



Research article

A new class of generalized Apostol–type Frobenius–Euler polynomials

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Abstract: The paper presents a new type of generalized Apostol-type Frobenius–Euler polynomials and numbers with specific order κ and level m . We establish fundamental identities and properties using generating function techniques, such as summation formulas, differential and integral relations, and addition theorems. Additionally, we explore the connections between these polynomials and the Stirling numbers of the second kind, as well as other polynomial families. Lastly, we derive a differential equation and a recurrence relation for these new classes of polynomials. Finally, we show applications that can be obtained using these polynomials where the graphs of the zero functions and the meshes are displayed.

Keywords: the generalized Apostol Frobenius–Euler polynomials; the generalized Apostol–Euler polynomials; differential equations; recurrence relations

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1. Introduction

For $\kappa \in \mathbb{N}_0, u \in \mathbb{C}$, with $u \neq 1$, the Frobenius–Euler polynomials $\mathcal{H}_r^{(\kappa)}(x; u)$ of order κ , are defined by the generating function as follows (by (see e.g., [4, 14] and the references therein):

$$\left(\frac{1-u}{e^z-u}\right)^\kappa e^{xz} = \sum_{r=0}^\infty \mathcal{H}_r^{(\kappa)}(x; u) \frac{z^r}{r!}, \quad |z| < |\log(u)|.$$

When $\kappa = 1$, $\mathcal{H}_r^{(1)}(x; u) := \mathcal{H}_r(x; u)$ is called the r -th Frobenius–Euler polynomials. In the special case where $\kappa = 1$ and $u = -1$, $\mathcal{H}_r(x; -1) := E_r(x)$ denotes Euler polynomials (see [17] and the references therein).

A remarkable amount of research has been done on polynomial families and their various extensions (see, for example, [6, 8, 18, 21, 22]). The introduction of new generalizations has been accompanied by the establishment of several fundamental properties, recurrence formulas, differential equations, and relationships between these polynomials and other families of numbers and polynomials.

The polynomials ${}_F\mathcal{E}_r^{[m-1, \kappa]}(x; u; \gamma)$ represent an intriguing blend of two classes of special functions: the hypergeometric Euler polynomials and the Frobenius–Euler polynomials. The notable presence of these polynomial families across diverse fields, including physical mathematics, information theory, combinatorics, approximation theory, number theory, numerical analysis, and partial differential equations, is well-documented and widely recognized. The significance of investigating the properties of the polynomials ${}_F\mathcal{E}_r^{[m-1, \kappa]}(x; u; \gamma)$ lies in their fundamental role in mathematical analysis, number theory, and applied sciences. These polynomials generalize various well-known polynomial families, providing deeper insights into their structural properties, recurrence relations, and differential equations. Here are some key aspects of their significance:

Unification and generalization. The study connects the generalized Apostol-type Frobenius–Euler polynomials with well-established families, such as the Stirling numbers, Apostol–Bernoulli, Apostol–Euler, Apostol–Genocchi polynomials, and classical polynomials like Fubini, Bernstein, and Hermite polynomials. This broader framework allows for new interpretations and extensions of known results.

Structural properties and recurrence relations. Establishing recurrence relations and differential equations associated with these polynomials is crucial for their computational efficiency and theoretical applications. These relations provide a foundation for deriving explicit formulas and generating functions, which are instrumental in various mathematical fields.

Connection with other special functions and polynomials. The exploration of connection formulas highlights the interplay between different polynomial families. Such connections often lead to new combinatorial identities, algebraic properties, and functional equations, which can be applied in diverse mathematical and engineering problems.

Zero distribution and graphical applications. The graphical representation and analysis of the zeros of these polynomials offer insights into their asymptotic behavior and structural patterns. Understanding the distribution of zeros is crucial in approximation theory, spectral analysis, and numerical methods.

Applications in mathematical and applied fields. The study of these polynomials can have implications in combinatorial mathematics, quantum physics, and numerical analysis. Their recurrence relations and differential properties may lead to new results in solving differential equations, signal processing, and optimization problems.

Thus, the investigation of these polynomials enriches the theory of special functions, providing new tools and results that contribute to both theoretical mathematics and practical applications.

The objective of this study is to investigate a number of properties of the polynomials ${}_F\mathcal{E}_r^{[m-1, \kappa]}(x; u; \gamma)$, including their connection formulas and the distribution of their zeros. The organization of this paper is as follows: In Section 2, we provide definitions and review previous results concerning Stirling numbers of the second kind, as well as generalized Apostol–Bernoulli, Apostol–Euler, Apostol–Genocchi polynomials of level m , and the Fubini, Bernstein, and Hermite

polynomials. Section 3 delves into several properties of the generalized Apostol-type Frobenius–Euler polynomials, their recurrence relations, and differential equations. Section 4 presents various formulas establishing connections with other families of numbers and polynomials. Finally, Section 5 presents the graphical application for the polynomials so that the zeros of the function and how the nets are formed for different n can be studied.

2. Background and previous results

Throughout this paper, adherence is observed to the following standard conventions: The set of natural numbers is denoted by $\mathbb{N} = 1, 2, 3, \dots$. The set of non-negative integers is denoted by $\mathbb{N}_0 = 0, 1, 2, \dots$ represents the set of non-negative integers; \mathbb{Z} refers to the set of integers; \mathbb{R} denotes the set of real numbers, and \mathbb{C} stands for the set of complex numbers. When referring to the complex logarithm, we adopt the principal branch, and for expressions of the form $w = z^k$, we denote the single branch of the multi-valued function $w = z^k$ such that $1^k = 1$.

The Stirling numbers $S(r, s)$ of the second kind are defined by (see [19, p. 78]):

$$\begin{aligned} \frac{(e^z - 1)^s}{s!} &= \sum_{r=s}^{\infty} S(r, s) \frac{z^r}{r!}, \\ z^r &= \sum_{s=0}^r S(r, s) z(z-1) \cdots (z-s+1), \end{aligned} \quad (2.1)$$

so that

$$S(r, 1) = S(r, r) = 1, \quad S(r, r-1) = \binom{r}{2}.$$

On the other hand, Açıkgöz et al. [2, Eq (2.12)] defined the generalized (p, q) –Stirling numbers $S_{p,q}^{[m-1]}(r; k; \gamma)$ of the second kind by means of the generating function

$$\frac{\left(\gamma e_{p,q}^z - \sum_{h=0}^{m-1} \frac{z^h}{[h]_{p,q}!}\right)^u}{[u]_{p,q}!} = \sum_{r=0}^{\infty} S_{p,q}^{[m-1]}(r; u; \gamma) \frac{z^r}{[r]_{p,q}!},$$

when $q \rightarrow p = 1$, we obtain

$$\frac{\left(\gamma e^z - \sum_{h=0}^{m-1} \frac{z^h}{h!}\right)^u}{u!} = \sum_{r=0}^{\infty} S^{[m-1]}(r; u; \gamma) \frac{z^r}{r!}. \quad (2.2)$$

Note that, when $m = \gamma = 1$, the above expression reduces to Eq (2.1).

The generating function of the two-variable Fubini polynomials is given by (see [13]):

$$\frac{e^{xz}}{1 - y(e^z - 1)} = \sum_{r=0}^{\infty} F_r(x; y) \frac{z^r}{r!}.$$

Note that $F_r(0; y) := F_r(y)$ and $F_r(0; 1) := F_r$.

The Bernstein polynomials $B_{s,r}(x)$, of degree r , are defined by employing the following generating function (see [1, 3]):

$$\frac{(zx)^s}{s!} e^{(1-x)z} = \sum_{r=s}^{\infty} B_{s,r}(x) \frac{z^r}{r!}, \quad s \in \mathbb{N}_0, \quad (2.3)$$

where

$$B_{s,r}(x) = \binom{r}{s} x^s (1-x)^{r-s}.$$

For mathematical convention, we usually set $B_{s,r}(x) = 0$ if $s > r$.

