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Research article

Weighted composition operators on α-Bloch-Orlicz spaces over the unit polydisc

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Abstract: Let $\mathbb{U}^n$ be the unit polydisc in the complex vector space $\mathbb{C}^n$. We defined the α-Bloch-Orlicz space on $\mathbb{U}^n$ by using Young's function and showed that its norm is equivalent with a special μ-Bloch space. We also characterized the boundedness and compactness of the weighted composition operator on α-Bloch-Orlicz space. Our results generalized the corresponding results on the unit disk.

Keywords: weighted composition operator; boundedness; compactness; α-Bloch-Orlicz space; polydisc
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1. Introduction

Let $\mathbb{U}$ be the unit disk in the complex plane $\mathbb{C}$, $\mathbb{U}^n = \{z = (z_1, z_2, \cdots, z_n) : |z_i| < 1, i = 1, 2, \cdots, n\}$ the unit polydisc in the complex vector space $\mathbb{C}^n$, and $\partial\mathbb{U}^n = \{z = (z_1, z_2, \cdots, z_n) : |z_i| = 1, \ \ i = 1, 2, \cdots, n\}$ the distinguished boundary of $\mathbb{U}^n$. Let $H(\mathbb{U}^n)$ be the space of all holomorphic functions on $\mathbb{U}^n$ and $H^\infty(\mathbb{U}^n)$ the space of all bounded holomorphic functions on $\mathbb{U}^n$ (see, for example, [12, 14, 20]).

Let $\psi \in H(\mathbb{U}^n)$ and $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n)$ be a holomorphic self-mapping of $\mathbb{U}^n$. The *weighted composition operator* $W_{\psi,\varphi}$ on some subspaces of $H(\mathbb{U}^n)$ is defined by

$$W_{\psi,\varphi} f(z) = \psi(z) f(\varphi(z)), \ z \in \mathbb{U}^n.$$

If $\psi(z) \equiv 1$ on $\mathbb{U}^n$, the operator $W_{\psi,\varphi}$ is reduced to the composition operator C_φ, while if $\varphi(z) = z$, it is reduced to the multiplication operator M_ψ. The theory of the (weighted) composition operators on various spaces has quite a long and rich history (we will give some concrete studies. For example, see [1–3] or [5] for the studies of composition operators; see [7, 11, 17] from one space to Bloch type spaces; see [8, 9, 24] from one space to weighted spaces or Bloch spaces; see [28–30] for one on Hardy spaces, Zygmund-Orlicz spaces, or logarithmic Bloch-Orlicz spaces).

Recall that for $\alpha > 0$, the α-Bloch space on $\mathbb{U}^n$ denoted by $\mathcal{B}_\alpha(\mathbb{U}^n)$ consists of all $f \in H(\mathbb{U}^n)$ such that

$$\|f\|_\alpha = \sup_{z\in\mathbb{U}^n} \sum_{k=1}^{n} (1 - |z_k|^2)^\alpha \Big|\frac{\partial f}{\partial z_k}(z)\Big| < +\infty.$$

It is well-known that $\mathcal{B}_\alpha(\mathbb{U}^n)$ is a Banach space with the norm $\|f\|_{\mathcal{B}_\alpha(\mathbb{U}^n)} = |f(0)| + \|f\|_\alpha$. For this and related spaces and operators on them, see, for example, [4, 19, 21] and the references therein.

A positive continuous function on $\mathbb{U}$ is called *weight*. Let μ be a weight. The μ-Bloch space on $\mathbb{U}^n$ denoted by $\mathcal{B}_\mu(\mathbb{U}^n)$ consists of all $f \in H(\mathbb{U}^n)$ such that

$$\|f\|_\mu = \sup_{z\in\mathbb{U}^n} \sum_{k=1}^{n} \mu(z_k)\Big|\frac{\partial f}{\partial z_k}(z)\Big| < +\infty.$$

It is a Banach space endowed with the norm $\|f\|_{\mathcal{B}_\mu(\mathbb{U}^n)} = |f(0)| + \|f\|_\mu$. Clearly, the μ-Bloch space is a natural generalization of the α-Bloch space (see [22, 23, 25, 27] for other Bloch type spaces, and see [10, 13] for α-Bloch space). For some information on this space, also see [18].

As an important closed subspace of $\mathcal{B}_\mu(\mathbb{U}^n)$, the little μ-Bloch space $\mathcal{B}_{\mu,0}(\mathbb{U}^n)$ consists of all $f \in \mathcal{B}_\mu(\mathbb{U}^n)$ such that

$$\lim_{z\to\partial\mathbb{U}^n} \sum_{k=1}^{n} \mu(z_k)\Big|\frac{\partial f}{\partial z_k}(z)\Big| = 0.$$

In order to introduce the α-Bloch-Orlicz space on $\mathbb{U}^n$, we explain here the Bloch-Orlicz space on $\mathbb{U}$, which was introduced by Ramos Fernández in [18]. More precisely, let ϕ be a Young's function, that is, ϕ is a strictly increasing convex function on the interval $[0, +\infty)$ such that

$$\phi(0) = 0 \quad \text{and} \quad \lim_{t\to+\infty} \phi(t) = +\infty.$$

Since $\phi(0) = 0$, from the convexity of ϕ, it clearly follows that $\phi(st) \le s\phi(t)$ for $0 < s < 1$ and $t > 0$.

The Bloch-Orlicz space denoted by $\mathcal{B}_\phi(\mathbb{U})$ consists of all $f \in H(\mathbb{U})$ such that

$$\sup_{z\in\mathbb{U}} (1 - |z|^2)\phi(\lambda|f'(z)|) < +\infty$$

for some positive λ depending on f. The Minkowski's functional

$$\|f\|_\phi = \inf\Big\{k > 0 : S_\phi\Big(\frac{f'}{k}\Big) \le 1\Big\}$$

defines a semi-norm for $\mathcal{B}_\phi(\mathbb{U})$, where

$$S_\phi(f) = \sup_{z\in\mathbb{U}} (1 - |z|^2)\phi(|f(z)|).$$

$\mathcal{B}_\phi(\mathbb{U})$ becomes a Banach space with the norm

$$\|f\|_{\mathcal{B}_\phi(\mathbb{U})} = |f(0)| + \|f\|_\phi.$$

Motivated by the Bloch-Orlicz space, the α-Bloch-Orlicz space on $\mathbb{U}$ denoted by $\mathcal{B}_{\phi,\alpha}(\mathbb{U})$ was introduced by Liang in [15]. Since $\mathbb{U}^n$ is an important bounded symmetric domain of $\mathbb{C}^n$, it is natural to define a similar space on the domain and study some concrete operators on it.

The abovementioned facts motivate us to define the *α-Bloch-Orlicz space* on $\mathbb{U}^n$. The space consists of all $f \in H(\mathbb{U}^n)$ such that

$$\sup_{z\in\mathbb{U}^n} \sum_{k=1}^{n} (1-|z_k|^2)^\alpha \phi\Big(\lambda\Big|\frac{\partial f}{\partial z_k}(z)\Big|\Big) < +\infty$$

for some $\lambda > 0$ depending on f, and it is denoted by $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$. Since ϕ is convex, it is easy to see that the Minkowski's functional

$$\|f\|_{\phi,\alpha} = \inf\Big\{t > 0 : S_{\phi,\alpha}\Big(\frac{f}{t}\Big) \le 1\Big\}$$

defines a semi-norm on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$, which is known as Luxemburg's semi-norm, where

$$S_{\phi,\alpha}(f) = \sup_{z\in\mathbb{U}^n} \sum_{k=1}^{n} (1-|z_k|^2)^\alpha \phi\Big(\Big|\frac{\partial f}{\partial z_k}(z)\Big|\Big).$$

It can be easily proved that $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ is a Banach space with the norm

$$\|f\|_{\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)} = |f(0)| + \|f\|_{\phi,\alpha}.$$

Observe that when $\phi(t) = t$ with $t \ge 0$, we get back the α-Bloch space $\mathcal{B}_\alpha(\mathbb{U}^n)$. Furthermore, from [18] we can suppose that ϕ^{-1} is continuously differentiable on $[0, +\infty)$.

