuzzy ideal of S.

Definition 10. [29] A spherical fuzzy subset $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ of a semigroup S is called a spherical fuzzy interior ideal of S, if

1) $\mathcal{P}_X(s_1 x s_2) \ge \mathcal{P}_X(x)$;
2) $\mathcal{I}_X(s_1 x s_2) \ge \mathcal{I}_X(x)$;
3) $\mathcal{N}_X(s_1 x s_2) \le \mathcal{N}_X(x)$ for all $s_1, s_2, x \in S$.

Definition 11. [29] A spherical fuzzy subsemigroup $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ of a semigroup S is called a spherical fuzzy bi-ideal of S, if

1) $\mathcal{P}_X(s_1 x s_2) \geq \mathcal{P}_X(s_1) \wedge \mathcal{P}_X(s_2)$
2) $\mathcal{I}_X(s_1 x s_2) \geq \mathcal{I}_X(s_1) \wedge \mathcal{I}_X(s_2)$
3) $\mathcal{N}_X(s_1 x s_2) \leq \mathcal{N}_X(s_1) \vee \mathcal{N}_X(s_2)$ for all $s_1, s_2, x \in S$.

Definition 12. [28] Let $(\mathcal{C}, U)$ be a soft binary relation from S_1 to S_2 and $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ be a spherical fuzzy subset of S_2. Then lower approximation $\underline{\mathcal{C}}^X = (\underline{\mathcal{C}}^{\mathcal{P}_X}, \underline{\mathcal{C}}^{\mathcal{I}_X}, \underline{\mathcal{C}}^{\mathcal{N}_X})$ and upper approximation $\overline{\mathcal{C}}^X = (\overline{\mathcal{C}}^{\mathcal{P}_X}, \overline{\mathcal{C}}^{\mathcal{I}_X}, \overline{\mathcal{C}}^{\mathcal{N}_X})$ of X in terms of aftersets are defined as follows:

$$\underline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_1) = \begin{cases} \bigwedge_{x \in s_1\mathcal{C}(u)} \mathcal{P}_X(x) & if \quad s_1\mathcal{C}(u) \neq \phi, \\ 0 & if \quad s_1\mathcal{C}(u) = \phi. \end{cases} \tag{1}$$

$$\underline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_1) = \begin{cases} \bigwedge_{x \in s_1\mathcal{C}(u)} \mathcal{I}_X(x) & if \quad s_1\mathcal{C}(u) \neq \phi, \\ 0 & if \quad s_1\mathcal{C}(u) = \phi. \end{cases} \tag{2}$$

$$\underline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_1) = \begin{cases} \bigvee_{x \in s_1\mathcal{C}(u)} \mathcal{N}_X(x) & if \quad s_1\mathcal{C}(u) \neq \phi, \\ 0 & if \quad s_1\mathcal{C}(u) = \phi. \end{cases} \tag{3}$$

And

$$\overline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_1) = \begin{cases} \bigvee_{x \in s_1\mathcal{C}(u)} \mathcal{P}_X(x) & if \quad s_1\mathcal{C}(u) \neq \phi, \\ 0 & if \quad s_1\mathcal{C}(u) = \phi. \end{cases} \tag{4}$$

$$\overline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_1) = \begin{cases} \bigvee_{x \in s_1\mathcal{C}(u)} \mathcal{I}_X(x) & if \quad s_1\mathcal{C}(u) \neq \phi, \\ 0 & if \quad s_1\mathcal{C}(u) = \phi. \end{cases} \tag{5}$$

$$\overline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_1) = \begin{cases} \bigwedge_{x \in s_1\mathcal{C}(u)} \mathcal{N}_X(x) & if \quad s_1\mathcal{C}(u) \neq \phi, \\ 0 & if \quad s_1\mathcal{C}(u) = \phi, \end{cases} \tag{6}$$

where $s_1\mathcal{C}(u) = \{x \in S_2 : (s_1, x) \in \mathcal{C}(u)\}$ is called afterset of $s_1 \in S_1$ for $u \in U$.

Definition 13. [28] Let $(\mathcal{C}, U)$ be a soft binary relation from S_1 to S_2 and $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ be a spherical fuzzy subset of S_1. Then, the lower approximation ${}^X\underline{\mathcal{C}} = ({}^{\mathcal{P}_X}\underline{\mathcal{C}}, {}^{\mathcal{I}_X}\underline{\mathcal{C}}, {}^{\mathcal{N}_X}\underline{\mathcal{C}})$ and upper approximation ${}^X\overline{\mathcal{C}} = ({}^{\mathcal{P}_X}\overline{\mathcal{C}}, {}^{\mathcal{I}_X}\overline{\mathcal{C}}, {}^{\mathcal{N}_X}\overline{\mathcal{C}})$ of X in terms of foresets are defined as follows:

$${}^{\mathcal{P}_X}\underline{\mathcal{C}}(u)(s_2) = \begin{cases} \bigwedge_{x \in \mathcal{C}(u)s_2} \mathcal{P}_X(x) & if \quad \mathcal{C}(u)s_2 \neq \phi, \\ 0 & if \quad \mathcal{C}(u)s_2 = \phi. \end{cases} \tag{7}$$

$${}^{\mathcal{I}_X}\underline{\mathcal{C}}(u)(s_2) = \begin{cases} \bigwedge_{x \in \mathcal{C}(u)s_2} \mathcal{I}_X(x) & if \quad \mathcal{C}(u)s_2 \neq \phi, \\ 0 & if \quad \mathcal{C}(u)s_2 = \phi. \end{cases} \tag{8}$$

$${}^{\mathcal{N}_X}\underline{\mathcal{C}}(u)(s_2) = \begin{cases} \bigvee_{x \in \mathcal{C}(u)s_2} \mathcal{N}_X(x) & if \quad \mathcal{C}(u)s_2 \neq \phi, \\ 0 & if \quad \mathcal{C}(u)s_2 = \phi. \end{cases} \tag{9}$$

And

$$^{\mathcal{P}_X}\overline{\mathcal{C}}(u)(s_2) = \begin{cases} \bigvee_{x\in\mathcal{C}(u)s_2} \mathcal{P}_X(x) & if \quad \mathcal{C}(u)s_2 \neq \phi, \\ 0 & if \quad \mathcal{C}(u)s_2 = \phi. \end{cases} \tag{10}$$

$$^{\mathcal{I}_X}\overline{\mathcal{C}}(u)(s_2) = \begin{cases} \bigvee_{x\in\mathcal{C}(u)s_2} \mathcal{I}_X(x) & if \quad \mathcal{C}(u)s_2 \neq \phi, \\ 0 & if \quad \mathcal{C}(u)s_2 = \phi. \end{cases} \tag{11}$$

$$^{\mathcal{N}_X}\overline{\mathcal{C}}(u)(s_2) = \begin{cases} \bigwedge_{x\in\mathcal{C}(u)s_2} \mathcal{N}_X(x) & if \quad \mathcal{C}(u)s_2 \neq \phi, \\ 0 & if \quad \mathcal{C}(u)s_2 = \phi, \end{cases} \tag{12}$$

where $\mathcal{C}(u)s_2 = \{x \in S_1 : (s_2, x) \in \mathcal{C}(u)\}$ is called a foreset of $s_2 \in S_2$ for $u \in U$.

Also, $\underline{\mathcal{C}}^X : U \to SF(S_1)$, $\overline{\mathcal{C}}^X : U \to SF(S_1)$, $^X\underline{\mathcal{C}} : U \to SF(S_2)$ and $^X\overline{\mathcal{C}} : U \to SF(S_2)$ where $SF(S_1)$ is a collection of all spherical fuzzy subsets in S_1, and $SF(S_2)$ is a collection of all spherical fuzzy subsets in S_2.

Theorem 1. [28] *Let $(\mathcal{C}, U)$ be a soft binary relation from S_1 to S_2, that is, $\mathcal{C}: U \to P(S_1 \times S_2)$. Then, for any spherical fuzzy subsets, $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ and $Y = \langle \mathcal{P}_Y, \mathcal{I}_Y, \mathcal{N}_Y \rangle$* of S_2 *the following statements hold:*

1) *If $X \subseteq Y$, then $\underline{\mathcal{C}}^X \subseteq \underline{\mathcal{C}}^Y$;*
2) *If $X \subseteq Y$, then $\overline{\mathcal{C}}^X \subseteq \overline{\mathcal{C}}^Y$;*
3) $\underline{\mathcal{C}}^X \cap \underline{\mathcal{C}}^Y = \underline{\mathcal{C}}^{X\cap Y}$;
4) $\overline{\mathcal{C}}^X \cap \overline{\mathcal{C}}^Y \supseteq \overline{\mathcal{C}}^{X\cap Y}$;
5) $\underline{\mathcal{C}}^X \cup \underline{\mathcal{C}}^Y \subseteq \underline{\mathcal{C}}^{X\cup Y}$;
6) $\overline{\mathcal{C}}^X \cup \overline{\mathcal{C}}^Y = \overline{\mathcal{C}}^{X\cup Y}$;
7) $\underline{\mathcal{C}}^X = \left(\overline{\mathcal{C}}^{X^c}\right)^c$ *if* $s_1\mathcal{C}(u) \neq \phi$ *for all* $s_1 \in S_1$*;*
8) $\overline{\mathcal{C}}^X = \left(\underline{\mathcal{C}}^{X^c}\right)^c$ *if* $s_1\mathcal{C}(u) \neq \phi$ *for all* $s_1 \in S_1$*;*
9) $\underline{\mathcal{C}}^{1_{S_2}} = 1_{S_1}$ *if* $s_1\mathcal{C}(u) \neq \phi$ *for all* $s_1 \in S_1$*;*
10) $\overline{\mathcal{C}}^{1_{S_2}} = 1_{S_1}$ *if* $s_1\mathcal{C}(u) \neq \phi$ *for all* $s_1 \in S_1$*;*
11) $\underline{\mathcal{C}}^{0_{S_2}} = 0_{S_1} = \overline{\mathcal{C}}^{0_{S_2}}$ *if* $s_1\mathcal{C}(u) \neq \phi$ *for all* $s_1 \in S_1$.

