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Research article

Evolution of null Cartan and pseudo null curves via the Bishop frame in Minkowski space $\mathbb{R}^{2,1}$

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Abstract: In the present work, we focused on studying the evolution of null Cartan and pseudo null curves using the Bishop frame in Minkowski space $\mathbb{R}^{2,1}$. We obtained the necessary and sufficient conditions for the null Cartan and pseudo null curves to be inextensible curves (the arc length is preserved). In addition, we derived the time evolution equations of the Bishop frame (TEEsBF) for these curves. Moreover, we obtained the time evolution equations of Bishop curvatures (TEEsBCs) as partial differential equations in terms of Bishop velocities. Finally, we presented some applications.

Keywords: evolution equations; inextensible flows; null curves; null Cartan curves; pseudo null curves; Bishop frame
Mathematics Subject Classification: 51B20, 53A04, 53A35, 53C50

Abbreviations

The abbreviations that are used in this manuscript are illustrated by:

BC(s)	Bishop curvature(s)
IF	Inextensible flows
NCC(s)	Null Cartan curves
PDE(s)	Partial differential equation(s)
PNC(s)	pseudo null curves
TEEs	Time evolution equations
TEEsBF	Time evolution equation(s) for Bishop frame
TEEsBC(s)	Time evolution equation(s) for Bishop curvature(s)

1. Introduction

The flow of curves and surfaces in Euclidean and Minkowski spaces is a crucial issue in differential geometry. The flow of curves determines their time evolution. The evolution of curves is therefore referred to as a flow in this article. The flow of curves and surfaces has numerous applications in physics, computer graphics, and engineering. In physics, there is no strain energy generated by inextensible flows (IF) of curves. Inextensible curves and surface flows can describe the swinging motion of a cord of fixed length or the motion of a piece of paper carried by the wind. There are many physical applications in which such motions occur naturally. Chirikjian and Burdick [1] and Mochiyama et al. [2] investigated the shape control of hyper-redundant and snake-like robots.

Computer graphics and animation depend extensively on the flow of curves [3–6], where it is used to simulate the movement of flexible objects like clothing and hair, enhancing the realism of animated characters and scenes.

The IF of curves and surfaces have various applications in different fields of engineering like structural engineering [7], where the inextensible curves and surfaces are utilized in the design and analysis of structures like cable-supported bridges, suspension bridges, and tension systems. They help ensure the stability and load-bearing capacity of these structures, allowing engineers to create efficient and safe designs.

To emphasize the novelty of this study, we analyze previous works on the evolution of curves in Euclidean and Minkowski spaces according to various frames, such as the Frenet frame, the Darboux frame, the modified orthonormal frame, the Bishop frame, and the quasi-frame.

S. Gaber [8–10] investigated the IF of curves in 3-dimensional Euclidean space $\mathbb{R}^3$, spherical space S^3, and three-dimensional de-Sitter space $\mathbb{S}^{2,1}$. Additionally, the TEEs of the Frenet frame and for curvatures have been constructed as a set of partial differential equations (PDEs). In [11], S. Gaber derived the TEEs of curves in $\mathbb{R}^3$ by employing the type-1 Bishop frame. Gaber and Sorour [12] studied the IF of time-like curves according to a quasi-frame in Minkowski space $\mathbb{R}^{2,1}$. Gaber and Al Elaiw [13] discussed the flow of a null Cartan curve (NCC) characterized by velocity and acceleration functions. The binormal velocity impacts the tangential and normal velocities along the motion. Moreover, the TEEs for the Cartan frame of the null curve were constructed. The flows of an initial NCC were also utilized to create a family of inextensible NCCs.

In [14], N. Gurbuz provided several innovative transformations and established the relationships between the motion of non-null curves and soliton equations by using the Bishop frame in Minkowski three-space. Furthermore, the differential formulae of these transformations related to the nonlinear system of the heat and repulsive-type nonlinear Schrodinger equation were derived.

In [15], A. Ucum studied the IF of curves such as pseudo null and partially null curves in $\mathbb{R}^4_2$. Eren et al. [16] employed the modified orthogonal frame in investigating the motions of space curves and some special-ruled surfaces. Additionally, evolution equations for space curves with a modified orthogonal frame were obtained. In [17], D. Yang et al. studied a curve flow of fourth order for a smooth closed curve in $\mathbb{R}^2$, which arises as a nonlinear parabolic PDE. By using the evolution equations, and assuming some specific conditions, they proved that for any smooth closed initial curve in $\mathbb{R}^2$, the flow has a smooth solution for all time. In [18], K. Eren investigated the motion of a moving space curve using a modified orthogonal frame and derived the TEEs for curvatures.

Ergut et al. [19] investigated the flows of inextensible spacelike curves related to the Sabban frame on S^2_1 and derived the flows of spacelike curves in terms of PDEs. Korpinar et al. [20] constructed a novel approach for the flows of inextensible curves in three-dimensional Euclidean space by employing the Frenet frame of the given curve and provided certain characterizations for curvatures of the curve.

Bektas et al. [21] determined the necessary and sufficient inextensibility conditions for the spacelike curves in the three-dimensional lightlike cone in $\mathbb{E}^4_1$. Arroyo et al. [22] investigated the binormal flow of curves with curvatures depending on velocity and sweeping out immersed surfaces. Filaments evolving with constant torsion were constructed by using the Gauss-Codazzi equations.

Yuzbasi et al. [23] investigated the flow of an inextensible curve on a lightlike surface in Minkowski three-space. A necessary and sufficient inextensibility condition for the curve flow was obtained. It arose as a PDE involving the curvatures. Additionally, the lightlike ruled surfaces were characterized in three-dimensional Minkowski space, and the evolution of an inextensible lightlike curve on a lightlike tangent developable surface was described.

This article presents a novel approach for investigating the evolution of NCCs and pseudo null curves (PNCs) in Minkowski space $\mathbb{R}^{2,1}$. We derive the TEEs of NCCs and PNCs by using the Bishop frame. We obtain the necessary and sufficient conditions for NCCs and PNCs to be inextensible. We get the TEEsBCs for NCCs and PNCs as PDEs depending on Bishop velocities.

This research article is organized as follows: In Section 2, we study the geometry of curves in Minkowski space. We introduce the NCCs in $\mathbb{R}^{2,1}$ and the relationship between the Cartan frame and Bishop frame for NCCs. In addition, we discuss the PNCs in $\mathbb{R}^{2,1}$ and the relationship between the Frenet frame and Bishop frame for PNCs. In Section 3, we study the evolution of NCCs using the Bishop frame in $\mathbb{R}^{2,1}$. In Section 4, we present an application for the evolution of NCCs using the Bishop frame. In Section 5, we discuss the evolution of the PNCs employing the Bishop frame in $\mathbb{R}^{2,1}$. In Section 6, we provide two applications for the evolution of PNCs by using the Bishop frame. Finally, we present our conclusions.