The generating function of the ordinary Hermite polynomials is defined by (see [10]):

$$e^{2xz-z^2} = \sum_{r=0}^{\infty} H_r(x) \frac{z^r}{r!},$$

so that

$$H_r(x) = r! \sum_{s=0}^{\lfloor \frac{r}{2} \rfloor} \frac{(-1)^s (2x)^{r-2s}}{(r-2s)! s!},$$

where $\lfloor \frac{r}{2} \rfloor$ is the truncated part of $\frac{r}{2}$.

Let $\gamma \in \mathbb{C}$, $\kappa \in \mathbb{N}_0$, $a, c \in \mathbb{R}^+$ the generalized Apostol–Euler $\mathfrak{E}_r^{[m-1, \kappa]}(x; c, a; \gamma)$ polynomials of order κ , are defined respectively (cf. [5, 7, 15]):

$$\sum_{r=0}^{\infty} \mathfrak{E}_r^{[m-1, \kappa]}(x; c, a; \gamma) \frac{z^r}{r!} = \left(\frac{2^m}{\gamma c^z + \sum_{h=0}^{m-1} \frac{(z \ln a)^h}{h!}} \right)^\kappa c^{xz}. \quad (2.4)$$

When $c = a = e$, we arrive at the following:

$$\mathfrak{E}_r^{[m-1, \kappa]}(x; e, e; \gamma) := \mathfrak{E}_r^{[m-1, \kappa]}(x; \gamma).$$

Recently, Quintana et al. [9] introduced some properties and recurrence formula of generalized Euler polynomials $E_r^{[m-1]}(x)$ of level m . Also, they provided the following expression:

$$e^z + \sum_{h=0}^{m-1} \frac{z^h}{h!} = \sum_{n=0}^{\infty} (1 + a_{r,m}) \frac{z^r}{r!},$$

where

$$a_{r,m} = \begin{cases} 1, & \text{if } 0 \leq r < m, \\ 0, & \text{if } r \geq m. \end{cases}$$

Motivated by these papers, we define the generalized Apostol-type Frobenius–Euler polynomials of order κ and level m .

3. The generalized Apostol-type Frobenius–Euler polynomials and their properties

Definition 3.1. For a fixed $m \in \mathbb{N}$, $\kappa, r \in \mathbb{N}_0$, $\gamma \in \mathbb{C}$, $u \in \mathbb{C} \setminus \{1\}$, the generalized Apostol-type Frobenius–Euler polynomials of order κ and level m are defined by the following generating function, in a suitable neighborhood of $z = 0$:

$$\left(\frac{(1-u)^m}{\gamma e^z - u \sum_{l=0}^{m-1} \frac{z^l}{l!}} \right)^\kappa e^{xz} = \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1, \kappa]}(x; u; \gamma) \frac{z^r}{r!}. \quad (3.1)$$

Upon setting $x = 0$ in (3.1), we have

$${}_F\mathcal{E}_r^{[m-1, \kappa]}(0; u; \gamma) := {}_F\mathcal{E}_r^{[m-1, \kappa]}(u; \gamma),$$

called the generalized Apostol-type Frobenius–Euler numbers of order κ and level m .

Taking $u = -1$ in (3.1), we obtain the generalized Apostol–Euler polynomials $\mathfrak{E}_r^{[m-1, \kappa]}(x; c, a; \gamma)$; when $c = a = e$. Compare [15, p. 55],

$${}_F\mathcal{E}_r^{[m-1, \kappa]}(x; -1; \gamma) = \mathfrak{E}_r^{[m-1, \kappa]}(x; e, e; \gamma) := \mathfrak{E}_r^{[m-1, \kappa]}(x; \gamma).$$

According to Definition 3.1, we remark that

$$\begin{aligned} {}_F\mathcal{E}_r^{[m-1, 1]}(x; u; \gamma) &:= {}_F\mathcal{E}_r^{[m-1]}(x; u; \gamma), \\ {}_F\mathcal{E}_r^{[0, \kappa]}(x; u; \gamma) &:= {}_F\mathcal{E}_r^{(\kappa)}(x; u; \gamma) := \mathcal{H}_r^{(\kappa)}(x; u; \gamma), \\ {}_F\mathcal{E}_r^{[0, 1]}(x; u; \gamma) &:= {}_F\mathcal{E}_r(x; u; \gamma) := \mathcal{H}_r(x; u; \gamma). \end{aligned}$$

For example, the first four generalized Apostol-type Frobenius–Euler polynomials of order κ and level $m = 3$ are:

$$\begin{aligned} {}_F\mathcal{E}_0^{[2, \kappa]}(x; u; \gamma) &= \frac{(1-u)^{3\kappa}}{(\gamma-u)^\kappa}, \\ {}_F\mathcal{E}_1^{[2, \kappa]}(x; u; \gamma) &= \frac{(x-\kappa)(1-u)^{3\kappa}}{(\gamma-u)^\kappa}, \\ {}_F\mathcal{E}_2^{[2, \kappa]}(x; u; \gamma) &= \frac{(x-\kappa)^2(1-u)^{3\kappa}}{(\gamma-u)^\kappa}, \\ {}_F\mathcal{E}_3^{[2, \kappa]}(x; u; \gamma) &= \frac{\left[(u\kappa^3 - \gamma\kappa^3 - u\kappa) + (3\kappa^2\gamma - 3\kappa^2u)x + (3\kappa u - 3\kappa\gamma)x^2 + (\gamma - u)x^3 \right] (1-u)^{3\kappa}}{(\gamma-u)^{\kappa+1}}. \end{aligned}$$

Upon setting $\kappa = \gamma = 1$ and $u = -1$ above, we obtain the first four generalized Euler polynomials of level $m = 3$ (see [9, p. 47]).

Performing some manipulations on the generating function (3.1), we have

$$\left(\frac{(1-u)^m}{\gamma e^z - u \sum_{l=0}^{m-1} \frac{z^l}{l!}} \right)^\kappa e^{xz} = \left(\frac{(1-u)^m}{2^m(-u)} \right)^\kappa \left(\frac{2^m}{\left(\frac{-\gamma}{u}\right) e^z + \sum_{l=0}^{m-1} \frac{z^l}{l!}} \right)^\kappa e^{xz},$$

and thus,

$${}_F\mathcal{E}_r^{[m-1,\kappa]}(x; u; \gamma) = \left(\frac{(1-u)^m}{2^m(-u)} \right)^\kappa \mathfrak{E}_r^{[m-1,\kappa]} \left(x, \frac{-\gamma}{u} \right).$$

The proposition 3.1 provides some properties of the generalized Apostol-type Frobenius–Euler polynomials ${}_F\mathcal{E}_r^{[m-1,\kappa]}(x; u; \gamma)$ without proofs since they can easily be proved through Definition 3.1.