Theorem 2. [28] *Let $(\mathcal{C}, U)$ be a soft binary relations from S_1 to S_2, that is, $\mathcal{C}: U \to P(S_1 \times S_2)$. Then, for any spherical fuzzy subsets, $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ and $Y = \langle \mathcal{P}_Y, \mathcal{I}_Y, \mathcal{N}_Y \rangle$* of S_1 *the following statements hold:*

1) *If $X \subseteq Y$, then $^X\underline{\mathcal{C}} \subseteq {}^Y\underline{\mathcal{C}}$;*

2) *If* $X \subseteq Y$*, then* ${}^{X}\overline{\mathcal{C}} \subseteq {}^{Y}\overline{\mathcal{C}}$;

3) ${}^{X}\underline{\mathcal{C}} \cap {}^{Y}\underline{\mathcal{C}} = {}^{X\cap Y}\underline{\mathcal{C}}$;

4) ${}^{X}\overline{\mathcal{C}} \cap {}^{Y}\overline{\mathcal{C}} \supseteq {}^{X\cap Y}\overline{\mathcal{C}}$;

5) ${}^{X}\underline{\mathcal{C}} \cup {}^{Y}\underline{\mathcal{C}} \subseteq {}^{X\cup Y}\underline{\mathcal{C}}$;

6) ${}^{X}\overline{\mathcal{C}} \cup {}^{Y}\overline{\mathcal{C}} = {}^{X\cup Y}\overline{\mathcal{C}}$;

7) ${}^{X}\underline{\mathcal{C}} = \left({}^{X^c}\overline{\mathcal{C}}\right)^c$ *if* $\mathcal{C}(u)s_2 \neq \phi$ *for all* $s_2 \in S_2$*;*

8) ${}^{X}\overline{\mathcal{C}} = \left({}^{X^c}\underline{\mathcal{C}}\right)^c$ *if* $\mathcal{C}(u)s_2 \neq \phi$ *for all* $s_2 \in S_2$*;*

9) ${}^{1_{S_1}}\underline{\mathcal{C}} = 1_{S_2}$ *if* $\mathcal{C}(u)s_2 \neq \phi$ *for all* $s_2 \in S_2$*;*

10) ${}^{1_{S_1}}\overline{\mathcal{C}} = 1_{S_2}$ *if* $\mathcal{C}(u)s_2 \neq \phi$ *for all* $s_2 \in S_2$*;*

11) ${}^{0_{S_1}}\underline{\mathcal{C}} = 0_{S_2} = {}^{0_{S_1}}\overline{\mathcal{C}}$ *if* $\mathcal{C}(u)s_2 \neq \phi$ *for all* $s_2 \in S_2$.

To create a new model, we need some basic definitions and theorems. Here, the given definitions are of equivalence relations, rough set, soft set, spherical fuzzy subset, and spherical fuzzy soft set, and their properties in semigroup, lower, and upper approximations of the spherical fuzzy soft set reflect aftersets and foresets. This will be helpful to explore new work.

3. Approximation of spherical fuzzy ideals in semigroups by soft binary relation

In this section, prove that the upper approximations of the spherical fuzzy subsemigroup and spherical fuzzy (left, right, interior, bi-ideal) ideals of the semigroup are the spherical fuzzy soft subsemigroup and spherical fuzzy soft (left, right, interior, bi-ideal) ideals, respectively. We also discuss examples that show their converses are not true. Similarly, we do these for lower approximations.

Definition 14. A spherical fuzzy soft set (C, U) over a semigroup S is a called spherical fuzzy soft subsemigroup (left ideal, right ideal, interior ideal, bi-ideal) over S if $C(u)$ is a spherical fuzzy soft subsemigroup (left ideal, right ideal, interior ideal, bi-ideal) of S for all $u \in U$, respectively.

Definition 15. A soft binary relation $(\mathcal{C}, U)$ from S_1 to S_2 is a soft set over $S_1 \times S_2$, such as $\mathcal{C}: U \to P(S_1 \times S_2)$, where U is a subset of the set of parameters E.

$(\mathcal{C}, U)$ is a parameterized collection of binary relations from S_1 to S_2, so we have a binary relation $\mathcal{C}(u)$ from S_1 to S_2 for each $u \in U$.

Definition 16. For two semigroups S_1 and S_2, a soft binary relation $(\mathcal{C}, U)$ from S_1 to S_2 is a soft compatible regarding aftersets if $(s_1, s_2), (s_3, s_4) \in \mathcal{C}(u)$ implies $(s_1 s_3, s_2 s_4) \in \mathcal{C}(u)$ for all $s_1, s_3 \in S_1$ and $s_2, s_4 \in S_2$.

Generally, if $(\mathcal{C}, U)$ is a soft compatible relation from S_1 to S_2 regarding aftersets, then $s_1\mathcal{C}(u).s_2\mathcal{C}(u) \subseteq (s_1 s_2)\mathcal{C}(u)$ for all $u \in U$ and $s_1, s_2 \in S_1$. If $a \in s_1\mathcal{C}(u)$ and $b \in s_2\mathcal{C}(u)$, then $(s_1, a) \in \mathcal{C}(u)$ and $(s_2, b) \in \mathcal{C}(u)$, and by compatibility relation $(s_1 s_2, ab) \in \mathcal{C}(u)$ implies $ab \in s_1 s_2\mathcal{C}(u)$ for $s_1, s_2 \in S_1$ and $a, b \in S_2$. Similarly, for soft compatible relation $(\mathcal{C}, U)$ from

S_2 to S_1 regarding foresets, $\mathcal{C}(u)s_3.\mathcal{C}(u)s_4 \subseteq \mathcal{C}(u)(s_3s_4)$ for all $s_3, s_4 \in S_2$ and $u \in U$.

Definition 17. A soft compatible relation $(\mathcal{C}, U)$ from S_1 to S_2 is said to be soft complete regarding aftersets, if $s_1\mathcal{C}(u).\ s_2\mathcal{C}(u) = (s_1s_2)\mathcal{C}(u)$ for all $u \in U$ and $s_1, s_2 \in S_1$ and is said to be a soft complete relation regarding foresets if $\mathcal{C}(u)s_3.\mathcal{C}(u)s_4 = \mathcal{C}(u)(s_3s_4)$ for all $s_3, s_4 \in S_2$ and $u \in U$.

Theorem 3. *Let $(\mathcal{C}, U)$ be a soft compatible relation regarding aftersets from S_1 to S_2,*

1) *if $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ is a spherical fuzzy subsemigroup of S_2, then $(\overline{\mathcal{C}}^X, U)$ is a spherical fuzzy soft subsemigroup of S_1.*
2) *If $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ is a spherical fuzzy left (resp. right) ideal of S_2, then $(\overline{\mathcal{C}}^X, U)$ is a spherical fuzzy soft left (resp. right) ideal of S_1.*

Proof. 1) For a spherical fuzzy subsemigroup $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ of S_2, $\mathcal{P}_X(xy) \geq \mathcal{P}_X(x) \wedge \mathcal{P}_X(y)$, $\mathcal{I}_X(xy) \geq \mathcal{I}_X(x) \wedge \mathcal{I}_X(y)$ and $\mathcal{N}_X(xy) \leq \mathcal{N}_X(x) \vee \mathcal{N}_X(y)$ for all $x, y \in S_2$.

Now, for $s_1, s_2 \in S_1$,

$$\overline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_1) \wedge \overline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_2) = \Big(\bigvee_{x \in s_1\mathcal{C}(u)} \mathcal{P}_X(x)\Big) \wedge \Big(\bigvee_{y \in s_2\mathcal{C}(u)} \mathcal{P}_X(y)\Big)$$

$$= \bigvee_{x \in s_1\mathcal{C}(u)} \bigvee_{y \in s_2\mathcal{C}(u)} (\mathcal{P}_X(x) \wedge \mathcal{P}_X(y)) \leq \bigvee_{xy \in (s_1s_2)\mathcal{C}(u)} \mathcal{P}_X(xy)$$

$$= \bigvee_{s' \in (s_1s_2)\mathcal{C}(u)} \mathcal{P}_X(s') = \overline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_1s_2). \tag{13}$$

$$\overline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_1) \wedge \overline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_2) = \Big(\bigvee_{x \in s_1\mathcal{C}(u)} \mathcal{I}_X(x)\Big) \wedge \Big(\bigvee_{y \in s_2\mathcal{C}(u)} \mathcal{I}_X(y)\Big)$$

$$= \bigvee_{x \in s_1\mathcal{C}(u)} \bigvee_{y \in s_2\mathcal{C}(u)} (\mathcal{I}_X(x) \wedge \mathcal{I}_X(y)) \leq \bigvee_{xy \in (s_1s_2)\mathcal{C}(u)} \mathcal{I}_X(xy)$$

$$= \bigvee_{s' \in (s_1s_2)\mathcal{C}(u)} \mathcal{I}_X(s') = \overline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_1s_2). \tag{14}$$

Additionally,

$$\overline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_1) \vee \overline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_2) = \Big(\bigwedge_{x \in s_1\mathcal{C}(u)} \mathcal{N}_X(x)\Big) \vee \Big(\bigwedge_{y \in s_2\mathcal{C}(u)} \mathcal{N}_X(y)\Big)$$

$$= \bigwedge_{x \in s_1\mathcal{C}(u)} \bigwedge_{y \in s_2\mathcal{C}(u)} (\mathcal{N}_X(x) \vee \mathcal{N}_X(y)) \geq \bigwedge_{xy \in (s_1s_2)\mathcal{C}(u)} \mathcal{N}_X(xy)$$

$$= \bigwedge_{s' \in (s_1s_2)\mathcal{C}(u)} \mathcal{N}_X(s') = \overline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_1s_2). \tag{15}$$

Hence, $\overline{\mathcal{C}}^X(u)$ is a spherical fuzzy subsemigroup of S_1 for all $u \in U$. Thus, $(\overline{\mathcal{C}}^X, U)$ is a spherical fuzzy soft subsemigroup of S_1 regarding aftersets.

2) For a spherical fuzzy left ideal $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ of S_2, $\mathcal{P}_X(xy) \geq \mathcal{P}_X(y)$, $\mathcal{I}_X(xy) \geq \mathcal{I}_X(y)$ and $\mathcal{N}_X(xy) \leq \mathcal{N}_X(y)$ for all $x, y \in S_2$.

Now for $s_1, s_2 \in S_1$,

$$\overline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_2) = \bigvee_{y\in s_2\mathcal{C}(u)} \mathcal{P}_X(y) \leq \bigvee_{x\in s_1\mathcal{C}(u)} \bigvee_{y\in s_s\mathcal{C}(u)} \mathcal{P}_X(xy) \leq \bigvee_{xy\in(s_1s_2)\mathcal{C}(u)} \mathcal{P}_X(xy)$$

$$= \bigvee_{s'\in(s_1s_2)\mathcal{C}(u)} \mathcal{P}_X(s') = \overline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_1s_2). \quad (16)$$

$$\overline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_2) = \bigvee_{y\in s_2\mathcal{C}(u)} \mathcal{I}_X(y) \leq \bigvee_{x\in s_1\mathcal{C}(u)} \bigvee_{y\in s_s\mathcal{C}(u)} \mathcal{I}_X(xy) \leq \bigvee_{xy\in(s_1s_2)\mathcal{C}(u)} \mathcal{I}_X(xy)$$

$$= \bigvee_{s'\in(s_1s_2)\mathcal{C}(u)} \mathcal{I}_X(s') = \overline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_1s_2). \quad (17)$$

Additionally,

$$\overline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_2) = \bigwedge_{y\in s_2\mathcal{C}(u)} \mathcal{N}_X(y) \geq \bigwedge_{x\in s_1\mathcal{C}(u)} \bigwedge_{y\in s_s\mathcal{C}(u)} \mathcal{N}_X(xy) \geq \bigwedge_{xy\in(s_1s_2)\mathcal{C}(u)} \mathcal{N}_X(xy)$$

$$= \bigwedge_{s'\in(s_1s_2)\mathcal{C}(u)} \mathcal{N}_X(s') = \overline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_1s_2). \quad (18)$$

Hence, $\overline{\mathcal{C}}^{X}(u)$ is a spherical fuzzy left ideal of S_1 for all $u \in U$. Thus, $(\overline{\mathcal{C}}^{X}, U)$ is a spherical fuzzy soft left ideal of S_1 regarding aftersets.

