Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(2): 3652–3671.
DOI:10.3934/math.2025169
Received: 26 December 2024
Revised: 13 February 2025
Accepted: 17 February 2025
Published: 25 February 2025

Research article

Leader–follower dynamics: stability and consensus in a socially structured population

Hsin-Lun Li*

National Sun Yat-sen University, Kaohsiung 804, Taiwan

* **Correspondence:** Email: hsinlunl@math.nsysu.edu.tw.

Abstract: The original leader–follower model categorizes agents with opinions in $[-1, 1]$ into a follower group, a leader group with a positive target opinion in $[0, 1]$, and a leader group with a negative target opinion in $[-1, 0]$. Leaders maintain a constant attraction to their target, blending it with the average opinion of their group neighbors at each update. Followers, on the other hand, have a constant attraction to the average opinion of their leader group's opinion neighbors, also integrating it with their group neighbors' average opinion. This model was numerically studied.

This paper extends the leader–follower model to include a social relationship, variable degrees over time, high-dimensional opinions, and a flexible number of leader groups. We theoretically investigate conditions for asymptotic stability or consensus, particularly in scenarios where a few leaders can dominate the entire population.

Keywords: social network; leader–follower dynamics; consensus; Hegselmann–Krause dynamics; averaging dynamics
Mathematics Subject Classification: 91C20, 91D25, 91D30, 93D50, 94C15

1. Introduction

The leader–follower model contains the Hegselmann–Krause model and involves two types of individuals: leaders and followers. The Hegselmann–Krause (HK) model is widely studied in averaging opinion dynamics. Early work in this area, such as Hegselmann and Krause (2002), introduced the bounded-confidence model, where agents update their opinions by averaging those of their neighbors within a given confidence threshold [1]. This model was later extended in various directions. In 2005, Lorenz provided a stabilization theorem that characterized the conditions under which opinion dynamics under bounded confidence would converge [2]. Two years later, Lorenz (2007) offered a survey on continuous opinion dynamics under bounded confidence, discussing the theoretical underpinnings and extensions of the HK model [3]. Both works laid the foundation for understanding the mathematical stability of opinion dynamics models. In 2009, Castellano et al. provided a comprehensive review on the statistical physics of social dynamics, comparing different models of opinion dynamics, including the Hegselmann–Krause model [4]. This paper highlighted the universality of the HK model in describing consensus and fragmentation processes in social systems. The next decade saw significant advancements in understanding the behavior of opinion dynamics under heterogeneous conditions. In 2013, Bhattacharyya et al. analyzed the convergence properties of the HK model and proved mathematical results related to the stability of the model [5]. Their work showed that under certain conditions, the model would reach consensus, while under others, fragmentation would occur. In 2015, Fu et al. extended the HK model to study the effects of group-based populations with heterogeneous confidence thresholds, further emphasizing the role of heterogeneity in opinion dynamics [6]. This work demonstrated that the structure of social groups significantly influences the dynamics of opinion formation. A year later, Proskurnikov and Tempo (2017) presented a tutorial on the modeling and analysis of dynamic social networks from a control theory perspective, which opened up new avenues for studying opinion dynamics on complex networks [7]. Their approach enabled the use of control-theoretic tools to study stability and consensus in networked systems. In 2021, Bernardo et al. introduced heterogeneous opinion dynamics with adaptive confidence thresholds that evolve over time based on agents' interactions. This work added a layer of realism to the original HK model by acknowledging that agents' openness to influence may change dynamically [8]. Similarly, Vasca et al. (2021) focused on the practical consensus in bounded-confidence models, emphasizing the idea that perfect consensus is not always necessary, but rather, achieving agreement within acceptable bounds is sufficient for system stability [9]. The research by Fortunato (2005) explored the consensus threshold in the HK model, analyzing how network structure and the initial distribution of opinions impact the likelihood of achieving consensus in the system [10]. Finally, recent work by Lanchier and Li (2022) analyzed the consensus behavior of the HK model under various conditions, providing a rigorous mathematical analysis of when consensus is achieved and how it depends on the initial conditions and network topology [11]. The exploration of limited connectivity in opinion dynamics was addressed by Parasnis et al. (2018), who studied the Hegselmann–Krause dynamics under sparsely connected networks, showing how local interactions in such networks affect the formation of consensus [12]. Li (2022) introduced the mixed Hegselmann–Krause model [13], which generalizes the Hegselmann–Krause model, argued in [14] that it also encompasses the Deffuant model [15–18], and further investigated it on infinite graphs in [19].

The authors in [20] proposed a leader–follower model that partitions agents whose opinion is in $[-1, 1]$ to a follower group, a leader group with a positive target opinion in $[0, 1]$, and a leader group with a negative target opinion in $[-1, 0]$. Individual j is an opinion neighbor of individual i if their opinion distance does not exceed the confidence threshold of individual i. If all individuals share the same confidence threshold, two are opinion neighbors if their distance does not exceed that threshold. A leader's opinion depends on the opinion neighbors in its group and its group target, while a follower's opinion depends on all opinion neighbors. Define $[n] = \{1, 2, \ldots, n\}$. Say N agents, including N_1 followers, N_2 positive target agents, and N_3 negative target agents, set as $[N_1]$, $[N_1 + N_2] - [N_1]$, and $[N] - [N_1 + N_2]$. The mechanism is as follows:

$$
\begin{aligned}
x_i(t+1) &= \frac{1-\alpha_i-\beta_i}{|N_i^F(t)|}\sum_{j\in N_i^F(t)} x_j(t) + \frac{\alpha_i}{|N_i^P(t)|}\sum_{j\in N_i^P(t)} x_j(t) \\
&\quad + \frac{\beta_i}{|N_i^N(t)|}\sum_{j\in N_i^N(t)} x_j(t), \quad i = 1,\ldots,N_1, \\
x_i(t+1) &= \frac{(1-w_i)}{|N_i^P(t)|}\sum_{j\in N_i^P(t)} x_j(t) + w_i d, \quad i = N_1+1,\ldots,N_1+N_2, \\
x_i(t+1) &= \frac{1-z_i}{|N_i^N(t)|}\sum_{j\in N_i^N(t)} x_j(t) + z_i g, \quad i = N_1+N_2+1,\ldots,N,
\end{aligned} \tag{1.1}
$$

where

$$
\begin{aligned}
x_i(t) &= \text{opinion of agent } i \text{ at time } t, \\
d &\in [0,1] \text{ is the positive target opinion}, \\
g &\in [-1,0] \text{ is the negative target opinion}, \\
\epsilon_i &= \text{confidence threshold of agent } i, \\
N_i^F(t) &= \{j \in [N_1] : \|x_i(t) - x_j(t)\| \le \epsilon_i\}, \\
N_i^P(t) &= \{j \in [N_1+N_2] - [N_1] : \|x_i(t) - x_j(t)\| \le \epsilon_i\}, \\
N_i^N(t) &= \{j \in [N] - [N_1+N_2] : \|x_i(t) - x_j(t)\| \le \epsilon_i\}, \\
\alpha_i &= \text{degree to the average opinion of agent } i\text{'s} \\
&\quad\ \text{positive target neighbors}, \\
\beta_i &= \text{degree to the average opinion of agent } i\text{'s} \\
&\quad\ \text{negative target neighbors}, \\
w_i &= \text{degree to the positive target of agent } i, \\
z_i &= \text{degree to the negative target of agent } i, \\
\alpha_i,&\quad \beta_i, \quad w_i, \quad z_i \in [0,1].
\end{aligned}
$$

The authors in [21] pointed out that it can be an application in e-commerce. The Leader–Follower model we investigate now includes the following:

- There is a social relationship.
- The degree toward the average opinion of a group can vary over time.
- The number of leader groups is decidable.
- Opinions can be high-dimensional.

Let $\mathcal{N}_i^S(t)$ be the collection of all social and opinion neighbors of individual i in set S at time t and $x_i(t) \in \mathbb{R}^d$ be the opinion of individual i at time t where $x_i(0)$ is a random variable. The leader group L model with target $g \in \mathbb{R}^d$ is given by:

$$
x_i(t+1) = \frac{\alpha_i(t)}{|\mathcal{N}_i^L(t)|}\sum_{j\in\mathcal{N}_i^L(t)} x_j(t) + (1-\alpha_i(t))g, \quad i \in L, \tag{1.2}
$$

where $\alpha_i(t) \in [0,1]$ is a random variable indicating the degree of individual i toward the average opinion of its group neighbors at time t, and L is the collection of all leader group members. Say the leader groups are $L_1, \ldots, L_m$ with targets $g_1, \ldots, g_m$. The follower group F model is given by:

$$x_i(t+1) = \frac{(1 - \sum_{k=1}^{m} \beta_i^k(t))}{|\mathcal{N}_i^F(t)|} \sum_{k \in \mathcal{N}_i^F(t)} x_k(t) + \sum_{k=1}^{m} \frac{\beta_i^k(t)}{|\mathcal{N}_i^{L_k}(t)|} \sum_{k \in \mathcal{N}_i^{L_k}(t)} x_k(t), \quad i \in F, \tag{1.3}$$

where $\beta_i^k(t) \in [0, 1]$ is a random variable indicating the degree toward the average opinion of its social and opinion neighbors in leader group L_k at time t, and F consists of all followers. $\beta_i^k(t) = 0$ if $\mathcal{N}_i^{L_k}(t) = \emptyset$. Observe that (1.2) and (1.3) reduce to (1.1) when there are two leader groups and a follower group. (1.2) reduces to the synchronous Hegselmann–Krause model when $\alpha_i(t) = 1$ for all $i \in L$ at all times. Similarly, (1.3) reduces to the synchronous Hegselmann–Krause model when $\beta_i^k(t) = 0$ for all $k \in [m]$ and $i \in F$ all the time. In (1.2) and (1.3), a leader's opinion depends on the social and opinion neighbors in its group and its group target. In contrast, a follower's opinion depends on all social and opinion neighbors. Since finite time convergence holds in the synchronous Hegselmann–Krause model, finite time convergence also holds in (1.2) when it is reduced to the synchronous Hegselmann–Krause model. The same applies to (1.3). In particular, finite time convergence holds in (1.2) when all agents agree on their target at some time.