In this paper, we mainly study the boundedness and compactness of the operator $W_{\psi,\varphi}$ on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$. It can be regarded as a continuation of the investigation of concrete operators on these spaces.

Throughout the paper, we will write $W_{\psi,\varphi}f$ instead of $W_{\psi,\varphi}(f)$. The letter C will denote a positive constant, and the exact value may vary in each case. The notation $a \lesssim b$ means that there is a constant $C > 0$, such that $a \le Cb$. When $a \lesssim b$ and $b \lesssim a$, we write $a \asymp b$.

2. Preliminary results

First, we obtain the following result, which is similar to the corresponding result in [18].

Lemma 2.1. *For each $f \in \mathcal{B}_{\phi,\alpha}(\mathbb{U}^n) \setminus \{0\}$, it follows that*

$$S_{\phi,\alpha}\Big(\frac{f}{\|f\|_{\phi,\alpha}}\Big) \le 1. \tag{2.1}$$

Moreover, for each $f \in \mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ and $z \in \mathbb{U}^n$, the following holds:

$$\Big|\frac{\partial f}{\partial z_k}(z)\Big| \le \phi^{-1}\Big(\frac{1}{(1-|z_k|^2)^\alpha}\Big)\|f\|_{\phi,\alpha}. \tag{2.2}$$

Proof. For $f \in \mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)\backslash\{0\}$, by the definition of $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$, there exists a decreasing sequence $\{\lambda_i\} \subset \mathbb{R}^+$ with $S_{\phi,\alpha}(\frac{f}{\lambda_i}) \le 1$, such that $\lambda_i \to \|f\|_{\phi,\alpha}$ as $i \to \infty$. Since the function ϕ is increasing, we get

$$S_i := S_{\phi,\alpha}\Big(\frac{f}{\lambda_i}\Big) \le S_{\phi,\alpha}\Big(\frac{f}{\|f\|_{\phi,\alpha}}\Big) := S. \tag{2.3}$$

From the monotonicity of ϕ and (2.3), we obtain that $\{S_i\}$ is bounded and increasing. Hence, there is a real number S_0 such that

$$S_0 = \lim_{i\to\infty} S_i = \sup_{i\in N} S_i.$$

By (2.3) and $S_i \le 1$ for each $i \in \mathbb{N}$, we have $S_0 \le S$ and $S_0 \le 1$. So, for all $z \in \mathbb{U}^n$ and $i \in \mathbb{N}$, we have

$$S_i = \sum_{k=1}^{n} (1-|z_k|^2)^\alpha \phi\Big(\frac{|\frac{\partial f}{\partial z_k}(z)|}{\lambda_i}\Big) \le S_0. \tag{2.4}$$

Letting $i \to \infty$ in (2.4), we obtain

$$S = \sum_{k=1}^{n} (1-|z_k|^2)^\alpha \phi\Big(\frac{|\frac{\partial f}{\partial z_k}(z)|}{\|f\|_{\phi,\alpha}}\Big) \le S_0, \tag{2.5}$$

for all $z \in \mathbb{U}^n$. Consequently, we obtain that $S = S_0$, and then $S \le 1$. From this, for all $f \in \mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ and $z \in \mathbb{U}^n$, it follows that

$$\sum_{k=1}^{n} (1-|z_k|^2)^\alpha \phi\Big(\frac{|\frac{\partial f}{\partial z_k}(z)|}{\|f\|_{\phi,\alpha}}\Big) \le 1.$$

Then, for each $k \in \{1, 2, \cdots, n\}$ and all $z \in \mathbb{U}^n$, we have

$$\phi\Big(\frac{|\frac{\partial f}{\partial z_k}(z)|}{\|f\|_{\phi,\alpha}}\Big) \le \frac{1}{(1-|z_k|^2)^\alpha}. \tag{2.6}$$

Since ϕ is the strictly increasing convex function, it follows that

$$\Big|\frac{\partial f}{\partial z_k}(z)\Big| \le \phi^{-1}\Big(\frac{1}{(1-|z_k|^2)^\alpha}\Big)\|f\|_{\phi,\alpha}. \tag{2.7}$$

This completes the proof of the lemma. □

For the convenience, we write

$$\mu_{\phi,\alpha}(z) = \frac{1}{\phi^{-1}\big(\frac{1}{(1-|z|^2)^\alpha}\big)}, \quad z \in \mathbb{U}.$$

Noting that

$$f(z) - f(0) = \int_0^1 \frac{d}{dt} f(tz)dt = \sum_{k=1}^{n} \int_0^1 z_k \frac{\partial f}{\partial w_k}(tz)dt,$$

from Lemma 2.1 we obtain the following result.

Corollary 2.1. *Let $\alpha > 0$. If $f \in \mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$, then for all $z \in \mathbb{U}^n$, it follows that*

$$|f(z)| \leq \Big(1 + \sum_{k=1}^{n} \int_0^{|z_k|} \frac{1}{\mu_{\phi,\alpha}(t)} dt\Big)\|f\|_{\phi,\alpha}. \tag{2.8}$$

We also have the following result, which is similar to that in [18].

Lemma 2.2. *Let $\alpha > 0$. Then, $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n) = \mathcal{B}_{\mu_{\phi,\alpha}}(\mathbb{U}^n)$, where $\mathcal{B}_{\mu_{\phi,\alpha}}(\mathbb{U}^n)$ is the special $\mu_{\phi,\alpha}$-Bloch space with the weight $\mu_{\phi,\alpha}$. Moreover, for each $f \in \mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$, it follows that*

$$\|f\|_{\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)} \asymp \|f\|_{\mathcal{B}_{\mu_{\phi,\alpha}}(\mathbb{U}^n)}.$$

Proof. By Lemma 2.1, for all $f \in \mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ and $z = (z_1, z_2, \cdots, z_n) \in \mathbb{U}^n$, we have

$$\sum_{k=1}^{n} \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial f}{\partial z_k}(z)\Big| \leq n\|f\|_{\phi,\alpha},$$

which implies that $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n) \subseteq \mathcal{B}_{\mu_{\phi,\alpha}}(\mathbb{U}^n)$ and

$$\|f\|_{\mathcal{B}_{\mu_{\phi,\alpha}}(\mathbb{U}^n)} \leq n\|f\|_{\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)}. \tag{2.9}$$

Conversely, if $f \in \mathcal{B}_{\mu_{\phi,\alpha}}(\mathbb{U}^n)$, then we have

$$\sum_{k=1}^{n} \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial f}{\partial z_k}(z)\Big| \leq \|f\|_{\mu_{\phi,\alpha}},$$

for all $z \in \mathbb{U}^n$, that is,

$$\sum_{k=1}^{n} \frac{1}{\phi^{-1}(\frac{1}{(1-|z_k|^2)^\alpha})}\Big|\frac{\partial f}{\partial z_k}(z)\Big| \leq \|f\|_{\mu_{\phi,\alpha}},$$

which implies that

$$\frac{\big|\frac{\partial f}{\partial z_k}(z)\big|}{\|f\|_{\mu_{\phi,\alpha}}} \leq \phi^{-1}(\frac{1}{(1-|z_k|^2)^\alpha}).$$

From this, we get

$$(1-|z_k|^2)^\alpha \phi\Big(\frac{\big|\frac{\partial f}{\partial z_k}(z)\big|}{\|f\|_{\mu_{\phi,\alpha}}}\Big) \leq 1.$$

From this, and since $\phi(\frac{s}{n}) \leq \frac{1}{n}\phi(s)$ for $s \geq 0$ and $n \in \mathbb{N}$, it easily follows that

$$S_{\phi,\alpha}\Big(\frac{f}{n\|f\|_{\mu_{\phi,\alpha}}}\Big) \leq 1.$$

This shows that $\|f\|_{\phi,\alpha} \leq n\|f\|_{\mu_{\phi,\alpha}}$, and then

$$\|f\|_{\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)} \leq n\|f\|_{\mathcal{B}_{\mu_{\phi,\alpha}}(\mathbb{U}^n)} \tag{2.10}$$

and $\mathcal{B}_{\mu_{\phi,\alpha}}(\mathbb{U}^n) \subseteq \mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$.