For a spherical fuzzy right ideal $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ of S_2, $\mathcal{P}_X(xy) \geq \mathcal{P}_X(x)$, $\mathcal{I}_X(xy) \geq \mathcal{I}_X(x)$ and $\mathcal{N}_X(xy) \leq \mathcal{N}_X(x)$ for all $x, y \in S_2$.
Now, for $s_1, s_2 \in S_1$,

$$\overline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_2) = \bigvee_{x\in s_2\mathcal{C}(u)} \mathcal{P}_X(x) \leq \bigvee_{x\in s_1\mathcal{C}(u)} \bigvee_{y\in s_s\mathcal{C}(u)} \mathcal{P}_X(xy) \leq \bigvee_{xy\in(s_1s_2)\mathcal{C}(u)} \mathcal{P}_X(xy)$$

$$= \bigvee_{s'\in(s_1s_2)\mathcal{C}(u)} \mathcal{P}_X(s') = \overline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_1s_2). \quad (19)$$

$$\overline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_2) = \bigvee_{x\in s_2\mathcal{C}(u)} \mathcal{I}_X(x) \leq \bigvee_{x\in s_1\mathcal{C}(u)} \bigvee_{y\in s_s\mathcal{C}(u)} \mathcal{I}_X(xy) \leq \bigvee_{xy\in(s_1s_2)\mathcal{C}(u)} \mathcal{I}_X(xy)$$

$$= \bigvee_{s'\in(s_1s_2)\mathcal{C}(u)} \mathcal{I}_X(s') = \overline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_1s_2). \quad (20)$$

Additionally,

$$\overline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_2) = \bigwedge_{x\in s_2\mathcal{C}(u)} \mathcal{N}_X(x) \geq \bigwedge_{x\in s_1\mathcal{C}(u)} \bigwedge_{y\in s_s\mathcal{C}(u)} \mathcal{N}_X(xy) \geq \bigwedge_{xy\in(s_1s_2)\mathcal{C}(u)} \mathcal{N}_X(xy)$$

$$= \bigwedge_{s'\in(s_1s_2)\mathcal{C}(u)} \mathcal{N}_X(s') = \overline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_1s_2). \quad (21)$$

Hence, $\overline{\mathcal{C}}^{X}(u)$ is a spherical fuzzy right ideal of S_1 for all $u \in U$. Thus, $(\overline{\mathcal{C}}^{X}, U)$ is a spherical fuzzy soft right ideal of S_1 regarding aftersets.

Converse of Theorem 3, does not hold.

Example 1. Let $S_1 = \{s_1, s_2, s_3, s_4\}$ and $S_2 = \{1,2,3,4\}$ be two semigroups under multiplication as given in Tables 1 and 2 respectively.

Table 1. Multiplication on S_1.

$\cdot$	$s1$	$s2$	$s3$	$s4$
$s1$	$s2$	$s2$	$s4$	$s4$
$s2$	$s2$	$s2$	$s4$	$s4$
$s3$	$s4$	$s4$	$s3$	$s4$
$s4$	$s4$	$s4$	$s4$	$s4$

Table 2. Multiplication on S_2.

$\cdot$	1	2	3	4
1	1	1	3	4
2	1	2	3	4
3	1	3	3	4
4	1	4	3	1

Let $U = \{u_1, u_2\}$ and define $\mathcal{C} : U \to P(S_1 \times S_2)$as

$$\mathcal{C}(u_1) = \{(s_1,1),\ (s_2,2),\ (s_3,3),\ (s_4,4),\ (s_2,1),\ (s_4,3),\ (s_4,1)\},$$

$$\mathcal{C}(u_2) = \{(s_1,1),\ (s_2,2),\ (s_3,3),\ (s_4,4),\ (s_2,1),\ (s_4,3),\ (s_4,1),\ (s_2,3)\}.$$

Then, $(\mathcal{C}, U)$ is a soft compatible relation regarding aftersets from S_1 to S_2.
Now, $s_1\mathcal{C}(u_1) = \{1\}$, $s_2\mathcal{C}(u_1) = \{1,2\}$, $s_3\mathcal{C}(u_1) = \{3\}$, $s_4\mathcal{C}(u_1) = \{1,3,4\}$ and $s_1\mathcal{C}(u_2) = \{1\}$, $s_2\mathcal{C}(u_2) = \{1,2,3\}$, $s_3\mathcal{C}(u_2) = \{3\}$, $s_4\mathcal{C}(u_2) = \{1,3,4\}$.
Define $X : S_2 \to [0,1]$ as given in Table 3.

Table 3. Spherical fuzzy subset X.

X	1	2	3	4
$\mathcal{P}_X$	0.5	0.3	0.3	0.9
$\mathcal{I}_X$	0	0.3	0.1	0.1
$\mathcal{N}_X$	0.2	0.4	0.6	0

Then, X is not a spherical fuzzy subsemigroup of S_2, because for $x = 4,\ y = 4$ $\mathcal{P}_X(44) \ngeq \mathcal{P}_X(4) \wedge \mathcal{P}_X(4)$, $\mathcal{I}_X(44) \ngeq \mathcal{I}_X(4) \wedge \mathcal{I}_X(4)$ and $\mathcal{N}_X(44) \nleq \mathcal{N}_X(4) \vee \mathcal{N}_X(4)$. However, $\overline{\mathcal{C}}^X(u_1)$, $\overline{\mathcal{C}}^X(u_2)$ are spherical fuzzy subsemigroups of S_1 as given in Table 4. Therefore, $(\overline{\mathcal{C}}^X, U)$ is spherical fuzzy soft subsemigroup of S_1 regarding aftersets.

Table 4. Upper approximation of X.

	s_1	s_2	s_3	s_4
$\overline{\mathcal{C}}^X(u_1)$	(0.5,0,0.2)	(0.5,0.3,0.2)	(0.3,0.1,0.6)	(0.9,0,0.1)
$\overline{\mathcal{C}}^X(u_2)$	(0.5,0,0.2)	(0.5,0.3,0.2)	(0.3,0.1,0.6)	(0.9,0,0.1)

Also, X is not a spherical fuzzy left ideal of S_2, because for $x = 4,\ y = 4,\ \mathcal{P}_X(44) \ngeq \mathcal{P}_X(4)$, $\mathcal{I}_X(44) \ngeq \mathcal{I}_X(4)$ and $\mathcal{N}_X(44) \nleq \mathcal{N}_X(4)$. However, $\overline{\mathcal{C}}^X(u_1)$, $\overline{\mathcal{C}}^X(u_2)$ are spherical fuzzy left ideals of S_1 as given in Table 4. Therefore, $(\overline{\mathcal{C}}^X, U)$ is spherical fuzzy soft left ideal of S_1 regarding aftersets.

Theorem 4. *Let $(\mathcal{C}, U)$ be a soft compatible relation regarding foresets from S_1 to S_2,*

1) *if $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ is a spherical fuzzy subsemigroup of S_1, then $({}^{X}\overline{\mathcal{C}}, U)$ is a spherical fuzzy soft subsemigroup of S_2.*
2) *If $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ is a spherical fuzzy left (resp. right) ideal of S_1, then $({}^{X}\overline{\mathcal{C}}, U)$ is a spherical fuzzy soft left (resp. right) ideal of S_2.*

Proof. The proof is similar to proof of Theorem 3.

Theorem 5. *Let $(\mathcal{C}, U)$ be a soft complete relation regarding aftersets from S_1 to S_2,*

1) *if $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ is a spherical fuzzy subsemigroup of S_2, then $(\underline{\mathcal{C}}^X, U)$ is spherical fuzzy soft subsemigroup of S_1.*
2) *If $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ is a spherical fuzzy left (resp. right) ideal of S_2, then $(\underline{\mathcal{C}}^X, U)$ is spherical fuzzy soft left (resp. right) ideal of S_1.*

Proof. 1) For a spherical fuzzy subsemigroup $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ of S_2, $\mathcal{P}_X(xy) \geq \mathcal{P}_X(x) \wedge \mathcal{P}_X(y)$, $\mathcal{I}_X(xy) \geq \mathcal{I}_X(x) \wedge \mathcal{I}_X(y)$ and $\mathcal{N}_X(xy) \leq \mathcal{N}_X(x) \vee \mathcal{N}_X(y)$ for all $x, y \in S_2$.
Now for $s_1, s_2 \in S_1$,

$$\underline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_1) \wedge \underline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_2) = \Big(\bigwedge_{x \in s_1\mathcal{C}(u)} \mathcal{P}_X(x)\Big) \wedge \Big(\bigwedge_{y \in s_2\mathcal{C}(u)} \mathcal{P}_X(y)\Big)$$

$$\leq \bigwedge_{x \in s_1\mathcal{C}(u)} \bigwedge_{y \in s_2\mathcal{C}(u)} (\mathcal{P}_X(x) \wedge \mathcal{P}_X(y)) \leq \bigwedge_{xy \in (s_1 s_2)\mathcal{C}(u)} \mathcal{P}_X(xy)$$

$$= \bigwedge_{s' \in (s_1 s_2)\mathcal{C}(u)} \mathcal{P}_X(s') = \underline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_1 s_2). \tag{22}$$

$$\underline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_1) \wedge \underline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_2) = \Big(\bigwedge_{x \in s_1\mathcal{C}(u)} \mathcal{I}_X(x)\Big) \wedge \Big(\bigwedge_{y \in s_2\mathcal{C}(u)} \mathcal{I}_X(y)\Big)$$

$$\leq \bigwedge_{x \in s_1\mathcal{C}(u)} \bigwedge_{y \in s_2\mathcal{C}(u)} (\mathcal{I}_X(x) \wedge \mathcal{I}_X(y)) \leq \bigwedge_{xy \in (s_1 s_2)\mathcal{C}(u)} \mathcal{I}_X(xy)$$

$$= \bigwedge_{s' \in (s_1 s_2)\mathcal{C}(u)} \mathcal{I}_X(s') = \underline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_1 s_2). \tag{23}$$