2. Geometric preliminaries

Definition 1. *[24] The Minkowski space $\mathbb{R}^{2,1}$ is defined as the real vector space $\mathbb{R}^3$ with the Lorentzian inner product given by: $-dx_0^2 + dx_1^2 + dx_2^2$ with $\{X = (x_0, x_1, x_2) \mid x_0, x_1, x_2 \in \mathbb{R}\}$.*

Consider $X = (x_0, x_1, x_2), Y = (y_0, y_1, y_2) \in \mathbb{R}^{2,1}$ as two vectors in Minkowski space $\mathbb{R}^{2,1}$, and the following properties are defined in $\mathbb{R}^{2,1}$:

- *The Lorentzian inner product: $\langle X, Y\rangle = -x_0y_0 + x_1y_1 + x_2y_2$.*
- *The vector product: $X \times Y = (x_2y_1 - x_1y_2, x_2y_0 - x_0y_2, x_0y_1 - x_1y_0)$.*
- *The non-zero vector $U \in \mathbb{R}^{2,1}$ is spacelike, timelike, and null (lightlike) if $\langle U, U\rangle > 0$, $\langle U, U\rangle < 0$, and $\langle U, U\rangle = 0$, respectively.*

Definition 2. *[25, 26] Consider a regular parameterized curve $\alpha : I \to \mathbb{R}^{2,1}$ in Minkowski space $\mathbb{R}^{2,1}$. Let $u \in I$ be the parameter of the curve, and define $\alpha' = \frac{d\alpha}{du}$ to be the tangent vector of the curve α:*

- *If $\langle \alpha', \alpha'\rangle > 0$, then α is a spacelike curve for all $u \in I$.*
- *If $\langle \alpha', \alpha'\rangle < 0$, then α is a timelike curve for all $u \in I$.*
- *If $\langle \alpha', \alpha'\rangle = 0$, then α is a lightlike curve for all $u \in I$.*

2.1. Geometric concepts of the null Cartan curve (NCC)

Definition 3. *[27] Consider a regular curve $\alpha : I \in \mathbb{R} \to \mathbb{R}^{2,1}$ defined on $I \in \mathbb{R}$. The curve α is a null curve, if the tangent vector $\alpha'(u)$ for all $u \in I$ is a future-directed null vector. A curve is called a null Cartan curve (NCC), if it is parameterized by the pseudo-arc function s that is defined by:*

$$s(u) = \int_0^u \sqrt{\|\alpha''(u)\|}du = \int_0^u (\langle \alpha''(u), \alpha''(u)\rangle)^{\frac{1}{4}} du, \quad (\quad)' = \frac{d}{du} \tag{2.1}$$

Definition 4. *[28] Consider the NCC $\alpha : I \in \mathbb{R} \to \mathbb{R}^{2,1}$ parameterized by the pseudo-arc s. Assume that $\mathcal{F}_c = \{T, N, B\}$ is the Cartan frame at a point q along the curve α, where $\alpha'(s)$ for all $s \in I$ is a tangent null vector, N is a principal normal spacelike vector, and B is a principal binormal null vector. The Cartan frame equations along the NCC satisfy the following:*

$$\begin{pmatrix} \alpha' \\ T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\tau & 0 & 1 \\ 0 & 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ T \\ N \\ B \end{pmatrix} \quad , (\,)' = \frac{d}{ds}\,, \tag{2.2}$$

where $\tau = \tau(s)$ is the torsion of the NCC. The Cartan frame $\mathcal{F}_c$ satisfies the following properties:

$$\begin{aligned} &\langle T, T\rangle = 0, \ \ \langle N, N\rangle = 1, \ \ \langle B, B\rangle = 0, \\ &\langle T, B\rangle = -1, \quad \langle N, B\rangle = \ \langle N, T\rangle = 0. \end{aligned} \tag{2.3}$$

$$T \times N = -T, \ \ N \times B = -B, \quad B \times T = N. \tag{2.4}$$

Definition 5. *[29] Let $\mathcal{F}_B = \{T_1, N_1, N_2\}$ be a positively oriented Bishop frame of the NCC. The frame $\mathcal{F}_B$ is a pseudo-orthonormal frame containing the vector fields T_1, N_1, N_2, where T_1 is the tangential vector, N_1 is the relatively parallel spacelike normal vector, and N_2 is a relatively parallel lightlike transversal vector. The vector fields N_1 and N_2 are defined to be relatively parallel; if the normal component $T_1^{\perp} = \operatorname{span}\{T_1, N_1\}$ of their derivatives N_1' and N_2' vanish. So the vector fields N_1' and N_2' are collinear with N_2.*

Lemma 1. *[29] Let $\alpha = \alpha(s)$ be a NCC in $\mathbb{R}^{2,1}$ with a pseudo-arc parameter s, and let $\tau = \tau(s)$ be the torsion of the curve. The relationship between the Bishop frame $\mathcal{F}_B$ and the Cartan frame $\mathcal{F}_c$ is given by:*

$$\begin{pmatrix} T_1 \\ N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} k_1 & 0 & 0 \\ -k_2 & k_1 & 0 \\ \frac{k_2^2}{2} & -k_2 & 1 \end{pmatrix} . \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2.5}$$

where the functions $k_1(s) = 1$ and $k_2(s)$ are defined in this paper as Bishop curvatures (BCs).

Lemma 2. *[29] The Bishop frame $\mathcal{F}_B$ of the NCC satisfies the following equations:*

$$\frac{d}{ds}\begin{pmatrix} T_1 \\ N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} k_2 & k_1 & 0 \\ 0 & 0 & k_1 \\ 0 & 0 & -k_2 \end{pmatrix} . \begin{pmatrix} T_1 \\ N_1 \\ N_2 \end{pmatrix}, \tag{2.6}$$

where the first BC $k_1(s) = 1$, and then

$$\begin{pmatrix} T_1' \\ N_1' \\ N_2' \end{pmatrix} = \begin{pmatrix} k_2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -k_2 \end{pmatrix} . \begin{pmatrix} T_1 \\ N_1 \\ N_2 \end{pmatrix} , \ (\,)' = \frac{d}{ds} , \tag{2.7}$$

where the second BC k_2 satisfies the Riccati differential equation: $k_2'(s) = -\frac{1}{2}k_2^2(s) - \tau(s)$.