Proposition 3.1. For a fixed $m \in \mathbb{N}$, let $\left\{ {}_F\mathcal{E}_r^{[m-1,\kappa]}(x; u; \gamma) \right\}_{r=0}^\infty$ be the sequence of the generalized Apostol-type Frobenius–Euler polynomials, of order κ and level m . Then the following identities hold true:

(1) *Special value.* For every $r \in \mathbb{N}_0$

$${}_F\mathcal{E}_r^{[m-1,0]}(x; u; \gamma) = x^r. \quad (3.2)$$

(2) *Summation formula.* For every $r \in \mathbb{N}_0$

$${}_F\mathcal{E}_r^{[m-1,\kappa]}(x; u; \gamma) = \sum_{s=0}^r \binom{r}{s} {}_F\mathcal{E}_{r-s}^{[m-1,\kappa]}(u; \gamma) x^s.$$

(3) *Differential relation.* For a fixed $m \in \mathbb{N}$, $\kappa, \gamma \in \mathbb{C}$ and $r, j \in \mathbb{N}_0$ with $0 \leq j \leq r$, we have

$$\begin{aligned} D_x {}_F\mathcal{E}_{r+1}^{[m-1,\kappa]}(x; u; \gamma) &= (r+1) {}_F\mathcal{E}_r^{[m-1,\kappa]}(x; u; \gamma), \\ D_x^{(j)} {}_F\mathcal{E}_r^{[m-1,\kappa]}(x; u; \gamma) &= \frac{r!}{(r-j)!} {}_F\mathcal{E}_{r-j}^{[m-1,\kappa]}(x; u; \gamma). \end{aligned} \quad (3.3)$$

(4) *Integral formula.* For a fixed $m \in \mathbb{N}$, $\kappa, \gamma \in \mathbb{C}$, we have

$$\int_{x_0}^{x_1} {}_F\mathcal{E}_r^{[m-1,\kappa]}(x; u; \gamma) dx = \frac{{}_F\mathcal{E}_{r+1}^{[m-1,\kappa]}(x_1; u; \gamma) - {}_F\mathcal{E}_{r+1}^{[m-1,\kappa]}(x_0; u; \gamma)}{(r+1)}.$$

(5) *Addition theorems.*

$${}_F\mathcal{E}_r^{[m-1,\kappa]}(x+y; u; \gamma) = \sum_{s=0}^r \binom{r}{s} {}_F\mathcal{E}_{r-s}^{[m-1,\kappa]}(x; u; \gamma) y^s, \quad (3.4)$$

$${}_F\mathcal{E}_r^{[m-1,\kappa]}(x+y; u; \gamma) = \sum_{s=0}^r \binom{r}{s} {}_F\mathcal{E}_s^{[m-1,\kappa]}(u; \gamma) (x+y)^{r-s}.$$

Setting $y = 1$ in (3.4), we have

$${}_F\mathcal{E}_r^{[m-1,\kappa]}(x+1; u; \gamma) = \sum_{s=0}^r \binom{r}{s} {}_F\mathcal{E}_{r-s}^{[m-1,\kappa]}(x; u; \gamma).$$

(6) Addition theorems of the argument.

$${}_F\mathcal{E}_r^{[m-1, \kappa \pm \beta]}(x + y; u; \gamma) = \sum_{s=0}^r \binom{r}{s} {}_F\mathcal{E}_{r-s}^{[m-1, \kappa]}(x; u; \gamma) {}_F\mathcal{E}_s^{[m-1, \pm \beta]}(y; u; \gamma).$$

(7) We have

$$\mathfrak{E}_r^{[m-1]}(x; \gamma) = \frac{2^m \gamma^{m-1}}{(\gamma + 1)^m} {}_F\mathcal{E}_r^{[m-1]}(x; -\gamma^{-1}; 1).$$

Proposition 3.2. The generalized Apostol-type Frobenius–Euler polynomials satisfy the following inversion formula:

$$x^r = \frac{1}{(1-u)^m} \sum_{s=0}^r \binom{r}{s} (\gamma - u a_{s,m}) {}_F\mathcal{E}_{r-s}^{[m-1]}(x; u; \gamma), \quad (3.5)$$

where

$$a_{s,m} = \begin{cases} 1, & \text{if } 0 \leq s < m, \\ 0, & \text{if } s \geq m. \end{cases}$$

Proof. Setting $\kappa = 1$ in (3.1), we have

$$\begin{aligned} (1-u)^m e^{xz} &= \left(\gamma e^z - u \sum_{l=0}^{m-1} \frac{z^l}{l!} \right) \left(\sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1]}(x; u; \gamma) \frac{z^r}{r!} \right) \\ &= \left(\sum_{r=0}^{\infty} (\gamma - u a_{r,m}) \frac{z^r}{r!} \right) \left(\sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1]}(x; u; \gamma) \frac{z^r}{r!} \right), \end{aligned}$$

therefore,

$$(1-u)^m \sum_{r=0}^{\infty} x^r \frac{z^r}{r!} = \sum_{r=0}^{\infty} \sum_{s=0}^r \binom{r}{s} (\gamma - u a_{s,m}) {}_F\mathcal{E}_{r-s}^{[m-1]}(x; u; \gamma) \frac{z^r}{r!}.$$

By comparing the coefficients of $\frac{z^r}{r!}$ on both sides, we obtain the result. \square

Proposition 3.3. The generalized Apostol-type Frobenius–Euler polynomials satisfy the following relations:

$$\gamma {}_F\mathcal{E}_r^{[m-1, \kappa]}(x + 1; u; \gamma) - u \sum_{s=0}^{\min(r, m-1)} \binom{r}{s} {}_F\mathcal{E}_{r-s}^{[m-1, \kappa]}(x; u; \gamma) = (1-u)^m {}_F\mathcal{E}_r^{[m-1, \kappa-1]}(x; u; \gamma), \quad (3.6)$$

$$\gamma {}_F\mathcal{E}_r^{[m-1, \kappa]}(x; u; \gamma) - u \sum_{s=0}^{\min(r, m-1)} \binom{r}{s} {}_F\mathcal{E}_{r-s}^{[m-1, \kappa]}(x - 1; u; \gamma) = (1-u)^m {}_F\mathcal{E}_r^{[m-1, \kappa-1]}(x - 1; u; \gamma). \quad (3.7)$$

Upon setting $\kappa = 1$, in (3.6), we give the following corollary.

Corollary 3.1.

$$x^r = \frac{1}{(1-u)^m} \left[\gamma \sum_{s=0}^r \binom{r}{s} {}_F\mathcal{E}_k^{[m-1]}(x; u; \gamma) - u \sum_{s=0}^{\min(r, m-1)} \binom{r}{s} {}_F\mathcal{E}_{r-s}^{[m-1]}(x; u; \gamma) \right].$$

Proof. (3.6). We have

$$\begin{aligned} & \sum_{r=0}^{\infty} \left(\gamma {}_F\mathcal{E}_r^{[m-1, \kappa]}(x+1; u; \gamma) - u \sum_{s=0}^{\min(r, m-1)} \binom{r}{s} {}_F\mathcal{E}_{r-s}^{[m-1, \kappa]}(x; u; \gamma) \right) \frac{z^r}{r!} \\ &= \gamma \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1, \kappa]}(x+1; u; \gamma) \frac{z^r}{r!} - u \sum_{r=0}^{\infty} \sum_{s=0}^{\min(r, m-1)} \binom{r}{s} {}_F\mathcal{E}_{r-s}^{[m-1, \kappa]}(x; u; \gamma) \frac{z^r}{r!} \\ &= \gamma \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1, \kappa]}(x+1; u; \gamma) \frac{z^r}{r!} - u \sum_{l=0}^{m-1} \frac{z^l}{l!} \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1, \kappa]}(x; u; \gamma) \frac{z^r}{r!}. \end{aligned}$$

Now, using (3.1), we obtain

$$\begin{aligned} & \gamma \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1, \kappa]}(x+1; u; \gamma) \frac{z^r}{r!} - u \sum_{l=0}^{m-1} \frac{z^l}{l!} \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1, \kappa]}(x; u; \gamma) \frac{z^r}{r!} \\ &= \gamma \left(\frac{(1-u)^m}{\gamma e^z - u \sum_{l=0}^{m-1} \frac{z^l}{l!}} \right)^{\kappa} e^{(x+1)z} - u \sum_{l=0}^{m-1} \frac{z^l}{l!} \left(\frac{(1-u)^m}{\gamma e^z - u \sum_{l=0}^{m-1} \frac{z^l}{l!}} \right)^{\kappa} e^{xz} \\ &= \left(\frac{(1-u)^m}{\gamma e^z - u \sum_{l=0}^{m-1} \frac{z^l}{l!}} \right)^{\kappa} e^{xz} \left(\gamma e^z - u \sum_{l=0}^{m-1} \frac{z^l}{l!} \right) = (1-u)^m \left(\frac{(1-u)^m}{\gamma e^z - u \sum_{l=0}^{m-1} \frac{z^l}{l!}} \right)^{\kappa-1} e^{xz}, \end{aligned}$$

thus

$$(1-u)^m \left(\frac{(1-u)^m}{\gamma e^z - u \sum_{l=0}^{m-1} \frac{z^l}{l!}} \right)^{\kappa-1} e^{xz} = (1-u)^m \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1, \kappa-1]}(x; u; \gamma) \frac{z^r}{r!}.$$