And

$$\underline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_1) \vee \underline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_2) = \Big(\bigvee_{x \in s_1\mathcal{C}(u)} \mathcal{N}_X(x)\Big) \vee \Big(\bigvee_{y \in s_2\mathcal{C}(u)} \mathcal{N}_X(y)\Big)$$

$$\geq \bigvee_{x\in s_1\mathcal{C}(u)} \bigvee_{y\in s_2\mathcal{C}(u)} (\mathcal{N}_X(x) \vee \mathcal{N}_X(y)) \geq \bigvee_{xy\in(s_1s_2)\mathcal{C}(u)} \mathcal{N}_X(xy)$$

$$= \bigvee_{s'\in(s_1s_2)\mathcal{C}(u)} \mathcal{N}_X(s') = \underline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_1s_2). \tag{24}$$

Hence, $\underline{\mathcal{C}}^X(u)$ is a spherical fuzzy subsemigroup of S_1 for all $u \in U$. Thus, $(\underline{\mathcal{C}}^X, U)$ is spherical fuzzy soft subsemigroup of S_1 regarding aftersets.
2) For a spherical fuzzy left ideal $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ of S_2, $\mathcal{P}_X(xy) \geq \mathcal{P}_X(y)$, $\mathcal{I}_X(xy) \geq \mathcal{I}_X(y)$ and $\mathcal{N}_X(xy) \leq \mathcal{N}_X(y)$ for all $x, y \in S_2$.
Now, for $s_1, s_2 \in S_1$,

$$\underline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_2) = \bigwedge_{y\in s_2\mathcal{C}(u)} \mathcal{P}_X(y) \leq \bigwedge_{x\in s_1\mathcal{C}(u)} \bigwedge_{y\in s_2\mathcal{C}(u)} \mathcal{P}_X(xy) = \bigwedge_{xy\in(s_1s_2)\mathcal{C}(u)} \mathcal{P}_X(xy)$$

$$= \bigwedge_{s'\in(s_1s_2)\mathcal{C}(u)} \mathcal{P}_X(s') = \underline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_1s_2). \tag{25}$$

$$\underline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_2) = \bigwedge_{y\in s_2\mathcal{C}(u)} \mathcal{I}_X(y) \leq \bigwedge_{x\in s_1\mathcal{C}(u)} \bigwedge_{y\in s_2\mathcal{C}(u)} \mathcal{I}_X(xy) = \bigwedge_{xy\in(s_1s_2)\mathcal{C}(u)} \mathcal{I}_X(xy)$$

$$= \bigwedge_{s'\in(s_1s_2)\mathcal{C}(u)} \mathcal{I}_X(s') = \underline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_1s_2). \tag{26}$$

Additionally,

$$\underline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_2) = \bigvee_{y\in s_2\mathcal{C}(u)} \mathcal{N}_X(y) \geq \bigvee_{x\in s_1\mathcal{C}(u)} \bigvee_{y\in s_2\mathcal{C}(u)} \mathcal{N}_X(xy) = \bigvee_{xy\in(s_1s_2)\mathcal{C}(u)} \mathcal{N}_X(xy)$$

$$= \bigvee_{s'\in(s_1s_2)\mathcal{C}(u)} \mathcal{N}_X(s') = \underline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_1s_2). \tag{27}$$

Hence, $\underline{\mathcal{C}}^X(u)$ is a spherical fuzzy left ideal of S_1 for all $u \in U$. Thus, $(\underline{\mathcal{C}}^X, U)$ is a spherical fuzzy soft left ideal of S_1 regarding aftersets.

For a spherical fuzzy right ideal $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ of S_2, $\mathcal{P}_X(xy) \geq \mathcal{P}_X(x)$, $\mathcal{I}_X(xy) \geq \mathcal{I}_X(x)$ and $\mathcal{N}_X(xy) \leq \mathcal{N}_X(x)$ for all $x, y \in S_2$.
Now, for $s_1, s_2 \in S_1$,

$$\underline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_2) = \bigwedge_{x\in s_2\mathcal{C}(u)} \mathcal{P}_X(x) \leq \bigwedge_{x\in s_1\mathcal{C}(u)} \bigwedge_{y\in s_2\mathcal{C}(u)} \mathcal{P}_X(xy) = \bigwedge_{xy\in(s_1s_2)\mathcal{C}(u)} \mathcal{P}_X(xy)$$

$$= \bigwedge_{s'\in(s_1s_2)\mathcal{C}(u)} \mathcal{P}_X(s') = \underline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_1s_2). \tag{28}$$

$$\underline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_2) = \bigwedge_{x\in s_2\mathcal{C}(u)} \mathcal{I}_X(x) \leq \bigwedge_{x\in s_1\mathcal{C}(u)} \bigwedge_{y\in s_2\mathcal{C}(u)} \mathcal{I}_X(xy) = \bigwedge_{xy\in(s_1s_2)\mathcal{C}(u)} \mathcal{I}_X(xy)$$

$$= \bigwedge_{s'\in(s_1s_2)\mathcal{C}(u)} \mathcal{I}_X(s') = \underline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_1s_2). \tag{29}$$

Additionally,

$$\underline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_2) \;=\; \bigvee_{x\in s_2\mathcal{C}(u)} \mathcal{N}_X(x) \;\geq\; \bigvee_{x\in s_1\mathcal{C}(u)} \; \bigvee_{y\in s_2\mathcal{C}(u)} \mathcal{N}_X(xy) = \bigvee_{xy\in (s_1s_2)\mathcal{C}(u)} \mathcal{N}_X(xy)$$

$$= \bigvee_{s'\in (s_1s_2)\mathcal{C}(u)} \mathcal{N}_X(s') = \underline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_1s_2). \tag{30}$$

Hence, $\underline{\mathcal{C}}^X(u)$ is a spherical fuzzy right ideal of S_1 for all $u \in U$. Thus, $(\underline{\mathcal{C}}^X, U)$ is a spherical fuzzy soft right ideal of S_1 regarding aftersets.

Theorem 5 does not hold.

Example 2. Let $S_1 = \{s_1, s_2, s_3, s_4\}$ and $S_2 = \{1,2,3,4\}$ be two semigroups under multiplication as given in Tables 5 and 6 respectively.

Table 5. Multiplication on S_1.

$\cdot$	$\boldsymbol{s_1}$	$\boldsymbol{s_2}$	$\boldsymbol{s_3}$	$\boldsymbol{s_4}$
s_1	s_1	s_1	s_1	s_4
s_2	s_1	s_2	s_1	s_4
s_3	s_1	s_1	s_3	s_4
s_4	s_4	s_4	s_4	s_4

Table 6. Multiplication on S_2.

$\cdot$	**1**	**2**	**3**	**4**
1	1	2	3	4
2	2	2	2	2
3	3	3	3	3
4	4	3	2	1

Let $U = \{u_1, u_2\}$ and define $\mathcal{C} : U \rightarrow P(S_1 \times S_2)$ as

$$\mathcal{C}(u_1) = \{(s_1,2),\ (s_1,3),\ (s_2,2),\ (s_2,3),\ (s_3,2),\ (s_3,3),\ (s_4,2),\ (s_4,3)\},$$

$$\mathcal{C}(u_2) = \{(s_1,3),\ (s_2,3),\ (s_3,3),\ (s_4,3)\}.$$

Then, $(\mathcal{C}, U)$ is a soft complete relation regarding aftersets from S_1 to S_2.

Now, $s_1\mathcal{C}(u_1) = \{2,3\} = s_2\mathcal{C}(u_1) = s_3\mathcal{C}(u_1) = s_4\mathcal{C}(u_1)$ and

$s_1\mathcal{C}(u_2) = \{3\} = s_2\mathcal{C}(u_2) = s_3\mathcal{C}(u_2) = s_4\mathcal{C}(u_2)$.

Define X: $S_2 \rightarrow [0,1]$ as given in Table 7.

Table 7. Spherical fuzzy subset X.

$\boldsymbol{X}$	**1**	**2**	**3**	**4**
$\mathcal{P}_X$	0.2	0.3	0.4	0.5
$\mathcal{I}_X$	0.2	0.2	0.4	0.5
$\mathcal{N}_X$	0.5	0.4	0.1	0

Then, X is not a spherical fuzzy subsemigroup of S_2, because for $x = 4$, $y = 4$, $\mathcal{P}_X(44) \ngeq \mathcal{P}_X(4) \wedge \mathcal{P}_X$, $\mathcal{I}_X(44) \ngeq \mathcal{I}_X(4) \wedge \mathcal{I}_X(4)$ and $\mathcal{N}_X(44) \nleq \mathcal{N}_X(4) \vee \mathcal{N}_X(4)$. However, $\underline{\mathcal{C}}^X(u_1)$,

$\underline{\mathcal{C}}^X(u_2)$ are spherical fuzzy subsemigroups of S_1 as given in Table 8. Therefore, $(\underline{\mathcal{C}}^X, U)$ is spherical fuzzy soft subsemigroup of S_1 regarding aftersets.

Table 8. Upper approximation of X.

	s_1	s_2	s_3	s_4
$\underline{\mathcal{C}}^X(u_1)$	(0.3,0.2,0.4)	(0.3,0.2,0.4)	(0.3,0.2,0.4)	(0.3,0.2,0.4)
$\underline{\mathcal{C}}^X(u_2)$	(0.4,0.4,0.1)	(0.4,0.4,0.1)	(0.4,0.4,0.1)	(0.4,0.4,0.1)

Also, X is not a spherical fuzzy left ideal of S_2, because for $x = 3$, $y = 4$, $\mathcal{P}_X(34) = \mathcal{P}_X(4)$, $\mathcal{I}_X(34) \ngeq \mathcal{I}_X(4)$ and $\mathcal{N}_X(34) \nleq \mathcal{N}_X(4)$. However, $\underline{\mathcal{C}}^X(u_1)$, $\underline{\mathcal{C}}^X(u_2)$ are spherical fuzzy left ideals of S_1 as given in Table 8. Therefore, $(\underline{\mathcal{C}}^X, U)$ is a spherical fuzzy soft left ideal of S_1 regarding aftersets.

In the following example, we show that condition of complete relation is necessary for lower approximation.

Example 3. Consider the semigroups and soft binary relations of Example 1. Define $Y : S_2 \to [0,1]$ as given in Table 9.

Table 9. Spherical fuzzy subset Y.

Y	1	2	3	4
$\mathcal{P}_Y$	0.7	0.7	0.8	0
$\mathcal{I}_Y$	0.1	0.1	0.2	0
$\mathcal{N}_Y$	0.1	0.1	0	1

Then, Y is a spherical fuzzy left ideal of S_2. Also, lower approximations of Y are given in Table 10.

