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Research article

On the sum of matrices of special linear group over finite field

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Abstract: Let $\mathbb{F}_q$ be a finite field of q elements. For $n \in \mathbb{N}^*$ with $n \geq 2$, let $M_n := Mat_n(\mathbb{F}_q)$ be the ring of matrices of order n over $\mathbb{F}_q$, $G_{n,1} := Sl_n(\mathbb{F}_q)$ be the special linear group over $\mathbb{F}_q$. In this paper, by using the technique of Fourier transformation, we obtain a formula for the number of representations of any element of M_n as the sum of k matrices in $G_{n,1}$. As a corollary, we give another proof of the number of the third power moment of the classic Kloosterman sum.

Keywords: ring of matrices; finite field; sum of matrices; Fourier transformation; Kloosterman sum
Mathematics Subject Classification: 11B13, 11C20, 11T24

1. Introduction

Let R be a finite ring with $1 \in R$, and let R^* denote the multiplicative group of units in R. Let k be an integer with $k \geq 2$ and let $\sharp S$ denote the cardinality of any finite set S. For any $c \in R$, we define

$$S_k(R, c) := \left\{ (x_1, x_2, \ldots, x_k) \in (R^*)^k \;\middle|\; \sum_{i=1}^{k} x_i = c \right\},$$

and

$$N_k(R, c) := \sharp S_k(R, c).$$

For a positive integer n, let $\mathbb{Z}/n\mathbb{Z}$ be the ring of residue classes modulo n. In 2000, Deaconescu [3] obtained a formula for $N_2(\mathbb{Z}/n\mathbb{Z}, c)$. In 2009, Sander [13] gave a generalization of the above result. In fact, for any integer c, he determined the number of representations of c as a sum of two units, two nonunits, a unit and a nonunit, respectively, in $\mathbb{Z}/n\mathbb{Z}$.

For a positive integer n with divisors $k_1, k_2, ..., k_t (t \geqslant 2)$ and $c \in \mathbb{Z}$, let

$$S_{n;k_1,k_2,\ldots,k_t}(c) := \left\{ (x_1, x_2, \ldots, x_t) \;\middle|\; \begin{array}{l} 1 \leq x_i \leq n/k_i, (x_i, n/k_i) = 1,\ i = 1, 2, \ldots, t, \\ \sum_{i=1}^{t} k_i x_i \equiv c \pmod n \end{array} \right\}.$$

We define $N_{n;k_1,k_2,\ldots,k_t}(c) := \sharp S_{n;k_1,k_2,\ldots,k_t}(c)$.

In 2013, Sander and Sander [14] gave a formula for $N_{n;k_1,k_2}(c)$. In 2014, Sun and Yang [15] obtained a formula for $N_{n;k_1,k_2,\ldots,k_t}(c)$.

In 2017, Ji and Zhang [17] extended Sander's results to the residue ring of a Dedekind ring.

For a finite ring R with identity 1, a unit $u \in R^*$ is called an exunit if $1-u \in R^*$. We write R^{**} for the set of all exunits of R. We define $N'_k(R,c)$ to be the number of representations of any $c \in R$ as a sum of k exunits of R. Namely,

$$N'_k(R,c) := \sharp\left\{(x_1,x_2,\ldots,x_k) \in (R^{**})^k \;\middle|\; \sum_{i=1}^{k} x_i = c\right\}.$$

In 2017, Yang and Zhao [16] gave an explicit formula for $N'_k(R,c)$ with $R = \mathbb{Z}/n\mathbb{Z}$. In 2018, Miguel [11] generalized the Yang-Zhao results to any finite commutative ring R with identity.

In this paper, we shall extend the above results to the ring of matrices over a finite field $\mathbb{F}_q$ of q elements. The theory of matrices used in this paper can be found in [5]. What we focus on is the number of representations of a matrix as a sum of k matrices. Readers who are interested in algorithms can refer to [12]. The theory of matrices also has many applications in other fields, such as graph theory, for example, [1,6,8].