Comparing the coefficients of $\frac{z^r}{r!}$, we obtain (3.6). □

Proof. For the proof of (3.7), a similar scheme to the previous one is used. □

Theorem 3.1. *The generalized Apostol-type Frobenius–Euler polynomials satisfy the following relation, with $u \neq 1$:*

$$\frac{\gamma {}_F\mathcal{E}_r^{[m-1, \kappa]}(x+1; u; \gamma) - u {}_F\mathcal{E}_r^{[m-1, \kappa]}(x; u; \gamma)}{(1-u)} = \sum_{s=0}^r \binom{r}{s} {}_F\mathcal{E}_k^{[m-1, \kappa]}(x; u; \gamma) {}_F\mathcal{E}_{r-s}^{(-1)}(u; \gamma).$$

Proof.

$$\begin{aligned}
 & \sum_{r=0}^{\infty} \left(\gamma {}_F\mathcal{E}_r^{[m-1,\kappa]}(x+1; u; \gamma) - u {}_F\mathcal{E}_r^{[m-1,\kappa]}(x; u; \gamma) \right) \frac{z^r}{r!} \\
 &= \gamma e^z \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1,\kappa]}(x; u; \gamma) \frac{z^r}{r!} - u \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1,\kappa]}(x; u; \gamma) \frac{z^r}{r!} \\
 &= \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1,\kappa]}(x; u; \gamma) \frac{z^r}{r!} (\gamma e^z - u) \\
 &= (1-u) \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1,\kappa]}(x; u; \gamma) \frac{z^r}{r!} \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[0,-1]}(u; \gamma) \frac{z^r}{r!}.
 \end{aligned}$$

Applying the Cauchy product in the above equation and comparing the coefficients of $\frac{z^r}{r!}$ on both sides, we obtain the proof. \square

Theorem 3.2. *The generalized Apostol-type Frobenius–Euler polynomials satisfy the following identity:*

$$\gamma {}_F\mathcal{E}_r^{[m-1]}(x+1; u; \gamma) - u \sum_{s=0}^{\min(r,m-1)} \binom{r}{s} {}_F\mathcal{E}_{r-s}^{[m-1]}(x; u; \gamma) = \sum_{s=0}^r \binom{r}{s} (\gamma - u a_{s,m}) \times {}_F\mathcal{E}_{r-s}^{[m-1]}(x; u; \gamma),$$

where

$$a_{s,m} = \begin{cases} 1, & \text{if } 0 \leq s < m, \\ 0, & \text{if } s \geq m. \end{cases}$$

Proof. To prove Theorem 3.2, it is enough to use Definition 3.1 and perform the necessary mathematical calculations, which lead directly to the desired result. \square

Theorem 3.3. *The following implicit summation formula for the generalized Apostol-type Frobenius–Euler polynomials holds true:*

$${}_F\mathcal{E}_{s+l}^{[m-1,\kappa]}(w+y; u; \gamma) = \sum_{r,n=0}^{s,l} \binom{s}{r} \binom{l}{n} (w-x)^{r+n} {}_F\mathcal{E}_{s+l-r-n}^{[m-1,\kappa]}(x+y; u; \gamma). \quad (3.8)$$

Proof. From (3.1), we have

$$\left(\frac{(1-u)^m}{\gamma e^z - u \sum_{l=0}^{m-1} \frac{z^l}{l!}} \right)^{\kappa} e^{(x+y)z} = \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1,\kappa]}(x+y; u; \gamma) \frac{z^r}{r!},$$

substituting z by $z + a$ in the above equation, we obtain

$$\left(\frac{(1-u)^m}{\gamma e^{(z+a)} - u \sum_{l=0}^{m-1} \frac{(z+a)^l}{l!}} \right)^{\kappa} e^{x(z+a)} e^{y(z+a)} = \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1, \kappa]}(x+y; u; \gamma) \frac{(z+a)^r}{r!}.$$

Now, using the following formula [20, p. 52]:

$$\sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{s,l} f(s+l) \frac{x^s y^l}{s! l!}, \quad (3.9)$$

we obtain

$$\left(\frac{(1-u)^m}{\gamma e^{(z+a)} - u \sum_{l=0}^{m-1} \frac{(z+a)^l}{l!}} \right)^{\kappa} e^{y(z+a)} = e^{-x(z+a)} \sum_{s,l=0}^{\infty} {}_F\mathcal{E}_{s+l}^{[m-1, \kappa]}(x+y; u; \gamma) \frac{z^s a^l}{s! l!}.$$

Replacing x by w in the above equation and equating the resultant equation to the above equation, we obtain

$$\begin{aligned} e^{(w-x)(z+a)} \sum_{s,l=0}^{\infty} {}_F\mathcal{E}_{s+l}^{[m-1, \kappa]}(x+y; u; \gamma) \frac{z^s a^l}{s! l!} &= \sum_{s,l=0}^{\infty} {}_F\mathcal{E}_{s+l}^{[m-1, \kappa]}(w+y; u; \gamma) \frac{z^s a^l}{s! l!}, \\ \sum_{N=0}^{\infty} (w-x)^N \frac{(z+a)^N}{N!} \sum_{s,l=0}^{\infty} {}_F\mathcal{E}_{s+l}^{[m-1, \kappa]}(x+y; u; \gamma) \frac{z^s a^l}{s! l!} &= \sum_{s,l=0}^{\infty} {}_F\mathcal{E}_{s+l}^{[m-1, \kappa]}(w+y; u; \gamma) \frac{z^s a^l}{s! l!}. \end{aligned} \quad (3.10)$$

Recalling (3.9), the left-hand side of (3.10) becomes

$$\begin{aligned} &\sum_{N=0}^{\infty} (w-x)^N \frac{(z+a)^N}{N!} \sum_{s,l=0}^{\infty} {}_F\mathcal{E}_{s+l}^{[m-1, \kappa]}(x+y; u; \gamma) \frac{z^s a^l}{s! l!} \\ &= \sum_{r,n=0}^{\infty} (w-x)^{r+n} \frac{z^r a^n}{r! n!} \sum_{s,l=0}^{\infty} {}_F\mathcal{E}_{s+l}^{[m-1, \kappa]}(x+y; u; \gamma) \frac{z^s a^l}{s! l!} \\ &= \sum_{s,l=0}^{\infty} \sum_{r,n=0}^{s,l} \frac{(w-x)^{r+n}}{r! n!} {}_F\mathcal{E}_{s-r+l-n}^{[m-1, \kappa]}(x+y; u; \gamma) \frac{z^s a^l}{(s-r)!(l-n)!} \\ &= \sum_{s,l=0}^{\infty} \sum_{r,n=0}^{s,l} \binom{s}{r} \binom{l}{n} (w-x)^{r+n} {}_F\mathcal{E}_{s+l-r-n}^{[m-1, \kappa]}(x+y; u; \gamma) \frac{z^s a^l}{s! l!}. \end{aligned}$$

Comparing coefficients, we get the assertion (3.8). \square

Theorem 3.4. *The following relation for the generalized Apostol-type Frobenius–Euler polynomials holds true:*

$$\begin{aligned} & (2u - 1) \sum_{j=0}^r \sum_{s=0}^{\min(j, m-1)} \binom{r}{j} \binom{j}{s} {}_F\mathcal{E}_{r-j}^{[m-1]}(u; \gamma) {}_F\mathcal{E}_{j-s}^{[m-1]}(x; 1 - u; \gamma) \\ &= u^m {}_F\mathcal{E}_r^{[m-1]}(x; u; \gamma) - (1 - u)^m {}_F\mathcal{E}_r^{[m-1]}(x; 1 - u; \gamma). \end{aligned} \quad (3.11)$$