Table 10. Lower approximation of Y.

	s_1	s_2	s_3	s_4
$\underline{\mathcal{C}}^Y(u_1)$	(0.7,0.1,0.1)	(0.7,0.1,0.1)	(0.8,0.2,0)	(0,0,1)
$\underline{\mathcal{C}}^Y(u_2)$	(0.7,0.1,0.1)	(0.7,0.1,0.1)	(0.8,0.2,0)	(0,0,1)

Here, $\underline{\mathcal{C}}^Y(u_1)$ and $\underline{\mathcal{C}}^Y(u_2)$ are not spherical fuzzy left ideals of S_1. Take $x = s_2$, $y = s_3$, then $\underline{\mathcal{C}}^{\mathcal{P}_Y}(u_1)(s_2 s_3) \ngeq \underline{\mathcal{C}}^{\mathcal{P}_Y}(u_1)(s_3)$, $\underline{\mathcal{C}}^{\mathcal{I}_Y}(u_1)(s_2 s_3) \ngeq \underline{\mathcal{C}}^{\mathcal{I}_Y}(u_1)(s_3)$, and $\underline{\mathcal{C}}^{\mathcal{N}_Y}(u_1)(s_2 s_3) \nleq \underline{\mathcal{C}}^{\mathcal{N}_Y}(u_1)(s_3)$.

Moreover, $\underline{\mathcal{C}}^{\mathcal{P}_Y}(u_2)(s_2 s_3) \ngeq \underline{\mathcal{C}}^{\mathcal{P}_Y}(u_2)$, $\underline{\mathcal{C}}^{\mathcal{I}_Y}(u_2)(s_2 s_3) \ngeq \underline{\mathcal{C}}^{\mathcal{I}_Y}(u_2)$, and $\underline{\mathcal{C}}^{\mathcal{N}_Y}(u_2)(s_2 s_3) \nleq \underline{\mathcal{C}}^{\mathcal{N}_Y}(u_2)(s_3)$.

This example shows that if the soft relation is not complete, then the lower approximation of the spherical fuzzy left ideal is not spherical fuzzy soft left ideal of S_1.

Theorem 6. *Let $(\mathcal{C}, U)$ be a soft complete relation regarding foresets from S_1 to S_2,*

1) *if $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ is a spherical fuzzy subsemigroup of S_1, then $({}^X\underline{\mathcal{C}}, U)$ is a spherical fuzzy soft subsemigroup of S_2.*
2) *If $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ is a spherical fuzzy left (resp. right) ideal of S_1, then $({}^X\underline{\mathcal{C}}, U)$ is a spherical fuzzy soft left (resp. right) ideal of S_2.*

Proof. The proof is similar to proof of Theorem 5.

Theorem 7. *Suppose* $(\mathcal{C}, U)$ *is a soft binary relation from* S_1 *to* S_2*, that is,* $\mathcal{C}: U \rightarrow P(S_1 \times S_2)$*. Then for any spherical fuzzy right ideal* $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ *and spherical fuzzy left ideal* $Y = \langle \mathcal{P}_Y, \mathcal{I}_Y, \mathcal{N}_Y \rangle$ of S_2, $\overline{\mathcal{C}}^{XY} \subseteq \overline{\mathcal{C}}^{X} \cap \overline{\mathcal{C}}^{Y}$.

Proof. For a spherical fuzzy right ideal $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ and a spherical fuzzy left ideal $Y = \langle \mathcal{P}_Y, \mathcal{I}_Y, \mathcal{N}_Y \rangle$ of S_2, we have $XY \subseteq X \cap Y$. Now, from Theorem 1 (part 2 and 4),

$$\overline{\mathcal{C}}^{\mathcal{P}_X \mathcal{P}_Y} \subseteq \overline{\mathcal{C}}^{\mathcal{P}_X \cap \mathcal{P}_Y} \subseteq \overline{\mathcal{C}}^{\mathcal{P}_X} \cap \overline{\mathcal{C}}^{\mathcal{P}_Y};$$

$$\overline{\mathcal{C}}^{\mathcal{I}_X \mathcal{I}_Y} \subseteq \overline{\mathcal{C}}^{\mathcal{I}_X \cap \mathcal{I}_Y} \subseteq \overline{\mathcal{C}}^{\mathcal{I}_X} \cap \overline{\mathcal{C}}^{\mathcal{I}_Y};$$

$$\overline{\mathcal{C}}^{\mathcal{N}_X \mathcal{N}_Y} \supseteq \overline{\mathcal{C}}^{\mathcal{N}_X \cap \mathcal{N}_Y} \supseteq \overline{\mathcal{C}}^{\mathcal{N}_X} \cap \overline{\mathcal{C}}^{\mathcal{N}_Y}, \text{ implies } \overline{\mathcal{C}}^{XY} \subseteq \overline{\mathcal{C}}^{X} \cap \overline{\mathcal{C}}^{Y}.$$

Theorem 8. *Suppose* $(\mathcal{C}, U)$ *is a soft binary relation from* S_1 *to* S_2*, that is,* $\mathcal{C}: U \rightarrow P(S_1 \times S_2)$*. Then for any spherical fuzzy right ideal* $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ *and spherical fuzzy left ideal* $Y = \langle \mathcal{P}_Y, \mathcal{I}_Y, \mathcal{N}_Y \rangle$ *of* S_1, ${}^{XY}\overline{\mathcal{C}} \subseteq {}^{X}\overline{\mathcal{C}} \cap {}^{Y}\overline{\mathcal{C}}$.

Proof. The proof is similar to proof of Theorem 7.

Theorem 9. *Suppose* $(\mathcal{C}, U)$ *is a soft binary relation from* S_1 *to* S_2*, that is,* $\mathcal{C}: U \rightarrow P(S_1 \times S_2)$*. Then for any spherical fuzzy right ideal* $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ *and spherical fuzzy left ideal* $Y = \langle \mathcal{P}_Y, \mathcal{I}_Y, \mathcal{N}_Y \rangle$ *of* S_2, $\underline{\mathcal{C}}^{XY} \subseteq \underline{\mathcal{C}}^{X} \cap \underline{\mathcal{C}}^{Y}$.

Proof. For a spherical fuzzy right ideal $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ and a spherical fuzzy left ideal $Y = \langle \mathcal{P}_Y, \mathcal{I}_Y, \mathcal{N}_Y \rangle$ of S_2, we have $XY \subseteq X \cap Y$. Now, from Theorem 1 (part 1 and 3),

$$\underline{\mathcal{C}}^{\mathcal{P}_X \mathcal{P}_Y} \subseteq \underline{\mathcal{C}}^{\mathcal{P}_X \cap \mathcal{P}_Y} \subseteq \underline{\mathcal{C}}^{\mathcal{P}_X} \cap \underline{\mathcal{C}}^{\mathcal{P}_Y};$$

$$\underline{\mathcal{C}}^{\mathcal{I}_X \mathcal{I}_Y} \subseteq \underline{\mathcal{C}}^{\mathcal{I}_X \cap \mathcal{I}_Y} \subseteq \underline{\mathcal{C}}^{\mathcal{I}_X} \cap \underline{\mathcal{C}}^{\mathcal{I}_Y};$$

$$\underline{\mathcal{C}}^{\mathcal{N}_X \mathcal{N}_Y} \supseteq \underline{\mathcal{C}}^{\mathcal{N}_X \cap \mathcal{N}_Y} \supseteq \underline{\mathcal{C}}^{\mathcal{N}_X} \cap \underline{\mathcal{C}}^{\mathcal{N}_Y}, \text{ implies } \underline{\mathcal{C}}^{XY} \subseteq \underline{\mathcal{C}}^{X} \cap \underline{\mathcal{C}}^{Y}.$$

Theorem 10. *Suppose* $(\mathcal{C}, U)$ *is a soft binary relation from* S_1 *to* S_2*, that is,* $\mathcal{C}: U \rightarrow P(S_1 \times S_2)$*. Then for any spherical fuzzy right ideal* $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ *and spherical fuzzy left ideal* $Y = \langle \mathcal{P}_Y, \mathcal{I}_Y, \mathcal{N}_Y \rangle$ *of* S_1, ${}^{XY}\underline{\mathcal{C}} \subseteq {}^{X}\underline{\mathcal{C}} \cap {}^{Y}\underline{\mathcal{C}}$.

Proof. The proof is similar to proof of Theorem 9.

Theorem 11. *Let* $(\mathcal{C}, U)$ *be a soft compatible relation regarding aftersets from* S_1 *to* S_2*. If* $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ *is a spherical fuzzy interior ideal of* S_2*, then* $(\overline{\mathcal{C}}^{X}, U)$ *is a spherical fuzzy soft interior ideal of* S_1.

Proof. For a spherical fuzzy interior ideal $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ of S_2, $\mathcal{P}_X(xby) \geq \mathcal{P}_X(b)$, $\mathcal{I}_X(xby) \geq \mathcal{I}_X(b)$ and $\mathcal{N}_X(xby) \leq \mathcal{N}_X(b)$ for all $x, y, b \in S_2$.

Now for $s_1, s_2, a \in S_1$,

$$\overline{\mathcal{C}}^{\mathcal{P}_X}(u)(a) = \bigvee_{b \in a\mathcal{C}(u)} \mathcal{P}_X(b) \leq \bigvee_{x \in s_1\mathcal{C}(u)} \bigvee_{b \in a\mathcal{C}(u)} \bigvee_{y \in s_s\mathcal{C}(u)} \mathcal{P}_X(xby) \leq \bigvee_{xby \in (s_1 a s_2)\mathcal{C}(u)} \mathcal{P}_X(xby)$$

$$= \bigvee_{s' \in (s_1 a s_2)\mathcal{C}(u)} \mathcal{P}_X(s') = \overline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_1 a s_2). \tag{31}$$

$$\overline{\mathcal{C}}^{\mathcal{I}_X}(u)(a) = \bigvee_{b\in a\mathcal{C}(u)} \mathcal{I}_X(b) \leq \bigvee_{x\in s_1\mathcal{C}(u)} \bigvee_{b\in a\mathcal{C}(u)} \bigvee_{y\in s_s\mathcal{C}(u)} \mathcal{I}_X(xby) \leq \bigvee_{xby\in(s_1as_2)\mathcal{C}(u)} \mathcal{I}_X(xby)$$

$$= \bigvee_{s'\in(s_1as_2)\mathcal{C}(u)} \mathcal{I}_X(s') = \overline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_1as_2). \tag{32}$$

Additionally,

$$\overline{\mathcal{C}}^{\mathcal{N}_X}(u)(a) = \bigwedge_{b\in a\mathcal{C}(u)} \mathcal{N}_X(b) \geq \bigwedge_{x\in s_1\mathcal{C}(u)} \bigwedge_{b\in a\mathcal{C}(u)} \bigwedge_{y\in s_s\mathcal{C}(u)} \mathcal{N}_X(xby) \geq \bigwedge_{xby\in(s_1as_2)\mathcal{C}(u)} \mathcal{N}_X(xby)$$

$$= \bigwedge_{s'\in(s_1as_2)\mathcal{C}(u)} \mathcal{N}_X(s') = \overline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_1as_2). \tag{33}$$

Hence, $\overline{\mathcal{C}}^X(u)$ is a spherical fuzzy interior ideal of S_1 for all $u \in U$. Thus, $(\overline{\mathcal{C}}^X, U)$ is a spherical fuzzy soft interior ideal of S_1 regarding aftersets.