Let $M_n := \mathrm{Mat}_n(\mathbb{F}_q)$, $G_n := \mathrm{Gl}_n(\mathbb{F}_q)$, i.e., the general linear group over $\mathbb{F}_q$. For any $u \in \mathbb{F}_q^*$ and $0 \le r \le n$, define

$$G_{n,u} := \left\{x \in M_n \;\middle|\; \det(x) = u\right\},\ M_{n,r} := \left\{x \in M_n \;\middle|\; \mathrm{rank}(x) = r\right\}.$$

Specifically, $G_n = M_{n,n}$, $G_{n,1} = \mathrm{Sl}_n(\mathbb{F}_q)$, i.e., the special linear group over $\mathbb{F}_q$. For any matrix $A \in M_n$ and $k \in \mathbb{N}^*$, we define

$$\mathcal{S}_{n,k}(A) := \left\{(x_1,x_2,\ldots,x_k) \in G_{n,1}^k \;\middle|\; \sum_{i=1}^{k} x_i = A\right\},$$

and

$$\mathcal{N}_{n,k}(A) = \sharp\mathcal{S}_{n,k}(A).$$

Let a, b be non-negative integers with $a \ge b$. The q-binomial coefficient is defined as:

$$\binom{a}{b}_q = \frac{(a)_q!}{(b)_q!(a-b)_q!},$$

where $(0)_q! = 1, (a)_q = \frac{q^a-1}{q-1}$ and $(a)_q! = (1)_q(2)_q\cdots(a)_q$ when $a \geqslant 1$. Let ψ be a fixed nontrivial additive character of $\mathbb{F}_q$, e.g., take

$$\psi(x) = \exp\left(\frac{2\pi i}{p}\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(x)\right),\ \ \forall\ x \in \mathbb{F}_q.$$

Define

$$K_n(\psi,y) := \sum_{x_1x_2\cdots x_n=y} \psi(x_1+x_2+\cdots+x_n),\ \text{for } y \in \mathbb{F}_q^*$$

be the Kloosterman sum over $\mathbb{F}_q$. Our first main result is:

Theorem 1.1. *Let $k \in \mathbb{N}^*$ and $A \in M_{n,r}$ with determinant u. Then, we have*

$$\mathcal{N}_{n,k}(A) = \frac{q^{k\binom{n}{2}}}{\sharp M_n}\left(\sum_{v\in F^*} q^{\binom{n}{2}} K_n(\psi, v)^k K_n(\psi, uv) + \frac{1}{(q-1)^k}\sum_{l=0}^{n-1}(-1)^{l(k+1)} q^{\binom{l}{2}} \binom{n}{l}_q \prod_{i=1}^{n-l}(q^i - 1)^k\right),$$

if $u \neq 0$ and

$$\mathcal{N}_{n,k}(A) = \frac{q^{k\binom{n}{2}}}{\sharp M_n}\left(\frac{(-1)^r q^{\binom{n}{2}}}{q-1}\prod_{i=1}^{n-r}(q^i - 1)\cdot\sum_{v\in F^*} K_n(\psi, v)^k \right.$$
$$\left. + \sum_{l=0}^{n-1}\frac{(-1)^{kl}}{(q-1)^k}\prod_{i=1}^{n-l}(q^i-1)^k \cdot \left(\sum_{i=\max\{0,l-n+r\}}^{\min\{r,l\}} (-1)^i q^{\binom{i}{2}+r(l-i)}\binom{r}{i}_q \cdot \sharp M_{n-r,l-i}\right)\right),$$

if $u = 0$.

Let k be a positive integer and q be an odd prime power. Define

$$V(k) := \sum_{u\in\mathbb{F}_q^*} K_2(\psi, u)^k$$

to be the k-th power moment of the classic Kloosterman sum. Let $\eta(-)$ be the Legendre symbol over $\mathbb{F}_q$. We also give another proof of the number of $V(3)$ (see [7], Section 4.4):

Theorem 1.2. $V(3) = \eta\left(\frac{-3}{q}\right) q^2 + 2q + 1$.

This paper is organized as follows: In Section 2, we shall prove some lemmas that will be used in the proofs of our main results. In Sections 3 and 4, we shall give the proofs of Theorem 1.1 and Theorem 1.2, respectively.