Definition 6. *[27] The Bishop frame $\mathcal{F}_B$ of the NCC has the following characteristics:*

$$\begin{aligned} &\langle T_1, T_1 \rangle = 0, \ \ \langle N_1, N_1 \rangle = 1, \ \ \langle N_2, N_2 \rangle = 0, \\ &\langle T_1, N_1 \rangle = 0, \ \langle T_1, N_2 \rangle = -1, \ \langle N_1, N_2 \rangle = 0. \end{aligned} \tag{2.8}$$

$$T_1 \times N_1 = -T_1, \ T_1 \times N_2 = -N_1, \ N_1 \times N_2 = -N_2. \tag{2.9}$$

2.2. Geometric concepts of the pseudo null curves (PNCs)

Definition 7. *[27] Consider the pseudo null curve (PNC) $\alpha : I \in \mathbb{R} \to \mathbb{R}^{2,1}$. Assume that at a point q on the PNC, the Frenet frame is defined by $\mathbb{F}_f = \{T, N, B\}$, where the tangent vector T is a spacelike vector, N is the principal normal vector, and B is the principal binormal vector. The vectors N and B are null vectors with the condition $\langle N, B \rangle = 1$. The Frenet frame equations along the PNC satisfy the following:*

$$\frac{d}{ds}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ 0 & \tau & 0 \\ -k & 0 & -\tau \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2.10}$$

where the functions $k(s) = 1$ and $\tau = \tau(s)$ are the curvature and torsion of the PNC.

The Frenet frame for the PNC satisfies the following properties:

$$\begin{aligned} &\langle T, T \rangle = 1, \ \ \langle N, N \rangle = 0, \ \ \langle B, B \rangle = 0, \\ &\langle T, B \rangle = 0, \quad \langle N, B \rangle = 1, \ \langle N, T \rangle = 0. \end{aligned} \tag{2.11}$$

$$T \times N = N, \ N \times B = T, \quad B \times T = B. \tag{2.12}$$

Definition 8. *[27] Assume that $\mathbb{F}_B = \{T_1, N_1, N_2\}$ is the positively oriented pseudo-orthonormal Bishop frame of the PNC, where T_1 represents the tangential vector, and the vectors N_1 and N_2 are relatively parallel lightlike normal vector fields. The vectors N_1 and N_2 are defined to be relatively parallel if the normal component $T_1^{\perp} = \operatorname{span}\{N_1, N_2\}$ of their derivatives N_1' and N_2' is zero. So the vector fields N_1' and N_2' are collinear with T_1.*

Definition 9. *The $\mathbb{F}_B$ of the PNC satisfies the following properties:*

$$\begin{aligned} &\langle T_1, T_1 \rangle = 1, \ \ \langle N_1, N_1 \rangle = 0, \ \ \langle N_2, N_2 \rangle = 0, \\ &\langle T_1, N_2 \rangle = 0, \quad \langle N_1, N_2 \rangle = 1, \ \langle N_1, T_1 \rangle = 0. \end{aligned} \tag{2.13}$$

$$T_1 \times N_1 = N_1, \ N_1 \times N_2 = T_1, \quad N_2 \times T_1 = N_2. \tag{2.14}$$

Theorem 3. *[30] Let α be a PNC in $\mathbb{R}^{2,1}$ parameterized by the arc-length parameter s with the curvature $k(s) = 1$ and the torsion $\tau(s)$. The relationship between the Bishop frame $\{T_1, N_1, N_2\}$ and the Frenet frame $\{T, N, B\}$ is given by:*

- *(i) Case (1):*

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{k_2} & 0 \\ 0 & 0 & k_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{2.15}$$

In this case, the equations of the $\mathbb{F}_B$ for the PNC are given by:

$$\frac{d}{ds}\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 0 & k_2 & k_1 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}, \tag{2.16}$$

where the functions $k_1(s) = 0$ and $k_2(s) = c_0 e^{\int \tau(s)ds}$, $c_0 \in \mathbb{R}_0^+$, are the first and second BCs.

- *(ii) Case (2):*

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -k_1 \\ 0 & -\frac{1}{k_1} & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.17}$$

In this case, the equations of the $\mathbb{F}_B$ for the PNC are given by:

$$\frac{d}{ds}\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 0 & k_2 & k_1 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} \tag{2.18}$$

where the the first and second BCs are $k_1(s) = c_0 e^{\int \tau(s)ds}$, $c_0 \in \mathbb{R}_0^-$, and $k_2(s) = 0$.

3. Main results on the evolution of null Cartan curves (NCCs) using the Bishop frame

Let $\alpha : I = [0, l] \times [0, t] \longrightarrow \mathbb{R}^{2,1}$ be the family of NCCs according to Bishop frame, where l and t are the parameters of the initial curve and time, respectively. Let u be the parameter variable of the NCC, where $0 \leq u \leq l$. Define the pseudo-arc-length parameter of the NCC by:

$$s(u,t) = \int_0^u \sqrt{\|\alpha_{uu}(u,t)\|}du = \int_0^u \sigma du \quad , \quad \sigma(u,t) = (\langle \alpha_{uu}(u), \alpha_{uu}(u)\rangle)^{\frac{1}{4}} \ , \ \sigma = \frac{\partial s}{\partial u}. \tag{3.1}$$

The time evolution of the NCC with the Bishop frame is characterized by the velocities w_1, w_2, and w_3 as follows:

$$\frac{\partial \alpha}{\partial t} = w_1\, T_1 + w_2\, N_1 + w_3\, N_2\ . \tag{3.2}$$

In this paper, we call w_1, w_2, and w_3 the Bishop velocities.

Theorem 4. *The NCC is inextensible with the Bishop frame $\mathcal{F}_B$ if the Bishop velocities satisfy the following conditions:*

$$\begin{aligned} w_2 &= -w_{3,s} + k_2 w_3 \quad , \quad (\quad)_s = \frac{\partial}{\partial s} \quad , \\ w_1 &= \frac{1}{2}(-w_{2,s} + \int k_2 w_{2,s}\, ds)\ . \end{aligned} \tag{3.3}$$

Proof. From (3.1), we have

$$\sigma^4 = \langle \alpha_{uu}, \alpha_{uu} \rangle. \tag{3.4}$$

If the NCC with a Bishop frame is inextensible, then $\frac{\partial s}{\partial t} = \sigma_t = 0$, by taking the t-derivative of (3.4), and then:

$$0 = \langle \alpha_{uut}, \alpha_{uu} \rangle. \tag{3.5}$$

Since $\alpha_u = \sigma\alpha_s = \sigma T_1$, by using (2.7), then we get:

$$\alpha_{uu} = \sigma(\sigma_s + \sigma k_2)T_1 + \sigma^2 N_1. \tag{3.6}$$

Taking the u-derivative of (3.2), then we obtain:

$$\alpha_{tu} = \sigma\left(w_{1,s} + k_2 w_1\right) T_1 + \sigma\left(w_{2,s} + w_1\right) N_1 + \sigma\left(w_{3,s} + w_2 - k_2 w_3\right) N_2.$$

For simplicity, we put:

$$\begin{aligned} \Omega_1 &= w_{1,s} + k_2 w_1 \ , \\ \Omega_2 &= w_{2,s} + w_1 \ , \\ \Omega_3 &= w_{3,s} + w_2 - k_2 w_3 \ . \end{aligned} \tag{3.7}$$