From the above proof, (2.9), and (2.10), we obtain that $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n) = \mathcal{B}_{\mu_{\phi,\alpha}}(\mathbb{U}^n)$ and $\|f\|_{\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)} \asymp \|f\|_{\mathcal{B}_{\mu_{\phi,\alpha}}(\mathbb{U}^n)}$. This completes the proof of the lemma. □

The following result is a version of [6, Lemma 3.1]. For the completeness, we give its proof.

Lemma 2.3. *For a fixed $a \in \mathbb{U}$, there exists a function $f_{a,\alpha} \in H(\mathbb{U})$ such that*

$$\phi(|f_{a,\alpha}(z)|) = \Big(\frac{1-|a|^2}{|1-\bar{a}z|^2}\Big)^{\alpha}. \tag{2.11}$$

Proof. We set

$$u(z) = \phi^{-1}\Big(\Big(\frac{1-|a|^2}{|1-\bar{a}z|^2}\Big)^{\alpha}\Big), \quad z \in \mathbb{U}.$$

Therefore, u is a real and continuously differentiable function in the sense that its partial derivatives exist and are continuous on $\mathbb{U}$. Furthermore, for all $z \in \mathbb{U}$, the function u satisfies

$$u(z) \geq \phi^{-1}\Big(\Big(\frac{1}{4}\Big)^{\alpha}\Big(1-|a|^2\Big)^{\alpha}\Big) > 0.$$

Now, we let $f_{a,\alpha}(z) = u(z)e^{iv(z)}$, where v is a real function defined on $\mathbb{U}$. In the order that $f_{a,\alpha}$ is a holomorphic function on $\mathbb{U}$, then its real part and its imaginary part must satisfy the Cauchy-Riemann equations, that is,

$$\begin{cases} u_x\cos(v) - u\sin(v)v_x = u_y\sin(v) + u\cos(v)v_y, \\ u_y\cos(v) - u\sin(v)v_y = -u_x\sin(v) - u\cos(v)v_x. \end{cases} \tag{2.12}$$

It is easy to see that if

$$uv_x = -u_y \text{ and } uv_y = u_x, \tag{2.13}$$

then (2.12) holds. To find a function v that satisfies (2.13), we define

$$v(x,y) = -\int_0^x \frac{1}{u(s,y)}\frac{\partial u(s,y)}{\partial y}ds + h(y),$$

where h is a real function that satisfies

$$h'(y) = \frac{1}{u(x,y)}\frac{\partial u(x,y)}{\partial x} + \frac{\partial}{\partial y}\Big\{\int_0^x \frac{1}{u(s,y)}\frac{\partial u(s,y)}{\partial y}ds\Big\}.$$

Then, by a computation, we see that v satisfies (2.13). This completes the proof. □

Using the function $f_{a,\alpha}$, we can construct some special functions in the space $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$.

Lemma 2.4. *For the fixed $a \in \mathbb{U}$, the function*

$$g_{a,\alpha}(z) = \int_0^{z_l} f_{a,\alpha}(t)dt \tag{2.14}$$

belongs to $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ for $l \in \{1,2,\cdots,n\}$. Moreover, $\|g_{a,\alpha}\|_{\phi,\alpha} = 1$.

Proof. It is clear that $g_{a,\alpha}$ is holomorphic on $\mathbb{U}^n$. The result holds due to the following equality:

$$S_{\phi,\alpha}(g_{a,\alpha}) = \sup_{z\in\mathbb{U}^n}(1-|z_l|^2)^{\alpha}\Big(\frac{1-|a|^2}{|1-\bar{a}z_l|^2}\Big)^{\alpha} = \sup_{z\in\mathbb{U}^n}\Big(1-|\sigma_a(z_l)|^2\Big)^{\alpha} = 1, \tag{2.15}$$

where

$$\sigma_a(w) = \frac{a-w}{1-\bar{a}w} \tag{2.16}$$

is the automorphism of $\mathbb{U}$. From (2.15), we obtain that $\|g_{a,\alpha}\|_{\phi,\alpha} = 1$. This completes the proof. □

The following result characterizes the compactness of the operator $W_{\psi,\varphi}$. The proof is standard, so it is omitted (see Proposition 3.11 in [5] or Theorem 3.1 in [16]).

Lemma 2.5. *Let $\alpha > 0$. Then, the operator $W_{\psi,\varphi}$ is compact on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ if, and only if, for each bounded sequence $\{f_i\} \subset \mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ such that $\{f_i\} \to 0$ uniformly on every compact subset of $\mathbb{U}^n$ as $i \to \infty$, it follows that $\lim_{i\to\infty} \|W_{\psi,\varphi} f_i\|_{\phi,\alpha} = 0$.*

3. Main results and proofs

In this section, we assume that the function $\mu_{\phi,\alpha}$ satisfies the following condition:

$$c_0 = \int_0^1 \frac{1}{\mu_{\phi,\alpha}(t)} dt < +\infty. \tag{3.1}$$

In this case, we will give some examples that satisfy the condition (3.1). Moreover, we will characterize the boundedness and compactness of the operator $W_{\psi,\varphi}$ on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$. As the applications, the characterizations of the boundedness and compactness of C_φ and M_ψ on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ are obtained.

Theorem 3.1. *Let $\alpha > 0$, $\psi \in H(\mathbb{U}^n)$, and φ be a holomorphic self-mapping of $\mathbb{U}^n$. Then, the operator $W_{\psi,\varphi}$ is bounded on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ if, and only if,*

$$L = \sup_{z\in\mathbb{U}^n} \sum_{k,l=1}^{n} \frac{\mu_{\phi,\alpha}(z_k)}{\mu_{\phi,\alpha}(\varphi_l(z))} \Big|\psi(z) \frac{\partial \varphi_l}{\partial z_k}(z)\Big| < +\infty, \tag{3.2}$$

and

$$M = \sup_{z\in\mathbb{U}^n} \sum_{k=1}^{n} \mu_{\phi,\alpha}(z_k) \Big|\frac{\partial \psi(z)}{\partial z_k}\Big| < +\infty. \tag{3.3}$$

Proof. For each fixed $k \in \{1, 2, \cdots, n\}$, we write

$$L_k = \sup_{z\in\mathbb{U}^n} \mu_{\phi,\alpha}(z_k) \big|\psi(z)\big| \sum_{l=1}^{n} \Big|\frac{1}{\mu_{\phi,\alpha}(\varphi_l(z))} \frac{\partial \varphi_l}{\partial z_k}(z)\Big|$$

and

$$M_k = \sup_{z\in\mathbb{U}^n} \mu_{\phi,\alpha}(z_k) \Big|\frac{\partial \psi(z)}{\partial z_k}\Big|.$$

Suppose that $L, M < +\infty$. Then, we clearly have $L_k \le L$ and $M_k \le M$. For every $f \in \mathcal{B}_{\phi,\alpha}(\mathbb{U}^n) \setminus \{0\}$, from the convexity of ϕ, Lemmas 2.1 and 2.2, it follows that