2. Preliminaries

Lemma 2.1. *Let $A, B \in M_n$ and $k \in \mathbb{N}^*$. If there exist $C, D \in M_n$ such that $B = CAD$ and* $\det(C)\cdot\det(D) = 1$, *then, we have $\mathcal{N}_{n,k}(A) = \mathcal{N}_{n,k}(B)$.*

Consider the map

$$f : \mathcal{S}_{n,k}(A) \to \mathcal{S}_{n,k}(B),$$
$$(x_1, x_2, \ldots, x_k) \mapsto (Cx_1D, Cx_2, D, \ldots, Cx_kD).$$

Clearly, f is bijective. So we have $\mathcal{N}_{n,k}(A) = \mathcal{N}_{n,k}(B)$.

Corollary 2.2. *Let $k \in \mathbb{N}^*$ and $A, B \in M_n$ with* $\det(A) = \det(B) = u \neq 0$. *Then, we have $\mathcal{N}_{n,k}(A) = \mathcal{N}_{n,k}(B)$.*

The following two results (Lemma 2.3 and Theorem 2.4) are well-known.

Lemma 2.3. [9] *For any $u \in \mathbb{F}_q^*$ and $1 \le r < n$, we have*

$$\sharp G_n = \prod_{i=0}^{n-1}(q^n - q^i),\ \sharp G_{n,u} = \frac{\sharp G_n}{q-1},\ \sharp M_{n,r} = \prod_{i=0}^{r-1}\frac{(q^n-q^i)^2}{q^r-q^i}.$$

Theorem 2.4. *Let $A \in M_n$. Then there exist $P, Q \in G_{n,1}(\mathbb{F}_q)$ such that*

$$PAQ = \mathrm{diag}(d_1, d_2, \ldots, d_r, 0, \ldots, 0),$$

where $r = \mathrm{rank}(A)$.

Lemma 2.5. *Let $A, B \in M_{n,r}$ with $r < n$. Then, there exist $P, Q \in G_{n,1}$, such that $PAQ = B$. Hence, we have $\mathcal{N}_{n,k}(A) = \mathcal{N}_{n,k}(B)$ for any $k \in \mathbb{N}^*$.*

By Theorem 2.4, there exist $P_1, Q_1, P_2, Q_2 \in \mathrm{Sl}_n(\mathbb{F}_q)$, such that

$$P_1AQ_1 = A' := \mathrm{diag}(d_1, d_2, \ldots, d_r, 0, \ldots, 0),$$

$$P_2BQ_2 = B' := \mathrm{diag}(e_1, e_2, \ldots, e_r, 0, \ldots, 0),$$

where $e_i, d_i \neq 0, i = 1, \ldots, r$. Set

$$C = \mathrm{diag}(e_1d_1^{-1}, e_2d_2^{-1}, \ldots, e_rd_r^{-1}, \prod_{i=1}^{r} d_ie_i^{-1}, 1, \ldots, 1).$$

Then, $C \in G_{n,1}$, $A'C = B'$. Let $P = P_2^{-1}P_1$, $Q = Q_1CQ_2^{-1}$. Then $P, Q \in G_{n,1}$ and $PAQ = B$. Then, by Lemma 2.1, we have $\mathcal{N}_{n,k}(A) = \mathcal{N}_{n,k}(B)$.

Next, we consider the Gauss sum over some matrix groups. Let $\mathbb{S}$ be a subset of M_n and let ψ be a fixed nontrivial additive character of $\mathbb{F}_q$. For any $A \in M_n$, define

$$G_{\mathbb{S}}(\psi, A) := \sum_{x \in \mathbb{S}} \psi(tr(xA)).$$

If there exist $P, Q \in G_n$ such that $A = PBQ$, then, for any $r \leq n$, we have

$$\begin{aligned} G_{M_{n,r}}(\psi, A) &= \sum_{x \in M_{n,r}} \psi(tr(xPBQ)) \\ &= \sum_{x \in M_{n,r}} \psi(tr(QxPB)) \\ &= \sum_{y \in M_{n,r}} \psi(tr(yB)) \\ &= G_{M_{n,r}}(\psi, B). \end{aligned}$$

Similarly, if there exist $P, Q \in G_{n,1}$ such that $A = PBQ$, then we have $G_{G_{n,1}}(\psi, A) = G_{G_{n,1}}(\psi, B)$.