Interpreting in a graph, a vertex represents an individual, and an edge symbolizes a relationship between two individuals. Saying

- $G(t) = (V, E(t))$ is the social graph at time t with vertex set and edge set V and $E(t)$;
- $\mathcal{G}(t) = (V, \mathcal{E}(t))$ is the opinion graph at time t with vertex set and edge set V and $\mathcal{E}(t)$.

Edge $(i, j) \in E(t)$ if individual i is socially connected with individual j, or if individual j is a social neighbor of individual i. Similarly, edge $(i, j) \in \mathcal{E}(t)$ if individual i is opinion connected with individual j, or if individual j is an opinion neighbor of individual i, i.e., $\|x_i(t) - x_j(t)\| \leq \epsilon_i$. We can interpret a social relationship with an undirected social graph if every edge $(i, j) \in E(t)$ implies $(j, i) \in E(t)$. For instance, if individual i is a relative of individual j, then the reverse is also true. However, not all social relationships are reciprocal. For example, if individual i knows individual j, it does not necessarily imply that individual j knows individual i. In such cases, we use a directed social graph to represent the relationship. On the other hand, if all individuals share the same confidence threshold in an opinion relationship, we can interpret this opinion relationship with an undirected graph. Both $G(t)$ and $\mathcal{G}(t)$ can be directed graphs. The social graph $G(t)$ is considered undirected if every $(i, j) \in E(t)$ implies $(j, i) \in E(t)$. Similarly, the opinion graph $\mathcal{G}(t)$ is considered undirected if every $(i, j) \in \mathcal{E}(t)$ implies $(j, i) \in \mathcal{E}(t)$. A graph is *$\delta$-trivial* if the opinion distance between any two vertices does not exceed δ. Denote $\mathbf{B}(a, r)$ as the open ball centered at a with radius r, i.e., $\mathbf{B}(a, r) = \{x : \|x - a\| < r\}$. A *profile* $G \cap \mathcal{G}$ is the intersection of the social and opinion graphs.

2. Main results

Since leader groups are independent, and similarly for the follower groups, we respectively investigate sufficient conditions for asymptotic stability or a consensus in the leader group L and the follower group F with m leader groups. The sufficient condition in Theorem 2.1 is independent of social and opinion relationships. $\max_{i \in L} \alpha_i(t)$ represents the degree of the leader in L with the largest degree toward the average opinion of its group neighbors at time t. If there are infinitely many times at which $\max_{i\in L}\alpha_i(t)$ is less than some random variable smaller than 1, a consensus equal to the target is achieved within L. In fact, even a slight tendency toward the target by all leaders in L guarantees a consensus equal to the target.

Theorem 2.1. There is a consensus equal to the target in (1.2) when

$$\liminf_{t\to\infty}\max_{i\in L}\alpha_i(t) < 1.$$

The sufficient condition in Theorem 2.2 assumes an undirected social graph and an undirected opinion graph on L. Specifically, the synchronous Hegselmann–Krause model meets this condition, thus ensuring asymptotic stability. Asymptotic stability of the synchronous Hegselmann–Krause model illustrates finite time convergence. $\min_{i\in L}\alpha_i(t)$ represents the degree of the leader in L with the smallest degree toward the average opinion of its group neighbors at time t. $\sum_{t\geq 0}(1/\min_{i\in L}\alpha_i(t)-1)<\infty$ implies that $\min_{i\in L}\alpha_i(t)$ approaches 1 as $t\to\infty$.

Theorem 2.2. Assume that the social graph and opinion graph are undirected on L; the social graph becomes constant after some time, and

$$\sum_{t\geq 0}(1/\min_{i\in L}\alpha_i(t)-1)<\infty.$$ Then, asymptotic stability holds in (1.2).

The sufficient condition in Theorem 2.3 specifies that the social graph and opinion graph can be directed, provided that all followers are socially connected with a leader in each leader group. This condition also identifies circumstances under which a few leaders can dominate the entire population. $\beta_i^k(t)=\beta_i^k$ for all $k\in[m]$ and $i\in F$ at all times indicates that the degree of a follower toward the average opinion of its neighbors in a leader group remains constant over time. $\max_{i\in F}\left(1-\sum_{k=1}^m\beta_i^k\right)$ represents the degree of the follower in F with the maximum degree toward the average opinion of its group neighbors. Similarly, $\max_{i\in L_k;k\in[m]}\alpha_i^k(s)$ represents the degree of the leader with the maximum degree toward the average opinion of its group neighbors at time s. Following the assumptions in Theorem 2.3, all leaders in a leader group approach their target, and all followers approach their weighted average of the targets of the leader groups.

Theorem 2.3. Assume that all followers have one social neighbor in each leader group, that

$$\{x_i(t),\ g_k\}_{i\in(\bigcup_{k=1}^m L_k)\cup F,\ k\in[m]}\subset \mathbf{B}(g_j,\min_{i\in F}\epsilon_i/(1+\mathbb{1}\{m\geq 2\}))$$

for some $j\in[m]$ and $t\geq 0$, that $\beta_i^k(t)=\beta_i^k$ for all $k\in[m]$ and $i\in F$ all the time, and that

$$\max_{i\in F}(1-\sum_{k=1}^m\beta_i^k)<1 \quad\text{and}\quad \sup_{s\geq t}\{\max_{i\in L_k;k\in[m]}\alpha_i^k(s)\}<1.$$

Then,

$$\lim_{t\to\infty}\max_{i\in L_k;k\in[m]}\|x_i(t)-g_k\|=0 \quad\text{and}\quad \lim_{t\to\infty}\max_{i\in F}\|x_i(t)-\frac{\sum_{k=1}^m\beta_i^k g_k}{\sum_{j=1}^m\beta_i^j}\|=0.$$

The sufficient condition in Theorem 2.4 assumes an undirected social graph and an undirected opinion graph on F, allowing the social graph and opinion graph on leader groups to be directed.

Specifically, the synchronous Hegselmann–Krause model satisfies this condition. $\max_{i\in F;k\in[m]}\beta_i^k(t)$ represents the degree of the individual in F with the maximum degree toward the average opinion of its social and opinion neighbors in a leader group at time t. $\sum_{t\geq 0}\max_{i\in F;k\in[m]}\beta_i^k(t)<\infty$ implies that $\max_{i\in F;k\in[m]}\beta_i^k(t)$ approaches 0 as $t\to\infty$.

Theorem 2.4. Assume that the social graph and opinion graph are undirected on F, the social graph on F becomes constant after some time, and

$$\sum_{t\geq 0}\max_{i\in F;k\in[m]}\beta_i^k(t)<\infty.$$

Then, asymptotic stability holds in (1.3).

Theorems 2.1 and 2.3 impose no restrictions on the social graph and opinion graph being undirected. In other words, the social graph and opinion graph in Theorems 2.1 and 2.3 can be directed. However, Theorem 2.2 assumes that the social graph and opinion graph on L are undirected, while Theorem 2.4 assumes that the social graph and opinion graph on F are undirected. The critical steps in deriving the results of Theorems 2.2 and 2.4 involve constructing a function to establish an inequality and applying Cheeger's inequality. These derived inequalities, along with Cheeger's inequality, are valid under the assumption of undirected graphs. When the graphs are directed, the challenge arises because these techniques are no longer applicable.

3. The leader group model

All leader groups are independent. Therefore, we investigate the behavior of a leader group. Let $y_i(t)=x_i(t)-g$, (1.2) becomes

$$y_i(t+1)=\frac{\alpha_i(t)}{|\mathcal{N}_i^L(t)|}\sum_{j\in\mathcal{N}_i^L(t)}y_j(t). \tag{3.1}$$

It is clear that (3.1) is the synchronous Hegselmann–Krause model when $\alpha_i(t)=1$ for all $i\in L$ all the time. Asymptotic stability holding in (1.2) is equivalent to holding in (3.1).

Lemma 3.5. We derive $x_i(t)\to g$ if $\limsup_{t\to\infty}\alpha_i(t)=0$ for all $i\in L$.