Then, we have

$$\alpha_{tu} = \sigma\left(\Omega_1 T_1 + \Omega_2 N_1 + \Omega_3 N_2\right). \tag{3.8}$$

Taking the u-derivative of (3.8), we get:

$$\alpha_{tuu} = \sigma(\sigma_s\Omega_1 + \sigma\Omega_{1,s} + \sigma k_2\Omega_1)T_1 + \sigma(\sigma_s\Omega_2 + \sigma\Omega_{2,s} + \sigma\Omega_1)N_1 + \sigma(\sigma_s\Omega_3 + \sigma\Omega_{3,s} + \sigma\Omega_2 - \sigma k_2\Omega_3)N_2. \tag{3.9}$$

Since $\alpha_{ut} = \alpha_{tu}$, then $\alpha_{uut} = \alpha_{tuu}$. Substitute from (3.6) and (3.9) into (3.5), and using the properties of the Bishop frame for NCC that are defined by (2.8), then

$$(\sigma_s + \sigma k_2)(\sigma_s\Omega_3 + \sigma\Omega_{3,s} + \sigma\Omega_2 - \sigma k_2\Omega_3) = \sigma(\sigma_s\Omega_2 + \sigma\Omega_1 + \sigma\Omega_{2,s}). \tag{3.10}$$

Equate the coefficients of σ_s^2, $\sigma\sigma_s$, and σ^2 on both sides of equation (3.10), and then we respectively obtain:

$$\begin{aligned} \Omega_3 &= 0, \\ \Omega_{3,s} &= 0, \\ k_2\Omega_{3,s} + k_2\Omega_2 - k_2^2\Omega_3 &= \Omega_{2,s} + \Omega_1. \end{aligned}$$

Explicitly, by using (3.7), we get:

$$\begin{aligned} w_2 &= (-w_{3,s} + k_2 w_3), \\ w_1 &= \frac{1}{2}(-w_{2,s} + \int_0^s k_2 w_{2,s}\, ds). \end{aligned}$$

Hence, the theorem holds. □

Theorem 5. *Consider the motion of an inextensible NCC with a Bishop frame, and then the TEEsBFs are:*

$$\begin{pmatrix} T_1 \\ N_1 \\ N_2 \end{pmatrix}_t = \begin{pmatrix} w_{1,s} + k_2 w_1 & w_{2,s} + w_1 & 0 \\ 0 & 0 & w_{2,s} + w_1 \\ 0 & 0 & -(w_{1,s} + k_2 w_1) \end{pmatrix} \begin{pmatrix} T_1 \\ N_1 \\ N_2 \end{pmatrix}, \tag{3.11}$$

and the TEEsBC for k_2 is given by:

$$k_{2,t} = (w_{1,s} + k_2 w_1)_{,s}. \tag{3.12}$$

Proof. Since

$$\alpha_{ts} = (w_1 T_1 + w_2 N_1 + w_3 N_2)_s\,,$$

then

$$\alpha_{t\,s} = (w_{1,s} + k_2 w_1) T_1 + (w_{2,s} + w_1) N_1 + (w_{3,s} + w_2 - k_2 w_3) N_2. \tag{3.13}$$

Since the curve is inextensible, by using the first condition of (3.3) and substituting into (3.13), we have

$$\alpha_{t\,s} = (w_{1,s} + k_2 w_1)\, T_1 + (w_{2,s} + w_1)\, N_1. \tag{3.14}$$

Since $\alpha_s = T_1$, $\alpha_{st} = T_{1,t}$, and the curve is inextensible, so the commutative law $\frac{\partial}{\partial s}\frac{\partial}{\partial t}() = \frac{\partial}{\partial t}\frac{\partial}{\partial s}()$ is satisfied. So $\alpha_{ts} = \alpha_{st}$, hence

$$T_{1,t} = (w_{1,s} + k_2 w_1)\, T_1 + (w_{2,s} + w_1)\, N_1. \tag{3.15}$$

Taking the s-derivative for (3.15), we have

$$T_{1,ts} = \Big((w_{1,s} + k_2 w_1)_s + k_2(w_{1,s} + k_2 w_1)\Big) T_1 + \Big((w_{2,s} + w_1)_s + (w_{1,s} + k_2 w_1)\Big) N_1 + (w_{2,s} + w_1) N_2. \tag{3.16}$$

Taking the s-derivative of the second condition of (3.3), and substituting in (3.16), we have

$$T_{1,ts} = \Big((w_{1,s} + k_2 w_1)_s + k_2(w_{1,s} + k_2 w_1)\Big) T_1 + k_2(w_{2,s} + w_1) N_1 + (w_{2,s} + w_1) N_2. \tag{3.17}$$

Since $T_{1,s} = k_2 T_1 + N_1$, then by taking the t-derivative, we have

$$T_{1,st} = \Big(k_{2,t} + k_2(w_{1,s} + k_2 w_1)\Big) T_1 + k_2(w_{2,s} + w_1) N_1 + N_{1,t}. \tag{3.18}$$

Since $T_{ts} = T_{st}$, then we obtain the TEEsBC for k_2 and the time evolution equation of the normal vector $N_{1,t}$:

$$k_{2,t} = (w_{1,s} + k_2 w_1)_s\ ,$$

and

$$N_{1,t} = (w_{2,s} + w_1) N_2\ .$$

The evolution equation of the second curvature ($k_{2,t}$) explains how the non-constant curvature of the NCC bends and twists over time based on the Bishop velocities. This equation offers valuable insights into the dynamic behavior of NCCs and provides theoretical advancements and practical improvements in modeling geometric phenomena.

Assume that $N_{2,t}$ can be written as follows:

$$N_{2,t} = c_{11}\, T_1 + c_{12}\, N_1 + c_{13}\, N_2. \tag{3.19}$$

Since N_2 is a lightlike transversal vector, $\langle N_2, N_2\rangle = 0$, then $\langle N_{2,t}, N_2\rangle = 0$, hence $c_{11} = 0$. Using $\langle N_1, N_2\rangle = 0$, then $\langle N_1, N_{2,t}\rangle = -\langle N_{1,t}, N_2\rangle$, hence $c_{12} = 0$.