Theorem 2.6. ([10], Theorem 2.4) *Let $A \in M_{n,r}$ with $\det(A) = u$. Then, we have*

$$G_{G_{n,1}}(\psi, A) = \begin{cases} q^{\binom{n}{2}} K_n(\psi, u), & \text{if } r = n, \\ \frac{(-1)^r}{q-1} q^{\binom{n}{2}} \prod_{i=1}^{n-r} (q^i - 1), & \text{if } r < n. \end{cases}$$

Theorem 2.7. ([2], Theorem 1.1) *Let $A \in M_{n,r}$ and $0 \leq s \leq n$. Then, we have*

$$G_{M_{n,s}}(\psi, A) = \sum_{i=max\{0, s-n+r\}}^{min\{r,s\}} (-1)^i q^{\binom{i}{2} + r(s-i)} \binom{r}{i}_q \cdot \sharp M_{n-r, s-i}.$$

In particular, if $r = n$, then, we have

$$G_{M_{n,s}}(\psi, A) = (-1)^s q^{\binom{s}{2}} \binom{n}{s}_q.$$

Next, we recall some facts on the Fourier transformation. Let H be a finite abelian group, and let $\widehat{H} =: \mathrm{Hom}(H, \mathbb{C}^*)$ be the character group of H. Clearly, $H \cong \widehat{H}$. For any function $f : H \to \mathbb{C}$, the function

$$\widehat{f} : \widehat{H} \to \mathbb{C},\ \chi \mapsto \sum_{x \in H} f(x)\overline{\chi(x)}, \quad \forall\, \chi \in \widehat{H}$$

is called the Fourier transformation of f.

Lemma 2.8. ([4], Proposition 2.1.1.2) *Let $\widehat{f}$ be the Fourier transformation of $f : H \to \mathbb{C}$. Then, we have*

$$f = \frac{1}{\sharp \widehat{H}} \sum_{\chi \in \widehat{H}} \widehat{f}(\overline{\chi})\chi.$$

Now, we consider the case $n = k = 2$ and q is odd.

Lemma 2.9. *Let $A \in M_2$ with $\det(A) = u$ and O be the zero matrix of M_2. Assume q is odd. Then, we have*

$$\mathcal{N}_{2,2}(A) = \begin{cases} q(q-1)(q+1), & \text{if } A = O, \\ q(q-1), & \text{if } u = 0,\ A \neq O, \\ q\left(q + \eta\left(\frac{u^2-4u}{q}\right)\right), & \text{if } u \neq 0. \end{cases}$$

Consider the equation

$$x_1 + x_2 = A, \quad x_1, x_2 \in G_{2,1}, A \in M_2.$$

Case 1. $A = O$. For any $x_1 \in G_{2,1}$, $O - x_1 = x_1 \in G_{2,1}$. By Lemma 2.3, we have

$$\mathcal{N}_{2,2}(O) = \sharp G_{2,1} = q(q-1)(q+1).$$

Case 2. $u = 0$, $A \neq O$. By Lemma 2.5, it is sufficient to compute $\mathcal{N}_{2,2}(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix})$. Let $x_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then, $x_2 = \begin{bmatrix} 1-a & -b \\ -c & -d \end{bmatrix}$. We have $\det(x_1) = 1, \det(x_2) = 1$, i.e.,

$$ad - bc = 1, ad - d - bc = 1.$$

So, we have $d = 0$. For any $a \in \mathbb{F}_q$ and $b \in \mathbb{F}_q^*$, then c is uniquely determined by b. Hence, $\mathcal{N}_{2,2}(A) = q(q-1)$.

Case 3. $u \neq 0$. By Corollary 2.2, it is sufficient to compute $\mathcal{N}_{2,2}(\begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix})$. Let $x_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then, $x_2 = \begin{bmatrix} u-a & -b \\ -c & 1-d \end{bmatrix}$. We have

$$ad - bc = 1,\ u - ud - a + ad - bc = 1.$$

So,

$$a = u - ud,\ bc = ad - 1 = -ud^2 + ud - 1.$$

Let us consider the equation about d:

$$-ud^2 + ud - 1 = 0. \tag{2.1}$$

The determinant of Eq (2.1) is $u^2 - 4u$.