Proof. It follows from the triangle inequality that

$$\|y_i(t+1)\|\leq\alpha_i(t)\max_{j\in\mathcal{N}_i^L(t)}\|y_j(t)\|.$$

Taking lim sup on both sides, we obtain

$$\limsup_{t\to\infty}\|y_i(t+1)\|\leq\max_{i\in L}\|y_i(0)\|\limsup_{t\to\infty}\alpha_i(t)=0.$$

This indicates that $y_i(t)\to 0$ as $t\to\infty$. Hence, $x_i(t)\to g$ as $t\to\infty$. □

Lemma 3.6. Let $Z_t=\max_{i\in L}\|x_i(t)-g\|$. Then, $(Z_t)_{t\geq 0}$ is nonincreasing and

$$Z_t-Z_{t+1}\geq(1-\max_{i\in L}\alpha_i(t))Z_t. \tag{3.2}$$

Proof. By the triangle inequality, we obtain

$$Z_{t+1} = \max_{i\in L} \|y_i(t+1)\| \le \max_{i\in L} \alpha_i(t) \max_{i\in L} \|y_i(t)\| = \max_{i\in L} \alpha_i(t) Z_t.$$

It turns out that $(Z_t)_{t\ge 0}$ is nonincreasing and

$$Z_t - Z_{t+1} \ge (1 - \max_{i\in L} \alpha_i(t)) Z_t.$$

□

Next, we show circumstances in which social relationships and opinion relationships do not influence the achievement of consensus.

Proof of Theorem 2.1. It follows from Lemma 3.6 that $(Z_t)_{t\ge 0}$ is a nonnegative supermartingale. Via the martingale convergence theorem, Z_t converges to some random variable Z_∞ with finite expectation as $t \to \infty$. Letting $\alpha_t = \max_{i\in L} \alpha_i(t)$ and taking lim sup on (3.2), we derive

$$0 = \limsup_{t\to\infty} (Z_t - Z_{t+1}) \ge Z_\infty \limsup_{t\to\infty} (1 - \alpha_t) = (1 - \liminf_{t\to\infty} \alpha_t) Z_\infty.$$

This implies $Z_\infty = 0$. □

Lemma 3.7. Assume that the social graph is undirected on L, the opinion graph is undirected on L with a confidence threshold of ϵ, and $E(t) \subset E(t+1)$. Let $W_t = \sum_{i,j\in L} (\|x_i(t) - x_j(t)\|^2 \wedge \epsilon^2) \vee \epsilon^2 \mathbb{1}\{(i,j) \notin E(t)\}$. Then, we derive

$$\begin{aligned} W_t - W_{t+1} \ge\ & 4 \sum_{i\in L} \|x_i(t) - x_i(t+1)\|^2 - 4|L|^2 (1/\min_{i\in L} \alpha_i(t) - 1) \\ & \times \max_{i\in L} \|x_i(0) - g\| \Big(\max_{i\in L} \|x_i(0) - g\| \vee \max_{i,j\in L} \|x_i(0) - x_j(0)\| \Big). \end{aligned} \tag{3.3}$$

Proof. Let $\mathcal{N}_i^L = \mathcal{N}_i^L(t)$, $\alpha_i = \alpha_i(t)$, $x_i = x_i(t)$, $x_i^\star = x_i(t+1)$, $y_i = y_i(t)$, $y_i^\star = y_i(t+1)$, $E = E(t)$, and $E^\star = E(t+1)$. It turns out that

$$\begin{aligned} W_t - W_{t+1} &= \sum_{i\in L} \Big\{ \sum_{j\in \mathcal{N}_i^L} (\|x_i - x_j\|^2 - \|x_i^\star - x_j^\star\|^2 \wedge \epsilon^2) \\ &\qquad + \sum_{j\in (\mathcal{N}_i^L)^c} \big[\epsilon^2 - (\|x_i^\star - x_j^\star\|^2 \wedge \epsilon^2) \vee \epsilon^2 \mathbb{1}\{(i,j) \notin E^\star\}\big] \Big\} \\ &\ge \sum_{i\in L} \sum_{j\in \mathcal{N}_i^L} (\|x_i - x_j\|^2 - \|x_i^\star - x_j^\star\|^2) = \sum_{i\in L} \sum_{j\in \mathcal{N}_i^L} (\|y_i - y_j\|^2 - \|y_i^\star - y_j^\star\|^2) \\ &= \sum_{i\in L} \sum_{j\in \mathcal{N}_i^L} (2 < y_i - y_i^\star, y_i^\star - y_j > -2 < y_i^\star - y_j, y_j - y_j^\star >) \\ &= 2 \sum_{i\in L} |\mathcal{N}_i^L| < y_i - y_i^\star, y_i^\star > (1 - 1/\alpha_i) - 2 \sum_{i\in L} \sum_{j\in \mathcal{N}_i^L} < y_i^\star - y_i, y_j - y_j^\star > \\ &\quad - 2 \sum_{j\in L} \sum_{i\in \mathcal{N}_j^L} < y_i - y_j, y_j - y_j^\star > \end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{i\in L}|\mathcal{N}_i^L| < y_i - y_i^\star, y_i^\star > (1-1/\alpha_i) + 2\sum_{i\in L}\|y_i - y_i^\star\|^2 \\
&\quad - 2\sum_{i\in L}\sum_{j\in\mathcal{N}_i^L-\{i\}} < y_i^\star - y_i, y_j - y_j^\star > -2\sum_{j\in L}|\mathcal{N}_j^L| < y_j^\star/\alpha_j - y_j, y_j - y_j^\star > \\
&\geq 2\sum_{i\in L}|\mathcal{N}_i^L| < y_i - y_i^\star, y_i^\star > (1-1/\alpha_i) + 2\sum_{i\in L}\|y_i - y_i^\star\|^2 \\
&\quad - 2\sum_{i\in L}\sum_{j\in\mathcal{N}_i^L-\{i\}} \|y_i^\star - y_i\|\|y_j^\star - y_j\| \\
&\quad - 2\sum_{j\in L}|\mathcal{N}_j^L|(1/\alpha_j - 1) < y_j^\star, y_j - y_j^\star > +2\sum_{j\in L}|\mathcal{N}_j^L|\|y_j - y_j^\star\|^2 \\
&= -4\sum_{i\in L}|\mathcal{N}_i^L|(1/\alpha_i - 1) < y_i^\star, y_i - y_i^\star > +2\sum_{i\in L}\|y_i - y_i^\star\|^2 \\
&\quad + \sum_{i\in L}\sum_{j\in\mathcal{N}_i^L-\{i\}} [(\|y_i^\star - y_i\| - \|y_j^\star - y_j\|)^2 - \|y_i^\star - y_i\|^2 - \|y_j^\star - y_j\|^2] \\
&\quad + 2\sum_{j\in L}|\mathcal{N}_j^L|\|y_j - y_j^\star\|^2 \\
&\geq -2\sum_{i\in L}(|\mathcal{N}_i^L| - 1)\|y_i^\star - y_i\|^2 - 4\sum_{i\in L}|\mathcal{N}_i^L|(1/\alpha_i - 1) < y_i^\star, y_i - y_i^\star > \\
&\quad + 2\sum_{j\in L}|\mathcal{N}_j^L|\|y_j - y_j^\star\|^2 + 2\sum_{i\in L}\|y_i - y_i^\star\|^2 \\
&= 4\sum_{i\in L}\|y_i - y_i^\star\|^2 - 4\sum_{i\in L}|\mathcal{N}_i^L|(1/\alpha_i - 1) < y_i^\star, y_i - y_i^\star > \\
&\geq 4\sum_{i\in L}\|y_i - y_i^\star\|^2 - 4|L|^2(1/\min_{i\in L}\alpha_i - 1)\max_{i\in L}\|x_i(0) - g\| \\
&\qquad\qquad \times\left(\max_{i\in L}\|x_i(0) - g\| \vee \max_{i,j\in L}\|x_i(0) - x_j(0)\|\right).
\end{aligned}
$$

□

By finiteness of the social graph, the social graph monotone after some time is equivalent to the social graph constant after some time.

Lemma 3.8. Cheeger's inequality [22] Assume that $G = (V, E)$ is an undirected graph with the Laplacian $\mathscr{L}$. Define

$$i(G) = \min\left\{\frac{|\partial S|}{|S|} : S \subset V, 0 < |S| \leq \frac{|G|}{2}\right\},$$

where $\partial S = \{(u, v) \in E : u \in S, v \in S^c\}$. Then,

$$2i(G) \geq \lambda_2(\mathscr{L}) \geq \frac{i^2(G)}{2\Delta(G)} \quad \text{where} \quad \Delta(G) = \text{ maximum degree of } G.$$

Lemma 3.9. [13] Assume that Q is a real square matrix and that V is invertible such that the matrix $VQ = \mathscr{L}$ is the Laplacian of some connected graph. Then, 0 is a simple eigenvalue of $Q'Q$ corresponding to the eigenvector $\mathbb{1} = (1, 1, \ldots, 1)'$. In particular, we have

$$\lambda_2(Q'Q) = \min\{x'Q'Qx : \|x\| = 1 \text{ and } x \perp \mathbb{1}\}.$$

Lemma 3.10. Assume that the social graph and opinion graph are undirected on L. If some component H of the profile $G(t) \cap \mathscr{G}(t)$ on L is δ-nontrivial, then,

$$\sqrt{\sum_{i\in L} \|x_i(t) - x_i(t+1)\|^2} \geq \sqrt{2}\delta \min_{i\in L} \alpha_i(t)/|L|^4 - (1 - \min_{i\in L} \alpha_i(t)) \sqrt{|L|} \max_{i\in L} \|x_i(0) - g\|.$$