Proof. We set

$$\frac{(2u - 1)(1 - u)^m u^m e^{xz} \sum_{l=0}^{m-1} \frac{z^l}{l!}}{(\gamma e^z - u \sum_{l=0}^{m-1} \frac{z^l}{l!})(\gamma e^z - (1 - u) \sum_{l=0}^{m-1} \frac{z^l}{l!})} = \frac{(1 - u)^m u^m e^{xz}}{(\gamma e^z - u \sum_{l=0}^{m-1} \frac{z^l}{l!})} - \frac{(1 - u)^m u^m e^{xz}}{(\gamma e^z - (1 - u) \sum_{l=0}^{m-1} \frac{z^l}{l!})}. \quad (3.12)$$

On the left-hand side of (3.12), we deduce

$$\begin{aligned} & \frac{(2u - 1)(1 - u)^m u^m e^{xz} \sum_{l=0}^{m-1} \frac{z^l}{l!}}{(\gamma e^z - u \sum_{l=0}^{m-1} \frac{z^l}{l!})(\gamma e^z - (1 - u) \sum_{l=0}^{m-1} \frac{z^l}{l!})} \\ &= (2u - 1) \frac{(1 - u)^m \sum_{l=0}^{m-1} \frac{z^l}{l!} (1 - (1 - u))^m e^{xz}}{(\gamma e^z - u \sum_{l=0}^{m-1} \frac{z^l}{l!})(\gamma e^z - (1 - u) \sum_{l=0}^{m-1} \frac{z^l}{l!})} \\ &= (2u - 1) \sum_{r=0}^{\infty} \sum_{j=0}^r \binom{r}{j} {}_F\mathcal{E}_{r-j}^{[m-1]}(u; \gamma) \sum_{s=0}^{\min(j, m-1)} \binom{j}{s} {}_F\mathcal{E}_{j-s}^{[m-1]}(x; 1 - u; \gamma) \frac{z^r}{r!}. \end{aligned}$$

On the right-hand side of (3.12), we obtain

$$\begin{aligned} & \frac{(1 - u)^m u^m e^{xz}}{(\gamma e^z - u \sum_{l=0}^{m-1} \frac{z^l}{l!})} - \frac{(1 - u)^m u^m e^{xz}}{(\gamma e^z - (1 - u) \sum_{l=0}^{m-1} \frac{z^l}{l!})} \\ &= u^m \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1]}(x; u; \gamma) \frac{z^r}{r!} - (1 - u)^m \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1]}(x; 1 - u; \gamma) \frac{z^r}{r!} \\ &= \sum_{r=0}^{\infty} (u^m {}_F\mathcal{E}_r^{[m-1]}(x; u; \gamma) - (1 - u)^m {}_F\mathcal{E}_r^{[m-1]}(x; 1 - u; \gamma)) \frac{z^r}{r!}. \end{aligned}$$

Comparing coefficients yields (3.11). □

Theorem 3.5. *For the generalized Apostol-type Frobenius–Euler polynomials, we have the following recurrence relation:*

$$\begin{aligned} {}_F\mathcal{E}_{r+1}^{[m-1, \kappa]}(x; u; \gamma) &= (x - \kappa) {}_F\mathcal{E}_r^{[m-1, \kappa]}(x; u; \gamma) - \frac{ukr!}{(1 - u)^m (m - 1)! (r + 1 - m)!} \\ &\quad \times {}_F\mathcal{E}_{r+1-m}^{[m-1, \kappa+1]}(x; u; \gamma). \end{aligned} \quad (3.13)$$

Upon setting $m = \gamma = 1$ and $u = -1$ in (3.13), we obtain:

$$E_{r+1}^{(\kappa)}(x) = (x - \kappa) E_r^{(\kappa)}(x) + \frac{\kappa}{2} E_r^{(\kappa+1)}(x),$$

where $E_{r+1}^{(\kappa)}(x)$, is the Euler polynomial of order κ and degree $r + 1$ (see [11, p. 3258]).

Proof. Let us consider the following generating function

$$G^{[m-1,\kappa]}(x; z; u; \gamma) = \left(\frac{(1-u)^m}{\gamma e^z - u \sum_{h=0}^{m-1} \frac{z^h}{h!}} \right)^\kappa e^{xz}.$$

Then, the differentiation of both sides of the above equation

$$\begin{aligned} \frac{\partial}{\partial z} G^{[m-1,\kappa]}(x; z; u; \gamma) &= \kappa \left(\frac{(1-u)^m}{\gamma e^z - u \sum_{k=0}^{m-1} \frac{z^k}{k!}} \right)^{\kappa-1} \left[\frac{-(1-u)^m (\gamma e^z - u \sum_{k=0}^{m-2} \frac{z^k}{k!})}{(\gamma e^z - u \sum_{h=0}^{m-1} \frac{z^h}{h!})^2} \right] e^{xz} \\ &\quad + x \left(\frac{(1-u)^m}{\gamma e^z - u \sum_{h=0}^{m-1} \frac{z^h}{h!}} \right)^\kappa e^{xz} \\ &= (x - \kappa) G^{[m-1,\kappa]}(x; z; u; \gamma) - \frac{u \kappa z^m G^{[m-1,\kappa]}(x; z; u; \gamma) G^{[m-1]}(0; z; u; \gamma)}{(1-u)^m (m-1)! z}, \end{aligned}$$

and consequently

$$\begin{aligned} &(1-u)^m (m-1)! z \frac{\partial}{\partial z} G^{[m-1,\kappa]}(x; z; u; \gamma) \\ &= (1-u)^m (m-1)! z (x - \kappa) G^{[m-1,\kappa]}(x; z; u; \gamma) - u \kappa z^m G^{[m-1,\kappa]}(x; z; u; \gamma) G^{[m-1]}(0; z; u; \gamma). \end{aligned} \quad (3.14)$$

Now, differentiating with respect to z on the right-hand side of (3.1) and using (3.14), we obtain

$$(1-u)^m (m-1)! z \frac{\partial}{\partial z} G^{[m-1,\kappa]}(x; z; u; \gamma) = (1-u)^m (m-1)! \sum_{r=0}^{\infty} r {}_F \mathcal{E}_r^{[m-1,\kappa]}(x; u; \gamma) \frac{z^r}{r!}. \quad (3.15)$$

On the other hand, we have

$$\begin{aligned} &(1-u)^m (m-1)! z (x - \kappa) G^{[m-1,\kappa]}(x; z; u; \gamma) \\ &= (1-u)^m (m-1)! (x - \kappa) \times \sum_{r=0}^{\infty} r {}_F \mathcal{E}_{r-1}^{[m-1,\kappa]}(x; u; \gamma) \frac{z^r}{r!} \end{aligned} \quad (3.16)$$

and

$$\begin{aligned} &u \kappa z^m G^{[m-1,\kappa]}(x; z; u; \gamma) G^{[m-1]}(0; z; u; \gamma) \\ &= u \kappa \sum_{r=m}^{\infty} \frac{r!}{(r-m)!} \sum_{s=0}^{r-m} \binom{r-m}{s} {}_F \mathcal{E}_{r-m-s}^{[m-1]}(u; \gamma) {}_F \mathcal{E}_s^{[m-1,\kappa]}(x; u; \gamma) \frac{z^r}{r!}. \end{aligned} \quad (3.17)$$

Substitution of (3.15)–(3.17) into (3.14) and equating coefficients of power series on both sides, the proof is complete. \square

Theorem 3.6. For the generalized Apostol-type Frobenius–Euler polynomials, we have the following differential equation:

$$\left[(x - \kappa) D_x - \frac{u \kappa}{(1-u)^m (m-1)!} \sum_{s=0}^{r+1-m} \frac{{}_F \mathcal{E}_{r+1-m-s}^{[m-1]}(u; \gamma)}{(r+1-m-s)!} D_x^{r-s+1} - r \right] \times {}_F \mathcal{E}_r^{[m-1,\kappa]}(x; u; \gamma) = 0. \quad (3.18)$$