Assume $\eta(\frac{u^2-4u}{q}) = 0$, i.e., $u^2 - 4u = 0$, $u = 4$. Then, Eq (2.1) has only one solution $\frac{1}{2} \in \mathbb{F}_q$. If $d = \frac{1}{2}$, there are $2q - 1$ such pairs (b, c). If $d \neq \frac{1}{2}$, for any $b \in \mathbb{F}_q^*$, c is uniquely determined by b. So,

$$\mathcal{N}_{2,2}(A) = 2q - 1 + (q - 1)(q - 1) = q^2.$$

Assume $\eta(\frac{u^2-4u}{q}) = 1$, i.e., Eq (2.1) has two solutions $d_1, d_2 \in \mathbb{F}_q$. If $d = d_1, d_2$, there are $2q - 1$ pairs (b, c). If $d \neq d_1, d_2$, there are $q - 1$ pairs (b, c). So,

$$\mathcal{N}_{2,2}(A) = 2(2q - 1) + (q - 2)(q - 1) = q(q + 1).$$

Assume $\eta(\frac{u^2-4u}{q}) = -1$, i.e., Eq (2.1) has no solutions. It is obvious that

$$\mathcal{N}_{2,2}(A) = q(q - 1).$$

3. Proof of Theorem 1.1

Let S be a finite set. For any map $f : S \to M_n$ and $x \in M_n$, we define

$$P_f(x) := \frac{\sharp f^{-1}(x)}{\sharp S},$$

where $f^{-1}(x)$ is the set of all the inverse images of x. Let $\widehat{M_n} := \mathrm{Hom}(M_n, \mathbb{C}^*)$ be the additive character group of M_n. Then, we have

$$\widehat{P}_f(\chi) = \sum_{x \in M_n} P_f(x)\overline{\chi(x)} = \frac{1}{\sharp S} \sum_{s \in S} \overline{\chi(f(s))}, \quad \chi \in \widehat{M_n}. \tag{3.1}$$

By Lemma 2.8, we have

$$P_f(x) = \frac{1}{\sharp \widehat{M_n}} \sum_{\chi \in \widehat{M_n}} \widehat{P}_f(\overline{\chi})\chi(x). \tag{3.2}$$

Fix $k \geq 1$. Let $\phi : G_{n,1} \to M_n$ be the inclusion map, and

$$\varphi : G_{n,1}^k \to M_n, (x_1, x_2, \ldots, x_k) \mapsto x_1 + x_2 + \cdots + x_k.$$

Clearly,

$$\mathcal{N}_{n,k}(A) = (\sharp G_{n,1})^k \cdot P_\varphi(A), \quad \forall\, A \in M_n. \tag{3.3}$$

By Eq (3.1), for all $\chi \in \widehat{M_n}$, we have

$$\widehat{P}_\varphi(\chi) = \frac{1}{(\sharp G_{n,1})^k} \sum_{(x_1, x_2, \ldots, x_k) \in G_{n,1}^k} \overline{\chi(x_1 + x_2 + \cdots + x_k)}$$

$$
\begin{aligned}
&= \frac{1}{(\sharp G_{n,1})^k} \sum_{(x_1,x_2,\ldots,x_k)\in G_{n,1}^k} \overline{\chi(x_1)}\cdot\overline{\chi(x_2)}\cdots\overline{\chi(x_k)} \\
&= \left(\frac{1}{(\sharp G_{n,1})} \sum_{x_1\in G_{n,1}} \overline{\chi(x_1)}\right)^k \\
&= \widehat{P_\phi}(\chi)^k.
\end{aligned}
$$

Next, we consider $\widehat{P_\phi}(\chi)$. Let ψ be the canonical additive character of $\mathbb{F}_q$. Then the map

$$\langle \_, \_\rangle : M_n \times M_n \to \mathbb{F}_q \to \mathbb{C}^*, (x_1, x_2) \mapsto tr(x_1 x_2) \mapsto \psi(tr(x_1 x_2))$$

is a non-degenerated symmetric bilinear map. Hence, $\langle \_, \_\rangle$ induces an group isomorphism:

$$\rho : M_n \to \widehat{M_n}, y \mapsto \chi_y := \langle \_, y\rangle .$$

So, we have

$$\widehat{P_\phi}(\overline{\chi_y}) = \frac{1}{\sharp G_{n,1}} \sum_{x\in G_{n,1}} \overline{\overline{\chi_y}(x)} = \frac{1}{\sharp G_{n,1}} \sum_{x\in G_{n,1}} \chi_y(x) = \frac{1}{\sharp G_{n,1}} G_{G_{n,1}}(\psi, y).$$