$$\begin{aligned}
&S_{\phi,\alpha}\Big(\frac{W_{\psi,\varphi} f}{n\Big(L_k + M_k\Big(1 + \sum_{l=1}^{n} \int_0^{|\varphi_l(z)|} \frac{dt}{\mu_{\phi,\alpha}(t)}\Big)\Big)\|f\|_{\phi,\alpha}}\Big) \\
&= \sup_{z\in\mathbb{U}^n} \sum_{k=1}^{n} (1 - |z_k|^2)^\alpha \phi\Big(\frac{\big|\frac{\partial W_{\psi,\varphi} f}{\partial z_k}(z)\big|}{n\Big(L_k + M_k\Big(1 + \sum_{l=1}^{n} \int_0^{|\varphi_l(z)|} \frac{dt}{\mu_{\phi,\alpha}(t)}\Big)\Big)\|f\|_{\phi,\alpha}}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{z\in\mathbb{U}^n}\sum_{k=1}^{n}(1-|z_k|^2)^{\alpha}\phi\Big(\frac{\Big|\frac{\partial\psi(z)}{\partial z_k}f(\varphi(z))+\psi(z)\sum\limits_{l=1}^{n}\frac{\partial f}{\partial w_l}(\varphi(z))\frac{\partial\varphi_l}{\partial z_k}(z)\Big|}{n\Big(L_k+M_k\Big(1+\sum\limits_{l=1}^{n}\int_0^{|\varphi_l(z)|}\frac{dt}{\mu_{\phi,\alpha}(t)}\Big)\Big)\|f\|_{\phi,\alpha}}\Big)\\
&\le \sup_{z\in\mathbb{U}^n}\sum_{k=1}^{n}(1-|z_k|^2)^{\alpha}\phi\Big(\frac{\Big|\frac{\partial\psi(z)}{\partial z_k}\Big|\Big|f(\varphi(z))\Big|+\Big|\psi(z)\Big|\sum\limits_{l=1}^{n}\Big|\frac{\partial f}{\partial w_l}(\varphi(z))\Big|\Big|\frac{\partial\varphi_l}{\partial z_k}(z)\Big|}{n\Big(L_k+M_k\Big(1+\sum\limits_{l=1}^{n}\int_0^{|\varphi_l(z)|}\frac{dt}{\mu_{\phi,\alpha}(t)}\Big)\Big)\|f\|_{\phi,\alpha}}\Big)\\
&\le \sup_{z\in\mathbb{U}^n}\sum_{k=1}^{n}(1-|z_k|^2)^{\alpha}\phi\Big(\frac{\Big|\frac{\partial\psi(z)}{\partial z_k}\Big|\Big(1+\sum\limits_{l=1}^{n}\int_0^{|\varphi_l(z)|}\frac{dt}{\mu_{\phi,\alpha}(t)}\Big)\Big\|f\Big\|_{\phi,\alpha}+|\psi(z)|\sum\limits_{l=1}^{n}\frac{1}{\mu_{\phi,\alpha}(\varphi_l(z))}\Big|\frac{\partial\varphi_l}{\partial z_k}(z)\Big|\Big\|f\Big\|_{\phi,\alpha}}{n\Big(L_k+M_k\Big(1+\sum\limits_{l=1}^{n}\int_0^{|\varphi_l(z)|}\frac{dt}{\mu_{\phi,\alpha}(t)}\Big)\Big)\|f\|_{\phi,\alpha}}\Big)\\
&= \sup_{z\in\mathbb{U}^n}\sum_{k=1}^{n}(1-|z_k|^2)^{\alpha}\phi\Big(\frac{\Big|\frac{\partial\psi(z)}{\partial z_k}\Big|\Big(1+\sum\limits_{l=1}^{n}\int_0^{|\varphi_l(z)|}\frac{dt}{\mu_{\phi,\alpha}(t)}\Big)\Big\|f\Big\|_{\phi,\alpha}+\Big|\psi(z)\Big|\sum\limits_{l=1}^{n}\Big|\frac{1}{\mu_{\phi,\alpha}(\varphi_l(z))}\frac{\partial\varphi_l}{\partial z_k}(z)\Big|\Big\|f\Big\|_{\phi,\alpha}}{n\Big(L_k+M_k\Big(1+\sum\limits_{l=1}^{n}\int_0^{|\varphi_l(z)|}\frac{dt}{\mu_{\phi,\alpha}(t)}\Big)\Big)\|f\|_{\phi,\alpha}}\Big)\\
&\le \sup_{z\in\mathbb{U}^n}\sum_{k=1}^{n}(1-|z_k|^2)^{\alpha}\phi\Big(\frac{1}{n\mu_{\phi,\alpha}(z_k)}\Big)\\
&\le \sup_{z\in\mathbb{U}^n}\sum_{k=1}^{n}(1-|z_k|^2)^{\alpha}\frac{1}{n}\phi\Big(\frac{1}{\mu_{\phi,\alpha}(z_k)}\Big)\\
&= \sup_{z\in\mathbb{U}^n}\sum_{k=1}^{n}(1-|z_k|^2)^{\alpha}\frac{1}{n}\frac{1}{(1-|z_k|^2)^{\alpha}}\\
&= 1.
\end{aligned}
\tag{3.4}
$$

Hence, from (3.4), we obtain

$$
\|W_{\psi,\varphi}f\|_{\phi,\alpha}\le n\Big[L_k+M_k\Big(1+\sum_{l=1}^{n}\int_0^{|\varphi_l(z)|}\frac{dt}{\mu_{\phi,\alpha}(t)}\Big)\Big]\|f\|_{\phi,\alpha}\le n[L+M(1+nc_0)]\|f\|_{\phi,\alpha}. \tag{3.5}
$$

On the other hand, it follows from Corollary 2.1 that

$$
|W_{\psi,\varphi}f(0)|=|\psi(0)||f(\varphi(0))|\le C\|f\|_{\phi,\alpha}. \tag{3.6}
$$

From (3.5) and (3.6), we obtain that

$$
\|W_{\psi,\varphi}\|_{\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)}\le C\|f\|_{\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)},
$$

which shows that the operator $W_{\psi,\varphi}$ is bounded on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$.

Conversely, assume that the operator $W_{\psi,\varphi}$ is bounded on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$. By setting $\hat{f}(z)\equiv 1$ (clearly, $1\in\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$), we obtain $\psi=W_{\psi,\varphi}\hat{f}\in\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$. Then,

$$
1\ge S_{\phi,\alpha}\Big(\frac{W_{\psi,\varphi}\hat{f}}{C\|f\|_{\phi,\alpha}}\Big)=S_{\phi,\alpha}\Big(\frac{\psi}{C\|\hat{f}\|_{\phi,\alpha}}\Big)=\sup_{z\in\mathbb{U}^n}\sum_{k=1}^{n}(1-|z_k|^2)^{\alpha}\phi\Big(\frac{\Big|\frac{\partial\psi}{\partial z_k}(z)\Big|}{C\|\hat{f}\|_{\phi,\alpha}}\Big). \tag{3.7}
$$

From (3.7), for each $k \in \{1, 2, \cdots, n\}$ and all $z \in \mathbb{U}^n$, it follows that

$$\mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi}{\partial z_k}(z)\Big| \leq C\|\hat{f}\|_{\phi,\alpha},$$

which shows

$$M = \sup_{z\in\mathbb{U}^n}\sum_{k=1}^{n}\mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi}{\partial z_k}(z)\Big| \leq C. \tag{3.8}$$

By Lemma 2.4, we see that the following function belongs to $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$,

$$g_{\varphi_l(w),\alpha}(z) = \int_0^{z_l} f_{\varphi_l(w),\alpha}(t)dt.$$

Moreover, $\|g_{\varphi_l(w),\alpha}\|_{\phi,\alpha} = 1$. Then, we have

$$\begin{aligned}
1 &\geq S_{\phi,\alpha}\Big(\frac{W_{\psi,\varphi}g_{\varphi_l(w),\alpha}}{C\|g_{\varphi_l(w),\alpha}\|_{\phi,\alpha}}\Big)\\
&= \sup_{z\in\mathbb{U}^n}\sum_{k=1}^{n}(1-|z_k|^2)^{\alpha}\phi\Big(\frac{\Big|\frac{\partial\psi}{\partial z_k}(z)g_{\varphi_l(w),\alpha}(\varphi(z)) + \psi(z)\sum\limits_{j=1}^{n}\frac{\partial g_{\varphi_l(w),\alpha}}{\partial w_j}(\varphi(z))\frac{\partial\varphi_j}{\partial z_k}(z)\Big|}{C}\Big)\\
&\geq (1-|z_k|^2)^{\alpha}\phi\Big(\frac{\Big|\frac{\partial\psi}{\partial z_k}(z)g_{\varphi_l(w),\alpha}(\varphi(z)) + \psi(z)\sum\limits_{j=1}^{n}\frac{\partial g_{\varphi_l(w),\alpha}}{\partial w_j}(\varphi(z))\frac{\partial\varphi_j}{\partial z_k}(z)\Big|}{C}\Big).
\end{aligned} \tag{3.9}$$