In general, converse of Theorem 11, this does not hold.

Example 4. Let $S_1 = \{s_1, s_2, s_3\}$ and $S_2 = \{1,2,3\}$ be two semigroups under multiplication as given in Tables 11 and 12, respectively.

Table 11. Multiplication on S_1.

$\cdot$	s_1	s_2	s_3
s_1	s_1	s_2	s_3
s_2	s_1	s_2	s_3
s_3	s_1	s_2	s_3

Table 12. Multiplication on S_2.

$\cdot$	1	2	3
1	1	1	3
2	1	2	3
3	1	3	3

Let $U = \{u_1, u_2\}$ and define $\mathcal{C} : U \to P(S_1 \times S_2)$ as

$$\mathcal{C}(u_1) = \{(s_1,1),\ (s_2,2),\ (s_3,3),\ (s_1,2),\ (s_2,1),\ (s_1,3),\ (s_3,1)\},$$

$$\mathcal{C}(u_2) = \{(s_1,1),\ (s_1,2),\ (s_1,3),\ (s_2,1),\ (s_2,2),\ (s_3,3)\}.$$

Then, $(\mathcal{C}, U)$ is a soft compatible relation regarding aftersets from S_1 to S_2.

Now, $s_1\mathcal{C}(u_1) = \{1,2,3\}$, $s_2\mathcal{C}(u_1) = \{1,2\}$, $s_3\mathcal{C}(u_1) = \{1,3\}$, and $s_1\mathcal{C}(u_2) = \{1,2,3\}$, $s_2\mathcal{C}(u_2) = \{1,2\}$, $s_3\mathcal{C}(u_2) = \{3\}$.

Define I: $S_2 \to [0,1]$ as given in Table 13.

Table 13. Spherical fuzzy subset I.

I	1	2	3
$\mathcal{P}_I$	0	0.2	0.2
$\mathcal{I}_I$	0	0.5	0.5
$\mathcal{N}_I$	1	0.1	0.1

Then, I is not a spherical fuzzy interior ideal of S_2, because for $x = 1,\ y = 2,\ z = 1,\ \mathcal{P}_I(121) \ngeq \mathcal{P}_I(2),\ \mathcal{I}_I(121) \ngeq \mathcal{I}_I(2)$ and $\mathcal{N}_I(121) \nleq \mathcal{N}_I(2)$. However, $\overline{\mathcal{C}}^I(u_1),\ \overline{\mathcal{C}}^I(u_2)$ are spherical fuzzy interior ideals of S_1 as given in Table 14. Therefore, $(\overline{\mathcal{C}}^I, U)$ is spherical fuzzy soft interior ideal of S_1 regarding aftersets.

Table 14. Upper approximation of I.

	s_1	s_2	s_3
$\overline{\mathcal{C}}^I(u_1)$	(0.2,0.5,0.1)	(0.2,0.5,0.1)	(0.2,0.5,0.1)
$\overline{\mathcal{C}}^I(u_2)$	(0.2,0.5,0.1)	(0.2,0.5,0.1)	(0.2,0.5,0.1)

Theorem 12. *Let $(\mathcal{C}, U)$ be a soft compatible relation regarding foresets from S_1 to S_2. If $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ is a spherical fuzzy interior ideal of S_1, then $({}^{X}\overline{\mathcal{C}}, U)$ is a spherical fuzzy soft interior ideal of S_2.*

Proof. The proof is similar to proof of Theorem 11.

Theorem 13. *Let $(\mathcal{C}, U)$ be a soft complete relation regarding aftersets from S_1 to S_2. If $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ is a spherical fuzzy interior ideal of S_2, then $(\underline{\mathcal{C}}^X, U)$ is a spherical fuzzy soft interior ideal of S_1.*

Proof. For a spherical fuzzy interior ideal $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ of S_2, $\mathcal{P}_X(xby) \geq \mathcal{P}_X(b)$, $\mathcal{I}_X(xby) \geq \mathcal{I}_X(b)$ and $\mathcal{N}_X(xby) \leq \mathcal{N}_X(b)$ for all $x, y, b \in S_2$.

Now for $s_1, s_2, a \in S_1$,

$$\underline{\mathcal{C}}^{\mathcal{P}_X}(u)(a) = \bigwedge_{b \in a\mathcal{C}(u)} \mathcal{P}_X(b) \leq \bigwedge_{x \in s_1\mathcal{C}(u)} \bigwedge_{b \in a\mathcal{C}(u)} \bigwedge_{y \in s_2\mathcal{C}(u)} \mathcal{P}_X(xby) = \bigwedge_{xby \in (s_1 a s_2)\mathcal{C}(u)} \mathcal{P}_X(xby)$$

$$= \bigwedge_{s' \in (s_1 a s_2)\mathcal{C}(u)} \mathcal{P}_X(s') = \underline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_1 a s_2). \quad (34)$$

$$\underline{\mathcal{C}}^{\mathcal{I}_X}(u)(a) = \bigwedge_{b \in a\mathcal{C}(u)} \mathcal{I}_X(b) \leq \bigwedge_{x \in s_1\mathcal{C}(u)} \bigwedge_{b \in a\mathcal{C}(u)} \bigwedge_{y \in s_2\mathcal{C}(u)} \mathcal{I}_X(xby) = \bigwedge_{xby \in (s_1 a s_2)\mathcal{C}(u)} \mathcal{I}_X(xby)$$

$$= \bigwedge_{s' \in (s_1 a s_2)\mathcal{C}(u)} \mathcal{I}_X(s') = \underline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_1 a s_2). \quad (35)$$

Additionally,

$$\underline{\mathcal{C}}^{\mathcal{N}_X}(u)(a) = \bigvee_{b \in a\mathcal{C}(u)} \mathcal{N}_X(b) \geq \bigvee_{x \in s_1\mathcal{C}(u)} \bigvee_{b \in a\mathcal{C}(u)} \bigvee_{y \in s_2\mathcal{C}(u)} \mathcal{N}_X(xby) = \bigvee_{xby \in (s_1 a s_2)\mathcal{C}(u)} \mathcal{N}_X(xby)$$

$$=\bigvee_{s'\in(s_1as_2)\mathcal{C}(u)} \mathcal{N}_X(s') = \underline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_1as_2). \quad (36)$$

Hence, $\underline{\mathcal{C}}^X(u)$ is a spherical fuzzy interior ideal of S_1 for all $u\in U$. So, $(\underline{\mathcal{C}}^X, U)$ is a spherical fuzzy soft interior ideal of S_1 regarding aftersets.

Theorem 13 does not hold.

Example 5. Consider the semigroups and soft binary relations of Example 2.
Define I: $S_2 \to [0,1]$ as given in Table 15.

Table 15. Spherical fuzzy subset I.

I	1	2	3	4
$\mathcal{P}_I$	0.1	0.3	0.4	0.5
$\mathcal{I}_I$	0.1	0.2	0.5	0.5
$\mathcal{N}_I$	0.8	0.5	0	0

Then, I is not a spherical fuzzy interior ideal of S_2, because for $x=1,\ y=4,\ z=4,\ \mathcal{P}_I(144) \ngeq \mathcal{P}_I(4)$, $\mathcal{I}_I(144) \ngeq \mathcal{I}_I(4)$ and $\mathcal{N}_I(144) \nleq \mathcal{N}_I(4)$. However, $\underline{\mathcal{C}}^I(u_1)$, $\underline{\mathcal{C}}^I(u_2)$ are spherical fuzzy interior ideal of S_1 as given in Table 16. Therefore, $(\underline{\mathcal{C}}^I, U)$ is spherical fuzzy soft interior ideal of S_1 regarding aftersets.

Table 16. Upper approximation of I.

	s_1	s_2	s_3	s_4
$\underline{\mathcal{C}}^I(u_1)$	(0.3,0.2,0.5)	(0.3,0.2,0.5)	(0.3,0.2,0.5)	(0.3,0.2,0.5)
$\underline{\mathcal{C}}^I(u_2)$	(0.4,0.5,0)	(0.4,0.5,0)	(0.4,0.5,0)	(0.4,0.5,0)

Theorem 14. *Let $(\mathcal{C}, U)$ be a soft complete relation regarding foresets from S_1 to S_2. If $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X\rangle$ is a spherical fuzzy interior ideal of S_1, then $({}^X\underline{\mathcal{C}}, U)$ is a spherical fuzzy soft interior ideal of S_2.*

Proof. The proof is similar to proof of Theorem 13.

Theorem 15. *Let $(\mathcal{C}, U)$ be a soft compatible relation regarding aftersets from S_1 to S_2. If $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X\rangle$ is a spherical fuzzy bi-ideal of S_2, then $(\overline{\mathcal{C}}^X, U)$ is a spherical fuzzy soft bi-ideal of S_1.*

Proof. If $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X\rangle$ is a spherical fuzzy bi-ideal of S_2, then X is spherical fuzzy subsemigroup of S_2, and $\mathcal{P}_X(xby) \geq \mathcal{P}_X(x) \wedge \mathcal{P}_X(y)$, $\mathcal{I}_X(xby) \geq \mathcal{I}_X(x) \wedge \mathcal{I}_X(y)$ and $\mathcal{N}_X(xby) \leq \mathcal{N}_X(x) \vee \mathcal{N}_X(y)$ for all $x, y, b \in S_2$. Moreover, by Theorem 3, $(\overline{\mathcal{C}}^X, U)$ is a spherical fuzzy soft subsemigroup of S_1.