Define

$$
\begin{aligned}
I_u &:= \mathrm{diag}(u, 1, \ldots, 1) \in G_{n,u}, \quad u \in \mathbb{F}_q^*, \\
J_l &:= \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0) \in M_{n,l}, \quad 0 \le l < n.
\end{aligned}
$$

By Theorem 2.6, we have

$$G_{G_{n,v}}(\psi, I_u) = \sum_{y\in G_{n,v}} \psi(tr(yI_u)) = \sum_{x\in G_{n,1}} \psi(tr(xI_v I_u)) = G_{G_{n,1}}(\psi, I_{uv}) = q^{\binom{n}{2}} K_n(\psi, uv), \tag{3.4}$$

$$G_{G_{n,u}}(\psi, J_r) = G_{G_{n,1}}(\psi, I_u J_r) = \frac{(-1)^r}{q-1} q^{\binom{n}{2}} \prod_{i=1}^{n-r} (q^i - 1). \tag{3.5}$$

By Corollary 2.2 and Lemma 2.5, it is sufficient to consider A as one of the I_u and J_r with $u \in \mathbb{F}_q^*$, $r < n$. Let $A = I_u$ or J_r. Then, we have

$$
\begin{aligned}
&\mathcal{N}_{n,k}(A)\\
&\xlongequal{(3.3)} (\sharp G_{n,1})^k \cdot P_{\varphi}(A)\\
&\xlongequal{(3.2)} \tfrac{(\sharp G_{n,1})^k}{\sharp M_n} \cdot \sum_{\chi\in\widehat{M_n}} \widehat{P}_{\varphi}(\overline{\chi})\chi(A)\\
&= \tfrac{(\sharp G_{n,1})^k}{\sharp M_n} \cdot \left(\sum_{v\in\mathbb{F}_q^*} \sum_{x\in G_{n,v}} \widehat{P}_{\phi}(\overline{\chi_x})^k \chi_x(A) + \sum_{l=0}^{n-1} \sum_{x\in M_{n,l}} \widehat{P}_{\phi}(\overline{\chi_x})^k \chi_x(A) \right)\\
&= \tfrac{(\sharp G_{n,1})^k}{\sharp M_n} \cdot \left(\sum_{v\in\mathbb{F}_q^*} \sum_{x\in G_{n,v}} \left(\tfrac{1}{\sharp G_{n,1}} G_{G_{n,1}}(\psi, x)\right)^k \chi_x(A) \right.\\
&\qquad\qquad \left. + \sum_{l=0}^{n-1} \sum_{x\in M_{n,l}} \left(\tfrac{1}{\sharp G_{n,1}} G_{G_{n,1}}(\psi, x)\right)^k \chi_x(A) \right)\\
&= \tfrac{1}{\sharp M_n} \cdot \left(\sum_{v\in\mathbb{F}_q^*} \left(G_{G_{n,1}}(\psi, I_v)\right)^k \sum_{x\in G_{n,v}} \chi_x(A) + \sum_{l=0}^{n-1} G_{G_{n,1}}(\psi, J_l)^k \sum_{x\in M_{n,l}} \chi_x(A) \right)\\
&= \tfrac{1}{\sharp M_n} \cdot \left(\sum_{v\in\mathbb{F}_q^*} \left(G_{G_{n,1}}(\psi, I_v)\right)^k G_{G_{n,v}}(\psi, A) + \sum_{l=0}^{n-1} G_{G_{n,1}}(\psi, J_l)^k G_{M_{n,l}}(\psi, A) \right)\\
&\xlongequal{\text{Theorem 2.6}} \tfrac{1}{\sharp M_n} \cdot \left(\sum_{v\in\mathbb{F}_q^*} \left(q^{\binom{n}{2}} K_n(\psi, v)\right)^k G_{G_{n,v}}(\psi, A) \right.\\
&\qquad\qquad \left. + \sum_{l=0}^{n-1} \left(\tfrac{(-1)^l q^{\binom{n}{2}}}{q-1} \prod_{i=1}^{n-l} (q^i - 1) \right)^k G_{M_{n,l}}(\psi, A) \right)\\
&\xlongequal[(3.4)(3.5)]{\text{Theorem 2.7}} \begin{cases}
\tfrac{q^{k\binom{n}{2}}}{\sharp M_n} \left(\sum_{v\in\mathbb{F}_q^*} q^{\binom{n}{2}} K_n(\psi, v)^k K_n(\psi, uv) \right. \\
\qquad \left. + \tfrac{1}{(q-1)^k} \sum_{l=0}^{n-1} (-1)^{l(k+1)} q^{\binom{l}{2}} \binom{n}{l}_q \prod_{i=1}^{n-l} (q^i - 1)^k \right), & \text{if } u \neq 0,\\
\tfrac{q^{k\binom{n}{2}}}{\sharp M_n} \left(\tfrac{(-1)^r q^{\binom{n}{2}}}{q-1} \prod_{i=1}^{n-r} (q^i - 1) \cdot \sum_{v\in\mathbb{F}_q^*} K_n(\psi, v)^k + \sum_{l=0}^{n-1} \tfrac{(-1)^{kl}}{(q-1)^k} \right.\\
\quad \left. \times \prod_{i=1}^{n-l} (q^i - 1)^k \left(\sum_{i=\max\{0,l-n+r\}}^{\min\{r,l\}} (-1)^i q^{\binom{i}{2}+r(l-i)} \binom{r}{i}_q \sharp M_{n-r,l-i} \right) \right), & \text{if } u = 0.
\end{cases}
\end{aligned}
$$