Since $\langle T_1, N_2\rangle = -1$, then $\langle T_1, N_{2,t}\rangle = -\langle T_{1,t}, N_2\rangle$, so

$$c_{13} = -(w_{1,s} + k_2\, w_1).$$

Hence, we have

$$N_{2,t} = -(w_{1,s} + k_2 w_1) N_2.$$

Hence, the theorem holds. □

4. Application on the evolution of null Cartan curves (NCCs) using the Bishop frame

We begin by considering the nonlinear partial differential equation (NPDE) (3.12) and transforming it into an ordinary differential equation using a suitable traveling wave transformation defined as:

$$k_2(s,t) = k_2(\zeta)\,,\ \zeta = s - \lambda t,$$

where λ denotes the velocity of the traveling wave. This transformation simplifies the NPDE (3.12) into the following ODE:

$$\lambda \frac{dk_2}{d\zeta} + \frac{d}{d\zeta}\left(\frac{dw_1}{d\zeta} + k_2 w_1\right) = 0. \tag{4.1}$$

Upon integrating (4.1), we obtain:

$$\lambda k_2 + \frac{dw_1}{d\zeta} + k_2 w_1 = 0. \tag{4.2}$$

For simplicity, the constant of integration is set to zero. Furthermore, under the traveling wave transformation, $\zeta = s - \lambda t$, the inextensibility conditions (3.3) are transformed into the following system of ODEs:

$$\begin{aligned} w_2 &= -\frac{dw_3}{d\zeta} + k_2 w_3,\\ w_1 &= \frac{1}{2}\left(-\frac{dw_2}{d\zeta} + \int k_2 \frac{dw_2}{d\zeta} d\zeta\right). \end{aligned} \tag{4.3}$$

Application 1. *Consider the normal velocity* $w_3 = 1$. *By utilizing Eq* (4.3)*, we can determine the tangential velocity* w_1 *and the normal velocity* w_2 *as follows:*

$$\begin{aligned} w_2 &= k_2,\\ w_1 &= \frac{1}{2}\left(-k_2' + \frac{1}{2}k_2^2\right), \qquad (\ )' = \frac{d}{d\zeta}. \end{aligned} \tag{4.4}$$

the given ODE (4.2) *that describes the second Bishop curvature takes the following form:*

$$k_2'' = \frac{1}{2}k_2^3 + 2\lambda k_2. \tag{4.5}$$

To solve this equation, we assume that

$$\begin{aligned} k_2' &= \frac{d\,k_2}{d\zeta} = Y(k_2(\zeta)), \\ k_2'' &= Y\frac{dY}{d\,k_2}. \end{aligned} \tag{4.6}$$

Substituting from (4.6) *into* (4.5), *hence:*

$$Y\frac{dY}{d\,k_2} = \frac{1}{2}k_2^3 + 2\lambda k_2.$$

By integrating this equation, we get

$$Y(k_2(\zeta)) = -\frac{k_2}{2}\sqrt{k_2^2 + 8\lambda} + c_1. \tag{4.7}$$

For simplicity, the constant of integration is set to zero. Since $Y(k_2(\zeta)) = \frac{d\,k_2}{d\zeta}$, *then we rewrite* (4.7) *as follows:*

$$\frac{d\,k_2}{d\,\zeta} = -\frac{k_2}{2}\sqrt{k_2^2 + 8\lambda}.$$

By integrating this equation, then we obtain:

$$k_2(\zeta) = \sqrt{8\,\lambda}\,\mathrm{csch}\,(\sqrt{2\lambda}(\zeta + c_1)).$$

Therefore, we have obtained the second Bishop curvature in the form:

$$k_2(s,t) = \sqrt{8\,\lambda}\,\mathrm{csch}\,(\sqrt{2\lambda}(s - \lambda\,t + c_1)). \tag{4.8}$$

The second Bishop curvature is illustrated in Figure 1.

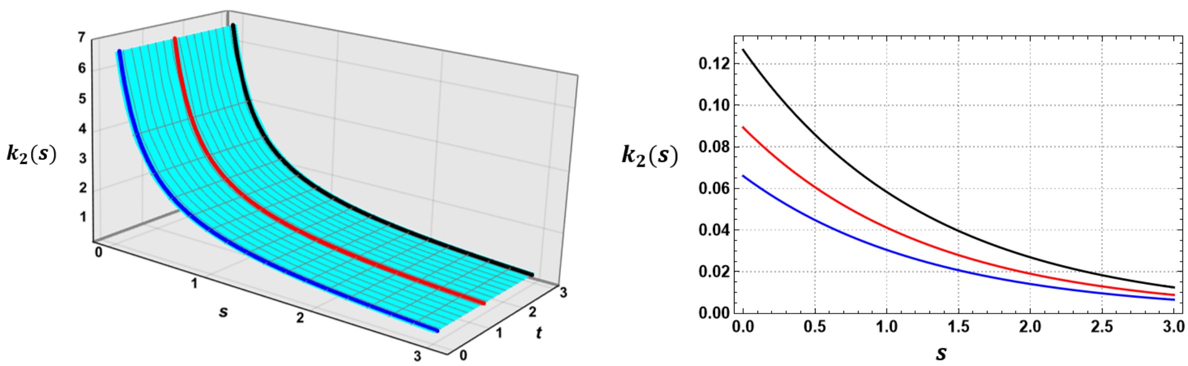

Figure 1. The evolution of the second Bishop curvature of the NCC with blue, red, and black curves at $t = 0.1, 1.4$, and 2.9, respectively, for $s \in [0, 3], t \in [0, 3]$.

Substituting from (4.8) *into the two systems* (2.7) *and* (3.11), *and solving them numerically, we can then visualize the evolution of the NCC at various values of time in Figure 2.*

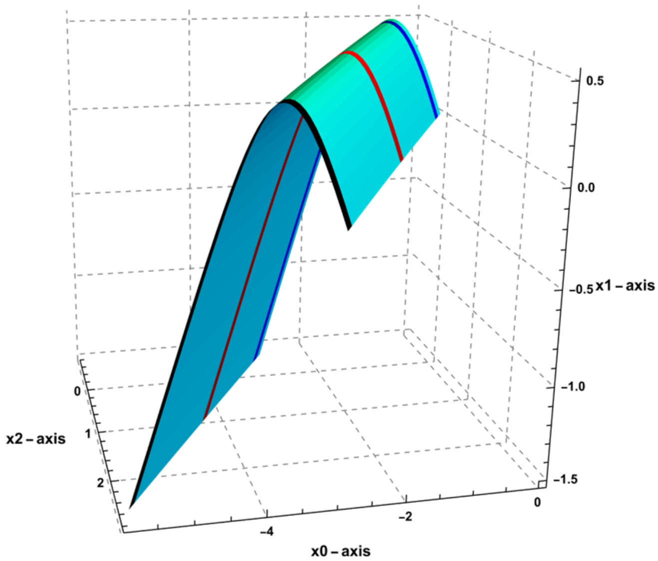

Figure 2. The evolution of the NCC with blue, red, and black curves at $t = 0.1, 1.4$, and 2.9, respectively, for $\lambda = 0.3, c_1 = 5$, $s \in [0, 3]$, and $t \in [0, 3]$.