Proof. Applying the factorization method (see [12, 16]) and using (3.3), we can rewrite (3.13) as follows:

$$\begin{aligned}
 & {}_F\mathcal{E}_{r+1}^{[m-1,\kappa]}(x; u; \gamma) \\
 &= \left[(x - \kappa) - \frac{u\kappa}{(1-u)^m(m-1)!} \sum_{s=0}^{r+1-m} \frac{{}_F\mathcal{E}_{r+1-m-s}^{[m-1]}(u; \gamma)}{(r+1-m-s)!} D_x^{r-s} \right] \times {}_F\mathcal{E}_r^{[m-1,\kappa]}(x; u; \gamma). \quad (3.19)
 \end{aligned}$$

Therefore, the operator

$$L_{r,m}^+ = \left[(x - \kappa) - \frac{u\kappa}{(1-u)^m(m-1)!} \sum_{s=0}^{r+1-m} \frac{{}_F\mathcal{E}_{r+1-m-s}^{[m-1]}(u; \gamma)}{(r+1-m-s)!} D_x^{r-s} \right],$$

where

$$D_x^{r-s} := \frac{d^{r-s}}{dx^{r-s}}.$$

Now, applying the operator $L_{r+1}^- := \frac{1}{r+1} D_x$ on both sides of (3.19), we have

$$(L_{r+1}^- L_{r,m}^+) {}_F\mathcal{E}_r^{[m-1,\kappa]}(x; u; \gamma) = {}_F\mathcal{E}_r^{[m-1,\kappa]}(x; u; \gamma).$$

Finally, after some rearrangements of terms, we achieve the desired result. \square

4. Some connection formulas for the generalized Apostol-type Frobenius–Euler polynomials

In this section, we present some formulas connecting the generalized Apostol-type Frobenius–Euler polynomials with generalized Stirling numbers of the second kind, the two-variable Fubini polynomials, Bernstein polynomials, and Hermite polynomials.

Theorem 4.1. *Let $m \in \mathbb{N}$, $v, j \in \mathbb{N}_0$. The following relations are demonstrated between the generalized Frobenius with Euler polynomials of the Apostol type and the generalized Stirling numbers of the second kind:*

$$x^n = v! \left(\frac{u}{(1-u)^m} \right)^v \sum_{j=0}^r \binom{r}{j} {}_F\mathcal{E}_j^{[m-1,v]}(x; u; \gamma) S^{[m-1]} \left(r - j, v, \frac{\gamma}{u} \right), \quad (4.1)$$

$${}_F\mathcal{E}_r^{[m-1,\kappa-v]}(x; u; \gamma) = v! \left(\frac{u}{(1-u)^m} \right)^v \sum_{j=0}^r \binom{r}{j} {}_F\mathcal{E}_j^{[m-1,\kappa]}(x; u; \gamma) S^{[m-1]} \left(r - j, v, \frac{\gamma}{u} \right). \quad (4.2)$$

Proof. (4.1). Indeed,

$$\left(\frac{(1-u)^m}{\gamma e^z - u \sum_{l=0}^{m-1} \frac{z^l}{l!}} \right)^v e^{xz} = \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1,v]}(x; u; \gamma) \frac{z^r}{r!},$$

$$\begin{aligned}
e^{xz} &= \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1, \nu]}(x; u; \gamma) \frac{z^r}{r!} \left(\gamma e^z - u \sum_{l=0}^{m-1} \frac{z^l}{l!} \right)^{\nu} \left(\frac{\nu!}{\nu!(1-u)^{m\nu}} \right) \\
&= \nu! \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1, \nu]}(x; u; \gamma) \frac{z^r}{r!} \frac{\left(\frac{\gamma}{u} e^z - \sum_{l=0}^{m-1} \frac{z^l}{l!} \right)^{\nu}}{\nu!} \left(\frac{u}{(1-u)^m} \right)^{\nu}.
\end{aligned}$$

Now, using (2.2), we have

$$\begin{aligned}
\sum_{r=0}^{\infty} x^r \frac{z^r}{r!} &= \nu! \left(\frac{u}{(1-u)^m} \right)^{\nu} \sum_{r=0}^{\infty} {}_F\mathcal{E}_r^{[m-1, \nu]}(x; u; \gamma) \frac{z^r}{r!} \sum_{r=0}^{\infty} S^{[m-1]} \left(r; \nu; \frac{\gamma}{u} \right) \frac{z^r}{r!} \\
&= \nu! \left(\frac{u}{(1-u)^m} \right)^{\nu} \sum_{r=0}^{\infty} \sum_{j=0}^r \binom{r}{j} {}_F\mathcal{E}_j^{[m-1, \nu]}(x; u; \gamma) S^{[m-1]} \left(r-j; \nu; \frac{\gamma}{u} \right) \frac{z^r}{r!}.
\end{aligned}$$

Finally, comparing the coefficients, we obtain the assertion (4.1). \square

Corollary 4.1. *We have*

$${}_F\mathcal{E}_r^{[m-1, -\nu]}(x; u; \gamma) = \nu! \left(\frac{u}{(1-u)^m} \right)^{\nu} \sum_{j=0}^r \binom{r}{j} S^{[m-1]} \left(r-j, \nu, \frac{\gamma}{u} \right) x^j.$$

Proof. If we set $\kappa = 0$ in (4.2) and use Eq (3.2), we complete the proof of the corollary. \square

Proposition 4.1. *Let $m \in \mathbb{N}$ and $r \in \mathbb{N}_0$. The following relationships have been established between the generalized Apostol-type Frobenius–Euler polynomials and the two-variable Fubini polynomials, the Bernstein polynomials, the Stirling polynomials of the second kind of level m , and the Hermite polynomials:*

$$\begin{aligned}
{}_F\mathcal{E}_r^{[m-1, \kappa]}(x; u; \gamma) &= \sum_{s=0}^r \binom{r}{s} {}_F\mathcal{E}_{r-s}^{[m-1, \kappa]}(u; \gamma) [(1+w)F_s(x, w) - wF_s(x+1, w)], \\
B_{s,r}(x) &= x^s \sum_{n=0}^{r-s} \binom{r-s}{n} \binom{r}{s} {}_F\mathcal{E}_{r-s-n}^{[m-1, -\nu]}(u; \gamma) {}_F\mathcal{E}_n^{[m-1, \nu]}(1-x; u; \gamma), \\
B_{r,s}(x) &= x^s \nu! \left(\frac{u}{(1-u)^m} \right)^{\nu} \sum_{j=0}^{r-s} \binom{r-s}{j} \binom{r}{s} {}_F\mathcal{E}_{r-s-j}^{[m-1, \kappa]}(1-x; u; \gamma) S^{[m-1]} \left(j; \nu, \frac{\gamma}{u} \right), \\
{}_F\mathcal{E}_r^{[m-1, \kappa]}(x; u; \gamma) &= \sum_{s=0}^{\lfloor \frac{r}{2} \rfloor} \sum_{j=0}^{r-2s} \binom{r-2s}{j} \binom{r}{2s} \frac{(2s)!}{s!} {}_F\mathcal{E}_j^{[m-1, \kappa]}(u; \gamma) H_{r-2s-j} \left(\frac{x}{2} \right).
\end{aligned}$$

5. Apostol-type Frobenius–Euler polynomials and their applications

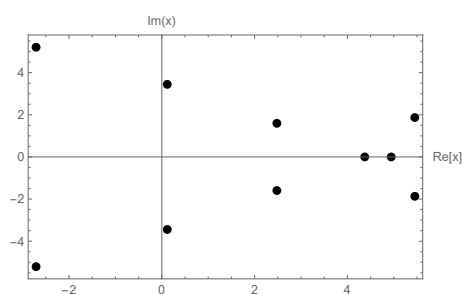
This section presents an application due to the zero distributions of this new class of generalized Apostol-type Frobenius–Euler polynomials.