Proof. Letting $V(H)$, the vertex set of H, be $[h]$ and $\mathbb{1} = (1, \ldots, 1)' \in \mathbb{R}^h$, express $\mathbb{R}^h = W \oplus W^\perp$ for $W = \mathrm{Span}(\{\mathbb{1}\})$. For $y(t) = (y_1(t), \ldots, y_h(t))'$, write

$$y(t) = [c_1\mathbb{1} \,|\, c_2\mathbb{1} \,|\, \cdots \,|\, c_d\mathbb{1}] + \left[\hat{c}_1 u^{(1)} \,|\, \hat{c}_2 u^{(2)} \,|\, \cdots \,|\, \hat{c}_d u^{(d)}\right],$$

where c_i and $\hat{c}_i$ are constants and $u^{(i)} \in \mathbb{1}^\perp$ is a unit vector for all $i \in [d]$. Observe that

$$\|y_i(t) - y_j(t)\|^2 = \sum_{k\in[d]} \hat{c}_k^2 (u_i^{(k)} - u_j^{(k)})^2 \leq 2 \sum_{k\in[d]} \hat{c}_k^2 ((u_i^{(k)})^2 + (u_j^{(k)})^2) \leq 2 \sum_{k\in[d]} \hat{c}_k^2$$

for all $i, j \in [h]$. Since $x_i - x_j = y_i - y_j$,

$$\text{component } H \ \delta\text{-nontrivial implies } \sum_{k\in[d]} \hat{c}_k^2 > \delta^2/2.$$

Letting $\alpha(t) = (\alpha_1(t), \ldots, \alpha_h(t))'$ and $B(t) = \mathrm{diag}(\alpha(t))A(t)$ for $A(t) \in \mathbb{R}^{h\times h}$ with $A_{i,j}(t) = \mathbb{1}\{j \in \mathscr{N}_i^L(t)\}/|\mathscr{N}_i^L(t)|$, we obtain

$$y(t) - y(t+1) = (I - B(t))y(t) = [C(t) + F(t)\mathscr{L}(t)]\, y(t),$$

where $C(t) = I - \mathrm{diag}(\alpha(t))$, $F(t) = \mathrm{diag}(\alpha(t))\left(\mathrm{diag}((d_i)_{i=1}^h) + I\right)^{-1}$ with d_i the degree of vertex i in component H, and $\mathscr{L}(t)$ is the Laplacian of component H. It follows from Lemmas 3.8 and 3.9 that

$$\lambda_2(\mathscr{L}) > \frac{(2/h)^2}{2h} = 2/h^3,$$

$$\begin{aligned}\|F(t)\mathscr{L}(t)y(t)\|^2 &= \sum_{k\in[d]} \hat{c}_k^2 \|F(t)\mathscr{L}(t)u^{(k)}\|^2 \geq \sum_{k\in[d]} \hat{c}_k^2 \lambda_2\left(\mathscr{L}(t)F^2(t)\mathscr{L}(t)\right) \\ &\geq (\delta^2/2)(\min_{i\in[h]} \alpha_i(t)/h)^2 \lambda_2^2(\mathscr{L}(t)) \geq 2\delta^2 \min_{i\in[h]} \alpha_i^2(t)/h^8.\end{aligned}$$

On the other hand, we derive

$$\|C(t)y(t)\| \leq (1 - \min_{i\in[h]} \alpha_i(t)) \sqrt{h} \max_{i\in[h]} \|x_i(0) - g\|.$$

It follows from the triangle inequality that

$$\sqrt{\sum_{i\in L} \|x_i(t) - x_i(t+1)\|^2} \geq \sqrt{\sum_{i\in[h]} \|y_i(t) - y_i(t+1)\|^2} = \|y(t) - y(t+1)\|$$

$$= \| [F(t)\mathscr{L}(t) + C(t)]\, y(t)\| \geq \|F(t)\mathscr{L}(t)y(t)\| - \|C(t)y(t)\|$$
$$\geq \sqrt{2}\delta \min_{i\in[h]} \alpha_i(t)/h^4 - (1 - \min_{i\in[h]} \alpha_i(t))\sqrt{h} \max_{i\in[h]} \|x_i(0) - g\|$$
$$\geq \sqrt{2}\delta \min_{i\in L} \alpha_i(t)/|L|^4 - (1 - \min_{i\in L} \alpha_i(t))\sqrt{|L|} \max_{i\in L} \|x_i(0) - g\|.$$

□

Proof of Theorem 2.2. We claim the following:

1) All components of profile $G \cap \mathscr{G}$ on L are δ-trivial after some time for all $\delta > 0$.
2) No components of profile $G \cap \mathscr{G}$ on L interact with each other after some time.

Without loss of generality, we assume the social graph on L remains constant over time, saying $G(t)|_L = G|_L = (L, E)$ for all $t \geq 0$. Observe that

$$\sum_{t\geq 0}(1/\min_{i\in L} \alpha_i(t) - 1) < \infty \implies \lim_{t\to\infty} \min_{i\in L} \alpha_i(t) = 1 \iff \lim_{t\to\infty} \alpha_i(t) = 1 \text{ for all } i \in L.$$

Hence, we derive

$$\sqrt{2}\delta \min_{i\in L} \alpha_i(t)/|L|^4 \to \sqrt{2}\delta/|L|^4 \text{ and } (1 - \min_{i\in L} \alpha_i(t))\sqrt{|L|} \max_{i\in L} \|x_i(0) - g\| \to 0$$

as $t \to \infty$. There is $t_0 \geq 0$ such that

$$\sqrt{2}\delta \min_{i\in L} \alpha_i(t)/|L|^4 - (1 - \min_{i\in L} \alpha_i(t))\sqrt{|L|} \max_{i\in L} \|x_i(0) - g\| \geq \delta/|L|^4$$

for all $t \geq t_0$. Assume that asymptotic stability does not hold in (1.2). Then, there are $\delta > 0$ and $(s_k)_{k\geq 0}$ increasing with $s_0 \geq t_0$ and some component in profile $G(t_k) \cap \mathscr{G}(t_k)$ on L δ-nontrivial for all $k \geq 0$. Letting

$$M_0 = 4|L|^2 \max_{i\in L} \|x_i(0) - g\|(\max_{i\in L} \|x_i(0) - g\| \vee \max_{i,j\in L} \|x_i(0) - x_j(0)\|),$$

it turns out from Lemma 3.7 that

$$\begin{aligned} W_0 + M_0 \sum_{t=0}^{m}(1/\min_{i\in L} \alpha_i(t) - 1) &\geq \sum_{t=0}^{m}(W_t - W_{t+1}) + M_0 \sum_{t=0}^{m}(1/\min_{i\in L} \alpha_i(t) - 1) \\ &\geq 4\sum_{t=0}^{m}\sum_{i\in L} \|x_i(t) - x_i(t+1)\|^2 \text{ for all } m \geq 0. \end{aligned}$$

As $m \to \infty$, we derive

$$\begin{aligned} \infty > W_0 + M_0 \sum_{t\geq 0}(1/\min_{i\in L} \alpha_i(t) - 1) &\geq 4\sum_{t\geq 0}\sum_{i\in L} \|x_i(t) - x_i(t+1)\|^2 \\ \geq 4\sum_{k\geq 0}\sum_{i\in L} \|x_i(s_k) - x_i(s_k+1)\|^2 &\geq 4\sum_{k\geq 0} \delta^2/|L|^8 = \infty, \text{ a contradiction.} \end{aligned}$$

Hence, all components of profile $G \cap \mathscr{G}$ on L are δ-trivial after some time for all $\delta > 0$.

Next, we claim that no components of profile $G \cap \mathscr{G}$ on L interact with each other after some time. It follows from claim 1) that all components of profile $G \cap \mathscr{G}$ on L are $\epsilon/4$-trivial after some time s_0.

Assume that claim 2) is not the case. By finiteness of the social graph, there are edge (i, j) and $(t_k)_{k\geq 0}$ increasing with $t_0 \geq s_0$ such that vertices i and j belong to distinct components of profile $G \cap \mathscr{G}(t_k)$ on L,

$$(i, j) \in E \cap \mathscr{E}(t_k)^c \text{ and } (i, j) \in E \cap \mathscr{E}(t_k + 1).$$

Letting $y_i = y_i(t_k)$, $y_i^\star = y_i(t_k+1)$ and $\alpha_i = \alpha_i(t_k)$ for all $i \in L$ and $k \geq 0$, it turns out from the triangle inequality that

$$\begin{aligned}\epsilon < \|y_i - y_j\| &\leq \|y_i - y_i^\star/\alpha_i\| + \|y_i^\star/\alpha_i - y_i^\star\| + \|y_i^\star - y_j^\star\| + \|y_j^\star - y_j^\star/\alpha_j\| \\ &+ \|y_j^\star/\alpha_j - y_j\| \leq \epsilon/2 + \|y_i^\star/\alpha_i - y_i^\star\| + \|y_i^\star - y_j^\star\| + \|y_j^\star - y_j^\star/\alpha_j\|\end{aligned}$$

for the last inequality following from $\|y_i - y_i^\star/\alpha_i\| \leq \epsilon/4$ and $\|y_j^\star/\alpha_j - y_j\| \leq \epsilon/4$. Since $\limsup_{k\to\infty} \|y_i^\star/\alpha_i - y_i^\star\| = 0 = \limsup_{k\to\infty} \|y_j^\star - y_j^\star/\alpha_j\|$, we derive

$$\epsilon/2 \leq \liminf_{k\to\infty} \|y_i^\star - y_j^\star\| = \liminf_{k\to\infty} \|x_i^\star - x_j^\star\|, \text{ a contradiction.}$$

□

Equation (1.2) reduces to the synchronous Hegselmann–Krause model when $\alpha_i(t) = 1$ for all $i \in L$ at all times. Therefore, $\sum_{t\geq 0} (1/\min_{i\in L} \alpha_i(t) - 1) = 0 < \infty$. From Theorem 2.2, it follows that all components of a profile on L become ϵ-trivial, and no components interact with each other after some time under undirected social and opinion graphs. This indicates that all components achieve their consensus at the next time step, substantiating the finite time convergence property of the synchronous Hegselmann–Krause model under undirected social and opinion graphs.