Now, for $s_1, s_2, a \in S_1$,

$$\overline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_1) \wedge \overline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_2) = \Big(\bigvee_{x\in s_1\mathcal{C}(u)} \mathcal{P}_X(x)\Big) \wedge \Big(\bigvee_{y\in s_2\mathcal{C}(u)} \mathcal{P}_X(y)\Big)$$

$$= \bigvee_{x\in s_1\mathcal{C}(u)} \bigvee_{y\in s_2\mathcal{C}(u)} (\mathcal{P}_X(x) \wedge \mathcal{P}_X(y)) \leq \bigvee_{x\in s_1\mathcal{C}(u)} \bigvee_{b\in a\mathcal{C}(u)} \bigvee_{y\in s_2\mathcal{C}(u)} (\mathcal{P}_X(xby))$$

$$\leq \bigvee_{xby\in(s_1as_2)\mathcal{C}(u)} \mathcal{P}_X(xby) = \bigvee_{s'\in(s_1as_2)\mathcal{C}(u)} \mathcal{P}_X(s')$$

$$= \overline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_1 a s_2). \tag{37}$$

$$\overline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_1) \wedge \overline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_2) = (\bigvee_{x \in s_1 \mathcal{C}(u)} \mathcal{I}_X(x)) \wedge (\bigvee_{y \in s_2 \mathcal{C}(u)} \mathcal{I}_X(y))$$

$$= \bigvee_{x \in s_1 \mathcal{C}(u)} \bigvee_{y \in s_2 \mathcal{C}(u)} (\mathcal{I}_X(x) \wedge \mathcal{I}_X(y)) \leq \bigvee_{x \in s_1 \mathcal{C}(u)} \bigvee_{b \in a \mathcal{C}(u)} \bigvee_{y \in s_2 \mathcal{C}(u)} (\mathcal{I}_X(xby))$$

$$\leq \bigvee_{xby \in (s_1 a s_2)\mathcal{C}(u)} \mathcal{I}_X(xby) = \bigvee_{s' \in (s_1 a s_2)\mathcal{C}(u)} \mathcal{I}_X(s')$$

$$= \overline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_1 a s_2). \tag{38}$$

And

$$\overline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_1) \vee \overline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_2) = (\bigwedge_{x \in s_1 \mathcal{C}(u)} \mathcal{N}_X(x)) \vee (\bigwedge_{y \in s_2 \mathcal{C}(u)} \mathcal{N}_X(y))$$

$$= \bigwedge_{x \in s_1 \mathcal{C}(u)} \bigwedge_{y \in s_2 \mathcal{C}(u)} (\mathcal{N}_X(x) \vee \mathcal{N}_X(y)) \geq \bigwedge_{x \in s_1 \mathcal{C}(u)} \bigwedge_{b \in a \mathcal{C}(u)} \bigwedge_{y \in s_2 \mathcal{C}(u)} (\mathcal{N}_X(xby))$$

$$\geq \bigwedge_{xby \in (s_1 a s_2)\mathcal{C}(u)} \mathcal{N}_X(xby) = \bigwedge_{s' \in (s_1 a s_2)\mathcal{C}(u)} \mathcal{N}_X(s')$$

$$= \overline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_1 a s_2). \tag{39}$$

Hence, $\overline{\mathcal{C}}^X(u)$ is a spherical fuzzy bi-ideal of S_1 for all $u \in U$. Thus, $(\overline{\mathcal{C}}^X, U)$ is a spherical fuzzy soft bi-ideal of S_1 regarding aftersets.

Theorem 15 does not hold.

Example 6. Consider the semigroups and soft binary relations of Example 1.
Define $B : S_2 \to [0,1]$ as given in Table 17.

Table 17. Spherical fuzzy subset B.

B	1	2	3	4
$\mathcal{P}_B$	0.3	0.5	0.3	0.9
$\mathcal{I}_B$	0	0.1	0.1	0.1
$\mathcal{N}_B$	0.4	0.1	0.6	0

Then, B is not a spherical fuzzy bi-ideal of S_2, because for $x = 2$, $y = 4$, $z = 4$, $\mathcal{P}_B(244) \ngeq \mathcal{P}_B(2) \wedge \mathcal{P}_B(4)$, $\mathcal{I}_B(244) \ngeq \mathcal{I}_B(2) \wedge \mathcal{I}_B(4)$, and $\mathcal{N}_B(244) \nleq \mathcal{N}_B(2) \vee \mathcal{N}_B(4)$. However, $\overline{\mathcal{C}}^B(u_1)$, $\overline{\mathcal{C}}^B(u_2)$ are spherical fuzzy bi-ideals of S_1 as given in Table 18. Therefore, $(\overline{\mathcal{C}}^B, U)$ is spherical fuzzy soft bi-ideal of S_1 regarding aftersets.

Table 18. Upper approximation of B.

	s_1	s_2	s_3	s_4
$\overline{\mathcal{C}}^B(u_1)$	(0.3,0,0.4)	(0.5,0.1,0.2)	(0.3,0.1,0.6)	(0.9,0.1,0)
$\overline{\mathcal{C}}^B(u_2)$	(0.3,0,0.4)	(0.5,0.1,0.2)	(0.3,0.1,0.6)	(0.9,0.1,0)

Theorem 16. *Let* $(\mathcal{C}, U)$ *be a soft compatible relation regarding foresets from* S_1 *to* S_2. *If* $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ *is a spherical fuzzy bi-ideal of* S_1, *then* $(\overline{\mathcal{C}}^X, U)$ *is a spherical fuzzy soft bi-ideal of* S_2.
Proof. The proof is similar to proof of Theorem 15.

Theorem 17. *Let* $(\mathcal{C}, U)$ *be a soft complete relation regarding aftersets from* S_1 *to* S_2. *If* $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ *is a spherical fuzzy bi-ideal of* S_2, *then* $(\underline{\mathcal{C}}^X, U)$ *is a spherical fuzzy soft bi-ideal of* S_1.
Proof. If $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ is a spherical fuzzy bi-ideal of S_2, then X is spherical fuzzy subsemigroup of S_2, and $\mathcal{P}_X(xby) \geq \mathcal{P}_X(x) \wedge \mathcal{P}_X(y)$, $\mathcal{I}_X(xby) \geq \mathcal{I}_X(x) \wedge \mathcal{I}_X(y)$ and $\mathcal{N}_X(xby) \leq \mathcal{N}_X(x) \vee \mathcal{N}_X(y)$ for all $x, y, b \in S_2$. Moreover, by Theorem 5, $(\underline{\mathcal{C}}^X, U)$ is a spherical fuzzy soft subsemigroup of S_1.
Now for $s_1, s_2, a \in S_1$,

$$\underline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_1) \wedge \underline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_2) = \Big(\bigwedge_{x \in s_1\mathcal{C}(u)} \mathcal{P}_X(x)\Big) \wedge \Big(\bigwedge_{y \in s_2\mathcal{C}(u)} \mathcal{P}_X(y)\Big)$$

$$\leq \bigwedge_{x \in s_1\mathcal{C}(u)} \bigwedge_{b \in a\mathcal{C}(u)} \bigwedge_{y \in s_2\mathcal{C}(u)} \mathcal{P}_X(xby) = \bigwedge_{xby \in (s_1 a s_2)\mathcal{C}(u)} \mathcal{P}_X(xby)$$

$$= \bigwedge_{s' \in (s_1 a s_2)\mathcal{C}(u)} \mathcal{P}_X(s') = \underline{\mathcal{C}}^{\mathcal{P}_X}(u)(s_1 a s_2). \quad (40)$$

$$\underline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_1) \wedge \underline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_2) = \Big(\bigwedge_{x \in s_1\mathcal{C}(u)} \mathcal{I}_X(x)\Big) \wedge \Big(\bigwedge_{y \in s_2\mathcal{C}(u)} \mathcal{I}_X(y)\Big)$$

$$\leq \bigwedge_{x \in s_1\mathcal{C}(u)} \bigwedge_{b \in a\mathcal{C}(u)} \bigwedge_{y \in s_2\mathcal{C}(u)} \mathcal{I}_X(xby) = \bigwedge_{xby \in (s_1 a s_2)\mathcal{C}(u)} \mathcal{I}_X(xby)$$

$$= \bigwedge_{s' \in (s_1 a s_2)\mathcal{C}(u)} \mathcal{I}_X(s') = \underline{\mathcal{C}}^{\mathcal{I}_X}(u)(s_1 a s_2). \quad (41)$$

And

$$\underline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_1) \vee \underline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_2) = \Big(\bigvee_{x \in s_1\mathcal{C}(u)} \mathcal{N}_X(x)\Big) \vee \Big(\bigvee_{y \in s_2\mathcal{C}(u)} \mathcal{N}_X(y)\Big)$$

$$\geq \bigvee_{x \in s_1\mathcal{C}(u)} \bigvee_{b \in a\mathcal{C}(u)} \bigvee_{y \in s_2\mathcal{C}(u)} \mathcal{N}_X(xby) = \bigvee_{xby \in (s_1 a s_2)\mathcal{C}(u)} \mathcal{N}_X(xby)$$

$$= \bigvee_{s' \in (s_1 a s_2)\mathcal{C}(u)} \mathcal{N}_X(s') = \underline{\mathcal{C}}^{\mathcal{N}_X}(u)(s_1 a s_2). \quad (42)$$

Hence, $\underline{\mathcal{C}}^X(u)$ is a spherical fuzzy bi-ideal of S_1 for all $u \in U$. So, $(\underline{\mathcal{C}}^X, U)$ is a spherical fuzzy soft bi-ideal of S_1 regarding aftersets.

In general, converse of Theorem 17, this does not hold.

Example 7. Consider the semigroups and soft binary relations of Example 2.
Define B: $S_2 \rightarrow [0,1]$ as given in Table 19.

Table 19. Spherical fuzzy subset B.

B	1	2	3	4
$\mathcal{P}_B$	0.6	0.1	0.2	0.4
$\mathcal{I}_B$	0.4	0.1	0.2	0.3
$\mathcal{N}_B$	0	0.7	0.5	0.3

Then, B is not a spherical fuzzy bi-ideal of S_2, because for $x = 1$, $y = 3$, $z = 4$, $\mathcal{P}_B(134) \ngeq \mathcal{P}_B(1) \wedge \mathcal{P}_B(4)$, $\mathcal{I}_B(134) \ngeq \mathcal{I}_B(1) \wedge \mathcal{I}_B(4)$, and $\mathcal{N}_B(134) \nleq \mathcal{N}_B(1) \vee \mathcal{N}_B(4)$. However, $\underline{\mathcal{C}}^B(u_1)$ and $\underline{\mathcal{C}}^B(u_2)$ are spherical fuzzy bi-ideals of S_1, as given in Table 20. Therefore, $(\underline{\mathcal{C}}^B, U)$ is a spherical fuzzy soft bi-ideal of S_1 regarding aftersets.

Table 20. Upper approximation of B.

	s_1	s_2	s_3	s_4
$\underline{\mathcal{C}}^B(u_1)$	(0.1,0.1,0.7)	(0.1,0.1,0.7)	(0.1,0.1,0.7)	(0.1,0.1,0.7)
$\underline{\mathcal{C}}^B(u_2)$	(0.2,0.2,0.5)	(0.2,0.2,0.5)	(0.2,0.2,0.5)	(0.2,0.2,0.5)

Theorem 18. *Let $(\mathcal{C}, U)$ be a soft complete relation regarding foresets from S_1 to S_2. If $X = \langle \mathcal{P}_X, \mathcal{I}_X, \mathcal{N}_X \rangle$ is a spherical fuzzy bi-ideal of S_1, then $({}^{X}\underline{\mathcal{C}}, U)$ is a spherical fuzzy soft bi-ideal of S_2.*
Proof. The proof is similar to proof of Theorem 17.