5. Main results on the evolution of pseudo null curves (PNCs) with the Bishop frame

Let $\alpha : J = [0, l] \times [0, t] \longrightarrow \mathbb{R}^{2,1}$ be the family of PNCs with Bishop frame $\mathbb{F}_B$, where l is the parameter of the initial curve and t represents the time parameter. Let u be the parameter variable of the PNC, where $0 \leq u \leq l$. Let s be the parameter of the arc length of the PNC and it is defined by:

$$s(u, t) = \int_0^u \|\alpha_u(u, t)\| du = \int_0^u g(u, t) du. \qquad g^2(u, t) = \|\alpha_u(u, t)\|^2 = |\langle \alpha_u, \alpha_u \rangle|. \tag{5.1}$$

The law describing the motion of the PNC with Bishop frame $\mathbb{F}_B$ is given by:

$$\frac{\partial \alpha}{\partial t} = \varpi_1\, T_1 + \varpi_2\, N_1 + \varpi_3\, N_2, \tag{5.2}$$

where ϖ_1, ϖ_2, and ϖ_3 are the tangential, normal, and binormal velocities, and we call them the Bishop velocities.

Theorem 6. *The PNC with the Bishop frame is inextensible if the Bishop velocities ϖ_1, ϖ_2, and ϖ_3 satisfy the following condition:*

$$\varpi_{1,s} = k_1 \varpi_2 + k_2 \varpi_3. \tag{5.3}$$

Proof. If the PNC with the Bishop frame is inextensible, then $\frac{\partial g(u,t)}{\partial t} = 0$, by taking the t-derivative of (5.1), and then:

$$2g\, \frac{\partial g}{\partial t} = 0 = \langle \alpha_{ut}, \alpha_u \rangle. \tag{5.4}$$

Using

$$\alpha_u = g\alpha_s = gT_1, \tag{5.5}$$

since $\alpha_{ut} = \alpha_{tu}$, and by taking the u-derivative of (5.2) and using (2.16), hence we obtain:

$$\alpha_{tu} = g\ (\varpi_{1,s} - k_1\varpi_2 - k_2\varpi_3)\, T_1 + g\ (\varpi_{2,s} + k_2\varpi_1)\, N_1 + g\ (\varpi_{3,s} + k_1\varpi_1)\, N_2. \tag{5.6}$$

Substituting from (5.5) and (5.6) into (5.4), and using the properties of the Bishop frame for the PNC that are defined by (2.13), we have

$$\varpi_{1,s} - k_1\varpi_2 - k_2\varpi_3 = 0.$$

Hence, the theorem holds. □

Lemma 7. *Consider the motion of the PNC with the Bishop frame. The curve is inextensible if the Bishop velocities ϖ_1, ϖ_2, and ϖ_3 satisfy the following condition:*

- *Case 1: In the case where the motion of the PNC with the Bishop frame defined by* (2.16) *with $k_1 = 0$, then the curve is inextensible if the Bishop velocities ϖ_1, ϖ_2, and ϖ_3 satisfy:*

$$\varpi_{1,s} = k_2\varpi_3. \tag{5.7}$$

- *Case 2: In the case where the motion of the PNC with the Bishop frame defined by* (2.18) *with $k_2 = 0$, then the curve is inextensible if the Bishop velocities ϖ_1, ϖ_2, and ϖ_3 satisfy:*

$$\varpi_{1,s} = k_1\varpi_2. \tag{5.8}$$

Theorem 8. *Consider the flow of the inextensible PNCs with a Bishop frame. Then the TEEsBFs are:*

$$\begin{pmatrix} T_1 \\ N_1 \\ N_2 \end{pmatrix}_t = \begin{pmatrix} 0 & \varpi_{2,s} + k_2\varpi_1 & \varpi_{3,s} + k_1\varpi_1 \\ -(\varpi_{3,s} + k_1\varpi_1) & 0 & 0 \\ -(\varpi_{2,s} + k_2\varpi_1) & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ N_1 \\ N_2 \end{pmatrix}. \tag{5.9}$$

In addition, the TEEsBCs for k_1 and k_2 are:

$$\begin{aligned} k_{1,t} &= (\varpi_{3,s} + k_1\varpi_1)_{,s}\ , \\ k_{2,t} &= (\varpi_{2,s} + k_2\varpi_1)_{,s}\ . \end{aligned} \tag{5.10}$$

Proof. Since $\alpha_s = T_1$, then

$$\alpha_{s\,t} = T_{1,t}. \tag{5.11}$$

Taking the s-derivative of (5.2), we have

$$\alpha_{t\,s} = \Big(\varpi_{1,s} - k_1\varpi_2 - k_2\varpi_3\Big)T_1 +\ (\varpi_{2,s} + k_2\varpi_1)\, N_1 +\ (\varpi_{3,s} + k_1\varpi_1)\, N_2\ . \tag{5.12}$$

Since the curve is inextensible, then the condition (5.3) is satisfied. Hence,

$$\alpha_{t\,s} =\ (\varpi_{2,s} + k_2\varpi_1)\, N_1 +\ (\varpi_{3,s} + k_1\varpi_1)\, N_2\ . \tag{5.13}$$

Since the curve is inextensible, then $\alpha_{ts} = \alpha_{st}$, so from (5.11) and (5.13), we obtain:

$$T_{1,t} = (\varpi_{2,s} + k_2\varpi_1)N_1 + (\varpi_{3,s} + k_1\varpi_1)\, N_2\ . \tag{5.14}$$

Applying the s-derivative of (5.14), we get

$$T_{1,ts} = \Big(- k_1(\varpi_{2,s} + k_2\varpi_1) - k_2(\varpi_{3,s} + k_1\varpi_1)\Big) T_1 + (\varpi_{2,s} + k_2\varpi_1)_{,s}\, N_1 + (\varpi_{3,s} + k_1\varpi_1)_{,s}\, N_2. \tag{5.15}$$

Since $T_{1,s} = k_2N_1 + k_1N_2$, by taking the t-derivative of this equation, we get:

$$T_{1,s\,t} = k_{2,t}\, N_1 + k_{1,t}\, N_2 + k_2\, N_{1,t} + k_1\, N_{2,t}. \tag{5.16}$$

Since $T_{1,ts} = T_{1,st}$, then from (5.15) and (5.16), we obtain that the TEEsBCs for k_1 and k_2 are:

$$\begin{aligned} k_{1,t} &= (\varpi_{3,s} + k_1\varpi_1)_{,s}, \\ k_{2,t} &= (\varpi_{2,s} + k_2\varpi_1)_{,s}, \end{aligned} \tag{5.17}$$

and

$$k_2\, N_{1,t} + k_1\, N_{2,t} = -(k_1\, (\varpi_{2,s} + k_2\varpi_1) + k_2\, (\varpi_{3,s} + k_1\varpi_1))T_1. \tag{5.18}$$

The PDEs (5.17) describe the evolution of curvatures $(k_{1,t}, k_{2,t})$, detailing how Bishop velocities affect the behavior of non-constant curvatures of the PNC (how the curve bends and twists over time).