4. The follower group model

Follower groups are independent. We first consider a leader group L and a follower group F. (1.3) becomes

$$x_i(t+1) = \frac{(1-\beta_i(t))}{|\mathcal{N}_i^F(t)|} \sum_{j\in\mathcal{N}_i^F(t)} x_j(t) + \frac{\beta_i(t)}{|\mathcal{N}_i^L(t)|} \sum_{j\in\mathcal{N}_i^L(t)} x_j(t), \quad i \in F,$$

which is equivalent to

$$y_i(t+1) = \frac{(1-\beta_i(t))}{|\mathcal{N}_i^F(t)|} \sum_{j\in\mathcal{N}_i^F(t)} y_j(t) + \frac{\beta_i(t)}{|\mathcal{N}_i^L(t)|} \sum_{j\in\mathcal{N}_i^L(t)} y_j(t), \quad i \in F. \tag{4.1}$$

Lemma 4.11. Assume that all followers have one social neighbor in leader group L with target g, that $\{x_i(t)\}_{i\in L\cup F} \subset \mathbf{B}(g, \min_{i\in F} \epsilon_i)$ for some $t \geq 0$ and that

$$\sup_{s\geq t} \{ \max_{i\in F}(1-\beta_i(s)), \max_{i\in L} \alpha_i(s)\} < 1.$$

Then,

$$\lim_{t\to\infty} \max_{i\in L\cup F} \|x_i(t) - g\| = 0.$$

Proof. $\{x_i(t)\}_{i\in L\cup F} \subset \mathbf{B}(g, \min_{i\in F} \epsilon_i)$ for some $t \geq 0$ implies $\{x_i(s)\}_{i\in L\cup F} \subset \mathbf{B}(g, \min_{i\in F} \epsilon_i)$ for all $s \geq t$. Via Theorem 2.1, $\lim_{t\to\infty} \max_{k\in L} \|y_k(t)\| = 0$; therefore,

$$\max_{k\in L} \|y_k(s)\| < \delta \text{ for all } \delta > 0, \text{ for some } p \geq t,$$

and for all $s \geq p$. For all $s \geq p$, $\delta > 0$, $i \in F$ and $j \in L$, we have

$$\|y_i(s) - y_j(s)\| \leq \|y_i(s)\| + \|y_j(s)\| < \min_{i\in F} \epsilon_i + \delta,$$

therefore, $\|x_i(s) - x_j(s)\| = \|y_i(s) - y_j(s)\| \leq \min_{i\in F} \epsilon_i \leq \epsilon_i$ and $\mathcal{N}_i^L(s) \neq \emptyset$. Let $\alpha_t = \max_{k\in L} \alpha_k(t)$, $\tilde{\beta}_t = \max_{k\in F}(1 - \beta_k(t))$, $\gamma = \sup_{s\geq t}\{\tilde{\beta}_s,\ \alpha_s\}$, $A_t = \max_{k\in F} \|y_k(t)\|$ and $Z_t = \max_{k\in L} \|y_k(t)\|$. Applying the triangle inequality on (4.1), for all $i \in F$ and $t > p$,

$$\begin{aligned} A_{t+1} &\leq \tilde{\beta}_t A_t + Z_t \\ &\leq \tilde{\beta}_t\tilde{\beta}_{t-1} \ldots \tilde{\beta}_p A_p + \tilde{\beta}_t \ldots \tilde{\beta}_{p+1} Z_p + \ldots + \tilde{\beta}_t Z_{t-1} + Z_t \\ &\leq \gamma^{t-p+1} A_p + (t - p + 1)\gamma^{t-p} Z_p, \end{aligned}$$

therefore,

$$\limsup_{t\to\infty} A_{t+1} \leq 0.$$

This completes the proof. □

We move on to the follower group model with m leader groups.

Proof of Theorem 2.3. $\{x_i(t),\ g_k\}_{i\in(\bigcup_{k=1}^m L_k)\cup F,\ k\in[m]} \subset \mathbf{B}(g_j, \min_{i\in F} \epsilon_i)$ for some $t \geq 0$ implies

$$\{x_i(s)\}_{i\in(\bigcup_{k=1}^m L_k)\cup F} \subset \mathbf{B}(g_j, \min_{i\in F} \epsilon_i) \text{ for all } s \geq t.$$

It follows from Theorem 2.1 that $\lim_{t\to\infty} \max_{k\in[m]} \max_{i\in L_k} \|x_i(t) - g_k\| = 0$; therefore,

$$\max_{k\in[m]} \max_{i\in L_k} \|x_i(s) - g_k\| < \delta \text{ for all } \delta > 0, \text{ for some } p \geq t \text{ and for all } s \geq p.$$

For all $s \geq p$, $\delta > 0$, $i \in F$ and $j \in L_k$, we have

$$\|x_i(s) - x_j(s)\| \leq \|x_i(s) - g_k\| + \|g_k - x_j(s)\| < \min_{i\in F} \epsilon_i + \delta,$$

therefore, $\|x_i(s) - x_j(s)\| \leq \min_{i\in F} \epsilon_i \leq \epsilon_i$ and $\mathcal{N}_i^{L_k}(s) \neq \emptyset$. Letting

$$\tilde{\beta} = \max_{i\in F}(1 - \sum_{k=1}^m \beta_i^k), \qquad \gamma = \sup_{s\geq t}\{\max_{k\in[m]} \max_{i\in L_k} \alpha_i^k(s),\ \tilde{\beta}\},\ g = \sum_{k=1}^m \beta_i^k g_k / \sum_{k=1}^m \beta_i^k,$$

$$A_t = \max_{i\in F} \|x_i(t) - g\|/m, \quad C_t = \sum_{k=1}^m \max_{i\in L_k} \|x_i(t) - g_k\|/m.$$

Letting

$$\bar{x}_i^F(t) = \frac{1}{|\mathcal{N}_i^F(t)|} \sum_{j\in\mathcal{N}_i^F(t)} x_j(t) \quad \text{and} \quad \bar{x}_i^{L_k}(t) = \frac{1}{|\mathcal{N}_i^{L_k}(t)|} \sum_{j\in\mathcal{N}_i^{L_k}(t)} x_j(t),$$

write (1.3) as

$$x_i(t+1) - g = (1 - \sum_{k\in[m]} \beta_i^k)(\bar{x}_i^F(t) - g) + \sum_{k\in[m]} \beta_i^k(\bar{x}_i^{L_k}(t) - g_k),$$

and apply the triangle inequality, for all $i \in F$ and $t > p$,

$$\begin{aligned} A_{t+1} &\le \tilde{\beta}_t A_t + C_t \\ &\le \tilde{\beta}_t\tilde{\beta}_{t-1}\ldots\tilde{\beta}_p A_p + \tilde{\beta}_t\ldots\tilde{\beta}_{p+1}C_p + \ldots + \tilde{\beta}_t C_{t-1} + C_t \\ &\le \gamma^{t-p+1}A_p + (t-p+1)\gamma^{t-p}C_p, \end{aligned}$$

therefore,

$$\limsup_{t\to\infty} A_{t+1} \le 0.$$

This completes the proof. □

Lemma 4.12. Assume that the social graph is undirected on F, and the opinion graph is undirected on F with a confidence threshold of ϵ. Let $X_t = \sum_{i,j\in F}(\|x_i(t) - x_j(t)\|^2 \wedge \epsilon^2) \vee \epsilon^2 \mathbb{1}\{(i,j) \notin E(t)\}$ and $E(t) \subset E(t+1)$. Then,

$$\begin{aligned} X_t - X_{t+1} \ge 4\sum_{i\in F}\|x_i(t) - x_i(t+1)\|^2 - 4m|F|^2 \max_{i\in F;k\in[m]} \beta_i^k(t) \\ \times \left(\max_{i,j\in\bigcup_{k\in[m]}L_k\cup F}\|x_i(0) - x_j(0)\| \vee \max_{i\in\bigcup_{k\in[m]}L_k\cup F}\|x_i(0) - g\|\right)^2. \end{aligned}$$