From (3.9), we obtain

$$\mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi}{\partial z_k}(z)g_{\varphi_l(w),\alpha}(\varphi(z)) + \psi(z)\sum_{j=1}^{n}\frac{\partial g_{\varphi_l(w),\alpha}}{\partial w_j}(\varphi(z))\frac{\partial\varphi_j}{\partial z_k}(z)\Big| \leq C,$$

which leads to

$$\mu_{\phi,\alpha}(z_k)\Big|\psi(z)\sum_{j=1}^{n}\frac{\partial g_{\varphi_l(w),\alpha}}{\partial w_j}(\varphi(z))\frac{\partial\varphi_j}{\partial z_k}(z)\Big| \leq C + \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi}{\partial z_k}(z)\Big|\Big|g_{\varphi_l(w),\alpha}(\varphi(z))\Big|. \tag{3.10}$$

By summing in (3.10) from 1 to n, we obtain

$$\sum_{k=1}^{n}\mu_{\phi,\alpha}(z_k)\Big|\psi(z)\sum_{j=1}^{n}\frac{\partial g_{\varphi_l(w),\alpha}}{\partial w_j}(\varphi(z))\frac{\partial\varphi_j}{\partial z_k}(z)\Big| \leq nC + \sum_{k=1}^{n}\mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi}{\partial z_k}(z)\Big|\Big|g_{\varphi_l(w),\alpha}(\varphi(z))\Big|. \tag{3.11}$$

Then, by Corollary 2.1 and the fact $\|g_{\varphi_l(w),\alpha}\|_{\phi,\alpha} = 1$, (3.11) becomes

$$\sum_{k=1}^{n}\mu_{\phi,\alpha}(z_k)\Big|\psi(z)\sum_{j=1}^{n}\frac{\partial g_{\varphi_l(w),\alpha}}{\partial w_j}(\varphi(z))\frac{\partial\varphi_j}{\partial z_k}(z)\Big| \leq nC + \sum_{k=1}^{n}\mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi}{\partial z_k}(z)\Big|\Big(1 + \sum_{l=1}^{n}\int_0^{|\varphi_l(z)|}\frac{1}{\mu_{\phi,\alpha}(t)}dt\Big). \tag{3.12}$$

Since

$$\frac{\partial g_{\varphi_l(w),\alpha}}{\partial z_l}(z) = f_{\varphi_l(w),\alpha}(z_l), \quad \frac{\partial g_{\varphi_l(w),\alpha}}{\partial z_j}(z) = 0, \quad j \neq l,$$

by Lemma 2.3 we get

$$\begin{aligned}&\sum_{k=1}^{n}\mu_{\phi,\alpha}(z_k)\Big|\psi(z)\Big|\Big|\phi^{-1}\Big(\Big(\frac{1-|\varphi_l(w)|^2}{|1-\overline{\varphi_l(w)}\varphi_l(z)|^2}\Big)^{\alpha}\Big)\frac{\partial\varphi_l}{\partial z_k}(z)\Big|\\ \le& nC+\sum_{k=1}^{n}\mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi}{\partial z_k}(z)\Big|\Big(1+\sum_{l=1}^{n}\int_0^{|\varphi_l(z)|}\frac{1}{\mu_{\phi,\alpha}(t)}dt\Big).\end{aligned} \tag{3.13}$$

Now, letting $z = w$ in (3.13), by (3.8) and $c_0 = \int_0^1 \frac{1}{\mu_{\phi,\alpha}(t)}dt < +\infty$, we have

$$\sum_{k=1}^{n}\frac{\mu_{\phi,\alpha}(w_k)}{\mu_{\phi,\alpha}(\varphi_l(w))}\Big|\psi(w)\frac{\partial\varphi_l}{\partial w_k}(w)\Big|\le n^2C+n\sum_{k=1}^{n}\mu_{\phi,\alpha}(w_k)\Big|\frac{\partial\psi}{\partial w_k}(w)\Big|\Big(1+\sum_{l=1}^{n}\int_0^{|\varphi_l(w)|}\frac{1}{\mu_{\phi,\alpha}(t)}dt\Big)\le nC+(1+nc_0)M,$$

which shows

$$L = \sup_{z\in\mathbb{U}^n}\sum_{k,l=1}^{n}\frac{\mu_{\phi,\alpha}(z_k)}{\mu_{\phi,\alpha}(\varphi_l(z))}\Big|\psi(z)\frac{\partial\varphi_l}{\partial z_k}(z)\Big| < +\infty.$$

The proof is finished. □

By directly applying Theorem 3.1, we derive the following two results.

Corollary 3.1. *Let $\alpha > 0$ and $\psi \in H(\mathbb{U}^n)$. Then, the operator M_ψ is bounded on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ if, and only if, $\psi \in H^\infty(\mathbb{U}^n) \cap \mathcal{B}_{\mu_{\phi,\alpha}}(\mathbb{U}^n)$.*

Corollary 3.2. *Let $\alpha > 0$ and φ be a holomorphic self-mapping of $\mathbb{U}^n$. Then, the operator C_φ is bounded on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ if, and only if,*

$$\sup_{z\in\mathbb{U}^n}\sum_{k,l=1}^{n}\frac{\mu_{\phi,\alpha}(z_k)}{\mu_{\phi,\alpha}(\varphi_l(z))}\Big|\frac{\partial\varphi_l}{\partial w_k}(z)\Big| < +\infty.$$

The compactness of the operator $W_{\psi,\varphi}$ has been characterized from p-Bloch space $\mathcal{B}_p(\mathbb{U}^n)$ to q-Bloch space $\mathcal{B}_q(\mathbb{U}^n)$ in [26], which motivates us to consider the same problem on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$.

Theorem 3.2. *Let $\alpha > 0$, $\psi \in H(\mathbb{U}^n)$, and φ be a holomorphic self-mapping of $\mathbb{U}^n$. Then, the operator $W_{\psi,\varphi}$ is compact on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ if, and only if, $W_{\psi,\varphi}$ is bounded on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$,*

$$\lim_{\varphi(z)\to\partial\mathbb{U}^n}\sum_{k,l=1}^{n}\frac{\mu_{\phi,\alpha}(z_k)}{\mu_{\phi,\alpha}(\varphi_l(z))}\Big|\psi(z)\frac{\partial\varphi_l(z)}{\partial z_k}\Big| = 0, \tag{3.14}$$

and

$$\lim_{\varphi(z)\to\partial\mathbb{U}^n}\sum_{k=1}^{n}\mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi(z)}{\partial z_k}\Big| = 0. \tag{3.15}$$

Proof. Assume that the operator $W_{\psi,\varphi}$ is bounded on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ and (3.14), (3.15) holds, respectively. To show that the operator $W_{\psi,\varphi}$ is compact on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$, using Lemma 2.5, we just need to prove that for each bounded sequence $\{f_j\}$ in $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ such that $f_j \to 0$ uniformly on any compact subset of $\mathbb{U}^n$ as $j \to \infty$, we have that $\lim_{j\to\infty} \|W_{\psi,\varphi} f_j\|_{\phi,\alpha} = 0$. Let $\{f_j\}$ be a such sequence in $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$. Set $C_0 = \sup_{j\in\mathbb{N}} \|f_j\|_{\phi,\alpha}$. For $\varepsilon > 0$, by (3.14) and (3.15), there exists a $\delta \in (0, 1)$ such that

$$\sum_{k,l=1}^{n} \frac{\mu_{\phi,\alpha}(z_k)}{\mu_{\phi,\alpha}(\varphi_l(z))}\Big|\psi(z)\frac{\partial\varphi_l(z)}{\partial z_k}\Big| < \varepsilon,$$

and

$$\sum_{k=1}^{n} \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi(z)}{\partial z_k}\Big| < \varepsilon,$$

for all $z \in E = \{z \in \mathbb{U}^n : dist(\varphi(z), \partial\mathbb{U}^n) < \delta\}$. From this, for all $z \in E$ we have

$$\begin{aligned}
&\sum_{k=1}^{n} \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial W_{\psi,\varphi} f_j}{\partial z_k}(z)\Big| \\
=&\sum_{k=1}^{n} \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi(z)}{\partial z_k} f_j(\varphi(z)) + \psi(z)\sum_{l=1}^{n} \frac{\partial f_j}{\partial w_l}(\varphi(z))\frac{\partial\varphi_l}{\partial z_k}(z)\Big| \\
\le&\sum_{k=1}^{n} \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi(z)}{\partial z_k}\Big|\Big|f_j(\varphi(z))\Big| + \sum_{k,l=1}^{n} \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial f_j}{\partial w_l}(\varphi(z))\Big|\Big|\psi(z)\frac{\partial\varphi_l}{\partial z_k}(z)\Big| \\
\le&\sum_{k=1}^{n} \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi(z)}{\partial z_k}\Big|\Big(1 + \sum_{l=1}^{n}\int_0^{|\varphi_l(z)|}\frac{1}{\mu_{\phi,\alpha}(t)}dt\Big)\big\|f_j\big\|_{\phi,\alpha} + \sum_{k,l=1}^{n}\frac{\mu_{\phi,\alpha}(z_k)}{\mu_{\phi,\alpha}(\varphi_l(z))}\Big|\psi(z)\frac{\partial\varphi_l(z)}{\partial z_k}\Big|\big\|f_j\big\|_{\phi,\alpha} \\
\le&(1 + nc_0)C_0\varepsilon + C_0\varepsilon \\
=&(2 + nc_0)C_0\varepsilon,
\end{aligned}$$

which shows

$$\sup_{z\in E}\sum_{k=1}^{n} \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial W_{\psi,\varphi} f_j}{\partial z_k}(z)\Big| \le (2 + nc_0)C_0\varepsilon.$$