Assume

$$\begin{aligned} N_{1,t} &= a_{11}\, T_1 + a_{12}\, N_1 + a_{13}\, N_2, \\ N_{2,t} &= b_{11}\, T_1 + b_{12}\, N_1 + b_{13}\, N_2. \end{aligned} \tag{5.19}$$

Since N_1 and N_2 are lightlike vectors, $\langle N_1, N_1\rangle = 0$ and $\langle N_2, N_2\rangle = 0$. Then by applying the derivative and using (2.13), we get:

$$a_{13} = 0\,, \quad b_{12} = 0.$$

Since $\langle N_1, T_1\rangle = 0$ and $\langle N_2, T_1\rangle = 0$, hence, by applying the t-derivative, we get:

$$a_{11} = -(\varpi_{3,s} + k_1\varpi_1)\,,\; b_{11} = -(\varpi_{2,s} + k_2\varpi_1).$$

Using the condition $\langle N_1, N_2\rangle = 1$, then by taking the t-derivative, we get:

$$b_{13} = -a_{12}.$$

Hence, we have

$$\begin{aligned} N_{1,t} &= -(\varpi_{3,s} + k_1\varpi_1)\, T_1 + a_{12}\, N_1\ , \\ N_{2,t} &= -(\varpi_{2,s} + k_2\varpi_1)\, T_1 - a_{12}\, N_2\ . \end{aligned} \tag{5.20}$$

Substitute from (5.20) in (5.18), then $a_{12} = 0$. Hence, we get

$$\begin{aligned} N_{1,t} &= -(\varpi_{3,s} + k_1\varpi_1)\, T_1, \\ N_{2,t} &= -(\varpi_{2,s} + k_2\varpi_1)\, T_1\ . \end{aligned} \tag{5.21}$$

Hence, the theorem holds. □

Lemma 9. *Consider the flows of the inextensible PNC with the Bishop frame that is defined by* (2.16) *and* (2.18) *with conditions* (5.7) *and* (5.8), *respectively. Then, the TEEsBFs are:*

- *Case 1: In the case of the motion of the inextensible PNC with Bishop frame* (2.16) *with* $k_1 = 0$ *and condition* (5.7), *we have*

$$\begin{pmatrix} T_1 \\ N_1 \\ N_2 \end{pmatrix}_t = \begin{pmatrix} 0 & \varpi_{2,s} + k_2\varpi_1 & \varpi_{3,s} \\ -\varpi_{3,s} & 0 & 0 \\ -(\varpi_{2,s} + k_2\varpi_1) & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ N_1 \\ N_2 \end{pmatrix}. \tag{5.22}$$

The TEEBC is:

$$k_{2,t} = (\varpi_{2,s} + k_2\varpi_1)_{,s}\,, \varpi_{3,ss} = 0. \tag{5.23}$$

- *Case 2: In the case of the motion of the inextensible PNC with Bishop frame* (2.18) *with* $k_2 = 0$ *and condition* (5.8), *we have*

$$\begin{pmatrix} T_1 \\ N_1 \\ N_2 \end{pmatrix}_t = \begin{pmatrix} 0 & \varpi_{2,s} & \varpi_{3,s} + k_1\varpi_1 \\ -(\varpi_{3,s} + k_1\varpi_1) & 0 & 0 \\ -\varpi_{2,s} & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ N_1 \\ N_2 \end{pmatrix}. \tag{5.24}$$

The TEEsBC is:

$$k_{1,t} = (\varpi_{3,s} + k_1\varpi_1)_{,s}\,,\ \ \varpi_{2,s} = 0. \tag{5.25}$$

6. Application on the evolution of pseudo null curves (PNCs) using the Bishop frame

Application 2. *Consider the constant tangential velocity* $\varpi_1 = a$, *and then the binormal velocity* $\varpi_3 = 0$. *Assume that the normal velocity* $\varpi_2 = \log\, k_2(s,t)$. *Therefore, we have*

$$k_{2,t} = (\frac{k_{2,s}}{k_2} + ak_2)_{,s}. \tag{6.1}$$

By solving the PDE (6.1), *we obtain the second Bishop curvature in the form:*

$$k_2(s,t) = \frac{C_1^2}{aC_1 - C_2}(-1 + \tanh(C_1\, s + C_2\, t + C_3)\,). \tag{6.2}$$

The second Bishop curvature is illustrated in Figure 3. Substituting from (6.2) *into* (2.16) *and* (5.22) *and solving the two systems* (2.16) *and* (5.22) *numerically for* $k_1 = 0$ *and* $k_2(s,t) = \frac{C_1^2}{aC_1 - C_2}(-1 + \tanh(C_1\, s + C_2\, t + C_3)\,)$, *then we can visualize the evolution of the PNC at various values of time (see Figure 4).*

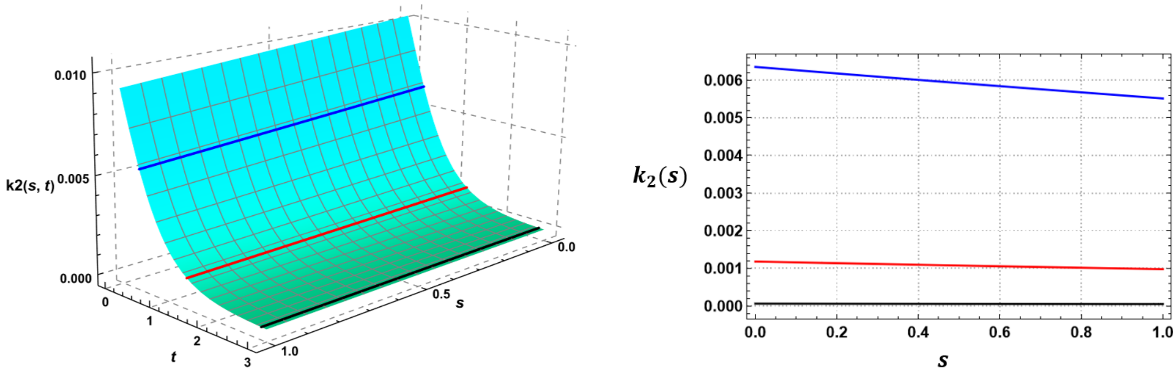

Figure 3. The evolution of the second Bishop curvature of the PNC with blue, red, and black curves at $t = 0.4, 1.4$, and 2.9 for $C_1 = 0.1, C_2 = 1, C_3 = -0.01, a = 0.1$, $s \in [0, 2]$, and $t \in [0, 3]$.