□

4. Proof of Theorem 1.2

Let O be the zero matrix of M_2 and I be the identity matrix of M_2. By Theorem 1.1, we have

$$
\mathcal{N}_{2,k}(O) = \frac{q^k}{\sharp M_2} \left((q^2 - 1)^k + (-1)^k (q-1)(q+1)^2 + q(q-1)(q+1) \sum_{u\in\mathbb{F}_q^*} K_2(\psi, u)^k \right),
$$

i.e.,

$$
V(k) = \sum_{u\in\mathbb{F}_q^*} K_2(\psi, u)^k
$$

$$
\begin{aligned}
&= \frac{1}{q(q-1)(q+1)}\left(\mathcal{N}_{2,k}(O)\cdot\frac{\sharp M_2}{q^k} - (q^2-1)^k - (-1)^k(q-1)(q+1)^2\right)\\
&= \frac{1}{q(q-1)(q+1)}\left(q^{4-k}\mathcal{N}_{2,k}(O) - (q^2-1)^k - (-1)^k(q-1)(q+1)^2\right).
\end{aligned}
$$

Consider the equation

$$x_1 + x_2 + \cdots + x_{k-1} = O - x_k, \quad x_i \in G_{2,1}, i = 1, \ldots, k.$$

For any $x_k \in G_{2,1}$, we have $O - x_k \in G_{2,1}$. So, we have

$$\mathcal{N}_{2,k+1}(O) = \sharp G_{2,1}\cdot\mathcal{N}_{2,k}(I).$$

By Lemmas 2.9 and 2.3, we have

$$\mathcal{N}_{2,3}(O) = \sharp G_{2,1}\cdot\mathcal{N}_{2,2}(I) = q^2(q-1)(q+1)\left(q + \eta\left(\frac{-3}{q}\right)\right).$$

Hence, we obtain

$$
\begin{aligned}
V(3) &= \frac{q^3(q-1)(q+1)\left(q+\eta\left(\frac{-3}{q}\right)\right) - (q^2-1)^3 - (-1)^3(q-1)(q+1)^2}{q(q-1)(q+1)}\\
&= \eta\left(\frac{-3}{q}\right)q^2 + 2q + 1.
\end{aligned}
$$

Author contributions

Yifan Luo: Methodology, Conceptualization, Writing-original draft preparation, Supervision, Writing, Formal analysis, Resources; Qingzhong Ji: Supervision, Conceptualization, Validation, Reviewing, Editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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