On the other hand, $\mathbb{U}^n \setminus E = \{z \in \mathbb{U}^n : dist(\varphi(z), \partial\mathbb{U}^n) \ge \delta\}$ is a compact subset of $\mathbb{U}^n$. Hence, $f_j \to 0$ uniformly on $\mathbb{U}^n \setminus E$ as $j \to \infty$. From Cauchy's estimate, it follows that $\frac{\partial f_j}{\partial z_l} \to 0$ uniformly on $\mathbb{U}^n \setminus E$ as $j \to \infty$ for each $l \in \{1, 2, \cdots, n\}$. Since $W_{\psi,\varphi}$ is bounded on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$, from Theorem 3.1 we have that $M < +\infty$.

Set $f_l(z) = z_l$, $z \in \mathbb{U}^n$. Then, $f_l \in \mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$. Since $W_{\psi,\varphi}$ is bounded on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$, $W_{\psi,\varphi} f_l = \psi\varphi_l \in \mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$. From the definition of $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$, we have

$$\begin{aligned}
1 \ge S_{\phi,\alpha}\Big(\frac{\psi\varphi_l}{C\|f_l\|_{\phi,\alpha}}\Big) &= \sup_{z\in\mathbb{U}^n}\sum_{k=1}^{n}(1 - |z_k|^2)^\alpha\phi\Big(\frac{\big|\frac{\partial\psi(z)}{\partial z_k}\varphi_l(z) + \psi(z)\frac{\partial\varphi_l}{\partial z_k}(z)\big|}{C\|f_l\|_{\phi,\alpha}}\Big) \\
&\ge (1 - |z_k|^2)^\alpha\phi\Big(\frac{\big|\frac{\partial\psi(z)}{\partial z_k}\varphi_l(z) + \psi(z)\frac{\partial\varphi_l}{\partial z_k}(z)\big|}{C\|f_l\|_{\phi,\alpha}}\Big).
\end{aligned} \tag{3.16}$$

Since $\mu_{\phi,\alpha}(z_k) = \frac{1}{\phi^{-1}(\frac{1}{(1-|z_k|^2)^\alpha})}$, from (3.16) we obtain

$$\mu_{\phi,\alpha}(z_k)\Big|\psi(z)\frac{\partial\varphi_l}{\partial z_k}(z)\Big| - \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi(z)}{\partial z_k}\varphi_l(z)\Big| \le \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi(z)}{\partial z_k}\varphi_l(z) + \psi(z)\frac{\partial\varphi_l}{\partial z_k}(z)\Big| \le C\|f_l\|_{\phi,\alpha}.$$

Therefore,

$$\mu_{\phi,\alpha}(z_k)\Big|\psi(z)\frac{\partial\varphi_l}{\partial z_k}(z)\Big| \le C\|f_l\|_{\phi,\alpha} + \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi(z)}{\partial z_k}\varphi_l(z)\Big|.$$

Since $|\varphi_l(z)| < 1,\ \ l \in \{1, 2, \cdots, n\}$, we get

$$\begin{aligned}\sup_{z\in\mathbb{U}^n\setminus E}\sum_{k=1}^n \mu_{\phi,\alpha}(z_k)\Big|\psi(z)\frac{\partial\varphi_l}{\partial z_k}(z)\Big| &\le C\|f_l\|_{\phi,\alpha} + \sup_{z\in\mathbb{U}^n\setminus E}\sum_{k=1}^n \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi(z)}{\partial z_k}\varphi_l(z)\Big| \\ &\le C\|f_l\|_{\phi,\alpha} + \sup_{z\in\mathbb{U}^n\setminus E}\sum_{k=1}^n \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi(z)}{\partial z_k}\Big| \\ &\le C\|f_l\|_{\phi,\alpha} + \|\psi\|_{\mu_{\phi,\alpha}} < +\infty.\end{aligned}$$

For the convenience, we write

$$C_l = \sup_{z\in\mathbb{U}^n\setminus E}\sum_{k=1}^n \mu_{\phi,\alpha}(z_k)\Big|\psi(z)\frac{\partial\varphi_l}{\partial z_k}(z)\Big|.$$

Then, $C_l < +\infty$ for each $l \in \{1, 2, \cdots, n\}$. Hence, for each $l \in \{1, 2, \cdots, n\}$, we have

$$\begin{aligned}\|W_{\psi,\varphi}f\|_{\phi,\alpha} &\asymp \sup_{z\in\mathbb{U}^n}\sum_{k=1}^n \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial W_{\psi,\varphi}f_j}{\partial z_k}(z)\Big| \\ &\le \sup_{z\in E}\sum_{k=1}^n \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial W_{\psi,\varphi}f_j}{\partial z_k}(z)\Big| + \sup_{z\in\mathbb{U}^n\setminus E}\sum_{k=1}^n \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial W_{\psi,\varphi}f_j}{\partial z_k}(z)\Big| \\ &\le (2 + c_0\varepsilon)C_0\varepsilon + \sup_{z\in\mathbb{U}^n\setminus E}\sum_{k=1}^n \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi(z)}{\partial z_k}f_j(\varphi(z)) + \psi(z)\sum_{l=1}^n\frac{\partial f_j}{\partial w_l}(\varphi(z))\frac{\partial\varphi_l}{\partial z_k}(z)\Big| \\ &\le (2 + c_0\varepsilon)C_0\varepsilon + \sup_{z\in\mathbb{U}^n\setminus E}\sum_{k=1}^n \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial\psi(z)}{\partial z_k}\Big|\Big|f_j(\varphi(z))\Big| \\ &\quad + \sup_{z\in\mathbb{U}^n\setminus E}\sum_{k,l=1}^n \mu_{\phi,\alpha}(z_k)\Big|\frac{\partial f_j}{\partial w_l}(\varphi(z))\Big|\Big|\psi(z)\frac{\partial\varphi_l}{\partial z_k}(z)\Big| \\ &\le (2 + c_0\varepsilon)C_0\varepsilon + M\sup_{z\in\mathbb{U}^n\setminus E}\Big|f_j(\varphi(z))\Big| + \sum_{l=1}^n C_l\Big|\frac{\partial f_j}{\partial w_l}(\varphi(z))\Big| \\ &\to 0, \end{aligned} \tag{3.17}$$

as $j \to \infty$. Moreover, it follows by (3.17) that the operator $W_{\psi,\varphi}$ is compact on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$.