Proof. Let $x_i = x_i(t)$, $x_i^\star = x_i(t+1)$, $\beta_i^k = \beta_i^k(t)$, $\mathcal{N}_i^F = \mathcal{N}_i^F(t)$, $\mathcal{N}_i^L = \mathcal{N}_i^L(t)$, $\bar{x}_i^F = \sum_{j\in\mathcal{N}_i^F} x_j/|\mathcal{N}_i^F|$ and $\bar{x}_i^L = \sum_{j\in\mathcal{N}_i^L} x_j/|\mathcal{N}_i^L|$. Observe that

$$\begin{aligned} X_t - X_{t+1} &\ge \sum_{i\in F}\sum_{j\in\mathcal{N}_i^F}(\|x_i - x_j\|^2 - \|x_i^\star - x_j^\star\|^2) \\ &= 2\sum_{i\in F}\sum_{j\in\mathcal{N}_i^F}(< x_i - x_i^\star, x_i^\star - x_j > - < x_i^\star - x_j, x_j - x_j^\star >) \\ &= 2\sum_{i\in F}|\mathcal{N}_i^F| < x_i - x_i^\star, x_i^\star - \bar{x}_i^F > +2\sum_{i\in F}\sum_{j\in\mathcal{N}_i^F} < x_i^\star - x_i, x_j^\star - x_j > \\ &\quad - 2\sum_{i\in F}\sum_{j\in\mathcal{N}_i^F} < x_i - x_j, x_j - x_j^\star > \\ &\ge 2\sum_{i\in F}|\mathcal{N}_i^F| < x_i - x_i^\star, x_i^\star - \bar{x}_i^F > +2\sum_{i\in F}\|x_i^\star - x_i\|^2 \\ &\quad - \sum_{i\in F}\sum_{j\in\mathcal{N}_i^F-\{i\}}(\|x_i^\star - x_i\|^2 + \|x_j^\star - x_j\|^2) - 2\sum_{j\in F}|\mathcal{N}_j^F| < \bar{x}_j^F - x_j, x_j - x_j^\star > \\ &= 4\sum_{i\in F}|\mathcal{N}_i^F| < x_i - x_i^\star, x_i^\star - \bar{x}_i^F > +2\sum_{i\in F}\|x_i^\star - x_i\|^2 \\ &\quad - 2\sum_{i\in F}(|\mathcal{N}_i^F| - 1)\|x_i - x_i^\star\|^2 + 2\sum_{i\in F}|\mathcal{N}_i^F|\|x_i - x_i^\star\|^2 \\ &= 4\sum_{i\in F}\sum_{k\in[m]}\beta_i^k|\mathcal{N}_i^F| < x_i - x_i^\star, \bar{x}_i^{L_k} - \bar{x}_i^F > +4\sum_{i\in F}\|x_i - x_i^\star\|^2 \end{aligned}$$

$$\geq 4\sum_{i\in F}\|x_i - x_i^{\star}\|^2 - 4m|F|^2 \max_{i\in F;k\in[m]} \beta_i^k$$
$$\times\left(\max_{i,j\in\bigcup_{k\in[m]}L_k\cup F}\|x_i(0)-x_j(0)\|\vee\max_{i\in\bigcup_{k\in[m]}L_k\cup F}\|x_i(0)-g\|\right)^2.$$

□

Lemma 4.13. Assume that the social graph and opinion graph are undirected on F. If some component H of the profile $G(t)\cap\mathscr{G}(t)$ on F is δ-nontrivial, then,

$$\sqrt{\sum_{i\in F}\|x_i(t)-x_i(t+1)\|^2} \geq \sqrt{2}\delta(1-\max_{i\in F}\sum_{k\in[m]}\beta_i^k(t))/|F|^4 - 2m|F|\max_{k\in[m];i\in F}\beta_i^k(t)$$
$$\times\left(\max_{i\in\bigcup_{k\in[m]}L_k\cup F}\|x_i(0)\|\vee\max_{k\in[m]}\|g_k\|\right).$$

Proof. Letting $V(H)$, the vertex set of H, be $[h]$ and $\mathbb{1}=(1,\dots,1)'\in\mathbb{R}^h$, express $\mathbb{R}^h = W\oplus W^{\perp}$ for $W=\mathrm{Span}(\{\mathbb{1}\})$. For $x(t)=(x_1(t),\dots,x_h(t))'$, write

$$x(t) = [c_1\mathbb{1}\,|\,c_2\mathbb{1}\,|\,\cdots\,|\,c_d\mathbb{1}] + \left[\hat{c}_1u^{(1)}\,|\,\hat{c}_2u^{(2)}\,|\,\cdots\,|\,\hat{c}_du^{(d)}\right],$$

where c_i and $\hat{c}_i$ are constants and $u^{(i)}\in\mathbb{1}^{\perp}$ is a unit vector for all $i\in[d]$. Observe that

$$\|x_i(t)-x_j(t)\|^2 = \sum_{k\in[d]}\hat{c}_k^2(u_i^{(k)}-u_j^{(k)})^2 \leq 2\sum_{k\in[d]}\hat{c}_k^2((u_i^{(k)})^2+(u_j^{(k)})^2) \leq 2\sum_{k\in[d]}\hat{c}_k^2$$

for all $i,j\in[h]$. Hence,

$$\text{component } H\ \delta\text{-nontrivial implies } \sum_{k\in[d]}\hat{c}_k^2 > \delta^2/2.$$

Letting $\tilde{\beta}(t) = (1-\sum_{k\in[m]}\beta_1^k(t),\dots,1-\sum_{k\in[m]}\beta_h^k(t))'$ and $B(t)=\mathrm{diag}(\tilde{\beta}(t))A(t)$ for $A(t)\in\mathbb{R}^{h\times h}$ with $A_{i,j}(t) = \mathbb{1}\{j\in\mathscr{N}_i^F(t)\}/|\mathscr{N}_i^F(t)|$, we obtain

$$x(t)-x(t+1) = (I-B(t))x(t) - O(t) = [C(t)+F(t)\mathscr{L}(t)]\,x(t) - O(t),$$

where $C(t) = I-\mathrm{diag}(\tilde{\beta}(t))$, $F(t)=\mathrm{diag}(\tilde{\beta}(t))\left(\mathrm{diag}((d_i)_{i=1}^h)+I\right)^{-1}$ with d_i the degree of vertex i in component H, $O(t)=\sum_{k\in[m]}\mathrm{diag}((\beta_i^k(t))_{i\in[h]})(\bar{x}_i^{L_k})'_{i\in[h]}$ and $\mathscr{L}(t)$ is the Laplacian of component H. It follows from Lemmas 3.8 and 3.9 that

$$\lambda_2(\mathscr{L}) > \frac{(2/h)^2}{2h} = 2/h^3,$$

$$\|F(t)\mathscr{L}(t)x(t)\|^2 = \sum_{k\in[d]}\hat{c}_k^2\|F(t)\mathscr{L}(t)u^{(k)}\|^2 \geq \sum_{k\in[d]}\hat{c}_k^2\lambda_2\left(\mathscr{L}(t)F^2(t)\mathscr{L}(t)\right)$$
$$\geq (\delta^2/2)(\min_{i\in[h]}\tilde{\beta}_i(t)/h)^2\lambda_2^2(\mathscr{L}(t)) \geq 2\delta^2(1-\max_{i\in[h]}\sum_{k\in[m]}\beta_i^k)^2/h^8.$$

On the other hand, it follows from the triangle inequality that

$$\begin{aligned}\|C(t)x(t)\| &\le \sum_{i\in[h]}\sum_{k\in[m]}\|\beta_i^k(t)x_i(t)\|\\ &\le mh\max_{i\in[h];k\in[m]}\beta_i^k(t)\left(\max_{i\in\bigcup_{k\in[m]}L_k\cup F}\|x_i(0)\|\vee\max_{k\in[m]}\|g_k\|\right),\\ \|O(t)\| &\le \sum_{i\in[h]}\sum_{k\in[m]}\|\beta_i^k\bar{x}_i^{L_k}\|\\ &\le mh\max_{k\in[m];i\in[h]}\beta_i^k(t)\left(\max_{i\in\bigcup_{k\in[m]}L_k}\|x_i(0)\|\vee\max_{k\in[m]}\|g_k\|\right),\end{aligned}$$

therefore,

$$\begin{aligned}&\sqrt{\sum_{i\in F}\|x_i(t)-x_i(t+1)\|^2}\ge\sqrt{\sum_{i\in[h]}\|x_i(t)-x_i(t+1)\|^2}=\|x(t)-x(t+1)\|\\ &=\|\,[F(t)\mathscr{L}(t)+C(t)]\,x(t)-O(t)\|\ge\|F(t)\mathscr{L}(t)y(t)\|-\|C(t)y(t)\|-\|O(t)\|\\ &\ge\sqrt{2}\delta(1-\max_{i\in[h]}\sum_{k\in[m]}\beta_i^k)/h^4-2mh\max_{k\in[m];i\in[h]}\beta_i^k(t)\\ &\qquad\qquad\times\left(\max_{i\in\bigcup_{k\in[m]}L_k\cup F}\|x_i(0)\|\vee\max_{k\in[m]}\|g_k\|\right)\\ &\ge\sqrt{2}\delta(1-\max_{i\in F}\sum_{k\in[m]}\beta_i^k)/|F|^4-2m|F|\max_{k\in[m];i\in F}\beta_i^k(t)\\ &\qquad\qquad\times\left(\max_{i\in\bigcup_{k\in[m]}L_k\cup F}\|x_i(0)\|\vee\max_{k\in[m]}\|g_k\|\right).\end{aligned}$$

□

Proof of Theorem 2.4. We claim the following:

1) All components of profile $G\cap\mathscr{G}$ on F are δ-trivial after some time for all $\delta>0$.
2) No components of profile $G\cap\mathscr{G}$ on F interact with each other after some time.