To study the zero distributions using (3.1) and Mathematica Wolfram, we show the zero patterns. We show plots of zero distributions; to plot these, we give values to the coefficients of the Apostol-type Frobenius–Euler polynomials $\{m \rightarrow 3, u \rightarrow 2, \kappa \rightarrow 2, \gamma \rightarrow 4\}$, and the equation used is:

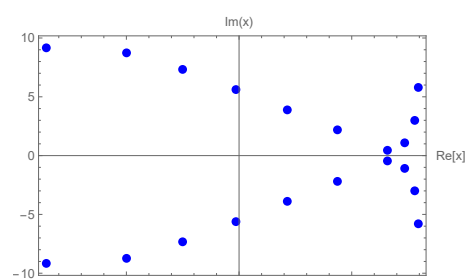
$$\frac{e^{xz}}{\left(4e^z - 2\left(1 + z + \frac{z^2}{2}\right)\right)^2}. \quad (5.1)$$

We give values for n (10, 20, 30, 50), and we have the plots.

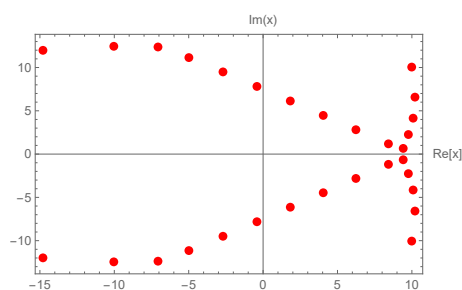
In Figure 1 the plots are organized as 1a black dots for 10 points, in 1b blue dots for 20 points, in 1c red dots for 30 points, and 1d green dots for 50 points, the shape of these plots seems like a fish. At the same time, as we increase the dots, we can see a more defined shape, and in the plot 1d, here we see two dots around $(-10, -20)$ and $(-10, 20)$.



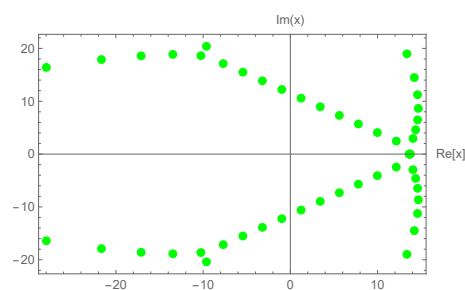
(a) Zeros of ${}_F\mathcal{E}_{10}^{[2,2]}(x; 2; 4) = 0$



(b) Zeros of ${}_F\mathcal{E}_{20}^{[2,2]}(x; 2; 4) = 0$



(c) Zeros of ${}_F\mathcal{E}_{30}^{[2,2]}(x; 2; 4) = 0$



(d) Zeros of ${}_F\mathcal{E}_{50}^{[2,2]}(x; 2; 4) = 0$

Figure 1. Roots of Apostol-type Frobenius–Euler generalized polynomials, plotted considering 10, 20, 30, and 50 points.

Another interesting application is the induced mesh of generalized Apostol-type Frobenius–Euler polynomials for different numbers of n (3.1) ${}_F\mathcal{E}_r^{[m-1, \kappa]} = 0$.

In Figure 2, we show four graphs representing the mesh distribution by z -coefficient equal to 10, 20, 30 and 50, respectively. As we increase the z -value, we notice that it forms a cone shaped. In Figure 2a, there are black dots for 10 points, and in Figure 2d green dots for 50 points, we see a more defined mesh shape.

Then, based on the Apostol-type Frobenius–Euler polynomials. We computed an approximate solution that satisfies them, ${}_F\mathcal{E}_r^{[m-1, \kappa]}$ for $n = 2, 3, \dots, 10$. The results are presented in Table 1.

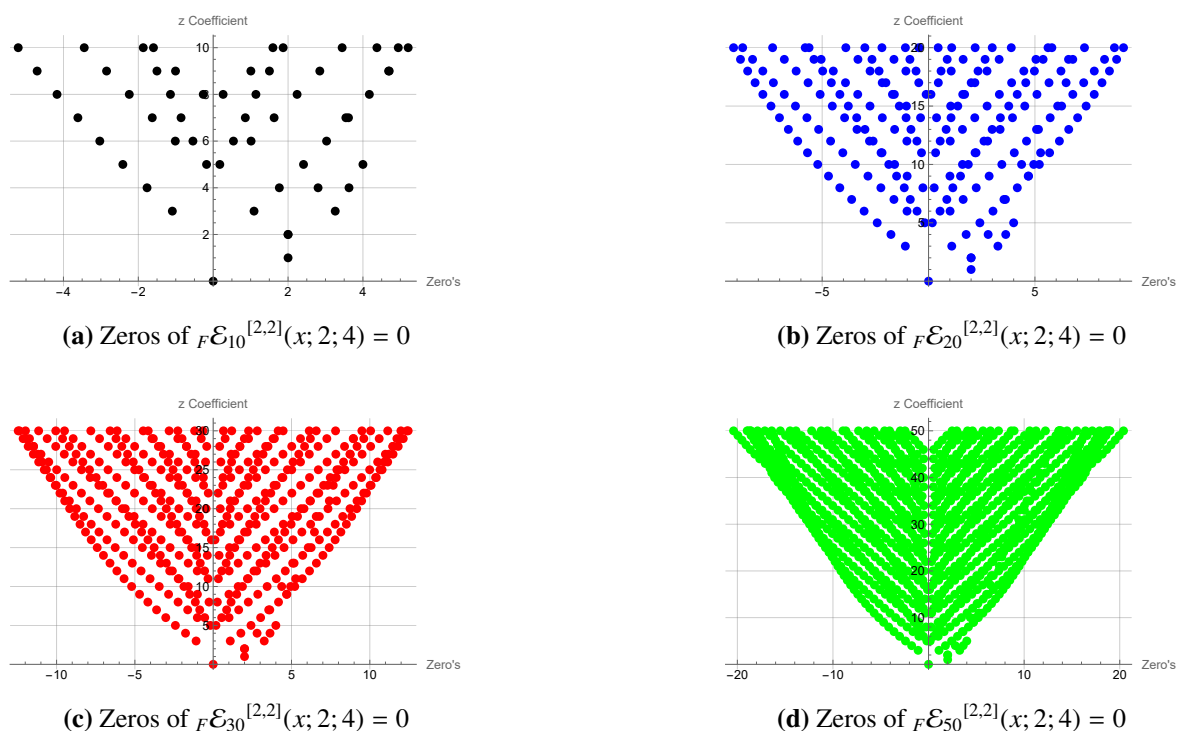


Figure 2. Mesh distribution of Apostol-type Frobenius–Euler generalized polynomials, plotted considering 10, 20, 30 and 50 points.