Now, suppose that the operator $W_{\psi,\varphi}$ is compact on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$. It is clear that $W_{\psi,\varphi}$ is bounded on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$. First, we prove that condition (3.14) holds. If the condition (3.14) is not right, then there is a constant $\varepsilon_0 > 0$ and a sequence $\{z^m\}$ in $\mathbb{U}^n$ with $w^m = \varphi(z^m) \to \partial\mathbb{U}^n$ as $m \to \infty$, such that

$$\sum_{k,l=1}^{n} \frac{\mu_{\phi,\alpha}(z_k^m)}{\mu_{\phi,\alpha}(\varphi_l(z^m))}\Big|\psi(z^m)\frac{\partial\varphi_l(z^m)}{\partial z_k^m}\Big| \geq \varepsilon_0, \tag{3.18}$$

for all $m \in \mathbb{N}$. Since $W_{\psi,\varphi}$ is bounded on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$, by the condition (3.2) in Theorem 3.1, we see that for all $l \in \{1, 2, \cdots, n\}$, the sequence

$$\Big\{\sum_{k=1}^{n} \frac{\mu_{\phi,\alpha}(z_k^m)}{\mu_{\phi,\alpha}(\varphi_l(z^m))}\Big|\psi(z^m)\frac{\partial\varphi_l(z^m)}{\partial z_k^m}\Big|\Big\}$$

is bounded. Hence, there is a subsequence of $\{z^m\}$ (for simplicity, here we assume that it is the sequence $\{z^m\}$) such that the following limit exists:

$$\lim_{m\to\infty}\sum_{k=1}^{n} \frac{\mu_{\phi,\alpha}(z_k^m)}{\mu_{\phi,\alpha}(\varphi_l(z^m))}\Big|\psi(z^m)\frac{\partial\varphi_l(z^m)}{\partial z_k^m}\Big|,$$

for every $l \in \{1, 2, \cdots, n\}$. Also, we may assume that for every $l \in \{1, 2, \cdots, n\}$, the following limit exists:

$$\lim_{m\to\infty} |w_l^m| = \lim_{m\to\infty} |\varphi_l(z^m)|.$$

From (3.18), there must be an $l_0 \in \{1, 2, \cdots, n\}$ (here, we can assume that $l_0 = 1$) such that

$$\lim_{m\to\infty}\sum_{k=1}^{n} \frac{\mu_{\phi,\alpha}(z_k^m)}{\mu_{\phi,\alpha}(\varphi_1(z^m))}\Big|\psi(z^m)\frac{\partial\varphi_1(z^m)}{\partial z_k^m}\Big| = \varepsilon_1 \neq 0.$$

In order to obtain a contradiction, we divide into the following two cases for consideration.
Case 1. Assume that $|w_1^m| \to 1$ as $m \to \infty$. Set

$$h_m(z) = \int_0^{z_1} |f_{w_1^m,\alpha}(t)|dt - \int_0^{w_1^m} |f_{w_1^m,\alpha}(t)|dt, \ \ z \in \mathbb{U}^n.$$

Then, $h_m \in \mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ and $h_m \to 0$ uniformly on compact subsets of $\mathbb{U}^n$ as $m \to \infty$. Moreover, by an easy computation,

$$\frac{\partial h_m}{\partial z_k}(z) = 0, \ \ k \neq 1, \tag{3.19}$$

$$h_m(w^m) = 0 \ \text{ and } \ \frac{\partial h_m(w^m)}{\partial z_1} = |f_{w_1^m,\alpha}(w_1^m)|. \tag{3.20}$$

Then, from (3.19) and (3.20), we get

$$\|W_{\psi,\varphi}h_m\|_{\mu_{\phi,\alpha}} \geq \sum_{k=1}^{n} \mu_{\phi,\alpha}(z_k^m)\Big|\psi(z^m)\frac{\partial h_m}{\partial w_1}(\varphi(z^m))\frac{\partial\varphi_1(z^m)}{\partial z_k^m}\Big| = \sum_{k=1}^{n} \frac{\mu_{\phi,\alpha}(z_k^m)}{\mu_{\phi,\alpha}(\varphi_1(z^m))}\Big|\psi(z^m)\frac{\partial\varphi_1(z^m)}{\partial z_k^m}\Big| \to \varepsilon_1 \neq 0,$$

as $m \to \infty$, which is a contradiction, since $\|W_{\psi,\varphi}h_m\|_{\mu_{\phi,\alpha}} \to 0$ as $m \to \infty$.
Case 2. Assume that $|w_1^m| \to \rho < 1$ as $m \to \infty$. Since $w^m \to \partial\mathbb{U}^n$, there is an $l \in \{2, \cdots, n\}$ such that $|w_l^m| \to 1$ as $m \to \infty$. If there is a $\varepsilon_2 > 0$ such that

$$\sum_{k=1}^{n} \frac{\mu_{\phi,\alpha}(z_k^m)}{\mu_{\phi,\alpha}(\varphi_l(z^m))}\Big|\psi(z^m)\frac{\partial\varphi_l(z^m)}{\partial z_k^m}\Big| \geq \varepsilon_2,$$

we can also assume that

$$\lim_{m\to\infty}\sum_{k=1}^{n} \frac{\mu_{\phi,\alpha}(z_k^m)}{\mu_{\phi,\alpha}(\varphi_l(z^m))}\Big|\psi(z^m)\frac{\partial\varphi_l(z^m)}{\partial z_k^m}\Big| = \varepsilon_3 \neq 0.$$

Similar to Case 1, we obtain a contradiction by using the functions

$$\widehat{h}_m(z) = \int_0^{z_l} |f_{w_l^m,\alpha}(t)|dt - \int_0^{w_l^m} |f_{w_l^m,\alpha}(t)|dt, \quad m \in \mathbb{N}.$$

Now, for the l chosen above, assume that

$$\lim_{m\to\infty}\sum_{k=1}^{n} \frac{\mu_{\phi,\alpha}(z_k^m)}{\mu_{\phi,\alpha}(\varphi_l(z^m))}\Big|\psi(z^m)\frac{\partial\varphi_l(z^m)}{\partial z_k^m}\Big| = 0.$$

Set $\widetilde{h}_m = h_m + \widehat{h}_m$. Then, $\{\widetilde{h}_m\} \subset \mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ and $\widetilde{h}_m \to 0$ uniformly on compact subsets of $\mathbb{U}^n$ as $m \to \infty$. We have

$$\|W_{\psi,\varphi}\widetilde{h}_m\|_{\mu_{\phi,\alpha}} \geq \sum_{k=1}^{n} \frac{\mu_{\phi,\alpha}(z_k^m)}{\mu_{\phi,\alpha}(\varphi_1(z^m))}\Big|\psi(z^m)\frac{\partial\varphi_1(z^m)}{\partial z_k^m}\Big| - \sum_{k=1}^{n} \frac{\mu_{\phi,\alpha}(z_k^m)}{\mu_{\phi,\alpha}(\varphi_l(z^m))}\Big|\psi(z^m)\frac{\partial\varphi_l(z^m)}{\partial z_k^m}\Big| \to \varepsilon_1 \neq 0, \quad (3.21)$$

as $m \to \infty$, which also is a contradiction, since $\|W_{\psi,\varphi}\widetilde{h}_m\|_{\mu_{\phi,\alpha}} \to 0$ as $m \to \infty$.

Now, we begin to prove that condition (3.15) holds. Assume that condition (3.15) is not right. Then, there exist a positive constant ε_4 and a sequence $\{z^m\}$ in $\mathbb{U}^n$ with $w^m = \varphi(z^m) \to \partial\mathbb{U}^n$ as $m \to \infty$, such that

$$\sum_{k=1}^{n} \mu_{\phi,\alpha}(z_k^m)\Big|\frac{\partial\psi(z^m)}{\partial z_k^m}\Big| \geq \varepsilon_4,$$

for all $m \in \mathbb{N}$. Since $W_{\psi,\varphi}$ is bounded on $\mathcal{B}_{\mu_{\phi,\alpha}}(\mathbb{U}^n)$, from (3.3) in Theorem 3.1, we know that the sequence

$$\Big\{\sum_{k=1}^{n} \mu_{\phi,\alpha}(z_k^m)\Big|\frac{\partial\psi(z^m)}{\partial z_k^m}\Big|\Big\}$$

is bounded. Hence, there is a subsequence of $\{z^m\}$ such that

$$0 \neq a_0 = \lim_{m\to\infty}\sum_{k=1}^{n} \mu_{\phi,\alpha}(z_k^m)\Big|\frac{\partial\psi(z^m)}{\partial z_k^m}\Big| < \infty.$$