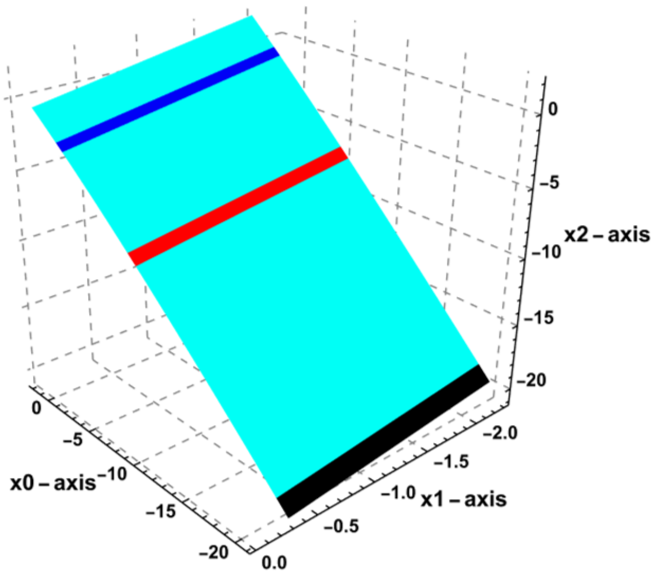

Figure 4. The evolution of the PNC with blue, red, and black curves at $t = 0.4, 1.4$, and 2.9 for $C_1 = 0.1, C_2 = 1, C_3 = -0.01, a = 0.1$, $s \in [0, 2]$, and $t \in [0, 3]$.

Application 3. *Consider the constant tangential velocity,* $\varpi_1 = a$*, and then the binormal velocity* $\varpi_3 = 0$*. Assume that the normal velocity is* $\varpi_2 = k_2(s, t)$*, and we have*

$$k_{2,t} = (k_{2,s} + ak_2)_{,s}. \tag{6.3}$$

By solving the PDE (6.3)*, then we obtain the second Bishop curvature in the form:*

$$k_2(s, t) = 1 + e^{C_1 s + t(aC_1 + C_2) + C_3}. \tag{6.4}$$

The second Bishop curvature is illustrated in Figure 5. Substituting from (6.4) *into the two systems* (2.16) *and* (5.22) *and solving them numerically for* $k_1 = 0$ *and* $k_2(s, t) = 1 + e^{C_1 s + t(aC_1 + C_2) + C_3}$*, then we can visualize the evolution of (PNC) at various values of time (see Figure 6).*

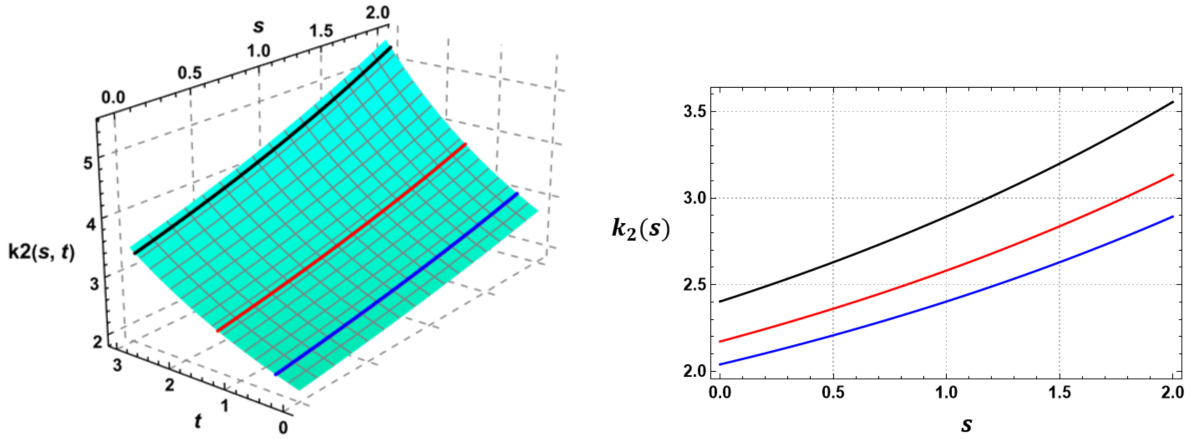

Figure 5. The evolution of the second Bishop curvature of the PNC with blue, red, and black curves at $t = 0.4, 1.4$, and 2.9, respectively, for $C_1 = 0.3, C_2 = 0.09, C_3 = -0.01, a = 0.1$, $s \in [0, 2]$, and $t \in [0, 3]$.

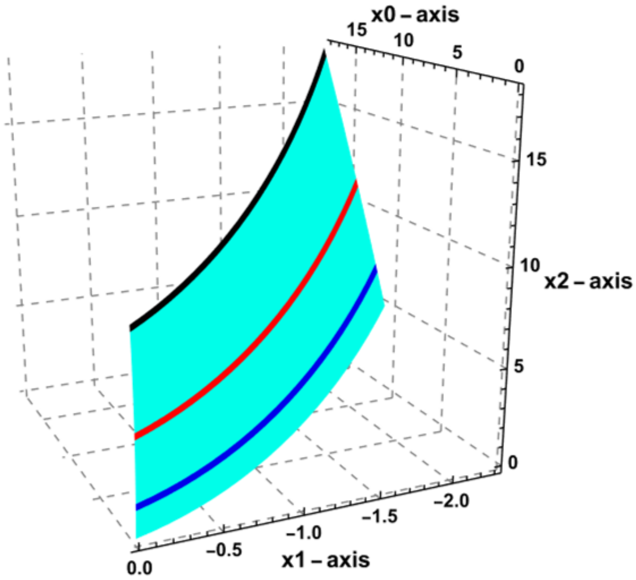

Figure 6. The evolution of the PNC with blue, red, and black curves at $t = 0.4, 1.4$, and 2.9, respectively, for $C_1 = 0.3, C_2 = 0.09, C_3 = -0.01, a = 0.1$, $s \in [0, 2]$, and $t \in [0, 3]$.

7. Conclusions

In the present paper, we study the evolution of null Cartan curves (NCCs) and pseudo null curves (PNCs) according to the Bishop frame. The new results are listed as follows:

- We obtained the necessary and sufficient inextensibility conditions for the NCCs (Theorem 4 by Eq (3.3)) and for the PNC according to the Bishop frame (Theorem 6 by Eq (5.3) and Lemma 7 by Eqs (5.7) and (5.8)).
- We derived the TEEsBF for the NCCs (Theorem 5 by Eq (3.11)) and the PNC in $\mathbb{R}^{2,1}$ (Theorem 8 by Eq (5.9) and Lemma 9 by Eqs (5.22) and (5.24)).

- We derived the TEEsBC of the NCCs (Theorem 5 by Eq (3.12)) and the PNCs as (PDEs) in terms of Bishop velocities (Theorem 8 by Eq (5.10) and Lemma 9 by Eqs (5.23) and (5.25)).
- The evolution of curvatures for both null Cartan curves (NCCs) and pseudo null curves (PNCs) were calculated and plotted for some applications. Additionally, the evolution of NCCs and PNCs is visualized.

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Authors contributions

Samah Gaber: Investigation, Writing the original draft, Writing the review and editing, Software; Abeer Al Elaiw: Investigation, Writing the original draft, Writing the review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in creating this article.

Conflict of interest

The authors declare no conflicts of interest.

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