Without loss of generality, we assume the social graph on F remains constant over time, saying $G(t)|_F=G|_F=(F,E)$ for all $t\ge 0$. Observe that

$$\sum_{t\ge 0}\max_{i\in F;k\in[m]}\beta_i^k(t)<\infty\implies\lim_{t\to\infty}\beta_i^k(t)=0\text{ for all }i\in F\text{ and }k\in[m].$$

Hence, we derive

$$\begin{aligned}a_t&=\sqrt{2}\delta(1-\max_{i\in F}\sum_{k\in[m]}\beta_i^k(t))/|F|^4\to\sqrt{2}\delta/|F|^4,\\ b_t&=2m|F|\max_{k\in[m];i\in F}\beta_i^k(t)\left(\max_{i\in\bigcup_{k\in[m]}L_k\cup F}\|x_i(0)\|\vee\max_{k\in[m]}\|g_k\|\right)\to 0\text{ as }t\to\infty.\end{aligned}$$

There is $t_0\ge 0$ such that

$$a_t-b_t\ge\delta/|F|^4\text{ for all }t\ge t_0.$$

Assume that asymptotic stability does not hold in (1.2). Then, there are $\delta > 0$ and $(s_k)_{k\geq 0}$ increasing with $s_0 \geq t_0$ and some component in profile $G(t_k) \cap \mathscr{G}(t_k)$ on F δ-nontrivial for all $k \geq 0$. Letting

$$M_0 = 4m|F|^2\Big(\max_{i,j\in\bigcup_{k\in[m]} L_k\cup F} \|x_i(0) - x_j(0)\| \vee \max_{i\in\bigcup_{k\in[m]} L_k\cup F} \|x_i(0) - g\|\Big)^2,$$

it turns out from Lemma 4.12 that

$$\begin{aligned} X_0 + M_0 \sum_{t=0}^{\hat{m}} \max_{i\in F;k\in[m]} \beta_i^k(t) &\geq \sum_{t=0}^{\hat{m}} (X_t - X_{t+1}) + M_0 \sum_{t=0}^{\hat{m}} \max_{i\in F;k\in[m]} \beta_i^k(t) \\ &\geq 4 \sum_{t=0}^{\hat{m}} \sum_{i\in F} \|x_i(t) - x_i(t+1)\|^2 \text{ for all } \hat{m} \geq 0. \end{aligned}$$

As $\hat{m} \to \infty$, we derive

$$\begin{aligned} \infty > W_0 + M_0 \sum_{t\geq 0} \max_{i\in F;k\in[m]} \beta_i^k(t) &\geq 4 \sum_{t\geq 0} \sum_{i\in F} \|x_i(t) - x_i(t+1)\|^2 \\ \geq 4 \sum_{k\geq 0} \sum_{i\in F} \|x_i(s_k) - x_i(s_k+1)\|^2 &\geq 4 \sum_{k\geq 0} \delta^2/|F|^8 = \infty, \text{ a contradiction.} \end{aligned}$$

Hence, all components of profile $G \cap \mathscr{G}$ on F are δ-trivial after some time for all $\delta > 0$.

Next, we claim that no components of profile $G \cap \mathscr{G}$ on F interact with each other after some time. It follows from claim 1) that all components of profile $G \cap \mathscr{G}$ on F are $\epsilon/4$-trivial after some time s_0. Assume that claim 2) is not the case. By finiteness of the social graph, there are edge (i, j) and $(t_k)_{k\geq 0}$ increasing with $t_0 \geq s_0$ such that vertices i and j belong to distinct components of profile $G \cap \mathscr{G}(t_k)$ on F,

$$(i, j) \in E \cap \mathscr{E}(t_k)^c \text{ and } (i, j) \in E \cap \mathscr{E}(t_k + 1).$$

Letting

$$\begin{aligned} \bar{x}_i^F &= \frac{1}{|\mathscr{N}_i^F(t_k)|} \sum_{j\in\mathscr{N}_i^F(t_k)} x_j(t_k), \quad x_i = x_i(t_k), \quad x_i^\star = x_i(t_k + 1), \\ \bar{x}_i^L &= \frac{1}{|\mathscr{N}_i^L(t_k)|} \sum_{j\in\mathscr{N}_i^L(t_k)} x_j(t_k), \quad \tilde{\beta}_i^j = \beta_i^j(t_k), \quad \tilde{\beta}_i = 1 - \sum_{j\in[m]} \beta_i^j(t_k) \end{aligned}$$

for all $i \in F$ and $k \geq 0$, it turns out from the triangle inequality that

$$\epsilon < \|x_i - x_j\| \leq \|x_i - x_i^\star\| + \|x_i^\star - x_j^\star\| + \|x_j^\star - x_j\|.$$

On top of that, we obtain

$$\begin{aligned} \|x_i - x_i^\star\| &\leq \tilde{\beta}_i \|x_i - \bar{x}_i^F\| + \|\sum_{j\in[m]} \beta_i^j (x_i - \bar{x}_i^L)\| \\ &\leq \tilde{\beta}_i \epsilon/4 + m \max_{j\in[m];i\in F} \beta_i^j \\ &\quad \times \Big(\max_{i,j\in\bigcup_{k\in[m]} L_k\cup F} \|x_i(0) - x_j(0)\| \vee \max_{i\in\bigcup_{k\in[m]} L_k\cup F} \|x_i(0) - g\|\Big), \end{aligned}$$

similarly for $\|x_j - x_j^\star\|$, therefore,

$$\liminf_{k\to\infty} \|x_i - x_i^\star\| \le \epsilon/4 \quad \text{and} \quad \liminf_{k\to\infty} \|x_j - x_j^\star\| \le \epsilon/4.$$

This implies

$$\epsilon/2 \le \liminf_{k\to\infty} \|x_i^\star - x_j^\star\|, \text{ a contradiction.}$$

It follows from claims 1) and 2) that $\sum_{j\in\mathcal{N}_i^F(t)} x_j(t)/|\mathcal{N}_j^F(t)|$ converges to some random variable $\tilde{x}_i$ as $t \to \infty$ for all $i \in F$. Since $\max_{k\in[m];i\in F} \beta_i^k(t) \to 0$ as $t \to \infty$ and $\sum_{j\in\mathcal{N}_i^L(t)} x_j(t)/|\mathcal{N}_i^L(t)|$ is bounded by $(\max_{i\in\bigcup_{k\in[m]} L_k} \|x_i(0)\| \vee \max_{k\in[m]} \|g_k\|)$, we get $x_i(t+1) \to \tilde{x}_i$ as $t \to \infty$ for all $i \in F$. □

5. Simulations

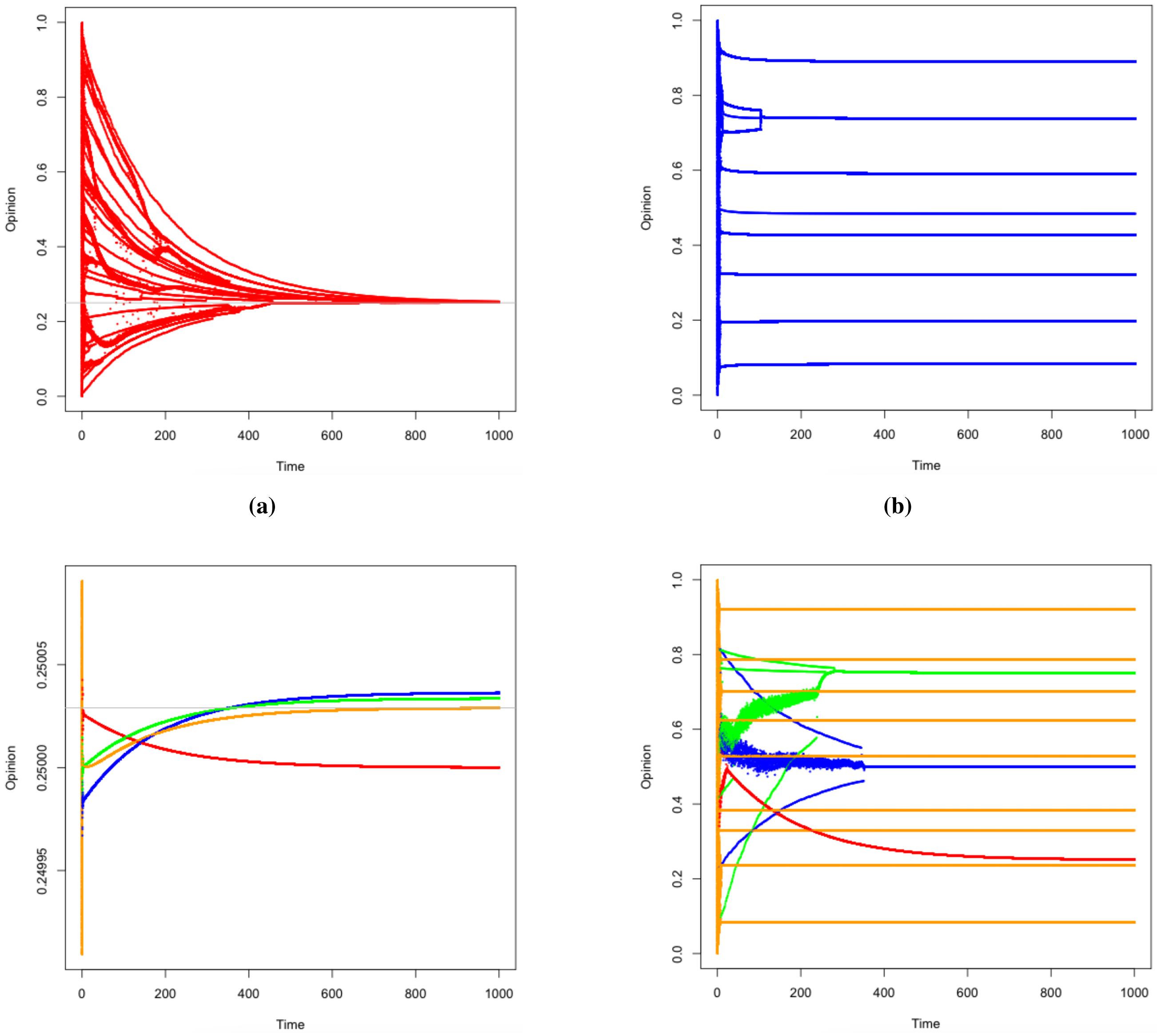
We conduct a numerical analysis of the theorems. For Theorem 2.1, consider the leader group L of size 2000 with a target $g = 0.25$. The initial opinions of all leaders are uniformly distributed random variables with values in $(0, 1)$. The threshold for each leader is a uniformly distributed random variable with values in $(0, 1)$. A directed social graph is randomly generated at each time step. $\alpha_i(t)$ is a uniformly distributed random variable with values in $(0.99, 0.999)$ for all $i \in L$ and $t \ge 0$, indicating that the leaders have little tendency toward their target. The result is shown in Figure 1(a), where all leaders approach the target $g = 0.25$. The gray line indicates the opinion equal to 0.25.