Also, we may assume that for every $l \in \{1, 2, \cdots, n\}$, there is a finite limit

$$\lim_{m\to\infty} |w_l^m| = \lim_{m\to\infty} |\varphi_l(z^m)|.$$

Case 3. Assume that $|w_1^m| \to 1$ as $m \to \infty$. Set

$$S_m(z) = \frac{1}{\ln(1-|w_1^m|^2)}\Big(\frac{(\ln(1-\overline{w_1^m}z_1))^2}{2\ln(1-|w_1^m|^2)} - \ln(1-\overline{w_1^m}z_1)\Big), \quad m \in \mathbb{N}.$$

Then, $\{S_m\} \subset \mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ and $S_m \to 0$ uniformly on compact subsets of $\mathbb{U}^n$ as $m \to \infty$. Moreover, we also have

$$\frac{\partial S_m(z)}{\partial z_1} = \frac{1}{\ln(1-|w_1^m|^2)}\Big(\frac{\ln(1-\overline{w_1^m}z_1)\frac{-\overline{w_1^m}}{1-\overline{w_1^m}z_1}}{\ln(1-|w_1^m|^2)} + \frac{\overline{w_1^m}}{1-\overline{w_1^m}z_1}\Big),$$

$$\frac{\partial S_m}{\partial z_k}(z) = 0, \ \ k \neq 1,$$

$$S_m(w^m) = -\frac{1}{2} \ \text{ and } \ \frac{\partial S_m(w^m)}{\partial z_1} = 0.$$

So, we get

$$\begin{aligned} \|W_{\psi,\varphi}S_m\|_{\mu_{\phi,\alpha}} &= \sup_{z\in\mathbb{U}^n} \sum_{k=1}^n \mu_{\phi,\alpha}(z_k^m)\Big|\frac{\partial W_{\psi,\varphi}S_m}{\partial z_k^m}\Big| \\ &\geq \sum_{k=1}^n \mu_{\phi,\alpha}(z_k^m)\Big|\frac{\partial\psi(z^m)}{\partial z_k^m}S_m(\varphi(z^m)) + \psi(z^m)\sum_{l=1}^n \frac{\partial S_m}{\partial w_l}(\varphi(z^m))\frac{\partial\varphi_l(z^m)}{\partial z_k^m}\Big| \\ &= \frac{1}{2}\sum_{k=1}^n \mu_{\phi,\alpha}(z_k^m)\Big|\frac{\partial\psi(z^m)}{\partial z_k^m}\Big| \to \frac{a_0}{2} \neq 0, \end{aligned}$$

as $m \to \infty$, which is a contradiction.

Case 4. Assume that $|w_1^m| \to \rho < 1$ as $m \to \infty$. Since $w^m \to \partial\mathbb{U}^n$, there is an $l \in \{2, \cdots, n\}$ such that $|w_l^m| \to 1$ as $m \to \infty$.

Set

$$\widehat{S}_m(z) = \frac{1}{\ln(1-|w_l^m|^2)}\Big(\frac{(\ln(1-\overline{w_l^m}z_l))^2}{2\ln(1-|w_l^m|^2)} - \ln(1-\overline{w_l^m}z_l)\Big), \quad m \in \mathbb{N}.$$

Similar to Case 3, we obtain

$$\lim_{m\to\infty} \sum_{k=1}^n \mu_{\phi,\alpha}(z_k^m)\Big|\frac{\partial\psi(z^m)}{\partial z_k^m}\Big| = a_0 \neq 0,$$

which also is a contradiction. Now, for the l chosen above, assume that

$$\lim_{m\to\infty} \sum_{k=1}^n \mu_{\phi,\alpha}(z_k^m)\Big|\frac{\partial\psi(z^m)}{\partial z_k^m}\Big| = 0.$$

Let

$$\widetilde{S}_m(z) = S_m(z) + \widehat{S}_m(z).$$

Then, $\{\widetilde{S}_m\} \subset \mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ and $\widetilde{S}_m \to 0$ uniformly on compact subsets of $\mathbb{U}^n$ as $m \to \infty$. For such sequence, we have

$$\|W_{\psi,\varphi}\widetilde{S}_m\|_{\mu_{\phi,\alpha}} \geq \sum_{k=1}^{n} \mu_{\phi,\alpha}(z_k^m)\Big|\frac{\partial\psi(z^m)}{\partial z_k^m} S_m(\varphi(z^m))\Big| - \sum_{k=1}^{n} \mu_{\phi,\alpha}(z_k^m)\Big|\frac{\partial\psi(z^m)}{\partial z_k^m}\widehat{S}_m(\varphi(z^m))\Big| \to \frac{a_0}{2},$$

as $m \to \infty$, from which we obtain a contradiction. Hence, the proof is finished. □

According to Theorem 3.2, we get the following two results.

Corollary 3.3. *Let $\alpha > 0$ and $\psi \in H(\mathbb{U}^n)$. Then, the operator M_ψ is compact on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ if, and only if, $\psi \in H_0^\infty(\mathbb{U}^n) \cap \mathcal{B}_{\mu_{\phi,\alpha},0}(\mathbb{U}^n)$.*

Corollary 3.4. *Let $\alpha > 0$ and φ be a holomorphic self-mapping of $\mathbb{U}^n$. Then, the operator C_φ is compact on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ if, and only if, C_φ is bounded on $\mathcal{B}_{\phi,\alpha}(\mathbb{U}^n)$ and*

$$\lim_{\varphi(z)\to\partial\mathbb{U}^n} \sum_{k,l=1}^{n} \frac{\mu_{\phi,\alpha}(z_k)}{\mu_{\phi,\alpha}(\varphi_l(z))}\Big|\frac{\partial\varphi_l}{\partial z_k}(z)\Big| = 0.$$

In the final of the paper, we give the following example in order to show that there exists ϕ such that $\mu_{\phi,\alpha}(t)$ satisfies the condition (3.1).

Example 3.1. Let $\phi(t) = t^p$ and $p > \max\{1, \alpha\}$. Then, $\mu_{\phi,\alpha}(t)$ satisfies the condition (3.1).

Proof. From the condition $p > 1$, it follows that ϕ is a strictly increasing convex function in $[0, +\infty)$. Now, we prove that

$$\int_0^1 \frac{1}{\mu_{\phi,\alpha}(t)} dt < +\infty.$$

It is not hard to see that $\phi^{-1}(t) = t^{\frac{1}{p}}$, and then $\mu_{\phi,\alpha}(t) = (1-t^2)^{\frac{\alpha}{p}}$. Since the function $\mu_{\phi,\alpha}(t)$ is unbounded in the point $t = 1$, we just need to show that the following integral is convergent,

$$\int_0^1 \frac{1}{\mu_{\phi,\alpha}(t)} dt = \int_0^1 \frac{1}{(1-t^2)^{\frac{\alpha}{p}}} dt. \tag{3.22}$$

In fact, we have

$$\lim_{t\to 1^-}(1-t)^{\frac{\alpha}{p}} \frac{1}{(1-t^2)^{\frac{\alpha}{p}}} = \lim_{t\to 1^-}(1-t)^{\frac{\alpha}{p}} \frac{1}{(1-t)^{\frac{\alpha}{p}}(1+t)^{\frac{\alpha}{p}}} = \lim_{t\to 1^-} \frac{1}{(1+t)^{\frac{\alpha}{p}}} = \Big(\frac{1}{2}\Big)^{\frac{\alpha}{p}}.$$

Since $0 < \alpha/p < 1$ and $0 < (\frac{1}{2})^{\frac{\alpha}{p}} < +\infty$, by the comparison rule, the integral (3.22) is convergent. □

4. Conclusions

In this paper, we define the α-Bloch-Orlicz space on $\mathbb{U}^n$ by using Young's function and show that its norm is equivalent with a special μ-Bloch space. We completely characterize the boundedness and compactness of the weighted composition operator $W_{\psi,\varphi}$ on the α-Bloch-Orlicz space in terms of the behaviors of the symbols ψ and φ. In addition, we give an example that satisfies the condition (3.1), which shows the rationality of this condition. As some applications, the corresponding results of the operators M_ψ and C_φ are obtained. This paper can be viewed as a continuation and extension of our previous studies.

Author contributions

Fuya Hu: Writing and editing, formal analysis and methodology; Chengshi Huang: Commenting and reviewing; Zhijie Jiang: Writing-original draft and investigation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

We declare we have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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