Table 1. Approximate solutions for ${}_F E_r^{[m-1, \kappa]} = 0$.

n	x
1	0.817219
2	0.820808 - 0.817211 i, 0.820808 + 0.817211 i
3	0.585556 - 1.48212 i, 0.585556 + 1.48212 i, 1.31145
4	0.239933 - 2.06742 i, 0.239933 + 2.06742 i, 1.42216 - 0.730256 i, 1.42216 + 0.730256 i
5	-0.181558 - 2.59823 i, -0.181558 + 2.59823 i, 1.3582 - 1.38743 i, 1.3582 + 1.38743 i, 1.79663
6	-0.762338 - 3.3425 i, -0.762338 + 3.3425 i, -0.567434, 1.92081 - 2.62112 i, 1.92081 + 2.62112 i, 3.28321
7	-1.27326 - 3.64882 i, -1.27326 + 3.64882 i, -0.567429, 1.26675 - 3.3269 i, 1.26675 + 3.3269 i, 3.28321 + 1.43895 i, 3.29972 + 1.43895 i
8	-1.73694 - 4.00473 i, -1.73694 + 4.00473 i, -0.567428, 0.608126 - 3.82671 i, 0.608126 + 3.82671 i, 2.90345 + -2.584355 i, 2.90345 + 2.58435 i, 3.91179
9	-2.23827 - 4.43454 i, -2.23827 + 4.43454 i, -0.567428, -0.0245166 - 4.13057 i, -0.0245166 + 4.13057 i, 2.34366 + -3.41153 i, 2.34366 + 3.41153 i, 3.99888 - 1.34432 i, 3.99888 + 1.34432 i
10	-2.70965 - 5.20815 i, -2.70965 + 5.20815 i, 0.117982 - 3.44461 i, 0.117982 + 3.44461 i, 2.47995 + -1.59536 i, 2.47995 + 1.59536 i, 4.37324, 4.94372, 5.45324 - 1.86857 i, 5.45324 + 1.86857 i

6. Conclusions

This paper introduces a novel category of generalized Apostol-type Frobenius with Euler polynomials and investigates their intricate mathematical properties in detail using generating function techniques. A significant contribution of this study is the establishment of their associated differential equation through the factorization method, complemented by the formulation of a recurrence relation for these polynomials. Furthermore, the study derives several correlation formulas that establish a connection between these generalized polynomials and classical special functions, including the Bernstein, Fubini, and Hermite polynomials.

Moreover, graphical representations are provided by analysing the behavior of polynomial zeros and visualising their networks for specific values of n . These graphs offer insights into the structure and behavior of the polynomials and demonstrate their practical utility in modeling and numerical computations.

The results presented in this paper establish a solid foundation for future explorations. In the field of approximation theory, these generalized polynomials have the potential to enhance the efficacy of techniques such as spectral methods, orthogonal expansions, and the approximation of complex functions. Similarly, their associations with special polynomials demonstrate fascinating avenues for the advancement of number theory, particularly concerning modular forms, integer sequences, and combinatorial identities. Finally, the factorization and recurrence approaches adopted here could inspire new developments in classical analysis and symbolic computation, rendering these polynomials a promising tool for further interdisciplinary research.

Author contributions

Letelier Castilla: Conceptualization, Data curation, Investigation, Resources, Validation, Writing original draft; William Ramírez: Conceptualization, Investigation, Methodology, Project administration, Software, Supervision, Writing original draft; Clemente Cesarano: Conceptualization, Data curation, Formal analysis, Funding, Supervision; Shahid Ahmad Wani: Conceptualization, Investigation, Project administration, Supervision, Writing the final version; Maria-Fernanda Heredia-Moyano: Conceptualization, Investigation, Methodology, Project administration, Software, Supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Prof. Clemente Cesarano is the Guest Editor of Special Issue “Special functions and related applications” for AIMS Mathematics. Prof. Clemente Cesarano was not involved in the editorial review and the decision to publish this article.

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References

1. M. Acikgoz, S. Araci, A study on the integral of the product of several type Bernstein polynomials, *IST Trans. Appl. Math.-Modell. Simul.*, **1** (2010), 10–14.
2. M. Açıkgöz, S. Araci, U. Duran, Some (p, q) -analogues of Apostol–type numbers and polynomials, *Acta Commentat. Univ. Tartuensis Math.*, **23** (2019), 37–50. <https://doi.org/10.12697/ACUTM.2019.23.04>
3. M. Açıkgöz, S. Araci, On the generating function for Bernstein polynomials, *AIP Conf. Proc.*, **1281** (2010), 1141–1143. <https://doi.org/10.1063/1.3497855>
4. L. Carlitz, Eulerian numbers and polynomials, *Math. Mag.*, **32** (1959), 247–260. <https://doi.org/10.2307/3029225>
5. C. Cesarano, W. Ramírez, S. Díaz, New results for degenerated generalized Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi Polynomials, *WSEAS Trans. Math.*, **21** (2022), 604–608. <https://doi.org/10.37394/23206.2022.21.69>
6. C. Cesarano, B. Germano, P. E. Ricci, Laguerre-type Bessel functions, *Integr. Transf. Spec. Funct.*, **16** (2005), 315–322. <https://doi.org/10.1080/10652460412331270629>
7. C. Cesarano, W. Ramírez, Some new classes of degenerated generalized Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials, *Carpathian Math. Publ.*, **14** (2022), 354–363. <https://doi.org/10.15330/cmp.14.2.354-363>
8. C. Cesarano, W. Ramírez, S. Khan, A new class of degenerate Apostol–type Hermite polynomials and applications, *Dolomites Res. Notes Approx.*, **15** (2022), 1–10.
9. C. Cesarano, Y. Quintana, W. Ramírez, Degenerate versions of hypergeometric Bernoulli–Euler polynomials, *Lobachevskii J. Math.*, **45** (2024), 3509–3521. <https://doi.org/10.1134/S1995080224604235>
10. G. Dattoli, C. Chiccoli, S. Lorenzutta, G. Maino, A. Torre, Theory of generalized Hermite polynomials, *Comput. Math. Appl.*, **28** (1994), 71–83. [https://doi.org/10.1016/0898-1221\(94\)00128-6](https://doi.org/10.1016/0898-1221(94)00128-6)
11. R. Dere, Y. Simsek, H. M. Srivastava, A unified presentation of three families of generalized Apostol type polynomials based upon the theory of the umbral calculus and the umbral algebra, *J. Number Theory*, **133** (2013), 3245–3263. <https://doi.org/10.1016/j.jnt.2013.03.004>
12. M. X. He, P. E. Ricci, Differential equation of Appell polynomials via the factorization method, *J. Comput. Appl. Math.*, **139** (2022), 231–237. [https://doi.org/10.1016/S0377-0427\(01\)00423-X](https://doi.org/10.1016/S0377-0427(01)00423-X)
13. L. Kargin, Some formulae for products of Fubini polynomials with applications, *arXiv*, 2016. <https://doi.org/10.48550/arXiv.1701.01023>
14. B. Kurt, Y. Simsek, On the generalized Apostol–type Forbenius-Euler polynomials, *Adv. Differ. Equ.*, **2013** (2013), 1. <https://doi.org/10.1186/1687-1847-2013-1>
15. B. Kurt, Some relationships between the generalized Apostol–Bernoulli and Apostol–Euler polynomials, *Turk. J. Anal. Number Theory*, **1** (2013), 54–58. <https://doi.org/10.12691/tjant-1-1-11>

16. P. Natalini, A. Bernardini, A generalization of the Bernoulli polynomials, *J. Appl. Math.*, **3** (2003), 155–163. <https://doi.org/10.1155/S1110757X03204101>
17. Y. Quintana, W. Ramírez, A. Urieles, Euler matrices and their algebraic properties revisited, *Appl. Math. Inf. Sci.*, **14** (2020), 1–14.
18. W. Ramírez, C. Kızılateş, D. Bedoya, C. Cesarano, C. S. Ryou, On certain properties of three parametric kinds of Apostol–type unified Bernoulli–Euler polynomials, *AIMS Math.*, **10** (2025), 137–158. <https://doi.org/10.3934/math.2025008>
19. H. M. Srivastava, J. Choi, *Zeta and q-zeta functions and associated series and integrals*, Elsevier, 2012. <https://doi.org/10.1016/C2010-0-67023-4>
20. H. M. Srivastava, H. L. Manocha, *A treatise on generating functions*, Halsted Press, 1984.
21. S. A. Wani, K. S. Nisar, Quasi-monomiality and convergence theorem for the Boas–Buck–Sheffer polynomials, *AIMS Math.*, **5** (2020), 4432–4443. <https://doi.org/10.3934/math.2020283>
22. B. Yasar, M. A. Ozarslan, Frobenius–Euler and Frobenius–Genocchi polynomials and their differential equations, *New Trends Math. Sci.*, **3** (2015), 172–180.



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