For Theorem 2.2, consider a leader group of size 2000 with a target $g = 0.25$, a common threshold of 0.05, and a constant undirected social graph. The initial opinions of all leaders are uniformly distributed random variables with values in $(0, 1)$. To diversify $\alpha_i(t)$, let $\alpha_i(t) = \left[1 + (t^2 + n(i,t))^{-1}\right]^{-1}$ for all $i \in L$ and $t \ge 0$, where $n(i,t)$ is a uniformly distributed random variable with values in $[2000]$. The outcome is shown in Figure 1(b).

For Theorem 2.3, consider three leader groups of sizes 10, 20, and 30, and a follower group of size 1940. The threshold for each individual is a uniformly distributed random variable with values in $(0, 1)$. Given $g_1 = 0.25$, g_2, g_3, and the initial opinions of all individuals are uniformly distributed random variables with values in $(0.25 - \min_{i\in F} \epsilon_i/2, 0.25 + \min_{i\in F} \epsilon_i/2)$. A directed social graph is randomly generated at each time step. It follows from Theorem 2.3 that all followers achieve a consensus if they have the same propensity toward the average of their social and opinion neighbors in each of the leader groups. Say $\beta_i^1 = 0.01$, $\beta_i^2 = 0.02$, and $\beta_i^3 = 0.03$ for all $i \in F$. $\alpha_i(t)$ is a uniformly distributed random variable with values in $(0.99, 0.999)$ for all $k \in [3]$, $i \in L_k$, and $t \ge 0$. The result is shown in Figure 1(c), where all followers approach the weighted average of the targets of the leader groups. The opinions of the leaders are colored red, blue, or green, with each group sharing the same color. The opinions of the followers are colored orange. The gray line indicates the opinion equal to the weighted average of the targets of the leader groups.

For Theorem 2.4, consider three leader groups of sizes 10, 20, and 30 with targets 0.25, 0.5, and 0.75, respectively, and a follower group of size 1940. The threshold for each leader is a uniformly distributed random variable with values in $(0, 1)$. All followers share a common threshold of 0.05. The opinions of all individuals are uniformly distributed random variables with values in $(0, 1)$. The social graph on F is undirected and constant, while other social relationships can be directed and vary over time. $\alpha_i(t)$ is a uniformly distributed random variable with values in $(0.99, 0.999)$ for all $k \in [3]$, $i \in L_k$ and $t \ge 0$. To diversify $\beta_i^k(t)$, let $\beta_i^k(t) = (t^2 + n(i,k,t))^{-1}$ for all $i \in F$, $k \in [3]$, and $t \ge 0$, where $n(i,k,t)$

is a uniformly distributed random variable with values in 1940. The outcome is shown in Figure 1(d). The opinions of the leaders are colored red, blue, or green, with each group sharing the same color. The opinions of the followers are colored orange.



(a) (b) (c) (d)

Figure 1. Demonstration of the theorems.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author is supported by an NSTC grant.

Conflict of interest

The author declares no competing interests.

References

1. R. Hegselmann, U. Krause, Opinion dynamics and bounded confidence models, analysis, and simulation, *Journal of Artifical Societies and Social Simulation*, **5** (2002), 1–33.
2. J. Lorenz, A stabilization theorem for dynamics of continuous opinions, *Physica A*, **355** (2005), 217–223. https://doi.org/10.1016/j.physa.2005.02.086
3. J. Lorenz, Continuous opinion dynamics under bounded confidence: A survey, *Int. J. Mod. Phys. C*, **18** (2007), 1819–1838. https://doi.org/10.1142/S0129183107011789
4. C. Castellano, S. Fortunato, V. Loreto, Statistical physics of social dynamics, *Rev. Mod. Phys.*, **81** (2009), 591. https://doi.org/10.1103/RevModPhys.81.591
5. A. Bhattacharyya, M. Braverman, B. Chazelle, H. L. Nguyen, On the convergence of the Hegselmann–Krause syste, *The 4th Conference on Innovations in Theoretical Computer Science*, New York, USA, 2013, 61–66. https://doi.org/10.1145/2422436.2422446
6. G. Y. Fu, W. D. Zhang, Z. J. Li, Opinion dynamics of modified Hegselmann–Krause model in a group-based population with heterogeneous bounded confidence, *Physica A*, **419** (2015), 558–565. https://doi.org/10.1016/j.physa.2014.10.045
7. A. V. Proskurnikov, R. Tempo, A tutorial on modeling and analysis of dynamic social networks. Part I, *Annu. Rev. Control*, **43** (2017), 65–79. https://doi.org/10.1016/j.arcontrol.2017.03.002
8. C. Bernardo, F. Vasca, R. Iervolino, Heterogeneous opinion dynamics with confidence thresholds adaptation, *IEEE T. Control Netw.*, **9** (2022), 1068–1079. https://doi.org/10.1109/TCNS.2021.3088790
9. F. Vasca, C. Bernardo, R. Iervolino, Practical consensus in bounded confidence opinion dynamics, *Automatica*, **129** (2021), 109683. https://doi.org/10.1016/j.automatica.2021.109683
10. S. Fortunato, On the consensus threshold for the opinion dynamics of Krause–Hegselmann, *Int. J. Mod. Phys. C*, **16** (2005), 259–270. https://doi.org/10.1142/S0129183105007078
11. N. Lanchier, H.-L. Li, Consensus in the Hegselmann–Krause model, *J. Stat. Phys.*, **187** (2022), 20. https://doi.org/10.1007/s10955-022-02920-8
12. R. Parasnis, M. Franceschetti, B. Touri. Hegselmann–Krause dynamics with limited connectivity, *2018 IEEE Conference on Decision and Control (CDC)*, Miami, FL, USA, 2018, 5364–5369. https://doi.org/10.1109/CDC.2018.8618877
13. H.-L. Li, Mixed Hegselmann–Krause dynamics, *Discrete Cont. Dyn.-B*, **27** (2022), 1149–1162. https://doi.org/10.3934/dcdsb.2021084
14. H.-L. Li, Mixed Hegselmann–Krause dynamics II, *Discrete Cont. Dyn.-B*, **28** (2023), 2981–2993. https://doi.org/10.3934/dcdsb.2022200

15. G. Chen, W. Su, W. J. Mei, F. Bullo, Convergence properties of the heterogeneous Deffuant–Weisbuch model, *Automatica*, **114** (2020), 108825. https://doi.org/10.1016/j.automatica.2020.108825
16. G. Deffuant, D. Neau, F. Amblard, G. Weisbuch, Mixing beliefs among interacting agents, *Adv. Complex Syst.*, **3** (2000), 87–98. https://doi.org/10.1142/S0219525900000078
17. N. Lanchier, The critical value of the Deffuant model equals one half, *ALEA Lat. Am. J. Probab. Math. Stat.*, **9** (2012), 383–402.
18. N. Lanchier, H.-L. Li, Probability of consensus in the multivariate Deffuant model on finite connected graphs, *Electron. Commun. Probab.*, **25** (2020), 1–12. https://doi.org/10.1214/20-ECP359
19. H.-L. Li, Mixed Hegselmann–Krause dynamics on infinite graphs, *J. Stat. Mech-Theory E.*, **2024** (2024), 113404. https://doi.org/10.1088/1742-5468/ad8bb6
20. Y. Y. Zhao, G. Kou, Y. Peng, Y. Chen, Understanding influence power of opinion leaders in e-commerce networks: An opinion dynamics theory perspective, *Inform. Sciences*, **426** (2018), 131–147. https://doi.org/10.1016/j.ins.2017.10.031
21. Q. B. Zha, G. Kou, H. J. Zhang, H. M. Liang, X. Chen, C.-C. Li, Y. C. Dong, Opinion dynamics in finance and business: a literature review and research opportunities, *Financ. Innov.*, **6** (2020), 44. https://doi.org/10.1186/s40854-020-00211-3
22. L. W. Beineke, R. J. Wilson, P. J. Cameron, M. Doob, R. A. Brualdi, B. L. Shader, et al., *Topics in algebraic graph theory*, Cambridge: Cambridge University Press, 2004.