4. Comparative study

In this paper, rough approximations of spherical fuzzy ideals in semigroups using soft binary relations are studied. In [23], Kanwal and Shabir introduced the study of roughness of fuzzy ideals in terms of soft relations, which gives only one-sided information due to its membership degree [0,1]. Anwar et al. studied the roughness of the intuitionistic fuzzy set based on soft relations [25], which works on the membership and nonmembership degrees of given information. By extending previous studies, we work on the roughness of spherical fuzzy ideals of semigroups based on soft relations, which gives information about membership degree, nonmembership degree, and neutral membership degree.

Advantages:

This approach has potential impacts on fields like coding theory, automata theory, and cryptogr aphy, providing a flexible and robust method for handling imprecise data. The model explored in this paper is the most generalized form of existing techniques and handles incomplete and vague information in problems in a more accurate way.

1) If $\mathcal{N} = 0 = \mathcal{I}$ and $\mathcal{P} \in [0,1]$, then the roughness of spherical fuzzy ideals over a soft binary relation (RSISB) is reduced to rough approximations of fuzzy ideals over soft binary relation [23].
2) If $\mathcal{I} = 0$ and $\mathcal{P} + \mathcal{N} \in [0,1]$, then RSISB is reduced to the roughness of the intuitionistic fuzzy set over a soft binary relation [25].

3) If $\mathcal{I} = 0$ and $\mathcal{P}^2 + \mathcal{N}^2 \in [0,1]$, then RSISB is reduced to the roughness of the pythagorean fuzzy set over a soft binary relation [26].
4) If $\mathcal{P} + \mathcal{I} + \mathcal{N} \in [0,1]$, then RSISB is reduced to [27].

It is clear that RSISB is a generalization of the above models.

5. Conclusions

In this paper, we generalize the structure of [23, 25] to a spherical fuzzy set and introduce a model based on the roughness of spherical fuzzy ideals with the use of soft binary relations. We investigate the roughness of the spherical fuzzy subsemigroup using soft relations and obtain two soft sets. Upper approximations of the spherical fuzzy subsemigroup and spherical fuzzy ideals are studied using a soft compatible relation, while lower approximations of the spherical fuzzy subsemigroup and spherical fuzzy ideals are studied using a soft complete relation. Also, with the help of examples, we show that conditions of a complete relation are necessary for lower approximations. The main theorem of this study is that "If X is a spherical fuzzy ideal of S_2, then $(\overline{C}^X, U)$ and $(\underline{C}^X, U)$ are spherical fuzzy soft ideals of S_1" for upper approximations, (C, U) is a soft compatible relation, and for lower approximations, (C, U) is a soft complete relation. The converse of this theorem may not hold, which is shown by examples.

With a new model, to manage data having multiple granulations, might be challenging in many real-world problems. To tackle this, we need a technique that can approximate spherical fuzzy ideals with respect to multi-equivalence relations instead of single-equivalence relations. In the future, we will extend our work to optimistic multigranulation roughness of spherical fuzzy sets and optimistic multigranulation roughness of spherical fuzzy ideals in a semigroup using soft relations. Interval-valued Fermatean fuzzy set (IVFFS), an extension of Fermatean fuzzy sets, has many practical applications [30]. We will also work on this.

Authors Contributions

Rabia Mazhar: Methodology, formal analysis, writing-original draft, resources; Shahida Bashir: Supervision, reviewing and editing, project administration; Muhammad Shabir: Conceptualization, validation, reviewing and editing; Mohammed Al-Shamiri: Reviewing and editing, data curation, funding acquisition. All authors have thoroughly reviewed and approved the final version of this research paper.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

The authors extend their appreciation to the Deanship of Research and Graduate Studies at King Khalid University for funding this work through Large Research Project under grant number RGP2/302/45.

Conflict of interest

We declare that we have no conflict of interest in this paper.

References

1. L. A. Zadeh, Fuzzy sets, *Inf. control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
2. K. T. Atanassov, S. Stoeva, Intuitionistic fuzzy sets, *Fuzzy set. Syst.*, **20** (1986), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3
3. S. Ashraf, S. Abdullah, Spherical aggregation operators and their application in multiattribute group decision-making, *Int. J. Intell. Syst.*, **34** (2019), 493–523. https://doi.org/10.1002/int.22062
4. F. Karaaslan, F. Karamaz, Interval-valued (p, q, r)-spherical fuzzy sets and their applications in MCGDM and MCDM based on TOPSIS method and aggregation operators, *Expert Syst. Appl.*, **255** (2024), 124575. https://doi.org/10.1016/j.eswa.2024.124575
5. M. Palanikumar, L. Mohan, M. M. Raj, A. Iampan, Real-life applications of new type spherical fuzzy sets and its extension using aggregation operators, *Int. J. Anal. Appl.*, **22** (2024), 131. https://doi.org/10.28924/2291-8639-22-2024-131
6. N. Jan, J. Gwak, D. Pamucar, H. Kang, An integrated complex T-spherical fuzzy set and soft set model for quantum computing and energy resource planning, *Inform. Sciences*, **661** (2024), 120101. https://doi.org/10.1016/j.ins.2024.120101
7. I. Bechar, R. Bechar, A. Benyettou, A novel score function for spherical fuzzy sets and its application to assignment problem, *Econ. Comput. Econ. Cyb.*, **58** (2024). https://doi.org/10.24818/18423264/58.3.24.13
8. A. Suschkewitsch, Über die Darstellung der eindeutig nicht umkehrbaren Gruppen mittelst der verallgemeinerten Substitutionen, *Matematicheskii Sbornik*, **33** (1926), 371–374.
9. D. Molodtsov, Soft set theory—first results, *Comput. math. Appl.*, **37** (1999), 19–31.
10. M. I. Ali, A note on soft sets, rough sets and fuzzy soft sets, *Appl. Soft Comput.*, **11** (2011), 3329–3332. https://doi.org/10.1016/j.asoc.2011.01.003
11. N. Cagman, S. Enginoglu, Soft matrix theory and its decision making, *Comput. Math. Appl.*, **59** (2010), 3308–3314. https://doi.org/10.1016/j.camwa.2010.03.015
12. N. Cagman, S. Enginoglu, Soft set theory and uni-int decision making, *Eur. J. Oper. Res.*, **207** (2010), 848–855. https://doi.org/10.1016/j.ejor.2010.05.004
13. N. Rehman, A. Ali, M. I. Ali, C. Park, SDMGRS soft dominance based multi granulation rough sets and their applications in conflict analysis problems, *IEEE Access*, 2018, 31399–31416. https://doi.org/10.1109/ACCESS.2018.2841876
14. Z. Pawlak, Rough sets, *Int. J. comput. Inf. Sci.*, **11** (1982), 341–356. https://doi.org/10.1007/BF01001956
15. Z. Pawlak, *Fuzzy logic for the management of uncertainty: Rough sets: A new approach to vagueness*, USA: John Wiley and Sons, Inc., 1992, 105–118.
16. L. Zheng, T. Mahmood, J. Ahmmad, U. U. Rehman, S. Zeng, Spherical fuzzy soft rough average aggregation operators and their applications to multi-criteria decision making, *IEEE Access*, **10** (2022), 27832–27852. https://doi.org/10.1109/ACCESS.2022.3150858
17. S. Zeng, A. Hussain, T. Mahmood, M. I. Ali, S. Ashraf, M. Munir, Covering-based spherical fuzzy rough set model hybrid with TOPSIS for multi-attribute decision-making, *Symmetry*, **11** (2019), 547. https://doi.org/10.3390/sym11040547

18. S. Ashraf, S. Abdullah, M. Aslam, M. Qiyas, M. A. Kutbi, Spherical fuzzy sets and its representation of spherical fuzzy t-norms and t-conorms, *J. Intell. Fuzzy Syst.*, **36** (2019), 6089–6102. https://doi.org/10.3233/JIFS-181941
19. P. A. F. Perveen, J. J. Sunil, K. V. Babitha, H. Garg, Spherical fuzzy soft sets and its applications in decision-making problems, *J. Intell. Fuzzy Syst.*, **37** (2019), 8237–8250. https://doi.org/10.3233/JIFS-190728
20. C. Kahraman, F. K. Gündogdu, *Decision making with spherical fuzzy sets*, Cham: Springer Cham, **392** (2021), 3–25. https://doi.org/10.1007/978-3-030-45461-6_1
21. A. B. Azim, A. ALoqaily, A. Ali, S. Ali, N. Mlaiki, F. Hussain, q-Spherical fuzzy rough sets and their usage in multi-attribute decision-making problems, *AIMS Math.*, **8** (2023), 8210–8248. https://doi.org/10.3934/math.2023415
22. D. Ajay, G. Selvachandran, J. Aldring, P. H. Thong, L. H. Son, B. C. Cuong, Einstein exponential operation laws of spherical fuzzy sets and aggregation operators in decision making, *Multimed. Tools Appl.*, **82** (2023), 41767–41790. https://doi.org/10.1007/s11042-023-14532-9
23. R. S. Kanwal, M. Shabir, Rough approximation of a fuzzy set in semigroups based on soft relations, *Comput. Appl. Math.*, **38** (2019), 1–23. https://doi.org/10.1007/s40314-019-0851-3
24. M. Shabir, A. Mubarak, M. Naz, Rough approximations of bipolar soft sets by soft relations and their application in decision making, *J. Intell. Fuzzy Syst.*, **40** (2021), 11845–11860. https://doi.org/10.3233/JIFS-202958
25. M. Z. Anwar, S. Bashir, M. Shabir, An efficient model for the approximation of intuitionistic fuzzy sets in terms of soft relations with applications in decision making, *Math. Probl. Eng.*, 2021, 1–19. https://doi.org/10.1155/2021/6238481
26. M. A. Bilal, M. Shabir, Approximations of pythagorean fuzzy sets over dual universes by soft binary relations, *J. Intell. Fuzzy Syst.*, **41** (2021), 2495–2511. https://doi.org/10.3233/JIFS-202725
27. R. Prasertpong, Roughness of soft sets and fuzzy sets in semigroups based on set-valued picture hesitant fuzzy relations, *AIMS Math.*, **7** (2022), 2891–2928. https://doi.org/10.3934/math.2022160
28. R. Mazhar, S. Bashir, M. Shabir, Approximations of spherical fuzzy sets by soft relations and its applications in decision making (submitted).
29. V. Chinnadurai, A. Bobin, A. Arulselvam, A study on spherical fuzzy ideals of semigroup, *TWMS J. Appl. Eng. Math.*, **(12)** 2022, 1202–1212.
30. M. Akram, S. M. U. Shah, M. M. A. Al-Shamiri, S. A. Edalatpanah, Fractional transportation problem under interval-valued Fermatean fuzzy sets, *AIMS Math.*, **7** (2022), 17327–17348. https://doi.org/10.3934/math.2022954

AIMS Press